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On Sevostyanov's Construction of Quantum Difference Toda Lattices

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We propose a natural generalization of the construction of the quantum difference Toda lattice [6, 22] associated with a simple Lie algebra $\mathfrak{g}$. Our construction depends on two orientations of the Dynkin diagram of $\mathfrak{g}$ and some other data (which we refer to as a pair of *Sevostyanov triples*). In types A and C, we provide an alternative construction via Lax matrix formalism, cf. [15]. We also show that the generating function of the pairing of Whittaker vectors in the Verma modules is an eigenfunction of the corresponding modified quantum difference Toda system and derive fermionic formulas for the former in spirit of [7]. We give a geometric interpretation of all Whittaker vectors in type A via line bundles on the Laumon moduli spaces and obtain an edge-weight path model for them, generalizing the construction of [4].

1 Introduction

In a recent work [10] of M. Finkelberg and the 2nd author, a family of 3^{n-1} commutative subalgebras in the algebra of difference operators on $(\mathbb{C}^{\times})^{n+1}$ was constructed, generalizing the type A quantum difference Toda lattice of [6, 22]. In this paper, we show how

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the construction of [22] for an arbitrary semisimple Lie algebra $\mathfrak{g}$ can be generalized to produce $3^{\mathrm{rk}(\mathfrak{g})-1}$ *integrable systems*, thus answering a question of P. Etingof. In types A and C, we identify these systems with the ones obtained via the Lax matrix formalism. We also discuss some generalizations of the basic results on the quantum difference Toda system to the current setting.

The importance of our generalization of q-Toda systems of [6, 22] is two-fold. First of all, as emphasized in [10] (historically this goes back at least to [20]) already, the quasi-classical limit of this construction (known as the *relativistic open Toda system*) crucially depends on a choice of a pair of Coxeter elements in the Weyl group of G (simply connected algebraic group associated with $\mathfrak{g}$). One of our main results, Theorem 3.1, gives an upper bound on the number of different integrable systems we obtain this way in the quantum case. Another motivation arises from the geometric representation theory, where Whittaker vectors (closely related to the Toda systems due to Theorem 4.9) often have natural geometric interpretations that unveil additional symmetry. We illustrate this in Section 5, where the universal Verma module over $U_v(\mathfrak{sl}_n)$ is realized as the equivariant K-theory of Laumon spaces due to [3] (see Theorem 5.3); one of the Whittaker vectors is realized as a sum of the structure sheaves (see (5.5) and Proposition 5.14(a)), while an extra symmetry noticed in Proposition 5.14(b) gives rise to a family of Whittaker vectors (see Theorem 5.5 and Proposition 5.17).

This paper is organized as follows:

- In Section 2, we construct *the modified quantum difference Toda systems* depending on a pair of *Sevostyanov triples* (following [21, 22]) and generalizing the q-Toda systems of [6, 22].
- In Section 3, we explain how to compute explicitly the corresponding hamiltonians using [13]. We write down the formulas for the hamiltonians corresponding to the 1st fundamental representation in the classical types and G_2. In the latter case of G_2, our formula seems to be new even in the simplest set-up of the standard q-Toda system of [6]; see (3.33).

  One of the key results of this section is that there are at most $3^{\mathrm{rk}(\mathfrak{g})-1}$ different modified quantum difference Toda systems; see Theorem 3.1. For the classical types and G_2, see Theorem 3.2, whose proof is more elementary and relies on Propositions 3.11, 3.14, 3.17, 3.20, and 3.38. We also show that these are maximal commutative subalgebras, determined by their 1st hamiltonians; see Theorem 3.3.

We also prove that in type A these integrable systems exactly match those of [10, 11(ii, iii)]; see Theorem 3.24. This generalizes the Lax matrix realization of the type A q-Toda system, due to [15]. In Theorem 3.31, we also provide a similar Lax matrix realization of the type C modified quantum difference Toda systems. Noticing that the *periodic* counterparts of these two constructions in the classical case (i.e., for $\vec{k} = \vec{0}$ in the notations of *loc.cit.*) match up with the hamiltonians of the affine q-Toda lattice of [6], see formulas (3.22, 3.29, 3.30, 3.31, 3.34), we propose a periodic analogue of the modified quantum difference Toda systems in types A, C; see Propositions 3.26 and 3.33 and Remarks 3.28(b,c) and 3.35(b).

- In Section 4, we study the Shapovalov pairing between a pair of Whittaker vectors (determined by Sevostyanov triples) in Verma modules. We obtain fermionic formulas for those in spirit of [7]; see Theorems 4.6 and 4.7. We also prove that their generating function is naturally an eigenfunction of the corresponding modified quantum difference Toda system; see Theorem 4.9.
- In Section 5, we provide a geometric interpretation of all type A Whittaker vectors and their Shapovalov pairing via the geometry of the Laumon moduli spaces, generalizing [3]; see Theorems 5.5 and 5.11.
  Following a suggestion of B. Feigin, we relate this family of Whittaker vectors to an eigen-property of the (geometrically) simplest one (5.5) with respect to the action of the quantum loop algebra $U_v(L\mathfrak{sl}_n)$ (via the evaluation homomorphism); see Propositions 5.14 and 5.17 and Corollary 5.16. This viewpoint also provides an edge-weight path model for a general type A Whittaker vector, generalizing the path model of [4] for a particular choice of a Sevostyanov triple; see Propositions 5.19 and 5.21.
- In Appendices, we prove Proposition 3.11 and Theorems 3.1, 3.2, 3.3, and 3.24.

2 Sevostyanov Triples and Whittaker Functions

2.1 Quantum groups

We fix the notations as follows. Let G be a simply connected complex algebraic group with a semisimple Lie algebra $\mathfrak{g}$. We denote by $H \subset B$ a pair of a Cartan torus and a Borel subgroup. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is defined as the Lie algebra of H, Δ denotes the set of roots of $(\mathfrak{g}, \mathfrak{h})$, and $\Delta_+ \subset \Delta$ denotes the set of positive roots corresponding to B. Let $n = \mathrm{rk}(\mathfrak{g})$ be the rank of $\mathfrak{g}$, $\alpha_1, \ldots, \alpha_n$ be the simple positive roots, and $\omega_1, \ldots, \omega_n$ be the fundamental weights. Let $P := \oplus_{i=1}^n \mathbb{Z}\omega_i$ be the weight lattice, $Q := \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be the root lattice, and set $P_+ := \oplus_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i$, $Q_+ := \oplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$. We write $\beta \geq \gamma$ if $\beta - \gamma \in Q_+$. We fix a nondegenerate invariant bilinear form $(\cdot,\cdot)\colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ and identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via $(\cdot,\cdot)$. We set $\mathsf{d}_i := \frac{(\alpha_i,\alpha_i)}{2}$. The choice of $(\cdot,\cdot)$ is such that $\mathsf{d}_i = 1$ for short roots α_i, in particular, $\mathsf{d}_i \in \{1, 2, 3\}$ for any i. We also define $\omega_i^\vee := \omega_i/\mathsf{d}_i$ so that $(\omega_i^\vee, \alpha_j) = \delta_{i,j}$, and $\rho := \sum_{i=1}^n \omega_i = \frac{1}{2}\sum_{\gamma \in \Delta_+} \gamma \in P$. Let $(a_{ij})_{i,j=1}^n$ be the corresponding Cartan matrix with $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. We define $b_{ij} = \mathsf{d}_i a_{ij} = (\alpha_i, \alpha_j)$, so that $(b_{ij})_{i,j=1}^n$ is symmetric.

Choose $\mathsf{N} \in \mathbb{Z}_{>0}$ so that $(P, P) \subset \frac{1}{\mathsf{N}}\mathbb{Z}$. The quantum group (of *adjoint type* in the terminology of [16]) $U_v(\mathfrak{g})$ is the unital associative $\mathbb{C}(v^{1/\mathsf{N}})$-algebra generated by $\{E_i, F_i, K_\mu\}_{1 \leq i \leq n}^{\mu \in P}$ with the following defining relations:

$$K_\mu K_{\mu'} = K_{\mu+\mu'},\ K_0 = 1,$$

$$K_\mu E_i K_\mu^{-1} = v^{(\mu,\alpha_i)} E_i,\ K_\mu F_i K_\mu^{-1} = v^{-(\mu,\alpha_i)} F_i,\ [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{v_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0,\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{v_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0\ (i \neq j),$$

where $K_i := K_{\alpha_i}$, $v_i := v^{\mathsf{d}_i}$, $[r]_v := \frac{v^r - v^{-r}}{v - v^{-1}}$, $[r]_v! := [1]_v \cdots [r]_v$, $\begin{bmatrix} m \\ r \end{bmatrix}_v := \frac{[m]_v!}{[r]_v! \cdot [m-r]_v!}$.

Set $L_i := K_{\omega_i}$. Since $P = \oplus_{i=1}^n \mathbb{Z}\omega_i$, we will alternatively view $U_v(\mathfrak{g})$ as the $\mathbb{C}(v^{1/\mathsf{N}})$-algebra generated by $\{E_i, F_i, L_i^{\pm 1}\}_{i=1}^n$ with the corresponding defining relations. In particular,

$$L_i E_j L_i^{-1} = v_i^{\delta_{i,j}} E_j,\ L_i F_j L_i^{-1} = v_i^{-\delta_{i,j}} F_j,\ K_i = \prod_{j=1}^n L_j^{a_{ji}}.$$

2.2 Sevostyanov triples

Let $\mathrm{Dyn}(\mathfrak{g})$ be the graph obtained from the Dynkin diagram of $\mathfrak{g}$ by replacing all multiple edges by simple ones, for example, $\mathrm{Dyn}(\mathfrak{sp}_{2n}) = \mathrm{Dyn}(\mathfrak{so}_{2n+1}) = \mathrm{Dyn}(\mathfrak{sl}_{n+1}) = A_n$. Given an orientation Or of $\mathrm{Dyn}(\mathfrak{g})$, define the associated matrix $\epsilon = (\epsilon_{ij})_{i,j=1}^n$ via

$$\epsilon_{ij} = \begin{cases} 0, & \text{if } a_{ij} = 0 \text{ or } i = j, \\ 1, & \text{if } a_{ij} < 0 \text{ and the edge is oriented } i \to j \text{ in Or}, \\ -1, & \text{if } a_{ij} < 0 \text{ and the edge is oriented } i \leftarrow j \text{ in Or}. \end{cases}$$

Definition 2.3. A **Sevostyanov triple** is a collection of the following data:

(a) an orientation Or of $\mathrm{Dyn}(\mathfrak{g})$,

(b) an integer matrix $\mathsf{n} = (\mathsf{n}_{ij})_{i,j=1}^n$ satisfying $\mathsf{d}_j\mathsf{n}_{ij} - \mathsf{d}_i\mathsf{n}_{ji} = \epsilon_{ij}b_{ij}$ for any i, j,

(c) a collection $c = (c_i)_{i=1}^n \in (\mathbb{C}(v^{1/\mathsf{N}})^\times)^n$.

We refer to this **Sevostyanov triple** by $(\epsilon, \mathsf{n}, c)$.

Fix a pair of integer matrices $\mathsf{n}^\pm = (\mathsf{n}^\pm_{ij})_{i,j=1}^n$ and collections $c^\pm = (c^\pm_i)_{i=1}^n \in (\mathbb{C}(v^{1/\mathsf{N}})^\times)^n$. Set $e_i := E_i \cdot \prod_{p=1}^n L_p^{\mathsf{n}^+_{ip}}$, $f_i := \prod_{p=1}^n L_p^{-\mathsf{n}^-_{ip}} \cdot F_i$, and let $U^+_{\mathsf{n}^+}(\mathfrak{g}), U^-_{\mathsf{n}^-}(\mathfrak{g})$ be the $\mathbb{C}(v^{1/\mathsf{N}})$-subalgebras of $U_v(\mathfrak{g})$ generated by $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$, respectively.

The following simple, but very important, observation is essentially due to [21].

Lemma 2.4. (a) The assignment $e_i \mapsto c^+_i$ $(1 \le i \le n)$ extends to an algebra homomorphism $\chi^+ \colon U^+_{\mathsf{n}^+}(\mathfrak{g}) \to \mathbb{C}(v^{1/\mathsf{N}})$ if and only if there exists an orientation Or^+ of $\mathrm{Dyn}(\mathfrak{g})$ with an associated matrix ϵ^+, such that $(\epsilon^+, \mathsf{n}^+, c^+)$ is a Sevostyanov triple.
(b) The assignment $f_i \mapsto c^-_i$ $(1 \le i \le n)$ extends to an algebra homomorphism $\chi^- \colon U^-_{\mathsf{n}^-}(\mathfrak{g}) \to \mathbb{C}(v^{1/\mathsf{N}})$ if and only if there exists an orientation Or^- of $\mathrm{Dyn}(\mathfrak{g})$ with an associated matrix ϵ^-, such that $(\epsilon^-, \mathsf{n}^-, c^-)$ is a Sevostyanov triple.

Proof. (a) As the "if" part is proved in [21, Theorem 4], let us now prove the "only if" part following similar arguments. Due to the triangular decomposition of $U_v(\mathfrak{g})$, the algebra $U^+_{\mathsf{n}^+}(\mathfrak{g})$ is generated by $\{e_i\}_{i=1}^n$ subject to $\sum_{r=0}^{1-a_{ij}} (-1)^r v^{r(\mathsf{d}_j\mathsf{n}^+_{ij} - \mathsf{d}_i\mathsf{n}^+_{ji})} \left[\begin{smallmatrix} 1-a_{ij} \\ r \end{smallmatrix}\right]_{v_i} e_i^{1-a_{ij}-r} e_j e_i^r = 0$ for $i \neq j$. Hence, there is a character $\chi^+ \colon U^+_{\mathsf{n}^+}(\mathfrak{g}) \to \mathbb{C}(v^{1/\mathsf{N}})$ with $\chi^+(e_i) \neq 0$ if and only if $\sum_{r=0}^{1-a_{ij}} (-1)^r v^{r(\mathsf{d}_j\mathsf{n}^+_{ij} - \mathsf{d}_i\mathsf{n}^+_{ji})} \left[\begin{smallmatrix} 1-a_{ij} \\ r \end{smallmatrix}\right]_{v_i} = 0$ for any $i \neq j$. If $a_{ij} = 0$, then we immediately get $\mathsf{d}_j\mathsf{n}^+_{ij} - \mathsf{d}_i\mathsf{n}^+_{ji} = 0$. If $a_{ij} = -1$, then we recover $\mathsf{d}_j\mathsf{n}^+_{ij} - \mathsf{d}_i\mathsf{n}^+_{ji} \in \{\pm \mathsf{d}_i\} = \{\pm b_{ij}\}$ and $\epsilon_{ij} \in \{\pm 1\}$. Finally, if $a_{ij} < -1$, then $a_{ji} = -1$ and we can apply the previous case.
(b) Analogous. ■

2.5. Whittaker functions

From now on, we fix a pair of Sevostyanov triples $(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$, which give rise to the subalgebras $U^\pm_{\mathsf{n}^\pm}(\mathfrak{g})$ of $U_v(\mathfrak{g})$ and the corresponding characters $\chi^\pm \colon U^\pm_{\mathsf{n}^\pm}(\mathfrak{g}) \to \mathbb{C}(v^{1/\mathsf{N}})$ of Lemma 2.4. We consider the quantum function algebra $\mathcal{O}_v(G)$ spanned by the matrix coefficients of integrable $U_v(\mathfrak{g})$-modules (with the highest weights in P_+). Let $\mathcal{D}_v(G)$ denote the corresponding Heisenberg double [19, Section 3]. It acts on $\mathcal{O}_v(G)$. It is equipped with a homomorphism $\mu_v \colon U_v(\mathfrak{g}) \otimes U_v(\mathfrak{g}) \to \mathcal{D}_v(G)$. Let $\mathcal{O}_v(Bw_0B)$ stand for the quantized coordinate ring of the big Bruhat cell [11, 8.2] (a localization of $\mathcal{O}_v(G)$). The action of $\mathcal{D}_v(G)$ on $\mathcal{O}_v(G)$ extends to the action on $\mathcal{O}_v(Bw_0B)$. In particular, $U^-_{\mathsf{n}^-}(\mathfrak{g}) \otimes U^+_{\mathsf{n}^+}(\mathfrak{g}) \subset U_v(\mathfrak{g}) \otimes U_v(\mathfrak{g})$ acts on $\mathcal{O}_v(Bw_0B)$. According to [11, (3.22), Theorem 4.7, Proposition 8.3], there are subalgebras $S_v^{\pm}$ of $\mathcal{O}_v(Bw_0B)$ (we note that $\mathcal{O}_v(G), \mathcal{O}_v(Bw_0B), S_v^{\pm}$ are denoted by $R_v[G], R_v[Bw_0B], \overline{S}_{w_0}^{\mp}$, respectively, in [11]) such that $\mathcal{O}_v(Bw_0B) \simeq S_v^- \otimes \mathcal{O}_v(H) \otimes S_v^+$ (as vector spaces) and $S_v^{\pm} \simeq U_v^{\pm}(\mathfrak{g})$, where $U_v^-(\mathfrak{g}), U_v^+(\mathfrak{g})$ are the subalgebras of $U_v(\mathfrak{g})$ generated by $\{F_i\}_{i=1}^n$ and $\{E_i\}_{i=1}^n$, respectively. Hence, there is an (vector space) isomorphism

$$\mathcal{O}_v(Bw_0B) \simeq U_{\mathsf{n}^-}^-(\mathfrak{g}) \otimes \mathcal{O}_v(H) \otimes U_{\mathsf{n}^+}^+(\mathfrak{g}), \tag{2.1}$$

under which the above actions of $U_{\mathsf{n}^-}^-(\mathfrak{g}), U_{\mathsf{n}^+}^+(\mathfrak{g})$ on $\mathcal{O}_v(Bw_0B)$ are via the left and the right multiplications. Let $U_{\mathsf{n}^\pm}^{\pm}(\mathfrak{g})^\wedge$ denote the completions of $U_{\mathsf{n}^\pm}^{\pm}(\mathfrak{g})$ with respect to the natural gradings with $\deg(e_i) = 1$ and $\deg(f_i) = 1$. In view of the identification (2.1), we define the completion of $\mathcal{O}_v(Bw_0B)$ via $\mathcal{O}_v(Bw_0B)^\wedge \simeq U_{\mathsf{n}^-}^-(\mathfrak{g})^\wedge \otimes \mathcal{O}_v(H) \otimes U_{\mathsf{n}^+}^+(\mathfrak{g})^\wedge$. Hence, the subspace of semi-invariants $\left(\mathcal{O}_v(Bw_0B)^\wedge\right)^{U_{\mathsf{n}^-}^-(\mathfrak{g})\otimes U_{\mathsf{n}^+}^+(\mathfrak{g}), \chi^-\otimes\chi^+}$ projects isomorphically onto $\mathcal{O}_v(H)$ under the restriction projection $\mathcal{O}_v(Bw_0B)^\wedge \to \mathcal{O}_v(H)$. We denote this projection by $\phi \mapsto \phi_{|H}$.

