Lecture #6 09/19/2017

- **Last time**

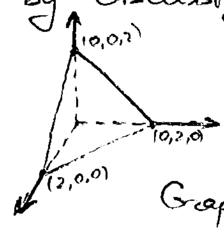
Last time we discussed functions of two variables and two ways to visualize them:
- through the graphs in $\mathbb{R}^3$
- through the level curves on $\mathbb{R}^2$ (! don't forget we put numbers over curves).

Rmk: By analogy with topographic maps of mountains, we see that the corresponding graph is steep when the level curves are close enough, while it is flatter once they are farther apart.

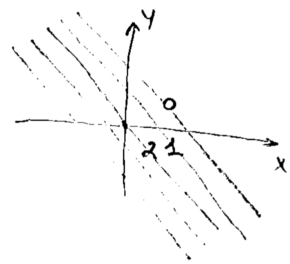
Last time we ended up by discussing 5 examples, recalled below:

(a) $f(x,y) = 2 - x - y$

(Plane)

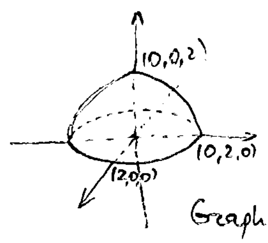


Graph

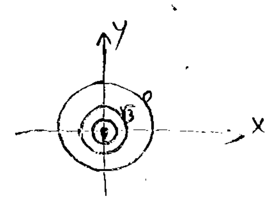

level curves

(b) $f(x,y) = \sqrt{4 - x^2 - y^2}$

(Upper half of a sphere)

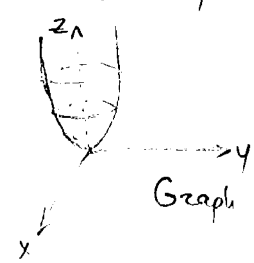


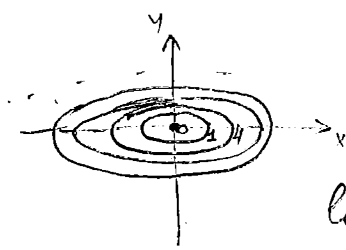
Graph


level curves

(c) $z = x^2 + 9y^2$

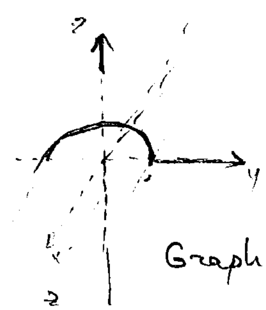
(Elliptic paraboloid)

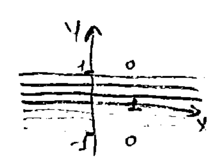

Graph


level curves

(d) $z = \sqrt{1 - y^2}$

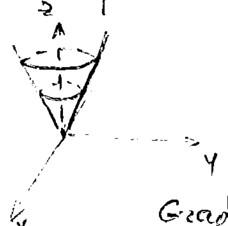
(Upper half of a cylinder)

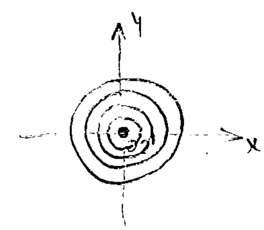

Graph


level curves

(e) $z = 2\sqrt{x^2 + y^2}$

(Cone)


Graph


level curves

Questions:

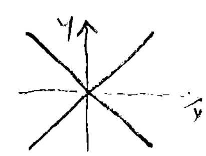
- How should we modify function $f(x,y)$ to flip upside down the corresponding graph?
  (replace $f(x,y) \rightsquigarrow -f(x,y)$)
- How should we modify function $f(x,y)$ to make it 3 times steeper?
  (replace $f(x,y) \rightsquigarrow 3f(x,y)$)
- How should we modify function $f(x,y)$ (e.g. from (b), (c) or (e)) to move the corresponding graph from the origin to another point (x_0, y_0)?
  (replace $f(x,y) \rightsquigarrow f(x-x_0, y-y_0)$)

Saddle point

One function we omitted last time as its graph is difficult to sketch is the following: $f(x,y) = x^2 - y^2$.

