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Def: Given a function of 3 variables, $F(x,y,z)$, the tangent plane to the level surface $F(x,y,z)=k$ at the point $P(x_0,y_0,z_0)$ (note: $k=F(x_0,y_0,z_0)$) is a plane passing through P and perpendicular to the gradient vector $\nabla F(x_0,y_0,z_0)$.
Explicitly, it has the following equation:

$$\boxed{F_x(x_0,y_0,z_0)\cdot(x-x_0)+F_y(x_0,y_0,z_0)\cdot(y-y_0)+F_z(x_0,y_0,z_0)\cdot(z-z_0)=0}$$

The key property is that if we consider any curve C: $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$ on the level surface, s.t. $\vec{r}(t_0)=P_0$, then the tangent vector to this curve at t_0 lies in tangent plane.

[Indeed, we have $F(x(t),y(t),z(t))=k \leftarrow$ constant. Hence, differentiating by t and using chain rule, we get $F_x\cdot\frac{dx}{dt}+F_y\cdot\frac{dy}{dt}+F_z\cdot\frac{dz}{dt}=0$
But the left-hand side is just $\nabla F\cdot\vec{r}\,'(t) \Rightarrow \vec{r}\,'(t_0)\perp\nabla F(x_0,y_0,z_0)$]

Def: The normal line to the level surface S at P is the line through P and perpendicular to the tangent plane.
In other words, the direction of it is $\nabla F(x_0,y_0,z_0)$.
In particular, its symmetric equation is:

$$\boxed{\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}}$$

Key Example: Surface $S\subset\mathbb{R}^3$ is given by $z=f(x,y)$ (i.e. S is a graph of $f(\cdot,\cdot)$)
We can realize it as a level zero surface of the function $\boxed{F(x,y,z):=-z+f(x,y).}$ Note that $\boxed{\nabla F(x_0,y_0,z_0)=\langle f_x(x_0,y_0),f_y(x_0,y_0),-1\rangle}$

Hence: Tangent Plane is $\boxed{f_x(x_0,y_0)\cdot(x-x_0)+f_y(x_0,y_0)\cdot(y-y_0)-(z-z_0)=0}$

This is an important f-la we skipped in Sect. 14.4.

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Ex 1: Find equation of the tangent plane and the normal line to the given surface at the given point:

(a) $z^2 - 2xy + e^x = 5$, $P(0,1,2)$

(b) $z = e^{\sin(x+y^2)}$, $P(\pi,0,1)$

(a) $F(x,y,z) = e^x - 2xy + z^2 - 5 \Rightarrow F_x = e^x - 2y,\ F_y = -2x,\ F_z = 2z$

$\Rightarrow F_x(0,1,2) = -1,\ F_y(0,1,2) = 0,\ F_z(0,1,2) = 4$

Tangent plane: $-1\cdot(x-0) + 0\cdot(y-1) + 4\cdot(z-2) = 0 \Leftrightarrow \boxed{-x+4z-8=0}$

Normal line: $\boxed{x = -t,\ y = 1,\ z = 2+4t.}$

(b) $F(x,y,z) = e^{\sin(x+y^2)} - z \Rightarrow F_x = e^{\sin(x+y^2)}\cdot\cos(x+y^2)$

$F_y = e^{\sin(x+y^2)}\cdot\cos(x+y^2)\cdot 2y$

$F_z = -1$

$\Rightarrow F_x(\pi,0,1) = -1,\ F_y(\pi,0,1) = 0,\ F_z = -1.$

Tangent plane: $-(x-\pi) - (z-1) = 0 \Leftrightarrow -x - z + \pi + 1 = 0 \Leftrightarrow \boxed{x+z-(\pi+1)=0}$

Normal line: $\boxed{x = \pi - t,\ y = 0,\ z = 1-t}$

Ex 2: In the setup of Ex1(a), at which point on this surface is the tangent plane parallel to xy-plane?

We computed $F_x = e^x - 2y$, $F_y = -2x$, $F_z = 2z$.