Definition 2.6. A **Whittaker function** is an element of $\left(\mathcal{O}_v(Bw_0B)^\wedge\right)^{U_{\mathsf{n}^-}^-(\mathfrak{g})\otimes U_{\mathsf{n}^+}^+(\mathfrak{g}), \chi^-\otimes\chi^+}$.

Remark 2.7. Following [6], we could alternatively work with the dual quantum formal group $\mathcal{A}_\hbar(\mathfrak{g}) = U_\hbar(\mathfrak{g})^*$, defined as the space of linear functions on $U_\hbar(\mathfrak{g})$. Here the quantum group $U_\hbar(\mathfrak{g})$ is defined over $\mathbb{C}[[\hbar]]$ with v replaced by $e^\hbar$. In this set-up, a **Whittaker function** is an element $\phi \in \mathcal{A}_\hbar(\mathfrak{g})$ such that $\phi(x^-xx^+) = \chi^-(x^-)\chi^+(x^+)\phi(x)$ for any $x^\pm \in U_{\mathsf{n}^\pm}^{\pm}(\mathfrak{g}), x \in U_\hbar(\mathfrak{g})$. Let us point out that this differs from the notion of Whittaker functions as defined in *loc.cit*.

We note that the character lattice $X^*(H) = P$ and the pairing $(Q, P) \subset \mathbb{Z}$, hence, we have the natural embedding of Q into the cocharacter lattice $X_*(H)$. Thus, for every $\lambda \in Q$ we can define the difference operators T_λ acting on $\mathcal{O}_v(H)$ via $(T_\lambda f)(x) = f(x \cdot v^\lambda)$. Moreover, since $v^{(P,P)} \subset \mathbb{C}(v^{1/\mathsf{N}})$, the difference operators T_λ are also well-defined for $\lambda \in P$. Let $\widetilde{\mathcal{D}}_v(H)$ be the algebra generated by $\{e^\lambda, T_\mu | \lambda, \mu \in P\}$, and $\mathcal{D}_v(H^{\mathrm{ad}})$ be its subalgebra generated by $\{e^\lambda, T_\mu | \lambda \in Q, \mu \in P\}$. The following is completely analogous to [6, Proposition 3.2].

Lemma 2.8. (a) For any $Y \in U_v(\mathfrak{g})$, there exists a unique difference operator $\tilde{\mathbf{D}}_Y = \tilde{\mathbf{D}}_Y(\epsilon^\pm, \mathsf{n}^\pm, c^\pm) \in \widetilde{\mathcal{D}}_v(H)$ such that $(Y\phi)_{|H} = \tilde{\mathbf{D}}_Y(\phi_{|H})$ for any Whittaker function ϕ.
(b) $\tilde{\mathbf{D}}_Y$ is an element of $\mathcal{D}_v(H^{\mathrm{ad}}) \subset \widetilde{\mathcal{D}}_v(H)$.
(c) If Y_1 and Y_2 are central elements of $U_v(\mathfrak{g})$, then $\tilde{\mathbf{D}}_{Y_1Y_2} = \tilde{\mathbf{D}}_{Y_1}\tilde{\mathbf{D}}_{Y_2}$.

Recall the element $\Theta \in (U_v(\mathfrak{g}) \otimes U_v(\mathfrak{g}))^\wedge$ of the completion of the vector space $U_v(\mathfrak{g}) \otimes U_v(\mathfrak{g})$ as defined in [16, 4.1.1]. Loosely speaking, the universal R-matrix is given by $R = \Theta^{\mathrm{op}} \cdot R^0$, where $R^0 = v^T$ and $T \in \mathfrak{h} \otimes \mathfrak{h}$ stand for the canonical element. Let $\pi_V \colon U_v(\mathfrak{g}) \to \mathrm{End}(V)$ be a finite-dimensional representation, $\{w_k\}_{k=1}^N$ be a weight basis of V, and $\mu_k \in P$ be the weight of w_k. First, we note that though $\Theta, \Theta^{\mathrm{op}}$ are defined as infinite sums, their images $(\mathrm{id}\otimes\pi_V)(\Theta), (\mathrm{id}\otimes\pi_V)(\Theta^{\mathrm{op}}) \in U_v(\mathfrak{g})\otimes\mathrm{End}(V)$ are well-defined. Second, the image $(\mathrm{id}\otimes\pi_V)(R^0) = (\mathrm{id}\otimes\pi_V)((R^0)^{\mathrm{op}}) \in U_v(\mathfrak{g})\otimes\mathrm{End}(V)$ is also well-defined via $(\mathrm{id}\otimes\pi_V)(R^0) = \sum_{k=1}^N K_{\mu_k} \otimes E_{k,k}$ with $E_{k,k} \in \mathrm{End}(V)$ given by $E_{k,k}(w_{k'}) = \delta_{k,k'} w_{k'}$ (this does not depend on the choice of a weight basis $\{w_k\}$). Hence, working over $\mathbb{C}(v^{1/\mathsf{N}})$ (rather than in the formal setting $\mathbb{C}[[\hbar]]$ as in [6, 22]), the elements $(\mathrm{id}\otimes\pi_V)(R)$ and $(\mathrm{id}\otimes\pi_V)(R^{\mathrm{op}})$ are still well-defined.

Due to [5, 18], the center of $U_v(\mathfrak{g})$ is spanned by elements C_V corresponding to finite-dimensional $U_v(\mathfrak{g})$-representations V via the formula

$$C_V = \mathrm{tr}_V(\mathrm{id}\otimes\pi_V)\left(R^{\mathrm{op}}R(1\otimes K_{2\rho})\right). \tag{2.2}$$

We define $\tilde{\mathbf{D}}_V, \mathbf{D}_V \in \mathcal{D}_v(H^{\mathrm{ad}})$ via $\tilde{\mathbf{D}}_V := \tilde{\mathbf{D}}_{C_V}$ and $\mathbf{D}_V := e^{\rho}\tilde{\mathbf{D}}_V e^{-\rho}$. Consider the fundamental representations $\{V_i\}_{i=1}^n$ of $U_v(\mathfrak{g})$ and set $\tilde{\mathbf{D}}_i := \tilde{\mathbf{D}}_{V_i}, \mathbf{D}_i := \mathbf{D}_{V_i}$. According to Lemma 2.8, $\{\tilde{\mathbf{D}}_i\}_{i=1}^n$ and therefore $\{\mathbf{D}_i\}_{i=1}^n$ are families of pairwise commuting elements of $\mathcal{D}_v(H^{\mathrm{ad}})$.

Definition 2.9. A **modified quantum difference Toda system** is the commutative subalgebra $\mathcal{T} = \mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ of $\mathcal{D}_v(H^{\mathrm{ad}})$ generated by $\{\mathbf{D}_i\}_{i=1}^n$.

Due to Theorem 3.3(c) below, $\mathbf{D}_V \in \mathcal{T}$ for any finite-dimensional $U_v(\mathfrak{g})$-representation V.

Remark 2.10. This construction is a q-deformed version of the Kazhdan–Kostant approach to the classical Toda system. In case the two Sevostyanov triples coincide, we recover the original construction of [22]. Let us point out right away that we do not know how to generalize an alternative approach of [6] to obtain our modified quantum difference Toda systems.

3 1st Hamiltonians, Classification, and Lax Realization in Types A,C

The main result of this section is the following.

Theorem 3.1. There are at most 3^{n-1} different modified quantum difference Toda systems, up to algebra automorphisms of $\mathcal{D}_v(H^{\mathrm{ad}})$.

The proof of this result is presented in Appendix D and crucially relies on Theorem 4.7.

We also provide a more straightforward proof for the classical types A_n, B_n, C_n, D_n as well as the exceptional type G_2. To state the result, we label the simple roots $\{\alpha_i\}_{i=1}^n$ as in [2, Chapter VI, Section 4] (here $n = 2$ for the type G_2). Given a pair of Sevostyanov triples $(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$, we define $\vec{\epsilon} = (\epsilon_{n-1}, \ldots, \epsilon_1) \in \{-1, 0, 1\}^{n-1}$ via

$$\epsilon_i := \begin{cases} \frac{\epsilon^+_{n-2,n} - \epsilon^-_{n-2,n}}{2}, & \text{if } i = n-1 \text{ in type } D_n, \\ \frac{\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1}}{2}, & \text{otherwise.} \end{cases}$$

Theorem 3.2. If $\mathfrak{g}$ is of type A_n, B_n, C_n, D_n or G_2, then up to algebra automorphisms of $\mathcal{D}_v(H^{\mathrm{ad}})$, the modified quantum difference Toda system $\mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ depends only on $\vec{\epsilon}$.

We present the proof of this result in Appendix B. The key ingredient in our proof is that the 1st hamiltonian $\mathbf{D}_1$ depends only on $\vec{\epsilon} \in \{-1, 0, 1\}^{n-1}$ up to an algebra automorphism of $\mathcal{D}_v(H^{\mathrm{ad}})$, which is established case-by-case in Propositions 3.11, 3.14, 3.17, 3.20, and 3.38. Following an elegant argument of P. Etingof, we show in Appendix B that the other hamiltonians $\mathbf{D}_i$ match as well under the same automorphism.

Let $\mathcal{D}^{\leq}_v(H^{\mathrm{ad}})$ be the subalgebra of $\mathcal{D}_v(H^{\mathrm{ad}})$, generated by $\{e^{-\alpha_i}, T_\mu\}_{1\leq i\leq n}^{\mu\in P}$. It follows from the construction that $\mathbf{D}_i \in \mathcal{D}^{\leq}_v(H^{\mathrm{ad}})$, so that $\mathcal{T} \subset \mathcal{D}^{\leq}_v(H^{\mathrm{ad}})$. Applying ideas similar to those from the proof of Theorem 3.2, we get another important result.

Theorem 3.3. Consider a modified quantum difference Toda system $\mathcal{T} = \mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$.

(a) The difference operators $\{\mathbf{D}_i\}_{i=1}^n \subset \mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ are algebraically independent.

(b) The centralizer of $\mathbf{D}_1$ in $\mathcal{D}^{\leq}_v(H^{\mathrm{ad}})$ coincides with $\mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$.

(c) We have $\mathbf{D}_V(\epsilon^\pm, \mathsf{n}^\pm, c^\pm) \in \mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ for any finite-dimensional $U_v(\mathfrak{g})$-module V.

The proof of Theorem 3.3 is presented in Appendix C.

3.4 R-matrix and convex orderings

Our computations are based on the explicit formula for the universal R-matrix R, due to [13]. First, let us recall the construction of Cartan–Weyl root elements $\{E_\gamma, F_\gamma\}_{\gamma\in\Delta_+}$, which is crucially based on the notion of a convex ordering on Δ_+.

Definition 3.5. An ordering $\prec$ on the set of positive roots Δ_+ is called **convex** (we note that such orderings are called *normal* in [13, 23]) if for any three roots $\alpha, \beta, \gamma \in \Delta_+$ such that $\gamma = \alpha + \beta$, we have either $\alpha \prec \gamma \prec \beta$ or $\beta \prec \gamma \prec \alpha$.

Fix a convex ordering $\prec$ on Δ_+. For a simple root α_i $(1 \leq i \leq n)$, set $E_{\alpha_i} := E_i, F_{\alpha_i} := F_i$. To construct the remaining root vectors, we apply the following inductive algorithm.

Let $\alpha, \beta, \gamma \in \Delta_+$ be such that $\gamma = \alpha + \beta, \alpha \prec \beta$, and there are no $\alpha \npreceq \alpha' \prec \beta' \npreceq \beta$ satisfying $\gamma = \alpha' + \beta'$. Suppose that $E_\alpha, F_\alpha, E_\beta, F_\beta$ have been already constructed. Then we define

$$E_\gamma := E_\alpha E_\beta - v^{(\alpha,\beta)} E_\beta E_\alpha, \ F_\gamma := F_\beta F_\alpha - v^{-(\alpha,\beta)} F_\alpha F_\beta.$$

According to [13], we have $[E_\gamma, F_\gamma] = a(\gamma)\frac{K_\gamma - K_\gamma^{-1}}{v_\gamma - v_\gamma^{-1}}$ for certain constants $a(\gamma) \in \mathbb{C}(v^{1/\mathsf{N}})$, where $v_\gamma := v^{(\gamma,\gamma)/2}$ (note that $a(\alpha_i) = 1$ and $v_{\alpha_i} = v_i$). For $\gamma \in \Delta_+$, define

$$R_\gamma := \exp_{v_\gamma^{-1}}\left(\frac{v_\gamma - v_\gamma^{-1}}{a(\gamma)} E_\gamma \otimes F_\gamma\right),$$

where $\exp_v(x) := \sum_{r=0}^\infty \frac{x^r}{(r)_v!}, (r)_v! := (1)_v \cdots (r)_v, (r)_v := \frac{1-v^r}{1-v}$. The following is due to [13].

Theorem 3.6. ([13]). Fix a convex ordering $\prec$ on Δ_+. Then $\Theta^{\mathrm{op}} = \prod_{\gamma\in\Delta_+} R_\gamma$, where the order in the product coincides with the ordering $\prec$.

The explicit computations of $\mathbf{D}_1$ below are based on the special choice of convex orderings. We choose two convex orderings $\prec_\pm$ on Δ_+ in such a way that $\epsilon_{ij}^\pm = -1 \Rightarrow \alpha_i \prec_\pm \alpha_j$ (as shown in [23], any ordering on simple positive roots can be extended to a convex ordering on Δ_+). This choice is motivated by Proposition 3.7 below. To state the result, define

$$C'_V = \mathrm{tr}_V(\mathrm{id} \otimes \pi_V)\left(\prod R_{\alpha_i}^{\mathrm{op}} \cdot (R^0)^{\mathrm{op}} \cdot \prod R_{\alpha_i} \cdot R^0 \cdot (1 \otimes K_{2\rho})\right), \tag{3.1}$$

where the 1st and the 2nd products are over all simple positive roots ordered according to $\prec_-$ and $\prec_+$, respectively, whereas $(\mathrm{id} \otimes \pi_V)(R^0) = (\mathrm{id} \otimes \pi_V)((R^0)^{\mathrm{op}})$ are understood as before. We define $\bar{\mathbf{D}}_V := \tilde{\mathbf{D}}_{C'_V}$.

Proposition 3.7. We have $\tilde{\mathbf{D}}_V = \bar{\mathbf{D}}_V$.

Proof. For $\gamma = \sum_{i=1}^n m_i \alpha_i \in \Delta_+$ $(m_i \in \mathbb{Z}_{\geq 0})$, define $e_\gamma, f_\gamma \in U_v(\mathfrak{g})$ via $e_\gamma := E_\gamma \cdot \prod_{i,k=1}^n L_k^{m_i \mathsf{n}_{ik}^+}$ and $f_\gamma := \prod_{i,k=1}^n L_k^{-m_i \mathsf{n}_{ik}^-} \cdot F_\gamma$, so that $e_{\alpha_i} = e_i, f_{\alpha_i} = f_i$ as defined in Section 2.2. The proof of Proposition 3.7 is based on the following properties of these elements $\{e_\gamma, f_\gamma\}_{\gamma\in\Delta_+}$ established in [22, Propositions 2.2.4 and 2.2.5].

Lemma 3.8. (a) For $\gamma \in \Delta_+$, we have $e_\gamma \in U^+_{\mathsf{n}^+}(\mathfrak{g})$ and $f_\gamma \in U^-_{\mathsf{n}^-}(\mathfrak{g})$.
(b) If $\gamma \in \Delta_+$ is not a simple root, then $\chi^+(e_\gamma) = 0$ and $\chi^-(f_\gamma) = 0$.

We recall the proof of this Lemma to make our exposition self-contained.

Proof. (a) The proof is by induction used in the above definition of the root vectors E_γ, F_γ. The claim is trivial when γ is a simple root. For the remaining cases, let $\alpha, \beta, \gamma \in \Delta_+$ be as above and assume that we have already established the inclusions $e_\alpha, e_\beta \in U^+_{\mathsf{n}^+}(\mathfrak{g})$ and $f_\alpha, f_\beta \in U^-_{\mathsf{n}^-}(\mathfrak{g})$. Let us write $\alpha = \sum_{i=1}^n m_i\alpha_i, \beta = \sum_{i=1}^n m'_i\alpha_i$. Then

$$E_\gamma = \left(v^{-\sum_{i,k=1}^n m_i\mathsf{n}^+_{ik}(\omega_k,\beta)}e_\alpha e_\beta - v^{(\alpha,\beta)-\sum_{i,k=1}^n m'_i\mathsf{n}^+_{ik}(\omega_k,\alpha)}e_\beta e_\alpha\right)\cdot \prod_{i,k=1}^n L_k^{-(m_i+m'_i)\mathsf{n}^+_{ik}},$$

$$F_\gamma = \prod_{i,k=1}^n L_k^{(m_i+m'_i)\mathsf{n}^-_{ik}}\cdot\left(v^{\sum_{i,k=1}^n m_i\mathsf{n}^-_{ik}(\omega_k,\beta)}f_\beta f_\alpha - v^{-(\alpha,\beta)+\sum_{i,k=1}^n m'_i\mathsf{n}^-_{ik}(\omega_k,\alpha)}f_\alpha f_\beta\right),$$

so that

$$e_\gamma = v^{-\sum_{i,k=1}^n m_i\mathsf{n}^+_{ik}(\omega_k,\beta)}e_\alpha e_\beta - v^{(\alpha,\beta)-\sum_{i,k=1}^n m'_i\mathsf{n}^+_{ik}(\omega_k,\alpha)}e_\beta e_\alpha \tag{3.2}$$

and

$$f_\gamma = v^{\sum_{i,k=1}^n m_i\mathsf{n}^-_{ik}(\omega_k,\beta)}f_\beta f_\alpha - v^{-(\alpha,\beta)+\sum_{i,k=1}^n m'_i\mathsf{n}^-_{ik}(\omega_k,\alpha)}f_\alpha f_\beta. \tag{3.3}$$

Thus, $e_\gamma \in U^+_{\mathsf{n}^+}(\mathfrak{g})$ and $f_\gamma \in U^-_{\mathsf{n}^-}(\mathfrak{g})$, which completes our inductive step. Part (a) follows.

(b) Due to the formulas (3.2, 3.3), it suffices to prove $\chi^+(e_\gamma) = 0$ and $\chi^-(f_\gamma) = 0$ for $\gamma = \alpha+\beta$ with $\alpha = \alpha_i, \beta = \alpha_j$.