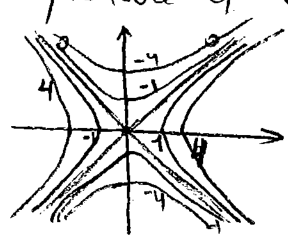
- Intersection of its graph with yz-plane is the parabola $z = -y^2$, looking "downstairs", while intersection of the same graph with xz-plane is the parabola $z = x^2$, looking "upstairs".
- Let us now examine level curves. For this we need to draw curves given by $x^2 - y^2 = k$ for each real number k.

$k = 0 \Rightarrow x^2 - y^2 = 0 \Rightarrow (x-y)(x+y) = 0 \Rightarrow x = \pm y$



$k \neq 0 \Rightarrow x^2 - y^2 = k \rightarrow (x-y)(x+y) = k \rightsquigarrow$ rotated hyperbola

So at the end the picture of level curves is the following:



Finally, the graph of this function is called a <u>hyperbolic paraboloid</u> (see wikipedia for a good picture). Its important feature is that $(0,0,0)$ on it is a saddle point: it is both a local minimum and local maximum for different directions

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• Today: Limits and Continuity

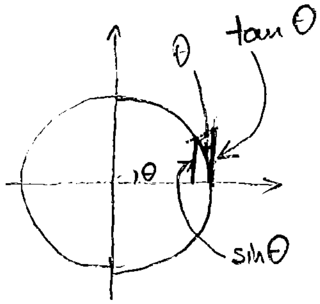
Today we will focus on functions in 2 variables, but the same reasoning can be applied in the case of >2 variables.

◦ First, recall classical limits (for functions in 1 variable).

! Note: $\lim_{x\to a} f(x)$ exists iff both limits $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ exist and coincide (in other words, approaching a from right or left, we get the same limit)

Ex1: Compute $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$.

► Well, applying the l'Hospital rule, we immediately get $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\cos\theta = 1$
But here we used derivatives, which is an overkill of the problem.
So the question is what is a straightforward proof of this result.



We obviously have
$$\sin\theta \le \theta \le \tan(\theta)$$
or to include also the case of $\theta<0$:
$$|\sin\theta| \le |\theta| \le \frac{|\sin\theta|}{|\cos\theta|}$$
$$\Downarrow$$
$$|\cos\theta| \le \frac{|\sin\theta|}{|\theta|} \le 1$$

As $\theta \to 0$, $\cos\theta \to 1$ and hence, $\frac{|\sin\theta|}{|\theta|}$ is "sandwiched" between two f-s which have limit 1 as $\theta\to 0$; therefore, $\frac{|\sin\theta|}{|\theta|}$ also has limit 1.
As $\left|\frac{\sin\theta}{\theta}\right| = \frac{\sin\theta}{\theta}$, we get a down-to-earth proof of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

When dealing with f-s of two variables, the computation of $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ is a bit more delicate as now we can approach (x_0,y_0) from different directions.

Ex2: Does the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

► First of all we check whether the f-n $f(x,y) = \frac{xy}{x^2+y^2}$ is well-defined at (0,0), but it is not as we get $\frac{0}{0}$.
Let us try to approach (0,0) along the path $y=x$: get $\frac{x^2}{2x^2} = \frac{1}{2}$
while approaching along $y=-x$ gives $\frac{-x^2}{2x^2} = -\frac{1}{2}$, and along $y=0$ gives 0

- (Continuation)

So we get different limits when approaching $(0,0)$ along different paths, and as a result the limit of $f(x,y)$ as $(x,y)\to(0,0)$ does not exist!

Ex3: Does the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist? $\left(f(x,y) := \frac{xy^2}{x^2+y^4} \right)$

- If we try to approach $(0,0)$ along the line $y=cx$, we get $\lim_{x\to 0} \frac{c^2x^3}{x^2+c^4x^4} = \lim_{x\to 0} \frac{c^2x}{1+c^4x^2} = 0$. So: the limits are all zero when approaching along the lines.

However, this is not enough to conclude an existence of limit.

Let us consider a path $x=y^2$:

$$f(x,y) = f(y^2,y) = \frac{y^4}{2y^4} = \frac{1}{2}$$

Therefore, approaching $(0,0)$ along this curve, we get a different limit.