Condition $\Leftrightarrow F_x(x_0,y_0,z_0) = F_y(x_0,y_0,z_0) = 0 \Leftrightarrow \begin{cases} x_0 = 0 \\ e^{x_0} - 2y_0 = 0 \end{cases} \Leftrightarrow \begin{cases} x_0 = 0 \\ y_0 = 1/2 \end{cases}$

Plug into $z^2 - 2xy + e^x = 5$ to find $z_0^2 = 4 \Leftrightarrow z_0 = \pm 2$.

Hence, there are two such points $\boxed{(0, \tfrac{1}{2}, \pm 2)}$

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- Max / Min Problems

Recall that for functions in one variables questions of finding local/global min/max points is attacked with the help of derivatives

Ex3: Find the maximal & minimal value of the function $f(x) = x^3 - 12x$ on $[-3,5]$

- First, we find critical points by solving $0 = f'(x) = 3x^2 - 12 \Leftrightarrow x = \pm 2$
  $f(2) = -16$, $f(-2) = 16$.
  Next we also evaluate the value at end points: $f(-3) = 9$, $f(5) = 65$.
  Hence, maximum is achieved at $x_{max} = 5$, minimum - at $x_{min} = 2$.

Today we will see how this approach can be used to find min/max values of functions of two variables.

Def: • A function f of two variables has a local maximum at the point (a,b) if $f(x,y) \le f(a,b)$ for (x,y) near (a,b). The number $f(a,b)$ is called a local maximum value

• Likewise f has a local minimum at (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) near (a,b) ; $f(a,b)$ - local minimum.

Restricting f to lines parallel to x- or y- axes, i.e. considering $g(x) := f(x,b)$, $h(y) := f(a,y)$, we see that points $x=a$ and $y=b$ are the local max/min for the corresponding function in 1 variable.
Hence they are critical, i.e. $g'(a) = 0$, $h'(b) = 0$. But $g'(a) = f_x(a,b)$, $h'(b) = f_y(a,b)$

Thm 1: If (a,b) - local max/min of f, then $f_x(a,b) = f_y(a,b) = 0$.

Def: All points (a,b) s.t. $f_x(a,b) = f_y(a,b) = 0$ (or one of them does not exist) are called critical points.

Rmk: Note that if $f_x(a,b) = f_y(a,b) = 0$, then the tangent plane to the graph of f at the point $(a,b,f(a,b))$ is parallel to xy-plane

However, not every critical point is a point of local min/max
[Recall the same remark for f-s of 1 variable, e.g. x^3]

Ex 4: Find $\underbrace{\text{local max/min}}_{\text{"extreme"}}$ values of $f(x,y) = y^2 - x^2$.

$f_x = -2x$, $f_y = 2y$ ⇒ the only critical point is $(0,0)$.
However, $f(x_0, 0) = -x_0^2 < 0$ as x_0 approaches to 0
$f(0, y_0) = y_0^2 > 0$ as y_0 approaches to 0.

Hence: f has no extreme values!

Rmk: Point $(0,0)$ is a saddle point.

Thm 2 (Second derivative test): Suppose second partial derivatives of f are continuous near (a,b), and assume $f_x(a,b) = f_y(a,b) = 0$.
Set $\boxed{D = D(a,b) := f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2.}$

(a) If $D > 0$ and $f_{xx}^{(a,b)} > 0$, then $f(a,b)$ is a local minimum
(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
(c) If $D < 0$, then $f(a,b)$ is not a local max/min.
(d) If $D = 0$ – nothing can be said.

Rmk: In case (c), point (a,b) is called a saddle point of f

Ex 5: Find the local min/max values and saddle points of
(a) $f(x,y) = y \sin x$
(b) $f(x,y) = 5 - x^4 + 2x^2 - y^2$

(a) $f_x = y \cos x$, $f_y = \sin x$. Critical points: (x,y) s.t. $\sin x = 0$, $y \cos x = 0$.
As $\sin x = 0 \Rightarrow \cos x \neq 0$, we see that critical points are $\{(\pi k, 0) | k \in \mathbb{Z}\}$
$$D := f_{xx}(\pi k, 0) f_{yy}(\pi k, 0) - (f_{xy}(\pi k, 0))^2 = -1 < 0$$

Hence: there are no local min/max values,
saddle points are $\{(\pi k, 0) | k - \text{integer}\}$

(part (b) of Ex 5)

(b) $f_x(x,y) = -4x^3 + 4x = 4x(1-x^2) = 4x(1-x)(1+x)$ — zero iff $x \in \{0, +1, -1\}$

$f_y(x,y) = -2y$ — zero iff $y = 0$.