In the former case, we get

$$e_\gamma = v^{-\mathsf{d}_j\mathsf{n}^+_{ij}}e_ie_j - v^{b_{ij}-\mathsf{d}_i\mathsf{n}^+_{ji}}e_je_i = v^{-\mathsf{d}_j\mathsf{n}^+_{ij}}[e_i,e_j],$$

since $\mathsf{d}_j\mathsf{n}^+_{ij} - \mathsf{d}_i\mathsf{n}^+_{ji} = \epsilon^+_{ij}b_{ij} = -b_{ij}$ as $\alpha_i \prec_+ \alpha_j$. Hence, $\chi^+(e_\gamma) = v^{-\mathsf{d}_j\mathsf{n}^+_{ij}}[\chi^+(e_i),\chi^+(e_j)] = 0$.

In the latter case, we get

$$f_\gamma = v^{\mathsf{d}_j\mathsf{n}^-_{ij}}f_jf_i - v^{-b_{ij}+\mathsf{d}_i\mathsf{n}^-_{ji}}f_if_j = v^{\mathsf{d}_j\mathsf{n}^-_{ij}}[f_j,f_i],$$

since $\mathsf{d}_j\mathsf{n}^-_{ij} - \mathsf{d}_i\mathsf{n}^-_{ji} = \epsilon^-_{ij}b_{ij} = -b_{ij}$ as $\alpha_i \prec_- \alpha_j$. Thus, $\chi^-(f_\gamma) = v^{\mathsf{d}_j\mathsf{n}^-_{ij}}[\chi^-(f_j),\chi^-(f_i)] = 0$. ■

Tracing back the definition of $\tilde{\mathbf{D}}_V$, Lemma 3.8 implies that $R_\gamma, R^{\mathrm{op}}_\gamma$ give trivial contributions to $\tilde{\mathbf{D}}_V$ unless $\gamma \in \Delta_+$ is a simple root, cf. [6, Lemma 5.2] and [7, Proposition 3.6]. Hence, the equality $\tilde{\mathbf{D}}_V = \bar{\mathbf{D}}_V$. ■

3.9. Explicit formulas and classification in type A_{n-1}

Recall explicit formulas for the action of $U_v(\mathfrak{sl}_n)$ on its 1st fundamental representation V_1. The space V_1 has a basis $\{w_1,\ldots,w_n\}$, in which the action is given by the following formulas:

$$E_i(w_j) = \delta_{j,i+1}w_{j-1},\ F_i(w_j) = \delta_{j,i}w_{j+1},\ L_i(w_j) = v^{-\frac{i}{n}+\delta_{j\le i}}w_j,\ K_i(w_j) = v^{\delta_{j,i}-\delta_{j,i+1}}w_j$$

for any $1 \leq i < n, 1 \leq j \leq n$. Let $\varpi_1, \ldots, \varpi_n$ be the weights of $w_1, \ldots, w_n$, respectively, so that $(\varpi_i, \varpi_j) = \delta_{i,j} - 1/n$. Recall that the simple roots are given by $\alpha_i = \varpi_i - \varpi_{i+1}$ $(1 \leq i \leq n-1)$, while $\rho = \sum_{j=1}^{n} \frac{n+1-2j}{2} \varpi_j$. According to Proposition 3.7, to compute $\mathbf{D}_1$ explicitly, we should

- evaluate C'_{V_1},
- replace E_i, F_i by e_i, f_i and L_p, moving the latter to the middle part,
- apply $\chi^{\pm}$ as in [6] to obtain the difference operator $\tilde{\mathbf{D}}_1 = \bar{\mathbf{D}}_{V_1}$,
- conjugate by e^{ρ}.

Note that the operators $\{E_i^r, F_i^r\}_{1 \leq i \leq n-1}^{r>1}$ act trivially on V_1. Hence, applying formula (3.1), we can replace R_{α_i} by $\bar{R}_{\alpha_i} := 1 + (v - v^{-1}) E_i \otimes F_i$. Let us now compute all the nonzero terms contributing to C'_{V_1}:

- Picking 1 out of each $\bar{R}^{\mathrm{op}}_{\alpha_i}, \bar{R}_{\alpha_i}$, we recover $\sum_{j=1}^{n} v^{n+1-2j} \cdot K_{2\varpi_j}$.
- Picking nontrivial terms only at $\bar{R}^{\mathrm{op}}_{\alpha_j}, \bar{R}_{\alpha_i}$, the result does not depend on $\mathrm{Or}^{\pm}$ (hence, the orderings $\prec_{\pm}$) and the total contribution is $\sum_{i=1}^{n-1} (v - v^{-1})^2 v^{n+1-2i} \cdot F_i K_{\varpi_{i+1}} E_i K_{\varpi_i}$. Rewriting in terms of e_i, f_i and L_p, we get $(v - v^{-1})^2 \sum_{i=1}^{n-1} v^{n-2i+(\mathsf{n}^+_{ii} - \mathsf{n}^-_{ii})} \cdot f_i K_{\varpi_i + \varpi_{i+1}} \prod_{p=1}^{n-1} L_p^{\mathsf{n}^-_{ip} - \mathsf{n}^+_{ip}} e_i$.
- The computation of the remaining terms is based on the following obvious formulas
$$F_{i_k} \cdots F_{i_2} F_{i_1}(w_i) = \delta_{i_1, i} \delta_{i_2, i_1+1} \cdots \delta_{i_k, i_{k-1}+1} w_{i+k},$$
$$E_{j_k} \cdots E_{j_2} E_{j_1}(w_j) = \delta_{j_1, j-1} \delta_{j_2, j_1-1} \cdots \delta_{j_k, j_{k-1}-1} w_{j-k}.$$

Hence, picking nontrivial terms only at $\bar{R}^{\mathrm{op}}_{\alpha_{j_1}}, \ldots, \bar{R}^{\mathrm{op}}_{\alpha_{j_{k'}}}, \bar{R}_{\alpha_{i_k}}, \ldots, \bar{R}_{\alpha_{i_1}}$ (in the order listed) is possible only if $i_k \prec_+ \cdots \prec_+ i_1$, $j_1 \prec_- \cdots \prec_- j_{k'}$, and gives a nonzero contribution to C'_{V_1} if and only if $k = k'$, $i_k = i_{k-1} + 1 = \ldots = i_1 + k - 1$, and $i_a = j_a$ for $1 \leq a \leq k$. Thus, the remaining terms contributing to C'_{V_1} depend on $\prec_{\pm}$ (only on $\mathrm{Or}^{\pm}$) and give in total

$$\sum_{1 \leq i < j-1 \leq n-1}^{\epsilon^{\pm}_{i,i+1} = \ldots = \epsilon^{\pm}_{j-2,j-1} = \pm 1} (v - v^{-1})^{2(j-i)} v^{n+1-2i} \cdot F_i \cdots F_{j-1} K_{\varpi_j} E_{j-1} \cdots E_i K_{\varpi_i}.$$

Rewriting this in terms of e_i, f_i and L_p, and moving the latter to the middle, we get

$$\sum (v - v^{-1})^{2(j-i)} v^{n-2i+\sum_{i \leq a \leq b \leq j-1} (\mathsf{n}^+_{ab} - \mathsf{n}^-_{ab})} \cdot f_i \cdots f_{j-1} \cdot K_{\varpi_i + \varpi_j} \prod_{p=1}^{n-1} L_p^{\sum_{s=i}^{j-1} (\mathsf{n}^-_{sp} - \mathsf{n}^+_{sp})} \cdot e_{j-1} \cdots e_i,$$

where the sum is over all $1 \leq i < j - 1 \leq n - 1$ such that $\epsilon^{\pm}_{i,i+1} = \ldots = \epsilon^{\pm}_{j-2,j-1} = \pm 1$. Note that $L_p = K_{\varpi_1 + \ldots + \varpi_p}$. Set $\mathsf{m}_{ik} := \sum_{p=k}^{n-1} (\mathsf{n}^-_{ip} - \mathsf{n}^+_{ip})$. Then the Cartan part above equals $K_{\varpi_i + \varpi_j} \prod_{p=1}^{n-1} L_p^{\sum_{s=i}^{j-1} (\mathsf{n}^-_{sp} - \mathsf{n}^+_{sp})} = K_{\sum_{k=1}^{n} (\sum_{s=i}^{j-1} \mathsf{m}_{sk} + \delta_{k,i} + \delta_{k,j}) \varpi_k}$.

We have listed all the nonzero terms contributing to C'_{V_1}. To obtain the desired difference operator $\tilde{\mathbf{D}}_1$, apply the characters $\chi^\pm$ with $\chi^+(e_i) = c_i^+, \chi^-(f_i) = c_i^-$ as in [6, Lemma 5.2]. Set $b_i := (v - v^{-1})^2 v^{\mathsf{n}_{ii}^+ - \mathsf{n}_{ii}^-} c_i^+ c_i^-$. Then we have

$$\tilde{\mathbf{D}}_1 = \sum_{j=1}^{n} v^{n+1-2j} T_{2\varpi_j} + \sum_{i=1}^{n-1} b_i v^{n-2i} \cdot e^{-\alpha_i} T_{\sum_{k=1}^n (\mathsf{m}_{ik} + \delta_{k,i} + \delta_{k,i+1})\varpi_k} + \sum_{1 \le i < j-1 \le n-1}^{\epsilon^\pm_{i,i+1} = \ldots = \epsilon^\pm_{j-2,j-1} = \pm 1} b_i \cdots b_{j-1} v^{n-2i+\sum_{i \le a < b \le j-1} (\mathsf{n}^+_{ab} - \mathsf{n}^-_{ab})} \times e^{-\sum_{s=i}^{j-1} \alpha_s} T_{\sum_{k=1}^n (\sum_{s=i}^{j-1} \mathsf{m}_{sk} + \delta_{k,i} + \delta_{k,j})\varpi_k}. \quad (3.4)$$

Conjugating this by e^ρ, we finally obtain the explicit formula for the 1st hamiltonian $\mathbf{D}_1$ of the type A_{n-1} modified quantum difference Toda system:

$$\mathbf{D}_1 = \sum_{j=1}^{n} T_{2\varpi_j} + \sum_{i=1}^{n-1} b_i v^{\sum_{k=1}^n \frac{2k-n-1}{2} \mathsf{m}_{ik}} \cdot e^{-\alpha_i} T_{\sum_{k=1}^n (\mathsf{m}_{ik} + \delta_{k,i} + \delta_{k,i+1})\varpi_k} + \sum_{1 \le i < j-1 \le n-1}^{\epsilon^\pm_{i,i+1} = \ldots = \epsilon^\pm_{j-2,j-1} = \pm 1} b_i \cdots b_{j-1} v^{j-i-1+\sum_{i \le a < b \le j-1} (\mathsf{n}^+_{ab} - \mathsf{n}^-_{ab}) + \sum_{k=1}^n \sum_{s=i}^{j-1} \frac{2k-n-1}{2} \mathsf{m}_{sk}} \times e^{-\sum_{s=i}^{j-1} \alpha_s} T_{\sum_{k=1}^n (\sum_{s=i}^{j-1} \mathsf{m}_{sk} + \delta_{k,i} + \delta_{k,j})\varpi_k}. \quad (3.5)$$

Remark 3.10. If $\epsilon^+ = \epsilon^-$, then the last sum is vacuous. If we also set $\mathsf{n}^+ = \mathsf{n}^-$ and $c_i^\pm = \pm 1$ for all i, then we recover the formula [6, (5.7)] for the 1st hamiltonian of the type A_{n-1} quantum difference Toda lattice:

$$\mathbf{D}_1 = \sum_{j=1}^{n} T_{2\varpi_j} - (v - v^{-1})^2 \sum_{i=1}^{n-1} e^{-\alpha_i} T_{\varpi_i + \varpi_{i+1}}. \quad (3.6)$$

Let $\mathcal{A}_n$ be the associative $\mathbb{C}(v^{1/\mathsf{N}})$-algebra generated by $\{\mathsf{w}_j^{\pm 1}, \mathsf{D}_j^{\pm 1}\}_{j=1}^n$ with the defining relations

$$[\mathsf{w}_i, \mathsf{w}_j] = [\mathsf{D}_i, \mathsf{D}_j] = 0,\ \mathsf{w}_i^{\pm 1} \mathsf{w}_i^{\mp 1} = \mathsf{D}_i^{\pm 1} \mathsf{D}_i^{\mp 1} = 1,\ \mathsf{D}_i \mathsf{w}_j = v^{\delta_{i,j}} \mathsf{w}_j \mathsf{D}_i. \quad (3.7)$$

Define $\bar{\mathcal{A}}_n$ as the quotient of the $\mathbb{C}(v^{1/\mathsf{N}})$-subalgebra generated by $\{\mathsf{w}_j^{\pm 1}, (\mathsf{D}_i/\mathsf{D}_{i+1})^{\pm 1}\}_{1 \le i < n}^{1 \le j \le n}$ by the relation $\mathsf{w}_1 \cdots \mathsf{w}_n = 1$. Consider the anti-isomorphism from the algebra $\bar{\mathcal{A}}_n$ to the algebra $\mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sl}_n})$ of Section 2.5, sending $\mathsf{w}_j \mapsto T_{-\varpi_j}, \mathsf{D}_i/\mathsf{D}_{i+1} \mapsto e^{-\alpha_i}$. Then the hamiltonian $\mathbf{D}_1$ is the image of the following element $\mathsf{H} = \mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ of $\bar{\mathcal{A}}_n$:

$$\mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm}) = \sum_{j=1}^{n} \mathsf{w}_j^{-2} + \sum_{i=1}^{n-1} b_i v^{\sum_{k=1}^{n} \frac{2k-n-1}{2} \mathsf{m}_{ik}} \cdot \prod_{k=1}^{n} \mathsf{w}_k^{-\mathsf{m}_{ik} - \delta_{k,i} - \delta_{k,i+1}} \cdot \frac{\mathsf{D}_i}{\mathsf{D}_{i+1}} +$$
$$\sum_{1 \le i < j-1 \le n-1}^{\epsilon^{\pm}_{i,i+1} = \ldots = \epsilon^{\pm}_{j-2,j-1} = \pm 1} b_i \cdots b_{j-1} v^{j-i-1 + \sum_{i \le a < b \le j-1} (\mathsf{n}^{+}_{ab} - \mathsf{n}^{-}_{ab}) + \sum_{k=1}^{n} \sum_{s=i}^{j-1} \frac{2k-n-1}{2} \mathsf{m}_{sk}} \times$$
$$\prod_{k=1}^{n} \mathsf{w}_k^{-\sum_{s=i}^{j-1} \mathsf{m}_{sk} - \delta_{k,i} - \delta_{k,j}} \cdot \frac{\mathsf{D}_i}{\mathsf{D}_j}. \quad (3.8)$$

The following is the key property of $\mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ in type A.

Proposition 3.11. $\mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ depends only on $\vec{\epsilon} = (\epsilon_{n-2}, \ldots, \epsilon_1) \in \{-1, 0, 1\}^{n-2}$ with $\epsilon_i := \frac{\epsilon^{+}_{i,i+1} - \epsilon^{-}_{i,i+1}}{2}$, up to algebra automorphisms of $\bar{\mathcal{A}}_n$.

This result implies that given two pairs of Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ and $(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$ with $\epsilon^{+}_{i,i+1} - \epsilon^{-}_{i,i+1} = \tilde{\epsilon}^{+}_{i,i+1} - \tilde{\epsilon}^{-}_{i,i+1}$, there exists an algebra automorphism of $\mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sl}_n})$ that maps the 1st hamiltonian $\mathbf{D}_1(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathbf{D}_1(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$. As we will see in Appendix B, the same automorphism maps the modified quantum Toda system $\mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathcal{T}(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$.

We present the proof of Proposition 3.11 in Appendix A.

3.12. Explicit formulas and classification in type C_n

Recall explicit formulas for the action of $U_v(\mathfrak{sp}_{2n})$ on its 1st fundamental representation V_1. The space V_1 has a basis $\{w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n}\}$, in which the action is given via

$$E_i(w_j) = \delta_{j,i+1} w_{j-1},\ E_i(w_{n+j}) = \delta_{j,i} w_{n+j+1},\ E_n(w_j) = 0,\ E_n(w_{n+j}) = \delta_{j,n} w_n,$$
$$F_i(w_j) = \delta_{j,i} w_{j+1},\ F_i(w_{n+j}) = \delta_{j,i+1} w_{n+j-1},\ F_n(w_j) = \delta_{j,n} w_{2n},\ F_n(w_{n+j}) = 0,$$
$$L_i(w_j) = v^{\delta_{j \le i}} w_j,\ L_i(w_{n+j}) = v^{-\delta_{j \le i}} w_j,\ L_n(w_j) = v w_j,\ L_n(w_{n+j}) = v^{-1} w_{n+j},$$
$$K_i(w_j) = v^{\delta_{j,i} - \delta_{j,i+1}} w_j,\ K_i(w_{n+j}) = v^{-\delta_{j,i} + \delta_{j,i+1}} w_{n+j},$$
$$K_n(w_j) = v^{2\delta_{j,n}} w_j,\ K_n(w_{n+j}) = v^{-2\delta_{j,n}} w_{n+j}$$

for any $1 \le i < n, 1 \le j \le n$. Let ϖ_j be the weight of w_j $(1 \le j \le n)$, so that the weight of w_{n+j} equals $-\varpi_j$, while $(\varpi_i, \varpi_j) = \delta_{i,j}$. Recall that the simple positive roots are given by $\alpha_i = \varpi_i - \varpi_{i+1}$ $(1 \leq i \leq n-1)$ and $\alpha_n = 2\varpi_n$, while $\rho = \sum_{i=1}^n (n+1-i)\varpi_i$ and $\mathsf{d}_1 = \ldots = \mathsf{d}_{n-1} = 1, \mathsf{d}_n = 2$.