Upshot: The limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist ∎

Ex4: Does the following limit $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exist? (set $f(x,y) := \frac{3x^2y}{x^2+y^2}$)

- If we try to approach $(0,0)$ along the line $y=cx$, we get

$$\lim_{x\to 0} \frac{3cx^3}{x^2+c^2x^2} = \lim_{x\to 0} \frac{3cx}{1+c^2} = 0 \Rightarrow \text{all these limits are zero!}$$

If we try to approach $(0,0)$ along $x=y^2$ as in Ex3, we get

$$\lim_{y\to 0} \frac{3y^5}{y^4+y^2} = \lim_{y\to 0} \frac{3y^3}{1+y^2} = 0 \Rightarrow \text{again get a zero limit!}$$

At the moment it seems like the limit may exist, but how are we gonna prove it?

Squeeze Theorem: Let $g(x,y)$, $h(x,y)$ be two functions s.t.
$g(x,y) \le f(x,y) \le h(x,y)$ for any $(x,y) \ne (0,0)$.
If $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} h(x,y) = L$, then also
$\lim_{(x,y)\to(0,0)} f(x,y) = L$.

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- (Continuation of Ex 4)

To apply this theorem, we need to find the corresponding two auxiliary functions $g(x,y)$, $h(x,y)$ satisfying the two requirements.

Observation: $0 \le \frac{x^2}{x^2+y^2} \le 1$

Hence $-3|y| \le \frac{3x^2y}{x^2+y^2} \le 3|y|$

Therefore, we can take $g(x,y) := -3|y|$, $h(x,y) := 3|y|$

As $(x,y) \to (0,0)$, obviously $|y| \to 0 \Rightarrow \lim_{(x,y)\to(0,0)} g(x,y) = 0 = \lim_{(x,y)\to(0,0)} h(x,y)$

Thus: by the squeeze thm $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ ∎

Ex 5 : Does the limit $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$ exist? (set $f(x,y) := \frac{x^4+y^4}{x^2+y^2}$)

- Let us use polar coordinates r, θ: $x = r\cos\theta$, $y = r\sin\theta$.

The fact that $(x,y) \to (0,0)$ is equivalent to $r \to 0$, while θ is irrelevant

$$f(x,y) = \frac{r^4(\cos^4\theta + \sin^4\theta)}{r^2(\cos^2\theta + \sin^2\theta)} = r^2(\cos^4\theta + \sin^4\theta)$$

But $0 \le \cos^4\theta + \sin^4\theta \le 2$

$\Rightarrow 0 \le f(x,y) \le 2r^2$

Applying the squeeze thm (with coordinates r, θ), we see that
$\lim_{(x,y)\to 0} f(x,y) = 0$ ∎

Upshot (strategy for computing limits):

- try to plug in: If you get a number, then this is your limit
  If you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then:
- try different paths, going through the given point, e.g. for computing limit at $(0,0)$ you may consider paths $y = cx$, $x = 0$, $y = x^2$, $x = y^2$, etc.
  If two limits differ, then the limit does not exist
  If the limits are the same, try to use the squeeze theorem
  - $\to$ need to find appropriate $g(x,y)$, $h(x,y)$
  - $\to$ may need to use polar coordinates.

Ex6: Does the limit $\lim_{(x,y)\to(1,0)} xe^{-y}$ exist?

The function xe^{-y} is well-defined at $(1,0)$ and is a product of continuous f-n in x and a continuous function in y. Hence, the limit exists and

$$\lim_{(x,y)\to(1,0)} xe^{-y} = 1\cdot e^{-0} = 1$$

Ex7: Does the limit $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ exist?

- Restricting to the lines $y=cx$, we get $\frac{cx^5}{x^2+c^6x^6} = \frac{cx^3}{1+c^6x^4} \xrightarrow{x\to 0} 0$
  Hence, the limits along any line (didn't check for $x=0$ - the y-axis) are ZERO
- Restricting to the curve $x=y^3$, we get $\frac{y^3\cdot y^3}{y^6+y^6} = \frac{1}{2} \xrightarrow{y\to 0} \frac{1}{2}$
  Hence, restricting to this curve, the limit is different (equals $\frac{1}{2}$)

Thus: $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does not exist.