Hence, the critical points are $(0,0)$, $(-1,0)$, $(1,0)$.

$f_{xx}(x,y) = 4 - 12x^2$, $f_{yy}(x,y) = -2$, $f_{xy}(x,y) = 0 \Rightarrow D = 2(12x^2 - 4)$.

- At point $(0,0)$: $D = -8 < 0 \Rightarrow (0,0)$ - saddle point
- At point $(-1,0)$: $D = 16 > 0$, $f_{xx}(-1,0) = -8 < 0 \Rightarrow (-1,0)$ - local maximum
- At point $(1,0)$: $D = 16 > 0$, $f_{xx}(1,0) = -8 < 0 \Rightarrow (1,0)$ - local maximum.

Note $f(\pm 1, 0) = 6$.

Thus: The local maximum value is 6 (achieved at two loc. max pts $(\pm 1, 0)$), there is only one saddle point: $(0,0)$ ■

Absolute Max/Min

Again, let us recall the situation for functions $f(x)$ of one variable. Over there, one can ask what is the absolute max/min value of $f(x)$ on $[a,b]$ (if is known to exist). To solve such problems, you find local min/max points and compare values of f at these points as well as boundary points a, b to find where f achieves absolute max/min.

Ex 6: Find the absolute maximum and minimum values of the function $x^2 - 4x + 1 + y^2 - 6y$ on $\mathbb{R}^2$.

Instead of using above procedures, we can just rewrite $f(x,y) = (x-2)^2 + (y-3)^2 - 12$. Hence, we immediately see that the absolute minimum is -12, while absolute maximum does not exist ■

As illustrated by the previous example, absolute max/min of $f(x,y)$ on $\mathbb{R}^2$ do not always exist! However, if we consider a continuous function $f(x,y)$ on a closed and bounded set $D \subseteq \mathbb{R}^2$, then it is a theorem that f achieves(!) its absolute max and min on D.

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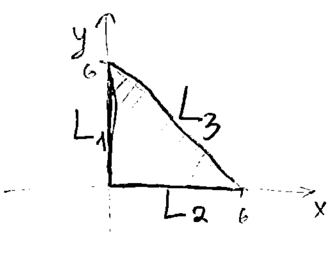
Algorithm (to find absolute max/min of $f(x,y)$ on the closed, bounded set $D \subseteq \mathbb{R}^2$)

- First, you find the values of f at the critical points of f in D
- Second, you find the extreme values of f on the boundary of D
- The largest/smallest of the values in the above two steps is the absolute max/min of f on D.

Warning: When finding extreme values of f on the boundary of D you cannot apply the 2^{nd} derivative test. Instead, you should reduce the question to the case of functions of 1 variable.

Ex 7: Find the absolute maximum and minimum values of $f(x,y) = x^2-4x+y^2-6y+1$ in the closed triangular region with vertices $(0,0)$, $(6,0)$, $(0,6)$.



- $f_x(x,y) = 2x-4 = 0$ iff $x=2$, $f_y(x,y) = 2y-6 = 0$ iff $y=3$ $\Rightarrow$ get only 1 critical point $(2,3) \Rightarrow f(2,3) = -12$
- Need to check if this point is in our region! (it is in our region as it is given by $x,y \geq 0$, $x+y \leq 6$ and $(2,3)$ satisfy this)

The boundary consists of 3 segments $L_1 = \{(0,y) \mid 0 \leq y \leq 6\}$, $L_2 = \{(x,0) \mid 0 \leq x \leq 6\}$, $L_3 = \{(x,6-x) \mid 0 \leq x \leq 6\}$.