To compute $\mathbf{D}_1$ explicitly, we use the same strategy as in type A. Note that the operators $\{E_i^r, F_i^r\}_{1\leq i\leq n}^{r>1}$ act trivially on V_1. Therefore, applying formula (3.1), we can replace R_{α_i} by $\bar{R}_{\alpha_i} := 1 + (v_i - v_i^{-1})E_i \otimes F_i$. Let us now compute all the nonzero terms contributing to C'_{V_1}:

- Picking 1 out of each $\bar{R}^{\mathrm{op}}_{\alpha_i}, \bar{R}_{\alpha_i}$, we recover $\sum_{i=1}^n \left(v^{2(n+1-i)} \cdot K_{2\varpi_i} + v^{-2(n+1-i)} \cdot K_{-2\varpi_i}\right)$
- Picking nontrivial terms only at $\bar{R}^{\mathrm{op}}_{\alpha_j}, \bar{R}_{\alpha_i}$, the result does not depend on $\mathrm{Or}^{\pm}$ (hence, the orderings $\prec_{\pm}$) and the total contribution of the nonzero terms equals

$$\sum_{i=1}^{n-1} (v - v^{-1})^2 \left(v^{2(n+1-i)} F_i K_{\varpi_{i+1}} E_i K_{\varpi_i} + v^{-2(n-i)} F_i K_{-\varpi_i} E_i K_{-\varpi_{i+1}}\right) +$$
$$(v^2 - v^{-2})^2 v^2 F_n K_{-\varpi_n} E_n K_{\varpi_n}.$$

- The other terms contributing to C'_{V_1} depend on $\prec_{\pm}$ (only on $\mathrm{Or}^{\pm}$). Picking nontrivial terms only at $\bar{R}^{\mathrm{op}}_{\alpha_{j_1}}, \ldots, \bar{R}^{\mathrm{op}}_{\alpha_{j_{k'}}}, \bar{R}_{\alpha_{i_k}}, \ldots, \bar{R}_{\alpha_{i_1}}$ (in the order listed) is possible only if $i_k \prec_+ \cdots \prec_+ i_1$, $j_1 \prec_- \cdots \prec_- j_{k'}$, and gives a nonzero contribution to C'_{V_1} if and only if $k = k', i_k = i_{k-1} \pm 1 = \ldots = i_1 \pm (k-1)$ (the sign stays the same everywhere), and $i_a = j_a$ for $1 \leq a \leq k$. When computing these contributions, we shall distinguish between the two cases: $\max(i_1, i_k) = n$ and $\max(i_1, i_k) < n$. The total contribution of such terms with $k > 1$ equals

$$\sum_{1\leq i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\pm 1} (v - v^{-1})^{2(j-i+1)} v^{2(n+1-i)} \cdot F_i \cdots F_j K_{\varpi_{j+1}} E_j \cdots E_i K_{\varpi_i} +$$

$$\sum_{1\leq i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\pm 1} (v - v^{-1})^{2(n-i)} (v^2 - v^{-2})^2 v^{2(n+1-i)} \cdot F_i \cdots F_n K_{-\varpi_n} E_n \cdots E_i K_{\varpi_i} +$$

$$\sum_{1\leq i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\mp 1} (v - v^{-1})^{2(j-i+1)} v^{-2(n-j)} \cdot F_j \cdots F_i K_{-\varpi_i} E_i \cdots E_j K_{-\varpi_{j+1}} +$$

$$\sum_{1\leq i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\mp 1} (v - v^{-1})^{2(n-i)} (v^2 - v^{-2})^2 v^2 \cdot F_n \cdots F_i K_{-\varpi_i} E_i \cdots E_n K_{\varpi_n}.$$

We have listed all the nonzero terms contributing to C'_{V_1}. To obtain $\tilde{\mathbf{D}}_1 = \bar{\mathbf{D}}_{V_1}$, we should rewrite the above formulas via e_i, f_i and $L_p = K_{\varpi_1+\ldots+\varpi_p}$ $(1 \le p \le n)$, moving all the Cartan terms to the middle, and then apply the characters $\chi^{\pm}$ with $\chi^+(e_i) = c_i^+, \chi^-(f_i) = c_i^-$. Conjugating further by e^{ρ}, we obtain the explicit formula for the 1st hamiltonian $\mathbf{D}_1$ of the type C_n modified quantum difference Toda system. To write it down, define constants b_i, m_{ik} via $b_i := (v_i - v_i^{-1})^2 v_i^{\mathsf{n}^+_{ii}-\mathsf{n}^-_{ii}} c_i^+ c_i^-$ and $\mathsf{m}_{ik} := \sum_{p=k}^n (\mathsf{n}^-_{ip} - \mathsf{n}^+_{ip})$. Then we have

$$
\begin{aligned}
\mathbf{D}_1 = & \sum_{i=1}^n (T_{2\varpi_i} + T_{-2\varpi_i}) + b_n v^{\sum_{k=1}^n (k-n-1)\mathsf{m}_{nk}} \cdot e^{-\alpha_n} T_{\sum_{k=1}^n \mathsf{m}_{nk}\varpi_k} + \\
& \sum_{i=1}^{n-1} b_i v^{\sum_{k=1}^n (k-n-1)\mathsf{m}_{ik}} \cdot e^{-\alpha_i} \left(T_{\sum_{k=1}^n (\mathsf{m}_{ik}+\delta_{k,i}+\delta_{k,i+1})\varpi_k} + T_{\sum_{k=1}^n (\mathsf{m}_{ik}-\delta_{k,i}-\delta_{k,i+1})\varpi_k} \right) + \\
& \sum_{1\le i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\pm 1} b_i \cdots b_j v^{j-i+\sum_{i\le a<b\le j}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})+\sum_{k=1}^n \sum_{s=i}^j (k-n-1)\mathsf{m}_{sk}} \times \\
& \qquad e^{-\alpha_i-\ldots-\alpha_j} T_{\sum_{k=1}^n (\sum_{s=i}^j \mathsf{m}_{sk}+\delta_{k,i}+\delta_{k,j+1})\varpi_k} + \\
& \sum_{1\le i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\pm 1} b_i \cdots b_n v^{n+1-i+\sum_{i\le a<b\le n}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})(1+\delta_{b,n})+\sum_{k=1}^n \sum_{s=i}^n (k-n-1)\mathsf{m}_{sk}} \times \\
& \qquad e^{-\alpha_i-\ldots-\alpha_n} T_{\sum_{k=1}^n (\sum_{s=i}^n \mathsf{m}_{sk}+\delta_{k,i}-\delta_{k,n})\varpi_k} + \\
& \sum_{1\le i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\mp 1} b_i \cdots b_j v^{j-i+\sum_{i\le a<b\le j}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^n \sum_{s=i}^j (k-n-1)\mathsf{m}_{sk}} \times \\
& \qquad e^{-\alpha_i-\ldots-\alpha_j} T_{\sum_{k=1}^n (\sum_{s=i}^j \mathsf{m}_{sk}-\delta_{k,i}-\delta_{k,j+1})\varpi_k} + \\
& \sum_{1\le i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\mp 1} b_i \cdots b_n v^{n+1-i+\sum_{i\le a<b\le n}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^n \sum_{s=i}^n (k-n-1)\mathsf{m}_{sk}} \times \\
& \qquad e^{-\alpha_i-\ldots-\alpha_n} T_{\sum_{k=1}^n (\sum_{s=i}^n \mathsf{m}_{sk}-\delta_{k,i}+\delta_{k,n})\varpi_k}. \qquad (3.9)
\end{aligned}
$$

Remark 3.13. If $\epsilon^+ = \epsilon^-$, then the last four sums are vacuous. If we also set $\mathsf{n}^+ = \mathsf{n}^-$ and $c_i^{\pm} = \pm 1$ for all i, then we obtain the formula for the 1st hamiltonian of the type C_n quantum difference Toda lattice as defined in [6] (we write down this formula as we could not find it in the literature, even though it can be derived completely analogously to [6, (5.7)], cf. [7, the end of Section 3]):

$$\mathbf{D}_1=\sum_{i=1}^{n}(T_{2\varpi_i}+T_{-2\varpi_i})-(v-v^{-1})^2\sum_{i=1}^{n-1}e^{-\alpha_i}\left(T_{\varpi_i+\varpi_{i+1}}+T_{-\varpi_i-\varpi_{i+1}}\right)-(v^2-v^{-2})^2e^{-\alpha_n}. \quad (3.10)$$

Let $\mathcal{C}_n$ be the $\mathbb{C}(v^{1/\mathsf{N}})$-subalgebra of $\mathcal{A}_n$ generated by $\{\mathsf{w}_j^{\pm 1},(\mathsf{D}_i/\mathsf{D}_{i+1})^{\pm 1},\mathsf{D}_n^{\pm 2}\}_{1\le i<n}^{1\le j\le n}$ (note that $\mathcal{C}_n$ can be abstractly defined as the associative algebra generated by $\{\tilde{\mathsf{w}}_i^{\pm 1},\tilde{\mathsf{D}}_i^{\pm 1}\}_{i=1}^n$ with the defining relations $[\tilde{\mathsf{w}}_i,\tilde{\mathsf{w}}_j]=[\tilde{\mathsf{D}}_i,\tilde{\mathsf{D}}_j]=0,\tilde{\mathsf{w}}_i^{\pm 1}\tilde{\mathsf{w}}_i^{\mp 1}=\tilde{\mathsf{D}}_i^{\pm 1}\tilde{\mathsf{D}}_i^{\mp 1}=1,\tilde{\mathsf{D}}_i\tilde{\mathsf{w}}_j=v^{\delta_{i,j}\mathsf{d}_i}\tilde{\mathsf{w}}_j\tilde{\mathsf{D}}_i$, where $\mathsf{d}_i=1+\delta_{i,n}$). Consider the anti-isomorphism from $\mathcal{C}_n$ to the algebra $\mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sp}_{2n}})$ of Section 2.5, sending $\mathsf{w}_j\mapsto T_{-\varpi_j},\mathsf{D}_i/\mathsf{D}_{i+1}\mapsto e^{-\alpha_i},\mathsf{D}_n^2\mapsto e^{-\alpha_n}$. Then the hamiltonian $\mathbf{D}_1$ is the image of the following element $\mathsf{H}=\mathsf{H}(\epsilon^{\pm},\mathsf{n}^{\pm},c^{\pm})$ of $\mathcal{C}_n$:

$$\mathsf{H}(\epsilon^{\pm},\mathsf{n}^{\pm},c^{\pm})=\sum_{i=1}^{n}(\mathsf{w}_i^{-2}+\mathsf{w}_i^2)+b_nv^{\sum_{k=1}^n(k-n-1)\mathsf{m}_{nk}}\cdot\prod_{k=1}^{n}\mathsf{w}_k^{-\mathsf{m}_{nk}}\cdot\mathsf{D}_n^2+$$
$$\sum_{i=1}^{n-1}b_iv^{\sum_{k=1}^n(k-n-1)\mathsf{m}_{ik}}\cdot\left(\prod_{k=1}^{n}\mathsf{w}_k^{-\mathsf{m}_{ik}-\delta_{k,i}-\delta_{k,i+1}}+\prod_{k=1}^{n}\mathsf{w}_k^{-\mathsf{m}_{ik}+\delta_{k,i}+\delta_{k,i+1}}\right)\cdot\frac{\mathsf{D}_i}{\mathsf{D}_{i+1}}+$$
$$\sum_{1\le i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\pm 1}b_i\cdots b_jv^{j-i+\sum_{i\le a<b\le j}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})+\sum_{k=1}^n\sum_{s=i}^j(k-n-1)\mathsf{m}_{sk}}\times$$
$$\prod_{k=1}^{n}\mathsf{w}_k^{-\sum_{s=i}^j\mathsf{m}_{sk}-\delta_{k,i}-\delta_{k,j+1}}\cdot\frac{\mathsf{D}_i}{\mathsf{D}_{j+1}}+$$
$$\sum_{1\le i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\mp 1}b_i\cdots b_jv^{j-i+\sum_{i\le a<b\le j}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^n\sum_{s=i}^j(k-n-1)\mathsf{m}_{sk}}\times$$
$$\prod_{k=1}^{n}\mathsf{w}_k^{-\sum_{s=i}^j\mathsf{m}_{sk}+\delta_{k,i}+\delta_{k,j+1}}\cdot\frac{\mathsf{D}_i}{\mathsf{D}_{j+1}}+$$
$$\sum_{1\le i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\pm 1}b_i\cdots b_nv^{n+1-i+\sum_{i\le a<b\le n}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})(1+\delta_{b,n})+\sum_{k=1}^n\sum_{s=i}^n(k-n-1)\mathsf{m}_{sk}}\times$$
$$\prod_{k=1}^{n}\mathsf{w}_k^{-\sum_{s=i}^n\mathsf{m}_{sk}-\delta_{k,i}+\delta_{k,n}}\cdot\mathsf{D}_i\mathsf{D}_n+$$
$$\sum_{1\le i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\mp 1}b_i\cdots b_nv^{n+1-i+\sum_{i\le a<b\le n}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^n\sum_{s=i}^n(k-n-1)\mathsf{m}_{sk}}\times$$
$$\prod_{k=1}^{n}\mathsf{w}_k^{-\sum_{s=i}^n\mathsf{m}_{sk}+\delta_{k,i}-\delta_{k,n}}\cdot\mathsf{D}_i\mathsf{D}_n. \quad (3.11)$$

The following is the key property of $\mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ in type C.

Proposition 3.14. $\mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ depends only on $\vec{\epsilon} = (\epsilon_{n-1}, \ldots, \epsilon_1) \in \{-1, 0, 1\}^{n-1}$ with $\epsilon_i := \frac{\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1}}{2}$, up to algebra automorphisms of $\mathcal{C}_n$.

The proof of this result is completely analogous to that of Proposition 3.11 given in Appendix A, see also Remark A.1; we leave the details to the interested reader. Proposition 3.14 implies that given two pairs of Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ and $(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$ with $\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1} = \tilde{\epsilon}^+_{i,i+1} - \tilde{\epsilon}^-_{i,i+1}$, there exists an algebra automorphism of $\mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sp}_{2n}})$ that maps the 1st hamiltonian $\mathbf{D}_1(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathbf{D}_1(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$. As we will see in Appendix B, the same automorphism maps the modified quantum Toda system $\mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathcal{T}(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$.

3.15 Explicit formulas and classification in type D_n

Recall explicit formulas for the action of $U_v(\mathfrak{so}_{2n})$ on its 1st fundamental representation V_1. The space V_1 has a basis $\{w_1, \ldots, w_{2n}\}$, in which the action is given via

$$E_i(w_{2j-1}) = \delta_{j,i+1} w_{2j-3},\ E_i(w_{2j}) = \delta_{j,i} w_{2j+2},$$

$$F_i(w_{2j-1}) = \delta_{j,i} w_{2j+1},\ F_i(w_{2j}) = \delta_{j,i+1} w_{2j-2},$$

$$L_p(w_{2j-1}) = v^{\delta_{j\le p}} w_{2j-1},\ L_p(w_{2j}) = v^{-\delta_{j\le p}} w_{2j},$$

$$L_{n-1}(w_{2j-1}) = v^{\frac{1}{2}-\delta_{j,n}} w_{2j-1},\ L_{n-1}(w_{2j}) = v^{-\frac{1}{2}+\delta_{j,n}} w_{2j},$$

$$K_i(w_{2j-1}) = v^{\delta_{j,i}-\delta_{j,i+1}} w_{2j-1},\ K_i(w_{2j}) = v^{-\delta_{j,i}+\delta_{j,i+1}} w_{2j},$$

$$E_n(w_{2j}) = \delta_{j,n-1} w_{2n-1} + \delta_{j,n} w_{2n-3},\ F_n(w_{2j-1}) = \delta_{j,n-1} w_{2n} + \delta_{j,n} w_{2n-2},$$

$$E_n(w_{2j-1}) = 0,\ F_n(w_{2j}) = 0,\ L_n(w_{2j-1}) = v^{\frac{1}{2}} w_{2j-1},\ L_n(w_{2j}) = v^{-\frac{1}{2}} w_{2j},$$

$$K_n(w_{2j-1}) = v^{\delta_{j,n}+\delta_{j,n-1}} w_{2j-1},\ K_n(w_{2j}) = v^{-\delta_{j,n}-\delta_{j,n-1}} w_{2j}$$

for any $1 \le p \le n-2, 1 \le i < n, 1 \le j \le n$. Let ϖ_j be the weight of w_{2j-1} $(1 \le j \le n)$, so that the weight of w_{2j} equals $-\varpi_j$, while $(\varpi_i, \varpi_j) = \delta_{i,j}$. Recall that the simple roots are given by $\alpha_i = \varpi_i - \varpi_{i+1}$ $(1 \le i \le n-1)$ and $\alpha_n = \varpi_{n-1} + \varpi_n$, while $\rho = \sum_{i=1}^n (n-i)\varpi_i$ and $\mathsf{d}_1 = \ldots = \mathsf{d}_n = 1$.

To compute $\mathbf{D}_1$ explicitly, we use the same strategy as in type A. Similarly to the types A and C treated above, we note that the operators $\{E_i^r, F_i^r\}_{1\le i\le n}^{r>1}$ act trivially on V_1; hence, applying formula (3.1), we can replace R_{α_i} by $\bar{R}_{\alpha_i} := 1 + (v - v^{-1}) E_i \otimes F_i$. Let us now compute all the nonzero terms contributing to C'_{V_1}:

- Picking 1 out of each $\bar{R}^{\mathrm{op}}_{\alpha_i}, \bar{R}_{\alpha_i}$, we recover $\sum_{i=1}^n \left(v^{2(n-i)} \cdot K_{2\varpi_i} + v^{-2(n-i)} \cdot K_{-2\varpi_i}\right)$.