Ex8: Does the limit $\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$ exist?

Once we see expressions with x^2+y^2, it may be a good idea to use polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$, so that $x^2+y^2=r^2$.

Note that $(x,y)\to(0,0) \iff r\to 0$ (while θ is arbitrary).

$$\frac{e^{-x^2-y^2}-1}{x^2+y^2} = \frac{e^{-r^2}-1}{r^2}$$

The q-n is whether $\lim_{r\to 0} \frac{e^{-r^2}-1}{r^2}$ exist or not?

At $r=0$, we get $\frac{0}{0}$, so we need to be more careful.

But since this is a function of one variable, we can apply l'Hospital rule

$$\lim_{r\to 0} \frac{e^{-r^2}-1}{r^2} = \lim_{r\to 0} \frac{e^{-r^2}\cdot(-2r)}{2r} = -1$$

So: $\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = -1$

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Ex 9: Does the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{\sqrt{x^2+y^2+1}-1}$ exist?

► First, we use the standard manipulation:

$$\left(\sqrt{x^2+y^2+1}-1\right)\left(\sqrt{x^2+y^2+1}+1\right) = x^2+y^2$$

$$\Downarrow$$

$$\frac{x^3+y^3}{\sqrt{x^2+y^2+1}-1} = \frac{(x^3+y^3)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2}$$

Now we can try to use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$.
Then the question boils to determining

$$\lim_{r\to 0} \frac{(r^3\cos^3\theta + r^3\sin^3\theta)(\sqrt{r^2+1}-1)}{r^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}} = \lim_{r\to 0} \underbrace{r\cdot(\sqrt{r^2+1}-1)}_{\downarrow r\to 0 \atop 0}\cdot\underbrace{(\cos^3\theta+\sin^3\theta)}_{-2\le\cos^3\theta+\sin^3\theta\le 2} = 0$$

To be more pedantic we squeeze:

$$-2r(\sqrt{r^2+1}-1) \le r(\sqrt{r^2+1}-1)(\cos^3\theta+\sin^3\theta) \le 2r(\sqrt{r^2+1}-1)$$

$r\to 0$ $\qquad$ $r\to 0$

0

$$\Rightarrow \lim_{r\to 0}\left(\begin{matrix} r(\sqrt{r^2+1}-1)\cdot \\ (\cos^3\theta+\sin^3\theta)\end{matrix}\right) = 0$$

Ex 10: Does the limit $\lim\limits_{(x,y)\to(0,0)} \underbrace{\dfrac{y^2\sin^2 x}{x^4+y^4}}_{f(x,y)}$ exist?

- Restricting to $y = x$, passing through $(0,0)$, we get

$$\left.\begin{aligned} f(x,x) &= \frac{x^2\sin^2 x}{2x^4} = \frac{1}{2}\cdot\frac{\sin^2 x}{x^2} = \frac{1}{2}\cdot\left(\frac{\sin x}{x}\right)^2 \\ \text{As recalled today } &\frac{\sin x}{x} \xrightarrow{x\to 0} 1 \end{aligned}\right\} \Rightarrow f(x,x) \xrightarrow{x\to 0} \frac{1}{2}$$

- Restricting to $x = 0$, get $f(0,y) = \dfrac{0}{y^4} = 0$ (as far as $y \neq 0$) $\xrightarrow{y\to 0} 0$

Thus, we got two different limits, hence, $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2\sin^2 x}{x^4+y^4}$ does not exist.

Ex 11: Does the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin^3 x\cdot y^4}{x^{10}+y^4}$ exist?

► Note that $0 \le \dfrac{y^4}{x^{10}+y^4} \le 1 \Rightarrow -|\sin^3 x| \le \dfrac{\sin^3 x\cdot y^4}{x^{10}+y^4} \le |\sin^3 x|$

But $\sin x \xrightarrow{x\to 0} 0 \Rightarrow -|\sin^3 x| \xrightarrow{x\to 0} 0$, $|\sin^3 x| \xrightarrow{x\to 0} 0$

$$\xRightarrow{\text{Squeeze Thm}} \lim_{(x,y)\to(0,0)} \frac{\sin^3 x\cdot y^4}{x^{10}+y^4} = 0$$