- on L_1: $f(0,y) = y^2-6y+1$. Critical pts: $\frac{\partial}{\partial y}(y^2-6y+1) = 0 \Leftrightarrow 2y-6=0 \Leftrightarrow y=3$.
  $f(0,3) = 3^2-6\cdot 3+1 = \underline{-8}$. But we also compute values at end-points: $f(0,0) = \underline{1}$, $f(0,6) = \underline{1}$
- on L_2: $f(x,0) = x^2-4x+1$. Critical pts: $\frac{\partial}{\partial x}(x^2-4x+1) = 0 \Leftrightarrow 2x-4=0 \Leftrightarrow x=2$
  This point $(2,0)$ is indeed on L_2, $f(2,0) = \underline{-3}$; also $f(6,0) = \underline{13}$.
- on L_3: $f(x,6-x) = \cancel{x^2}-4x+\cancel{36}-12x+\cancel{x^2}-\cancel{36}+6x+1 = 2x^2-10x+1$
  Critical pts: $\frac{d}{dx}(2x^2-10x+1) = 4x-10 = 0$ iff $x=\frac{5}{2}$ $\Rightarrow$ point $\left(\frac{5}{2}, \frac{7}{2}\right)$ - it is on L_3
  $f\left(\frac{5}{2}, \frac{7}{2}\right) = 2\cdot\left(\frac{5}{2}\right)^2 - 10\cdot\frac{5}{2}+1 = \frac{25}{2}-25+1 = \underline{-\frac{23}{2}}$

Among the above values maximal is 13 (at $(6,0)$), minimal is -12 (at $(2,3)$)

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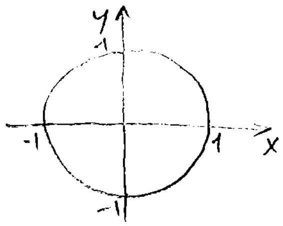
Ex 8: Find the absolute maximum and minimum values of $f(x,y) = 2x^4 + y^4$ on the set $D = \{(x,y) \mid x^2+y^2 \le 1\}$



- Critical pts : $f_x = 0 \Rightarrow 8x^3 = 0 \Rightarrow x = 0$, $f_y = 0 \Rightarrow 4y^3 = 0 \Rightarrow y = 0$ $\Big\} \Rightarrow$ point $(0,0)$ and it is indeed in D

$f(0,0) = 0$

- Boundary is a unit circle, hence, can be paremetrized by $\{(\cos\theta, \sin\theta) \mid 0 \le \theta \le 2\pi\}$.

$f(\cos\theta, \sin\theta) = 2\cos^4\theta + \sin^4\theta$ – this is a function of 1 variable and we need to find its extreme values on $[0, 2\pi]$

$$\frac{d}{d\theta}(2\cos^4\theta + \sin^4\theta) = -8\cos^3\theta \cdot \sin\theta + 4\sin^3\theta \cdot \cos\theta = 4\cos\theta\sin\theta(\sin^2\theta - 2\cos^2\theta)$$
$$= 4\cos\theta\sin\theta(3\sin^2\theta - 2)$$ (in the last equality : $\cos^2 = 1 - \sin^2$)

Hence it is zero if $\cos\theta = 0$, or $\sin\theta = 0$, or $\sin^2\theta = 2/3$.

The first four cases correspond to points $(1,0), (0,1), (-1,0), (0,-1)$.

Note $f(1,0) = f(-1,0) = 2$, $f(0,1) = f(0,-1) = 1$.

Finally, if $\sin^2\theta = 2/3 \Rightarrow \cos^2\theta = 1 - 2/3 = \frac{1}{3} \Rightarrow f(\cos\theta, \sin\theta) = 2 \cdot \left(\frac{1}{3}\right)^4 + \left(\frac{2}{3}\right)^4 = \frac{17}{81}$

Therefore : the absolute maximum is 2 (achieved at $(1,0)$ and $(-1,0)$)
the absolute minimum is 0 (achieved at $(0,0)$)

Ex 9: Find the point on the plane $x + y + 5z - 1 = 0$ that is closest to the point $(1,2,5)$.

Hint : Write down the f-la for the distance b/w your pt $(1,2,5)$ and points on the plane (use equation of the plane to express this distance as a function of two variables on $\mathbb{R}^2$).