- Picking nontrivial terms only at $\bar{R}^{\mathrm{op}}_{\alpha_j}, \bar{R}_{\alpha_i}$, the result does not depend on $\mathrm{Or}^{\pm}$ (hence, the orderings $\prec_{\pm}$) and the total contribution of the nonzero terms equals

$$\sum_{i=1}^{n-1}(v-v^{-1})^2\left(v^{2(n-i)}F_iK_{\varpi_{i+1}}E_iK_{\varpi_i}+v^{-2(n-i-1)}F_iK_{-\varpi_i}E_iK_{-\varpi_{i+1}}\right)+$$
$$(v-v^{-1})^2\left(F_nK_{-\varpi_{n-1}}E_nK_{\varpi_n}+F_nK_{-\varpi_n}E_nK_{\varpi_{n-1}}\right).$$

- In contrast to the types A, C considered above, there is one more summand independent of the orientations. It arises by picking nontrivial terms only at $\bar{R}^{\mathrm{op}}_{\alpha_{n-1}}, \bar{R}^{\mathrm{op}}_{\alpha_n}$ and $\bar{R}_{\alpha_{n-1}}, \bar{R}_{\alpha_n}$ (note that $E_{n-1}E_n=E_nE_{n-1}, F_{n-1}F_n=F_nF_{n-1}$, due to the v-Serre relations) and equals

$$(v-v^{-1})^4v^2F_nF_{n-1}K_{-\varpi_{n-1}}E_nE_{n-1}K_{\varpi_{n-1}}.$$

- The contribution of the remaining terms to C'_{V_1} depends on $\mathrm{Or}^{\pm}$. Tracing back explicit formulas for the action of $U_v(\mathfrak{so}_{2n})$ on V_1, we see that the total sum of such terms equals

$$\sum_{1\leq i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\pm 1}(v-v^{-1})^{2(j-i+1)}v^{2(n-i)}\cdot F_i\cdots F_jK_{\varpi_{j+1}}E_j\cdots E_iK_{\varpi_i}+$$
$$\sum_{1\leq i<n-1}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-3,n-2}=\epsilon^{\pm}_{n-2,n}=\pm 1}(v-v^{-1})^{2(n-i)}v^{2(n-i)}\cdot$$
$$F_i\cdots F_{n-2}F_nK_{-\varpi_n}E_nE_{n-2}\cdots E_iK_{\varpi_i}+$$
$$\sum_{1\leq i<n-1}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-2,n-1}=\epsilon^{\pm}_{n-2,n}=\pm 1}(v-v^{-1})^{2(n-i+1)}v^{2(n-i)}\cdot$$
$$F_i\cdots F_{n-1}F_nK_{-\varpi_{n-1}}E_nE_{n-1}\cdots E_iK_{\varpi_i}+$$
$$\sum_{1\leq i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\mp 1}(v-v^{-1})^{2(j-i+1)}v^{-2(n-j-1)}\cdot F_j\cdots F_iK_{-\varpi_i}E_i\cdots E_jK_{-\varpi_{j+1}}+$$
$$\sum_{1\leq i<n-1}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-3,n-2}=\epsilon^{\pm}_{n-2,n}=\mp 1}(v-v^{-1})^{2(n-i)}\cdot F_nF_{n-2}\cdots F_iK_{-\varpi_i}E_i\cdots E_{n-2}E_nK_{\varpi_n}+$$

$$\sum_{1\le i<n-1}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-2,n-1}=\epsilon^{\pm}_{n-2,n}=\mp 1} (v-v^{-1})^{2(n-i+1)}v^2\cdot$$

$$F_nF_{n-1}\cdots F_iK_{-\varpi_i}E_i\cdots E_{n-1}E_nK_{\varpi_{n-1}}.$$

We have listed all the nonzero terms contributing to C'_{V_1}. To obtain $\tilde{\mathbf{D}}_1=\bar{\mathbf{D}}_{V_1}$, we should rewrite the above formulas via e_i, f_i and $L_p=\begin{cases}K_{\varpi_1+\ldots+\varpi_p}, & \text{if } 1\le p\le n-2\\ K_{\frac{1}{2}(\varpi_1+\ldots+\varpi_{n-1}-\varpi_n)}, & \text{if } p=n-1\\ K_{\frac{1}{2}(\varpi_1+\ldots+\varpi_{n-1}+\varpi_n)}, & \text{if } p=n\end{cases}$, moving all the Cartan terms to the middle, and then apply the characters $\chi^{\pm}$ with $\chi^+(e_i)=c_i^+, \chi^-(f_i)=c_i^-$. Conjugating further by e^{ρ}, we obtain the explicit formula for the 1st hamiltonian $\mathbf{D}_1$ of the type D_n modified quantum difference Toda system. To write it down, define $\mathsf{m}_{ik}:=\begin{cases}\sum_{p=k}^{n-2}(\mathsf{n}^-_{ip}-\mathsf{n}^+_{ip})+\frac{1}{2}(\mathsf{n}^-_{i,n-1}-\mathsf{n}^+_{i,n-1})+\frac{1}{2}(\mathsf{n}^-_{in}-\mathsf{n}^+_{in}), & \text{if } k<n\\ -\frac{1}{2}(\mathsf{n}^-_{i,n-1}-\mathsf{n}^+_{i,n-1})+\frac{1}{2}(\mathsf{n}^-_{in}-\mathsf{n}^+_{in}), & \text{if } k=n\end{cases}$ and $b_i:=(v-v^{-1})^2v^{\mathsf{n}^+_{ii}-\mathsf{n}^-_{ii}}c_i^+c_i^-$. Then we have

$$\mathbf{D}_1=\sum_{i=1}^{n}(T_{2\varpi_i}+T_{-2\varpi_i})+b_nv^{\sum_{k=1}^n(k-n)\mathsf{m}_{nk}}\cdot e^{-\alpha_n}T_{\sum_{k=1}^n(\mathsf{m}_{nk}-\delta_{k,n-1}+\delta_{k,n})\varpi_k}+$$
$$b_nv^{-2+\sum_{k=1}^n(k-n)\mathsf{m}_{nk}}\cdot e^{-\alpha_n}T_{\sum_{k=1}^n(\mathsf{m}_{nk}+\delta_{k,n-1}-\delta_{k,n})\varpi_k}+$$
$$b_{n-1}b_nv^{(\mathsf{n}^+_{n-1,n}-\mathsf{n}^-_{n-1,n})+\sum_{k=1}^n(k-n)(\mathsf{m}_{n-1,k}+\mathsf{m}_{nk})}\cdot e^{-\alpha_{n-1}-\alpha_n}T_{\sum_{k=1}^n(\mathsf{m}_{n-1,k}+\mathsf{m}_{n,k})\varpi_k}+$$
$$\sum_{i=1}^{n-1}b_iv^{\sum_{k=1}^n(k-n)\mathsf{m}_{ik}}\cdot e^{-\alpha_i}\left(T_{\sum_{k=1}^n(\mathsf{m}_{ik}+\delta_{k,i}+\delta_{k,i+1})\varpi_k}+T_{\sum_{k=1}^n(\mathsf{m}_{ik}-\delta_{k,i}-\delta_{k,i+1})\varpi_k}\right)+$$
$$\sum_{1\le i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\pm 1} b_i\cdots b_jv^{j-i+\sum_{i\le a<b\le j}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})+\sum_{k=1}^n\sum_{s=i}^j(k-n)\mathsf{m}_{sk}}\times$$
$$e^{-\alpha_i-\ldots-\alpha_j}T_{\sum_{k=1}^n(\sum_{s=i}^j\mathsf{m}_{sk}+\delta_{k,i}+\delta_{k,j+1})\varpi_k}+$$
$$\sum_{1\le i<n-1}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-2,n}=\pm 1} b_i\cdots b_{n-2}b_nv^{n-i-1+\sum_{i\le a<b\le n}^{a,b\ne n-1}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})+\sum_{k=1}^n\sum_{i\le s\le n}^{s\ne n-1}(k-n)\mathsf{m}_{sk}}\times$$
$$e^{-\alpha_i-\ldots-\alpha_{n-2}-\alpha_n}T_{\sum_{k=1}^n(\sum_{i\le s\le n}^{s\ne n-1}\mathsf{m}_{sk}+\delta_{k,i}-\delta_{k,n})\varpi_k}+$$
$$\sum_{1\le i<n-1}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-2,n-1}=\epsilon^{\pm}_{n-2,n}=\pm 1} b_i\cdots b_nv^{n-i+\sum_{i\le a<b\le n}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})+\sum_{k=1}^n\sum_{s=i}^n(k-n)\mathsf{m}_{sk}}\times \qquad (3.12)$$

$$e^{-\alpha_i-\ldots-\alpha_n}T_{\sum_{k=1}^n(\sum_{s=i}^n \mathsf{m}_{sk}+\delta_{k,i}-\delta_{k,n-1})\varpi_k}+$$

$$\sum_{1\le i<j<n}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{j-1,j}=\mp 1} b_i\cdots b_j v^{j-i+\sum_{i\le a<b\le j}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^n\sum_{s=i}^j(k-n)\mathsf{m}_{sk}}\times$$

$$e^{-\alpha_i-\ldots-\alpha_j}T_{\sum_{k=1}^n(\sum_{s=i}^j \mathsf{m}_{sk}-\delta_{k,i}-\delta_{k,j+1})\varpi_k}+$$

$$\sum_{1\le i<n-1}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{n-2,n}=\mp 1} b_i\cdots b_{n-2}b_n v^{n-i-1+\sum_{i\le a<b\le n}^{a,b\neq n-1}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^n\sum_{i\le s\le n}^{s\neq n-1}(k-n)\mathsf{m}_{sk}}\times$$

$$e^{-\alpha_i-\ldots-\alpha_{n-2}-\alpha_n}T_{\sum_{k=1}^n(\sum_{i\le s\le n}^{s\neq n-1} \mathsf{m}_{sk}-\delta_{k,i}+\delta_{k,n})\varpi_k}+$$

$$\sum_{1\le i<n-1}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{n-2,n-1}=\epsilon^\pm_{n-2,n}=\mp 1} b_i\cdots b_n v^{n-i+\sum_{i\le a<b\le n}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^n\sum_{s=i}^n(k-n)\mathsf{m}_{sk}}\times$$

$$e^{-\alpha_i-\ldots-\alpha_n}T_{\sum_{k=1}^n(\sum_{s=i}^n \mathsf{m}_{sk}-\delta_{k,i}+\delta_{k,n-1})\varpi_k}.$$

Remark 3.16. If $\epsilon^+ = \epsilon^-$, then the last six sums are vacuous. If we further set $\mathsf{n}^+ = \mathsf{n}^-$ and $c_i^\pm = \pm 1$ for all i, then we obtain the formula for the 1st hamiltonian of the type D_n quantum difference Toda lattice as defined in [6] (we write down this formula as we could not find it in the literature, even though it can be derived completely analogously to [6, (5.7)], cf. [7, the end of Section 3]):

$$\mathbf{D}_1 = \sum_{i=1}^n (T_{2\varpi_i}+T_{-2\varpi_i}) - (v-v^{-1})^2\sum_{i=1}^{n-1} e^{-\alpha_i}\left(T_{\varpi_i+\varpi_{i+1}}+T_{-\varpi_i-\varpi_{i+1}}\right) -$$
$$(v-v^{-1})^2 e^{-\alpha_n}\left(T_{-\varpi_{n-1}+\varpi_n}+v^{-2}T_{\varpi_{n-1}-\varpi_n}\right)+(v-v^{-1})^4 e^{-\alpha_{n-1}-\alpha_n}. \quad (3.13)$$

Recall the algebra $\mathcal{C}_n$ from Section 3.12. Consider the anti-isomorphism from $\mathcal{C}_n$ to the algebra $\mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{so}_{2n}})$ of Section 2.5, sending $\mathsf{w}_j \mapsto T_{-\varpi_j}, \mathsf{D}_i/\mathsf{D}_{i+1} \mapsto e^{-\alpha_i}, \mathsf{D}_n^2 \mapsto e^{\alpha_{n-1}-\alpha_n}$. Let $\mathsf{H} = \mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ be the element of $\mathcal{C}_n$ that corresponds to $\mathbf{D}_1$ under this anti-isomorphism (to save space, we omit the explicit long formula for H). The following is the key property of $\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ in type D.

Proposition 3.17. $\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ depends only on $\vec{\epsilon} = (\epsilon_{n-1},\ldots,\epsilon_1) \in \{-1,0,1\}^{n-1}$ with $\epsilon_i := \frac{\epsilon^+_{i,i+1}-\epsilon^-_{i,i+1}}{2}$ $(1\le i\le n-2), \epsilon_{n-1} := \frac{\epsilon^+_{n-2,n}-\epsilon^-_{n-2,n}}{2}$, up to algebra automorphisms of $\mathcal{C}_n$.

The proof of this result is completely analogous to that of Proposition 3.11 given in Appendix A, see also Remark A.1; we leave the details to the interested reader. Proposition 3.17 implies that given two pairs of Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ and $(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$ with $\epsilon^{+}_{i,i+1} - \epsilon^{-}_{i,i+1} = \tilde{\epsilon}^{+}_{i,i+1} - \tilde{\epsilon}^{-}_{i,i+1}$ $(1 \leq i \leq n-2)$ and $\epsilon^{+}_{n-2,n} - \epsilon^{-}_{n-2,n} = \tilde{\epsilon}^{+}_{n-2,n} - \tilde{\epsilon}^{-}_{n-2,n}$, there exists an algebra automorphism of $\mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{so}_{2n}})$ that maps the 1st hamiltonian $\mathbf{D}_1(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathbf{D}_1(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$. As we will see in Appendix B, the same automorphism maps the modified quantum Toda system $\mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathcal{T}(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$.

3.18 Explicit formulas and classification in type B_n

Recall explicit formulas for the action of $U_v(\mathfrak{so}_{2n+1})$ on its 1st fundamental representation V_1. The space V_1 has a basis $\{w_0, \ldots, w_{2n}\}$, in which the action is given via

$$E_i(w_{2j-1}) = \delta_{j,i+1} w_{2j-3},\ E_i(w_{2j}) = \delta_{j,i} w_{2j+2},\ E_i(w_0) = 0,$$

$$F_i(w_{2j-1}) = \delta_{j,i} w_{2j+1},\ F_i(w_{2j}) = \delta_{j,i+1} w_{2j-2},\ F_i(w_0) = 0,$$

$$L_i(w_{2j-1}) = v^{2\delta_{j\leq i}} w_{2j-1},\ L_i(w_{2j}) = v^{-2\delta_{j\leq i}} w_{2j},\ L_i(w_0) = w_0,$$

$$K_i(w_{2j-1}) = v^{2\delta_{j,i} - 2\delta_{j,i+1}} w_{2j-1},\ K_i(w_{2j}) = v^{-2\delta_{j,i} + 2\delta_{j,i+1}} w_{2j},\ K_i(w_0) = w_0,$$

$$E_n(w_{2j-1}) = 0,\ E_n(w_{2j}) = \delta_{j,n} w_0,\ E_n(w_0) = w_{2n-1},$$

$$F_n(w_{2j-1}) = \delta_{j,n} w_0,\ F_n(w_{2j}) = 0,\ F_n(w_0) = w_{2n},$$

$$L_n(w_{2j-1}) = v w_{2j-1},\ L_n(w_{2j}) = v^{-1} w_{2j},\ L_n(w_0) = w_0,$$

$$K_n(w_{2j-1}) = v^{\delta_{j,n}} w_{2j-1},\ K_n(w_{2j}) = v^{-\delta_{j,n}} w_{2j},\ K_n(w_0) = w_0$$

for any $1 \leq i < n, 1 \leq j \leq n$.

Let ϖ_j be the weight of w_{2j-1} $(1 \leq j \leq n)$, so that the weight of w_{2j} equals $-\varpi_j$, while w_0 has the zero weight. We note that now $(\varpi_i, \varpi_j) = 2\delta_{i,j}$. Recall that the simple roots are given by $\alpha_i = \varpi_i - \varpi_{i+1}$ $(1 \leq i \leq n-1)$ and $\alpha_n = \varpi_n$, while $\rho = \sum_{i=1}^{n} (n + \frac{1}{2} - i)\varpi_i$ and $\mathsf{d}_1 = \ldots = \mathsf{d}_{n-1} = 2, \mathsf{d}_n = 1$.

To compute $\mathbf{D}_1$ explicitly, we use the same strategy as in type A. In contrast to the types A, C, D treated above, E_n^2 and F_n^2 act nontrivially on V_1, while $\{E_i^r, F_i^r\}_{1\leq i<n}^{r>1} \cup \{E_n^s, F_n^s\}_{s>2}$ still act by zero on V_1. Therefore, applying formula (3.1), we can replace R_{α_i} by $\bar{R}_{\alpha_i} := 1 + (v^2 - v^{-2}) E_i \otimes F_i$ for $1 \leq i < n$ and R_{α_n} by $\bar{R}_{\alpha_n} := 1 + (v - v^{-1}) E_n \otimes F_n + c E_n^2 \otimes F_n^2$ for $c := (1 - v^{-1})(v - v^{-1})$. Let us now compute all the non-zero terms contributing to C'_{V_1}:

- Picking 1 out of each $\bar{R}_i^{\mathrm{op}}, \bar{R}_i$, we recover $1 + \sum_{i=1}^{n} \Big(v^{4n+2-4i} \cdot K_{2\varpi_i} + v^{-4n-2+4i} \cdot K_{-2\varpi_i} \Big)$.

- Picking nontrivial terms only at $\bar{R}^{\mathrm{op}}_{\alpha_j}, \bar{R}_{\alpha_i}$, the result does not depend on $\mathrm{Or}^{\pm}$ (hence, the orderings $\prec_{\pm}$) and the total contribution of the nonzero terms equals

$$\sum_{i=1}^{n-1}(v^2-v^{-2})^2\left(v^{4n+2-4i}F_iK_{\varpi_{i+1}}E_iK_{\varpi_i}+v^{-4n+2+4i}F_iK_{-\varpi_i}E_iK_{-\varpi_{i+1}}\right)+$$
$$(v-v^{-1})^2(F_nK_{-\varpi_n}E_n+v^2F_nE_nK_{\varpi_n})+c^2v^2F_n^2K_{-\varpi_n}E_n^2K_{\varpi_n}.$$

- The contribution of the remaining terms to C'_{V_1} depends on $\mathrm{Or}^{\pm}$. Tracing back explicit formulas for the action of $U_v(\mathfrak{so}_{2n+1})$ on V_1, we see that the total sum of such terms equals

$$\sum_{1\le i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\pm 1}(v^2-v^{-2})^{2(j-i+1)}v^{4n+2-4i}\cdot F_i\cdots F_jK_{\varpi_{j+1}}E_j\cdots E_iK_{\varpi_i}+$$
$$\sum_{1\le i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\pm 1}(v^2-v^{-2})^{2(n-i)}(v-v^{-1})^2v^{4n+2-4i}\cdot F_i\cdots F_nE_n\cdots E_iK_{\varpi_i}+$$
$$\sum_{1\le i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\pm 1}c^2(v^2-v^{-2})^{2(n-i)}v^{4n+2-4i}\cdot F_i\cdots F_{n-1}F_n^2K_{-\varpi_n}E_n^2E_{n-1}\cdots E_iK_{\varpi_i}+$$
$$\sum_{1\le i<j<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{j-1,j}=\mp 1}(v^2-v^{-2})^{2(j-i+1)}v^{-4n+2+4j}\cdot F_j\cdots F_iK_{-\varpi_i}E_i\cdots E_jK_{-\varpi_{j+1}}+$$
$$\sum_{1\le i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\mp 1}(v^2-v^{-2})^{2(n-i)}(v-v^{-1})^2\cdot F_n\cdots F_iK_{-\varpi_i}E_i\cdots E_n+$$
$$\sum_{1\le i<n}^{\epsilon^{\pm}_{i,i+1}=\ldots=\epsilon^{\pm}_{n-1,n}=\mp 1}c^2(v^2-v^{-2})^{2(n-i)}v^2\cdot F_n^2F_{n-1}\cdots F_iK_{-\varpi_i}E_i\cdots E_{n-1}E_n^2K_{\varpi_n}.$$

We have listed all the nonzero terms contributing to C'_{V_1}. To obtain $\tilde{\mathbf{D}}_1=\bar{\mathbf{D}}_{V_1}$, we should rewrite the above formulas via e_i,f_i and $L_p=\begin{cases}K_{\varpi_1+\ldots+\varpi_p}, & \text{if } 1\le p<n\\ K_{\frac{1}{2}(\varpi_1+\ldots+\varpi_n)}, & \text{if } p=n\end{cases}$, moving all the Cartan terms to the middle, and then apply the characters $\chi^{\pm}$ with $\chi^+(e_i)=c_i^+,\chi^-(f_i)=c_i^-$. Conjugating further by e^{ρ}, we obtain the explicit formula for the 1st hamiltonian $\mathbf{D}_1$ of the type B_n modified quantum difference Toda

system. To write it down, define constants b_i, m_{ik} via $\mathsf{m}_{ik} := \sum_{p=k}^{n}(\mathsf{n}^-_{ip} - \mathsf{n}^+_{ip})(1 - \frac{1}{2}\delta_{p,n})$ and $b_i := (v_i - v_i^{-1})^2 v_i^{\mathsf{n}^+_{ii} - \mathsf{n}^-_{ii}} c_i^+ c_i^-$. Then we have

$$
\begin{aligned}
\mathbf{D}_1 = 1 &+ \sum_{i=1}^{n}(T_{2\varpi_i} + T_{-2\varpi_i}) + \\
& b_n v^{\sum_{k=1}^{n}(2k-2n-1)\mathsf{m}_{nk}} \cdot e^{-\alpha_n}\left(vT_{\sum_{k=1}^{n}(\mathsf{m}_{nk}-\delta_{k,n})\varpi_k} + v^{-1}T_{\sum_{k=1}^{n}(\mathsf{m}_{nk}+\delta_{k,n})\varpi_k}\right) + \\
& (1+v)^{-2} b_n^2 v^{-2+(\mathsf{n}^+_{nn}-\mathsf{n}^-_{nn})+\sum_{k=1}^{n}(4k-4n-2)\mathsf{m}_{nk}} \cdot e^{-2\alpha_n} T_{\sum_{k=1}^{n} 2\mathsf{m}_{n,k}\varpi_k} + \\
& \sum_{i=1}^{n-1} b_i v^{\sum_{k=1}^{n}(2k-2n-1)\mathsf{m}_{ik}} \cdot e^{-\alpha_i}\left(T_{\sum_{k=1}^{n}(\mathsf{m}_{ik}+\delta_{k,i}+\delta_{k,i+1})\varpi_k} + T_{\sum_{k=1}^{n}(\mathsf{m}_{ik}-\delta_{k,i}-\delta_{k,i+1})\varpi_k}\right) + \\
& \sum_{1\le i<j<n}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{j-1,j}=\pm 1} b_i\cdots b_j v^{2j-2i+2\sum_{i\le a<b\le j}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})+\sum_{k=1}^{n}\sum_{s=i}^{j}(2k-2n-1)\mathsf{m}_{sk}} \times \\
& \qquad e^{-\alpha_i-\ldots-\alpha_j} T_{\sum_{k=1}^{n}(\sum_{s=i}^{j}\mathsf{m}_{sk}+\delta_{k,i}+\delta_{k,j+1})\varpi_k} + \\
& \sum_{1\le i<n}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{n-1,n}=\pm 1} b_i\cdots b_n v^{2n-2i-1+\sum_{i\le a<b\le n}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})(2-\delta_{b,n})+\sum_{k=1}^{n}\sum_{s=i}^{n}(2k-2n-1)\mathsf{m}_{sk}} \times \\
& \qquad e^{-\alpha_i-\ldots-\alpha_n} T_{\sum_{k=1}^{n}(\sum_{s=i}^{n}\mathsf{m}_{sk}+\delta_{k,i})\varpi_k} + \\
& \sum_{1\le i<n}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{n-1,n}=\pm 1} (1+v)^{-2} b_i\cdots b_{n-1} b_n^2 v^{2n-2i+(\mathsf{n}^+_{nn}-\mathsf{n}^-_{nn})+2\sum_{i\le a<b\le n}(\mathsf{n}^+_{ab}-\mathsf{n}^-_{ab})} \times \\
& v^{\sum_{k=1}^{n}\sum_{s=i}^{j}(2k-2n-1)(1+\delta_{s,n})\mathsf{m}_{sk}} \cdot e^{-\alpha_i-\ldots-\alpha_{n-1}-2\alpha_n} T_{\sum_{k=1}^{n}(\sum_{s=i}^{n}\mathsf{m}_{sk}+\mathsf{m}_{nk}+\delta_{k,i}-\delta_{k,n})\varpi_k} + \\
& \sum_{1\le i<j<n}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{j-1,j}=\mp 1} b_i\cdots b_j v^{2j-2i+2\sum_{i\le a<b\le j}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^{n}\sum_{s=i}^{j}(2k-2n-1)\mathsf{m}_{sk}} \times \\
& \qquad e^{-\alpha_i-\ldots-\alpha_j} T_{\sum_{k=1}^{n}(\sum_{s=i}^{j}\mathsf{m}_{sk}-\delta_{k,i}-\delta_{k,j+1})\varpi_k} + \\
& \sum_{1\le i<n}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{n-1,n}=\mp 1} b_i\cdots b_n v^{2n-2i+1+2\sum_{i\le a<b\le n}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})+\sum_{k=1}^{n}\sum_{s=i}^{n}(2k-2n-1)\mathsf{m}_{sk}} \times \\
& \qquad e^{-\alpha_i-\ldots-\alpha_n} T_{\sum_{k=1}^{n}(\sum_{s=i}^{n}\mathsf{m}_{sk}-\delta_{k,i})\varpi_k} + \\
& \sum_{1\le i<n}^{\epsilon^\pm_{i,i+1}=\ldots=\epsilon^\pm_{n-1,n}=\mp 1} (1+v)^{-2} b_i\cdots b_{n-1} b_n^2 v^{2n-2i+(\mathsf{n}^+_{nn}-\mathsf{n}^-_{nn})+\sum_{i\le a<b\le n}(\mathsf{n}^+_{ba}-\mathsf{n}^-_{ba})(2+2\delta_{b,n})} \times \\
& v^{\sum_{k=1}^{n}\sum_{s=i}^{n}(2k-2n-1)(1+\delta_{s,n})\mathsf{m}_{sk}} \cdot e^{-\alpha_i-\ldots-\alpha_{n-1}-2\alpha_n} T_{\sum_{k=1}^{n}(\sum_{s=i}^{n}\mathsf{m}_{sk}+\mathsf{m}_{nk}-\delta_{k,i}+\delta_{k,n})\varpi_k}.
\end{aligned}
\tag{3.14}
$$

Remark 3.19. If $\epsilon^+ = \epsilon^-$, then the last six sums are vacuous. If we further set $\mathsf{n}^+ = \mathsf{n}^-$ and $c_i^\pm = \pm 1$ for all i, then we obtain the formula for the 1st hamiltonian of the type B_n quantum difference Toda lattice as defined in [6] (we write down this formula as we could not find it in the literature, even though it can be derived completely analogously to [6, (5.7)], cf. [7, the end of Section 3]):

$$\mathbf{D}_1 = 1 + \sum_{i=1}^{n} (T_{2\varpi_i} + T_{-2\varpi_i}) - (v^2 - v^{-2})^2 \sum_{i=1}^{n-1} e^{-\alpha_i} \left(T_{\varpi_i + \varpi_{i+1}} + T_{-\varpi_i - \varpi_{i+1}} \right) - \\ (v - v^{-1})^2 e^{-\alpha_n} \left(v T_{-\varpi_n} + v^{-1} T_{\varpi_n} \right) + v^{-2}(1 - v^{-1})^2 (v - v^{-1})^2 e^{-2\alpha_n}. \quad (3.15)$$

Recall the algebra $\mathcal{A}_n$ from Section 3.9. Consider the anti-isomorphism from $\mathcal{A}_n$ to the algebra $\mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{so}_{2n+1}})$ of Section 2.5, sending $\mathsf{w}_j \mapsto T_{-\varpi_j}, \mathsf{D}_j \mapsto e^{-\sum_{k=j}^{n} \alpha_k}$. Let $\mathsf{H} = \mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ be the element of $\mathcal{A}_n$ that corresponds to $\mathbf{D}_1$ under this anti-isomorphism (to save space, we omit the explicit long formula for H). The following is the key property of $\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ in type B.

Proposition 3.20. $\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ depends only on $\vec{\epsilon} = (\epsilon_{n-1}, \ldots, \epsilon_1) \in \{-1, 0, 1\}^{n-1}$ with $\epsilon_i := \frac{\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1}}{2}$, up to algebra automorphisms of $\mathcal{A}_n$.

The proof of this result is analogous to that of Proposition 3.11 given in Appendix A, see also Remark A.1; we leave the details to the interested reader. Proposition 3.20 implies that given two pairs of Sevostyanov triples $(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ and $(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ with $\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1} = \tilde{\epsilon}^+_{i,i+1} - \tilde{\epsilon}^-_{i,i+1}$, there exists an algebra automorphism of $\mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{so}_{2n+1}})$ that maps the 1st hamiltonian $\mathbf{D}_1(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ to $\mathbf{D}_1(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$. As we will see in Appendix B, the same automorphism maps the modified quantum Toda system $\mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ to $\mathcal{T}(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$.

3.21. Lax matrix realization in type A

In this section, we identify the type A_{n-1} modified quantum difference Toda systems with those discovered in [10, 11(ii, iii)] via the Lax matrix formalism.

Recall the algebra $\mathcal{A}_n$ of Section 3.9. Consider the following three *(local) Lax matrices*:

$$L_i^{v,-1}(z) := \begin{pmatrix} \mathsf{w}_i^{-1} - \mathsf{w}_i z^{-1} & \mathsf{w}_i \mathsf{D}_i^{-1} \\ -\mathsf{w}_i \mathsf{D}_i z^{-1} & \mathsf{w}_i \end{pmatrix} \in \mathrm{Mat}(2, z^{-1}\mathcal{A}_n[z]), \quad (3.16)$$

$$L_i^{v,0}(z) := \begin{pmatrix} \mathsf{w}_i^{-1} z^{1/2} - \mathsf{w}_i z^{-1/2} & \mathsf{D}_i^{-1} z^{1/2} \\ -\mathsf{D}_i z^{-1/2} & 0 \end{pmatrix} \in \mathrm{Mat}(2, z^{-1/2}\mathcal{A}_n[z]), \tag{3.17}$$

$$L_i^{v,1}(z) := \begin{pmatrix} \mathsf{w}_i^{-1} z - \mathsf{w}_i & \mathsf{w}_i^{-1}\mathsf{D}_i^{-1} z \\ -\mathsf{w}_i^{-1}\mathsf{D}_i & -\mathsf{w}_i^{-1} \end{pmatrix} \in \mathrm{Mat}(2, \mathcal{A}_n[z]). \tag{3.18}$$

For any $\vec{k} = (k_n, \ldots, k_1) \in \{-1, 0, 1\}^n$, define the *mixed complete monodromy matrix*

$$\mathsf{T}^v_{\vec{k}}(z) := L_n^{v,k_n}(z) \cdots L_1^{v,k_1}(z). \tag{3.19}$$

We also recall the standard trigonometric R-matrix

$$R_{\mathrm{trig}}(z) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z-1}{vz-v^{-1}} & \frac{z(v-v^{-1})}{vz-v^{-1}} & 0 \\ 0 & \frac{v-v^{-1}}{vz-v^{-1}} & \frac{z-1}{vz-v^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following key property of the complete monodromy matrices is established in [10, 11(ii)].

Proposition 3.22. For any $\vec{k} \in \{-1, 0, 1\}^n$, $\mathsf{T}^v_{\vec{k}}(z)$ satisfies the trigonometric RTT relation:

$$R_{\mathrm{trig}}(z/w)\left(\mathsf{T}^v_{\vec{k}}(z) \otimes 1\right)\left(1 \otimes \mathsf{T}^v_{\vec{k}}(w)\right) = \left(1 \otimes \mathsf{T}^v_{\vec{k}}(w)\right)\left(\mathsf{T}^v_{\vec{k}}(z) \otimes 1\right) R_{\mathrm{trig}}(z/w).$$

As an immediate corollary of this result, we obtained (see [10, 11(iii)]) the following.

Proposition 3.23. ([10]). Fix $\vec{k} = (k_n, \ldots, k_1) \in \{-1, 0, 1\}^n$.
(a) The coefficients in powers of z of the matrix element $\mathsf{T}^v_{\vec{k}}(z)_{11}$ generate a commutative subalgebra of $\mathcal{A}_n$. Moreover, they lie in the subalgebra of $\mathcal{A}_n$ generated by $\{\mathsf{w}_j^{\pm 1}, (\mathsf{D}_i/\mathsf{D}_{i+1})^{\pm 1}\}_{1 \le i < n}^{1 \le j \le n}$.
(b) $\mathsf{T}^v_{\vec{k}}(z)_{11} = (-1)^n \mathsf{w}_1 \cdots \mathsf{w}_n \left(z^s - \mathsf{H}_2^{\vec{k}} z^{s+1} + \text{higher powers of } z\right)$, where $s = \sum_{j=1}^n \frac{k_j - 1}{2}$. The hamiltonian $\mathsf{H}_2^{\vec{k}}$ equals

$$\mathsf{H}_2^{\vec{k}} = \sum_{j=1}^n \mathsf{w}_j^{-2} + \sum_{i=1}^{n-1} \mathsf{w}_i^{-k_i - 1} \mathsf{w}_{i+1}^{-k_{i+1} - 1} \cdot \frac{\mathsf{D}_i}{\mathsf{D}_{i+1}} + \sum_{1 \le i < j-1 \le n-1}^{k_{i+1} = \ldots = k_{j-1} = 1} \mathsf{w}_i^{-k_i - 1} \cdots \mathsf{w}_j^{-k_j - 1} \cdot \frac{\mathsf{D}_i}{\mathsf{D}_j}. \tag{3.20}$$

(c) Set $\vec{k}' = (0, k_{n-1}, \ldots, k_2, 0)$. Then $\mathsf{H}_2^{\vec{k}}$ is conjugate to $\mathsf{H}_2^{\vec{k}'}$.

Let $\mathcal{T}^{\vec{k}}$ denote the commutative subalgebra of $\bar{\mathcal{A}}_n$ generated by the images of the coefficients in powers of z of the matrix element $\mathsf{T}^v_{\vec{k}}(z)_{11}$, while $\bar{\mathsf{H}}^{\vec{k}}_2 \in \mathcal{T}^{\vec{k}}$ denote the image of $\mathsf{H}^{\vec{k}}_2$. The main result of this section identifies $\mathcal{T}^{\vec{k}}$ with the pre-images $\widetilde{\mathcal{T}}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ of $\mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ in type A_{n-1} under the anti-isomorphism $\bar{\mathcal{A}}_n \to \mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sl}_n})$ of Section 3.9. This provides a Lax matrix realization of the type A modified quantum difference Toda systems.

Theorem 3.24. Given a pair of type A_{n-1} Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ and $\vec{k} \in \{-1, 0, 1\}^n$ satisfying $k_{i+1} = \frac{\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1}}{2}$ for any $1 \leq i \leq n-2$, the following holds:

(a) There is an algebra automorphism of $\bar{\mathcal{A}}_n$ that maps $\mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\bar{\mathsf{H}}^{\vec{k}}_2$.
(b) The automorphism of part (a) maps $\widetilde{\mathcal{T}}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathcal{T}^{\vec{k}}$.

The proof of Theorem 3.24 is presented in Appendix E and closely follows our proofs of Proposition 3.11 (see Appendix A) and Theorem 3.2 (see Appendix B).

Remark 3.25. For $\vec{k} = \vec{0}$, we recover the Lax matrix realization of the type A quantum difference Toda lattice, due to [15].

Actually, the above construction admits a standard one-parameter deformation of commutative subalgebras of $\mathcal{A}_n$ as provided by the following result.

Proposition 3.26. (a) For any $\varepsilon \in \mathbb{C}(v^{1/\mathsf{N}})$, the coefficients in powers of z of the linear combination $\mathsf{T}^v_{\vec{k}}(z)_{11} + \varepsilon \mathsf{T}^v_{\vec{k}}(z)_{22}$ generate a commutative subalgebra of $\mathcal{A}_n$. Moreover, they lie in the subalgebra of $\mathcal{A}_n$ generated by $\{\mathsf{w}_j^{\pm 1}, (\mathsf{D}_i/\mathsf{D}_{i+1})^{\pm 1}\}^{1\leq j\leq n}_{1\leq i<n}$.
(b) We have $\mathsf{T}^v_{\vec{k}}(z)_{11} + \varepsilon \mathsf{T}^v_{\vec{k}}(z)_{22} = (-1)^n \mathsf{w}_1 \cdots \mathsf{w}_n \left(\hat{\mathsf{H}}^{\vec{k}}_1 z^s - \hat{\mathsf{H}}^{\vec{k}}_2 z^{s+1} + \text{higher powers of } z\right)$, where $s = \sum_{j=1}^n \frac{k_j - 1}{2}$. Here $\hat{\mathsf{H}}^{\vec{k}}_1 = 1 + \varepsilon \prod_{j=1}^n \delta_{k_j,1} \prod_{j=1}^n \mathsf{w}_j^{-2}$, while $\hat{\mathsf{H}}^{\vec{k}}_2$ is given by

$$
\begin{aligned}
\hat{\mathsf{H}}^{\vec{k}}_2 = \sum_{j=1}^n \mathsf{w}_j^{-2} + \sum_{i=1}^{n-1} \mathsf{w}_i^{-k_i-1} \mathsf{w}_{i+1}^{-k_{i+1}-1} \cdot \frac{\mathsf{D}_i}{\mathsf{D}_{i+1}} + \sum_{1\leq i<j-1\leq n-1}^{k_{i+1}=\ldots=k_{j-1}=1} \mathsf{w}_i^{-k_i-1} \cdots \mathsf{w}_j^{-k_j-1} \cdot \frac{\mathsf{D}_i}{\mathsf{D}_j} + \\
\varepsilon \left(\sum_{1\leq i<j\leq n}^{\substack{k_1=\ldots=k_{i-1}=1 \\ k_{j+1}=\ldots=k_n=1}} \mathsf{w}_1^{-k_1-1} \cdots \mathsf{w}_i^{-k_i-1} \mathsf{w}_j^{-k_j-1} \cdots \mathsf{w}_n^{-k_n-1} \cdot \frac{\mathsf{D}_j}{\mathsf{D}_i} + \sum_{1\leq j\leq n}^{\substack{k_j=-1 \\ k_i=1 (i\neq j)}} \prod_{k\neq j} \mathsf{w}_k^{-2} \right).
\end{aligned} \tag{3.21}
$$

Proof. (a) Follows from the equality $\left[\mathsf{T}^v_{\vec{k}}(z)_{11} + \varepsilon \mathsf{T}^v_{\vec{k}}(z)_{22}, \mathsf{T}^v_{\vec{k}}(w)_{11} + \varepsilon \mathsf{T}^v_{\vec{k}}(w)_{22}\right] = 0$, which is implied by the RTT relation of Proposition 3.22.
(b) Straightforward computation. ■

Definition 3.27. A **type A_{n-1} periodic modified quantum difference Toda system** is the commutative subalgebra $\hat{\mathcal{T}}^{\vec{k},\varepsilon}$ of $\bar{\mathcal{A}}_n$.

We note that $\hat{\mathcal{T}}^{\vec{k},0}$ coincides with $\mathcal{T}^{\vec{k}}$.

Remark 3.28. (a) In particular, $\hat{\mathsf{H}}_2^{\vec{0}}$ is conjugate (in the sense of (A1, A2)) to the element of $\bar{\mathcal{A}}_n$ that corresponds under the anti-isomorphism $\bar{\mathcal{A}}_n \to \mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sl}_n})$ of Section 3.9 to the 1st hamiltonian of the type $A^{(1)}_{n-1}$ quantum difference affine Toda system of [6, (5.9)]:

$$\hat{\mathbf{D}}_1 = \sum_{j=1}^{n} T_{2\varpi_j} - (v - v^{-1})^2 \sum_{i=1}^{n-1} e^{-\varpi_i + \varpi_{i+1}} T_{\varpi_i + \varpi_{i+1}} - \kappa (v - v^{-1})^2 e^{-\varpi_n + \varpi_1} T_{\varpi_n + \varpi_1} \tag{3.22}$$

with $\kappa = (-1)^n (v - v^{-1})^{-2n} \varepsilon$.

The quantum difference affine Toda systems are defined similarly to the (finite type) q-Toda systems of *loc.cit.*, but starting from a quantum affine algebra and its center at the critical level. The parameter $\kappa \in \mathbb{C}(v^{1/\mathsf{N}})$ is essential, that is, it cannot be removed, cf. [6, Remark 1].

(b) We expect that most of our results from this paper can be generalized to an affine setting. In particular, the type A_{n-1} periodic modified quantum difference Toda systems introduced above should be conjugate to the type $A^{(1)}_{n-1}$ modified quantum difference affine Toda systems, thus generalizing part (a) of the current Remark. To state this more precisely, let us first specify what we mean by a Sevostyanov triple $(\hat{\epsilon}, \hat{\mathsf{n}}, \hat{c})$ for $\hat{\mathfrak{g}}$ of an affine type, except $A^{(1)}_1$. Let $\alpha_0, \ldots, \alpha_n$ be simple positive roots of $\hat{\mathfrak{g}}$ (with α_0 the distinguished one) and $\{a_i\}_{i=0}^n$ be the labels on the Dynkin diagram of $\hat{\mathfrak{g}}$ as in [12, Chapter IV, Table Aff]. Then a Sevostyanov triple $(\hat{\epsilon}, \hat{\mathsf{n}}, \hat{c})$ is a collection of the following data: (1) $\hat{\epsilon} = (\epsilon_{ij})_{i,j=0}^n$ is the associated matrix of an orientation of $\mathrm{Dyn}(\hat{\mathfrak{g}})$ as before, (2) $\hat{c} = (c_i)_{i=0}^n \in (\mathbb{C}(v^{1/\mathsf{N}})^\times)^{n+1}$ is a collection of nonzero constants, and (3) $\hat{\mathsf{n}} = (\mathsf{n}_{ij})_{0\le i\le n}^{1\le j\le n}$ is an integer matrix satisfying $\mathsf{d}_j \mathsf{n}_{ij} - \mathsf{d}_i \mathsf{n}_{ji} = \epsilon_{ij} b_{ij}$ for $1 \le i,j \le n$ and $\mathsf{d}_j \mathsf{n}_{0j} + \sum_{p=1}^n \mathsf{d}_p \mathsf{n}_{jp} \frac{a_p}{a_0} = \epsilon_{0j} b_{0j}$ for $1 \le j \le n$.

In particular, given a pair of type $A^{(1)}_{n-1}$ $(n > 2)$ Sevostyanov triples $(\hat{\epsilon}^\pm, \hat{\mathsf{n}}^\pm, \hat{c}^\pm)$, the corresponding difference operator $\mathsf{H}(\hat{\epsilon}^\pm, \hat{\mathsf{n}}^\pm, \hat{c}^\pm) \in \bar{\mathcal{A}}_n$ should depend (up to algebra automorphisms of $\bar{\mathcal{A}}_n$) only on $\vec{\epsilon} = (\epsilon_{n-1}, \ldots, \epsilon_0) \in \{-1, 0, 1\}^n$, where $\epsilon_i := \frac{\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1}}{2}$ with indices considered modulo n. The resulting 3^n difference operators $\mathsf{H}(\hat{\epsilon}^\pm, \hat{\mathsf{n}}^\pm, \hat{c}^\pm)$ should be conjugate to the images of 3^n hamiltonians $\hat{\mathsf{H}}_2^{\vec{k}}$ in $\bar{\mathcal{A}}_n$. We have verified this result for $\epsilon^\pm_{0,1} = \epsilon^\pm_{0,n-1} = 1$.

(c) The type $A^{(1)}_1$ deserves a special treatment, since it is the only affine type for which the analogue of Lemma 2.4 fails to hold (as $a_{01} = a_{10} = -2$). Instead, such characters

$\chi^{\pm}$ exist if and only if $\mathsf{n}_{01}^{\pm} + \mathsf{n}_{11}^{\pm} \in \{-2, 0, 2\}$. Let $\mathsf{H}(\hat{\mathsf{n}}^{\pm}, \hat{c}^{\pm})$ be the element of $\bar{\mathcal{A}}_2$ that corresponds to the 1st hamiltonian of the type $A_1^{(1)}$ modified quantum difference affine Toda system associated with the pair $(\hat{\mathsf{n}}^{\pm}, \hat{c}^{\pm})$. We expect that $\mathsf{H}(\hat{\mathsf{n}}^{\pm}, \hat{c}^{\pm})$ depends (up to algebra automorphisms of $\bar{\mathcal{A}}_2$) only on $\frac{\mathsf{n}_{01}^{+}+\mathsf{n}_{11}^{+}-\mathsf{n}_{01}^{-}-\mathsf{n}_{11}^{-}}{2} \in \{\pm 2, \pm 1, 0\}$. On the other hand, it is easy to see that the equivalence class of the difference operator $\hat{\mathsf{H}}_2^{(k_2,k_1)}$ depends only on $k_2 - k_1 \in \{\pm 2, \pm 1, 0\}$. We expect that the resulting five difference operators in $\bar{\mathcal{A}}_2$ are conjugate to the aforementioned five difference operators $\mathsf{H}(\hat{\mathsf{n}}^{\pm}, \hat{c}^{\pm})$.

3.29. Lax matrix realization in type C

Motivated by the construction of the previous section, we provide a Lax matrix realization of the type C modified quantum difference Toda systems.

In addition to $L_i^{v,k}(z)$ $(k = \pm 1, 0)$ of (3.16–3.18), consider three more *(local) Lax matrices*:

$$\bar{L}_i^{v,-1}(z) := \begin{pmatrix} \mathsf{w}_i - \mathsf{w}_i^{-1} z^{-1} & \mathsf{w}_i^{-1} \mathsf{D}_i \\ -\mathsf{w}_i^{-1} \mathsf{D}_i^{-1} z^{-1} & \mathsf{w}_i^{-1} \end{pmatrix} \in \mathrm{Mat}(2, z^{-1}\mathcal{A}_n[z]), \tag{3.23}$$

$$\bar{L}_i^{v,0}(z) := \begin{pmatrix} \mathsf{w}_i z^{1/2} - \mathsf{w}_i^{-1} z^{-1/2} & \mathsf{D}_i z^{1/2} \\ -\mathsf{D}_i^{-1} z^{-1/2} & 0 \end{pmatrix} \in \mathrm{Mat}(2, z^{-1/2}\mathcal{A}_n[z]), \tag{3.24}$$

$$\bar{L}_i^{v,1}(z) := \begin{pmatrix} \mathsf{w}_i z - \mathsf{w}_i^{-1} & \mathsf{w}_i \mathsf{D}_i z \\ -\mathsf{w}_i \mathsf{D}_i^{-1} & -\mathsf{w}_i \end{pmatrix} \in \mathrm{Mat}(2, \mathcal{A}_n[z]). \tag{3.25}$$

For any $\vec{k} = (k_n, \ldots, k_1) \in \{-1, 0, 1\}^n$, define the *double mixed complete monodromy matrix*

$$\mathbb{T}_{\vec{k}}^{v}(z) := \bar{L}_1^{v,-k_1}(z) \cdots \bar{L}_n^{v,-k_n}(z) L_n^{v,k_n}(z) \cdots L_1^{v,k_1}(z). \tag{3.26}$$

Let us summarize the key properties of the double mixed complete monodromy matrices.

Proposition 3.30. Fix $\vec{k} = (k_n, \ldots, k_1) \in \{-1, 0, 1\}^n$.
(a) $\mathbb{T}_{\vec{k}}^{v}(z)$ satisfies the trigonometric RTT relation

$$R_{\mathrm{trig}}(z/w)\left(\mathbb{T}_{\vec{k}}^{v}(z) \otimes 1\right)\left(1 \otimes \mathbb{T}_{\vec{k}}^{v}(w)\right) = \left(1 \otimes \mathbb{T}_{\vec{k}}^{v}(w)\right)\left(\mathbb{T}_{\vec{k}}^{v}(z) \otimes 1\right) R_{\mathrm{trig}}(z/w).$$

(b) The coefficients in powers of z of the matrix element $\mathbb{T}_{\vec{k}}^{v}(z)_{11}$ generate a commutative subalgebra of $\mathcal{A}_n$. Moreover, they belong to the subalgebra $\mathcal{C}_n$ of $\mathcal{A}_n$, introduced in Section 3.12.

(c) We have $\mathbb{T}^{v}_{\vec{k}}(z)_{11} = z^{-n} - \mathbb{H}^{\vec{k}}_2 z^{-n+1} +$ higher powers of z. The hamiltonian $\mathbb{H}^{\vec{k}}_2$ equals

$$\mathbb{H}^{\vec{k}}_2 = \sum_{i=1}^{n}(\mathsf{w}_i^2 + \mathsf{w}_i^{-2}) + \sum_{i=1}^{n-1}\left(\mathsf{w}_i^{-k_i-1}\mathsf{w}_{i+1}^{-k_{i+1}-1} + \mathsf{w}_i^{-k_i+1}\mathsf{w}_{i+1}^{-k_{i+1}+1}\right)\cdot\frac{\mathsf{D}_i}{\mathsf{D}_{i+1}} + v^{-k_n}\mathsf{w}_n^{-2k_n}\cdot\mathsf{D}_n^2 +$$
$$\sum_{1\le i<j<n}^{k_{i+1}=\ldots=k_j=1} \mathsf{w}_i^{-k_i-1}\cdots\mathsf{w}_{j+1}^{-k_{j+1}-1}\cdot\frac{\mathsf{D}_i}{\mathsf{D}_{j+1}} + \sum_{1\le i<n}^{k_{i+1}=\ldots=k_n=1} v^{-1}\mathsf{w}_i^{-k_i-1}\cdots\mathsf{w}_n^{-k_n-1}\cdot\mathsf{D}_i\mathsf{D}_n +$$
$$\sum_{1\le i<j<n}^{k_{i+1}=\ldots=k_j=-1} \mathsf{w}_i^{-k_i+1}\cdots\mathsf{w}_{j+1}^{-k_{j+1}+1}\cdot\frac{\mathsf{D}_i}{\mathsf{D}_{j+1}} + \sum_{1\le i<n}^{k_{i+1}=\ldots=k_n=-1} v\mathsf{w}_i^{-k_i+1}\cdots\mathsf{w}_n^{-k_n+1}\cdot\mathsf{D}_i\mathsf{D}_n. \quad (3.27)$$

Proof. (a) Note that $\bar{L}_i^{v,k}(z)$ is obtained from $L_i^{v,k}(z)$ by applying the automorphism of $\mathcal{A}_n$ that maps $\mathsf{w}_i^{\pm 1}\mapsto \mathsf{w}_i^{\mp 1}, \mathsf{D}_i^{\pm 1}\mapsto \mathsf{D}_i^{\mp 1}$. Hence, each of them satisfies the trigonometric RTT relation. Thus, an arbitrary product of $L_i^{v,k}(z)$ and $\bar{L}_i^{v,k}(z)$ also satisfies the trigonometric RTT relation. (b) This is an immediate corollary of (a). (c) Straightforward computation. ■

Let $\mathcal{T}^{\vec{k}}$ denote the commutative subalgebra of $\mathcal{C}_n$ generated by the coefficients in powers of z of the matrix element $\mathbb{T}^{v}_{\vec{k}}(z)_{11}$. The main result of this section identifies $\mathcal{T}^{\vec{k}}$ with the pre-images $\widetilde{\mathcal{T}}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ of $\mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ in type C_n under the anti-isomorphism $\mathcal{C}_n \to \mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sp}_{2n}})$ of Section 3.12. This provides a Lax matrix realization of the type C modified quantum difference Toda systems.

Theorem 3.31. Given a pair of type C_n Sevostyanov triples $(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ and $\vec{k}\in\{-1,0,1\}^n$ satisfying $k_{i+1} = \frac{\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1}}{2}$ for any $1\le i\le n-1$, the following holds:
(a) There is an algebra automorphism of $\mathcal{C}_n$ which maps $\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ to $\mathbb{H}^{\vec{k}}_2$.
(b) The automorphism of part (a) maps $\widetilde{\mathcal{T}}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ to $\mathcal{T}^{\vec{k}}$.

The proof of Theorem 3.31 is completely analogous to that of Theorem 3.24 given in Appendix E, see also Remark A.1; we leave the details to the interested reader.

Remark 3.32. Recall that any nonsimply laced simple Lie algebra $\mathfrak{g}'$ admits a *folding realization* $\mathfrak{g}'\simeq\mathfrak{g}^\sigma$, where $\mathfrak{g}$ is a simply laced Lie algebra endowed with an outer automorphism σ of a finite order (arising as an automorphism of the corresponding Dynkin diagram). This observation allows to relate the classical Toda system of $\mathfrak{g}'$ to the Toda system of $\mathfrak{g}$; see [17]. The above construction of the modified quantum difference

Toda systems in types C_n and A_{2n-1} via Lax matrices exhibits the former as a folding of the latter, once we require that the orientations $\epsilon^{\pm}$ of $\mathrm{Dyn}(\mathfrak{sl}_{2n}) = A_{2n-1}$ satisfy $\epsilon^{\pm}_{i,i+1} = -\epsilon^{\pm}_{2n-i-1,2n-i}$ for all i. However, such a naive approach fails to work for the pair (B_n, D_{n+1}). It would be interesting to understand the explicit relation. Let us warn the interested reader that the folding for quantum groups is more elaborate than in the classical set-up; see [1].

Analogously to the type A case, the above construction admits a standard one-parameter deformation of commutative subalgebras of $\mathcal{C}_n$ as provided by the following result.

Proposition 3.33. (a) For any $\varepsilon \in \mathbb{C}(v^{1/\mathsf{N}})$, the coefficients in powers of z of the linear combination $\mathbb{T}^{v}_{\vec{k}}(z)_{11} + \varepsilon \mathbb{T}^{v}_{\vec{k}}(z)_{22}$ generate a commutative subalgebra of $\mathcal{A}_n$. Moreover, they belong to the subalgebra $\mathcal{C}_n$ of $\mathcal{A}_n$.
(b) We have $\mathbb{T}^{v}_{\vec{k}}(z)_{11} + \varepsilon \mathbb{T}^{v}_{\vec{k}}(z)_{22} = z^{-n} - \hat{\mathbb{H}}^{\vec{k}}_2 z^{-n+1}$ + higher powers of z. The hamiltonian $\hat{\mathbb{H}}^{\vec{k}}_2$ is given by the following formula:

$$\hat{\mathbb{H}}^{\vec{k}}_2 = \sum_{i=1}^{n} (\mathsf{w}_i^2 + \mathsf{w}_i^{-2}) + \sum_{i=1}^{n-1} \left(\mathsf{w}_i^{-k_i-1} \mathsf{w}_{i+1}^{-k_{i+1}-1} + \mathsf{w}_i^{-k_i+1} \mathsf{w}_{i+1}^{-k_{i+1}+1} \right) \cdot \frac{\mathsf{D}_i}{\mathsf{D}_{i+1}} + v^{-k_n} \mathsf{w}_n^{-2k_n} \cdot \mathsf{D}_n^2 +$$
$$\sum_{1 \le i < j < n}^{k_{i+1}=\ldots=k_j=1} \mathsf{w}_i^{-k_i-1} \cdots \mathsf{w}_{j+1}^{-k_{j+1}-1} \cdot \frac{\mathsf{D}_i}{\mathsf{D}_{j+1}} + \sum_{1 \le i < n}^{k_{i+1}=\ldots=k_n=1} v^{-1} \mathsf{w}_i^{-k_i-1} \cdots \mathsf{w}_n^{-k_n-1} \cdot \mathsf{D}_i \mathsf{D}_n +$$
$$\sum_{1 \le i < j < n}^{k_{i+1}=\ldots=k_j=-1} \mathsf{w}_i^{-k_i+1} \cdots \mathsf{w}_{j+1}^{-k_{j+1}+1} \cdot \frac{\mathsf{D}_i}{\mathsf{D}_{j+1}} + \sum_{1 \le i < n}^{k_{i+1}=\ldots=k_n=-1} v \mathsf{w}_i^{-k_i+1} \cdots \mathsf{w}_n^{-k_n+1} \cdot \mathsf{D}_i \mathsf{D}_n +$$
$$\varepsilon \Bigg(\delta_{n,1}(1-\delta_{k_1,0}) \mathsf{w}_1^{-2k_1} + v^{k_1} \mathsf{w}_1^{-2k_1} \cdot \mathsf{D}_1^{-2} + \prod_{i=1}^{n} \delta_{k_i,1} \cdot \prod_{i=1}^{n} \mathsf{w}_i^{-2} \cdot \frac{\mathsf{D}_n}{\mathsf{D}_1} + \prod_{i=1}^{n} \delta_{k_i,-1} \cdot \prod_{i=1}^{n} \mathsf{w}_i^{2} \cdot \frac{\mathsf{D}_n}{\mathsf{D}_1} +$$
$$\sum_{1 \le i < n}^{k_1=\ldots=k_i=1} v \mathsf{w}_1^{-k_1-1} \cdots \mathsf{w}_{i+1}^{-k_{i+1}-1} \cdot \frac{1}{\mathsf{D}_1 \mathsf{D}_{i+1}} + \sum_{1 \le i < n}^{k_1=\ldots=k_i=-1} v^{-1} \mathsf{w}_1^{-k_1+1} \cdots \mathsf{w}_{i+1}^{-k_{i+1}+1} \cdot \frac{1}{\mathsf{D}_1 \mathsf{D}_{i+1}} \Bigg). \tag{3.28}$$

Definition 3.34. A **type C_n periodic modified quantum difference Toda system** is the commutative subalgebra $\hat{\mathcal{T}}^{\vec{k},\varepsilon}$ of $\mathcal{C}_n$.

We note that $\hat{\mathcal{T}}^{\vec{k},0}$ coincides with $\mathcal{T}^{\vec{k}}$.

Remark 3.35. (a) In particular, $\hat{\mathbb{H}}_2^{\vec{0}}$ is conjugate to the element of $\mathcal{C}_n$ that corresponds under the anti-isomorphism $\mathcal{C}_n \to \mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sp}_{2n}})$ of Section 3.12 to the 1st hamiltonian of the type $C_n^{(1)}$ quantum difference affine Toda system of [6] (cf. Remark 3.28(a)):

$$\hat{\mathbf{D}}_1 = \sum_{i=1}^{n}(T_{2\varpi_i} + T_{-2\varpi_i}) - (v - v^{-1})^2 \sum_{i=1}^{n-1} e^{-\varpi_i+\varpi_{i+1}} \left(T_{\varpi_i+\varpi_{i+1}} + T_{-\varpi_i-\varpi_{i+1}}\right) - (v^2 - v^{-2})^2 e^{-2\varpi_n} - \kappa v^{-2n-2}(v^2 - v^{-2})^2 e^{2\varpi_1} \quad (3.29)$$

with $\kappa = v^{2n+2}(v - v^{-1})^{-4(n-1)}(v^2 - v^{-2})^{-4}\varepsilon$.

Here $\kappa \in \mathbb{C}(v^{1/\mathrm{N}})$ is an essential parameter. For $\kappa = 0$, we recover $\mathbf{D}_1$ of (3.10).

(b) Following our discussion and notations of Remark 3.28(b), we have also verified that the element of $\mathcal{C}_n$ corresponding under the anti-isomorphism $\mathcal{C}_n \to \mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sp}_{2n}})$ to $\mathsf{H}(\hat{\epsilon}^{\pm}, \hat{\mathsf{n}}^{\pm}, \hat{c}^{\pm})$ with $\epsilon^{\pm}_{01} = 1$, is conjugate to $\hat{\mathbb{H}}_2^{\vec{k}}$ with $k_1 = 0$ and $k_{i+1} = \frac{\epsilon^{+}_{i,i+1} - \epsilon^{-}_{i,i+1}}{2}$ for any $1 \le i \le n-1$.

(c) For completeness of our list (3.22, 3.29), let us present explicit formulas for the 1st hamiltonian $\hat{\mathbf{D}}_1$ of the quantum difference affine Toda systems (defined in [6]) for the remaining classical series $D_n^{(1)}$ and $B_n^{(1)}$ (as we could not find such formulas in the literature):

- In type $D_n^{(1)}$, we have

$$\hat{\mathbf{D}}_1 = \sum_{i=1}^{n}(T_{2\varpi_i} + T_{-2\varpi_i}) - (v - v^{-1})^2 \sum_{i=1}^{n-1} e^{-\varpi_i+\varpi_{i+1}} \left(T_{\varpi_i+\varpi_{i+1}} + T_{-\varpi_i-\varpi_{i+1}}\right) - (v - v^{-1})^2 e^{-\varpi_{n-1}-\varpi_n} \left(T_{-\varpi_{n-1}+\varpi_n} + v^{-2} T_{\varpi_{n-1}-\varpi_n}\right) + (v - v^{-1})^4 e^{-2\varpi_{n-1}} - \kappa v^{-2n+2}(v - v^{-1})^2 e^{\varpi_1+\varpi_2}(T_{-\varpi_1+\varpi_2} + T_{\varpi_1-\varpi_2}) + \kappa v^{-2n+2}(v - v^{-1})^4 e^{2\varpi_2}. \quad (3.30)$$

- In type $B_n^{(1)}$, we have

$$\hat{\mathbf{D}}_1 = 1 + \sum_{i=1}^{n}(T_{2\varpi_i} + T_{-2\varpi_i}) - (v^2 - v^{-2})^2 \sum_{i=1}^{n-1} e^{-\varpi_i+\varpi_{i+1}} \left(T_{\varpi_i+\varpi_{i+1}} + T_{-\varpi_i-\varpi_{i+1}}\right) - (v - v^{-1})^2 e^{-\varpi_n} \left(v T_{-\varpi_n} + v^{-1} T_{\varpi_n}\right) + v^{-2}(1 - v^{-1})^2 (v - v^{-1})^2 e^{-2\varpi_n} - \kappa v^{-4n+2}(v^2 - v^{-2})^2 e^{\varpi_1+\varpi_2}(T_{-\varpi_1+\varpi_2} + T_{\varpi_1-\varpi_2}) + \kappa v^{-4n+2}(v^2 - v^{-2})^4 e^{2\varpi_2}. \quad (3.31)$$

For $\kappa = 0$, these formulas recover $\mathbf{D}_1$ of (3.13) and (3.15), respectively.

3.36 Explicit formulas and classification in type G_2

Recall explicit formulas for the action of $U_v(\mathfrak{g}_2)$ on its 1st fundamental representation V_1. The space V_1 has a basis $\{w_i\}_{i=0}^6$, in which the action is given by the following formulas:

$$E_1\colon w_0 \mapsto (v+v^{-1})w_2,\ w_1 \mapsto 0,\ w_2 \mapsto 0,\ w_3 \mapsto 0,\ w_4 \mapsto w_3,\ w_5 \mapsto (v+v^{-1})w_0,$$
$$w_6 \mapsto w_1,$$
$$E_2\colon w_0 \mapsto 0,\ w_1 \mapsto 0,\ w_2 \mapsto w_6,\ w_3 \mapsto w_5,\ w_4 \mapsto 0,\ w_5 \mapsto 0,\ w_6 \mapsto 0,$$
$$F_1\colon w_0 \mapsto w_5,\ w_1 \mapsto w_6,\ w_2 \mapsto w_0,\ w_3 \mapsto w_4,\ w_4 \mapsto 0,\ w_5 \mapsto 0,\ w_6 \mapsto 0,$$
$$F_2\colon w_0 \mapsto 0,\ w_1 \mapsto 0,\ w_2 \mapsto 0,\ w_3 \mapsto 0,\ w_4 \mapsto 0,\ w_5 \mapsto w_3,\ w_6 \mapsto w_2,$$
$$L_1\colon w_0 \mapsto w_0, w_1 \mapsto v^2 w_1, w_2 \mapsto v w_2, w_3 \mapsto v^{-1} w_3, w_4 \mapsto v^{-2} w_4, w_5 \mapsto v^{-1} w_5,$$
$$w_6 \mapsto v w_6,$$

$$L_2\colon w_0 \mapsto w_0,\ w_1 \mapsto v^3 w_1,\ w_2 \mapsto w_2,\ w_3 \mapsto v^{-3} w_3,\ w_4 \mapsto v^{-3} w_4,\ w_5 \mapsto w_5,$$
$$w_6 \mapsto v^3 w_6,$$
$$K_1\colon w_0 \mapsto w_0, w_1 \mapsto v w_1, w_2 \mapsto v^2 w_2, w_3 \mapsto v w_3, w_4 \mapsto v^{-1} w_4, w_5 \mapsto v^{-2} w_5,$$
$$w_6 \mapsto v^{-1} w_6,$$
$$K_2\colon w_0 \mapsto w_0,\ w_1 \mapsto w_1,\ w_2 \mapsto v^{-3} w_2,\ w_3 \mapsto v^{-3} w_3,\ w_4 \mapsto w_4,\ w_5 \mapsto v^3 w_5,$$
$$w_6 \mapsto v^3 w_6.$$

Let ϖ_1 and ϖ_2 be the weights of w_2 and w_6, respectively, so that $(\varpi_1,\varpi_1) = (\varpi_2,\varpi_2) = 2, (\varpi_1,\varpi_2) = -1$. Then the weights of w_0, w_1, w_3, w_4, w_5 are equal to $0, \varpi_1 + \varpi_2, -\varpi_2, -\varpi_1 - \varpi_2, -\varpi_1$, respectively. We also note that the simple roots are given by $\alpha_1 = \varpi_1, \alpha_2 = -\varpi_1 + \varpi_2$. Finally, we have $\rho = 5\alpha_1 + 3\alpha_2 = 2\varpi_1 + 3\varpi_2$ and $\mathsf{d}_1 = 1, \mathsf{d}_2 = 3$.

To compute $\mathbf{D}_1$, we use the same strategy as for the classical types. Analogously to type B, the operators E_1^2 and F_1^2 act nontrivially on V_1, while $\{E_i^r, F_i^r\}_{i=1,2}^{r>1+\delta_{i,1}}$ still act by zero on V_1. Therefore, applying formula (3.1), we can replace R_{α_2} by $\bar{R}_{\alpha_2} := 1 + (v^3 - v^{-3})E_2 \otimes F_2$ and R_{α_1} by $\bar{R}_{\alpha_1} := 1 + (v - v^{-1})E_1 \otimes F_1 + cE_1^2 \otimes F_1^2$ for $c := (1 - v^{-1})(v - v^{-1})$. Let us now compute all the nonzero terms contributing to C'_{V_1}:

- Picking 1 out of each $\bar{R}_i^{\mathrm{op}}, \bar{R}_i$, we recover

$$1+v^2\cdot K_{2\varpi_1}+v^{-2}\cdot K_{-2\varpi_1}+v^8\cdot K_{2\varpi_2}+v^{-8}\cdot K_{-2\varpi_2}+v^{10}\cdot K_{2\varpi_1+2\varpi_2}+v^{-10}\cdot K_{-2\varpi_1-2\varpi_2}.$$

- Picking nontrivial terms only at $\bar{R}^{\mathrm{op}}_{\alpha_j}, \bar{R}_{\alpha_i}$, the result does not depend on $\mathrm{Or}^{\pm}$ (hence, the orderings $\prec_{\pm}$) and the total contribution of the nonzero terms equals

$$\begin{aligned}
&(v-v^{-1})^2\Big(v^2(v+v^{-1})F_1E_1K_{\varpi_1}+(v+v^{-1})F_1K_{-\varpi_1}E_1\\
&\qquad\qquad+v^{10}F_1K_{\varpi_2}E_1K_{\varpi_1+\varpi_2}+v^{-8}F_1K_{-\varpi_1-\varpi_2}E_1K_{-\varpi_2}\Big)\\
&+(v^3-v^{-3})^2\Big(v^8F_2K_{\varpi_1}E_2K_{\varpi_2}+v^{-2}F_2K_{-\varpi_2}E_2K_{-\varpi_1}\Big)\\
&+c^2v^2(v+v^{-1})^2F_1^2K_{-\varpi_1}E_1^2K_{\varpi_1}.
\end{aligned}$$

- The contribution of the remaining terms to C'_{V_1} depends on $\mathrm{Or}^{\pm}$. Tracing back explicit formulas for the action of $U_v(\mathfrak{g}_2)$ on V_1, let us evaluate the total contribution of such terms for each of the four possible pairs $(\mathrm{Or}^+, \mathrm{Or}^-)$.
  Case 1: if $\mathrm{Or}^+ = \mathrm{Or}^-$, then there are no other terms.
  Case 2: if $\mathrm{Or}^+ : \alpha_1 \leftarrow \alpha_2$ and $\mathrm{Or}^- : \alpha_1 \rightarrow \alpha_2$, then the total contribution equals

$$\begin{aligned}
&(v-v^{-1})^2(v^3-v^{-3})^2(v^8(v+v^{-1})F_2F_1E_1E_2K_{\varpi_2}+v^{-2}F_2F_1K_{-\varpi_1-\varpi_2}E_1E_2K_{-\varpi_1})+\\
&c^2(v^3-v^{-3})^2v^8(v+v^{-1})^2\cdot F_2F_1^2K_{-\varpi_1}E_1^2E_2K_{\varpi_2}.
\end{aligned}$$

  Case 3: if $\mathrm{Or}^+ : \alpha_1 \rightarrow \alpha_2$ and $\mathrm{Or}^- : \alpha_1 \leftarrow \alpha_2$, then the total contribution equals

$$\begin{aligned}
&(v-v^{-1})^2(v^3-v^{-3})^2(v^{10}F_1F_2K_{\varpi_1}E_2E_1K_{\varpi_1+\varpi_2}+(v+v^{-1})F_1F_2K_{-\varpi_2}E_2E_1)+\\
&c^2(v^3-v^{-3})^2v^2(v+v^{-1})^2\cdot F_1^2F_2K_{-\varpi_2}E_2E_1^2K_{\varpi_1}.
\end{aligned}$$

Thus, we have listed all the nonzero terms contributing to C'_{V_1}. To obtain $\tilde{\mathbf{D}}_1 = \bar{\mathbf{D}}_{V_1}$, we should rewrite the above formulas via e_i, f_i and $L_p = \begin{cases} K_{\varpi_1+\varpi_2}, & \text{if } p=1 \\ K_{\varpi_1+2\varpi_2}, & \text{if } p=2 \end{cases}$, moving all the Cartan terms to the middle, and then apply the characters $\chi^{\pm}$ with $\chi^+(e_i) = c_i^+, \chi^-(f_i) = c_i^-$. Conjugating further by e^{ρ}, we obtain the explicit formula for the 1st hamiltonian $\mathbf{D}_1$ of the type G_2 modified quantum difference Toda system. To write it down, define constants $\mathsf{m}_{i1} := (\mathsf{n}^-_{i1} - \mathsf{n}^+_{i1}) + (\mathsf{n}^-_{i2} - \mathsf{n}^+_{i2})$, $\mathsf{m}_{i2} := (\mathsf{n}^-_{i1} - \mathsf{n}^+_{i1}) + 2(\mathsf{n}^-_{i2} - \mathsf{n}^+_{i2})$ and $b_i := (v_i - v_i^{-1})^2 v_i^{\mathsf{n}^+_{ii} - \mathsf{n}^-_{ii}} c_i^+ c_i^-$. Then we have

$$\begin{aligned}
\mathbf{D}_1 = 1 &+ (T_{2\varpi_1} + T_{-2\varpi_1} + T_{2\varpi_2} + T_{-2\varpi_2} + T_{2\varpi_1+2\varpi_2} + T_{-2\varpi_1-2\varpi_2}) +\\
&b_1 v^{-\mathsf{m}_{11}-4\mathsf{m}_{12}} \cdot e^{-\alpha_1}\Big(v^{-1}(v+v^{-1})T_{(\mathsf{m}_{11}+1)\varpi_1+\mathsf{m}_{12}\varpi_2} + v(v+v^{-1})T_{(\mathsf{m}_{11}-1)\varpi_1+\mathsf{m}_{12}\varpi_2} +
\end{aligned}$$

$$
\begin{aligned}
& T_{(\mathsf{m}_{11}+1)\varpi_1+(\mathsf{m}_{12}+2)\varpi_2} + T_{(\mathsf{m}_{11}-1)\varpi_1+(\mathsf{m}_{12}-2)\varpi_2}\Big) + \\
& \qquad b_2 v^{-\mathsf{m}_{21}-4\mathsf{m}_{22}} \cdot e^{-\alpha_2} \left(T_{(\mathsf{m}_{21}+1)\varpi_1+(\mathsf{m}_{22}+1)\varpi_2} + T_{(\mathsf{m}_{21}-1)\varpi_1+(\mathsf{m}_{22}-1)\varpi_2}\right) + \\
& \qquad \frac{b_1^2(v+v^{-1})^2}{(1+v)^2} v^{-2+(\mathsf{n}_{11}^+-\mathsf{n}_{11}^-)-(2\mathsf{m}_{11}+8\mathsf{m}_{12})} \cdot e^{-2\alpha_1} T_{2\mathsf{m}_{11}\varpi_1+2\mathsf{m}_{12}\varpi_2} + \\
& \qquad \delta_{\epsilon_{12}^+,-1}\delta_{\epsilon_{12}^-,1} \cdot \Big\{ b_1 b_2 v^{-4+3(\mathsf{n}_{12}^+-\mathsf{n}_{12}^-)-(\mathsf{m}_{11}+\mathsf{m}_{21}+4\mathsf{m}_{12}+4\mathsf{m}_{22})} \times \\
& e^{-\alpha_1-\alpha_2}\left((v+v^{-1})T_{(\mathsf{m}_{11}+\mathsf{m}_{21})\varpi_1+(\mathsf{m}_{12}+\mathsf{m}_{22}+1)\varpi_2} + vT_{(\mathsf{m}_{11}+\mathsf{m}_{21}-2)\varpi_1+(\mathsf{m}_{12}+\mathsf{m}_{22}-1)\varpi_2}\right) + \\
& \qquad \frac{b_1^2 b_2 (v+v^{-1})^2}{(1+v)^2} v^{-8+(\mathsf{n}_{11}^+-\mathsf{n}_{11}^-)+6(\mathsf{n}_{12}^+-\mathsf{n}_{12}^-)-(2\mathsf{m}_{11}+\mathsf{m}_{21}+8\mathsf{m}_{12}+4\mathsf{m}_{22})} \times \\
& \qquad\qquad e^{-2\alpha_1-\alpha_2} T_{(2\mathsf{m}_{11}+\mathsf{m}_{21}-1)\varpi_1+(2\mathsf{m}_{12}+\mathsf{m}_{22}+1)\varpi_2} \Big\} + \\
& \qquad \delta_{\epsilon_{12}^+,1}\delta_{\epsilon_{12}^-,-1} \cdot \Big\{ b_1 b_2 v^{3+3(\mathsf{n}_{12}^+-\mathsf{n}_{12}^-)-(\mathsf{m}_{11}+\mathsf{m}_{21}+4\mathsf{m}_{12}+4\mathsf{m}_{22})} \times \\
& e^{-\alpha_1-\alpha_2}\left(v(v+v^{-1})T_{(\mathsf{m}_{11}+\mathsf{m}_{21})\varpi_1+(\mathsf{m}_{12}+\mathsf{m}_{22}-1)\varpi_2} + T_{(\mathsf{m}_{11}+\mathsf{m}_{21}+2)\varpi_1+(\mathsf{m}_{12}+\mathsf{m}_{22}+1)\varpi_2}\right) + \\
& \qquad \frac{b_1^2 b_2 (v+v^{-1})^2}{(1+v)^2} v^{4+(\mathsf{n}_{11}^+-\mathsf{n}_{11}^-)+6(\mathsf{n}_{12}^+-\mathsf{n}_{12}^-)-(2\mathsf{m}_{11}+\mathsf{m}_{21}+8\mathsf{m}_{12}+4\mathsf{m}_{22})} \times \\
& \qquad\qquad e^{-2\alpha_1-\alpha_2} T_{(2\mathsf{m}_{11}+\mathsf{m}_{21}+1)\varpi_1+(2\mathsf{m}_{12}+\mathsf{m}_{22}-1)\varpi_2} \Big\}. \qquad (3.32)
\end{aligned}
$$

Remark 3.37. If $\epsilon^+ = \epsilon^-$, then the terms with δ's are vacuous. If we further set $\mathsf{n}^+ = \mathsf{n}^-$ and $c_i^\pm = \pm 1$ for all i, then we obtain the formula for the 1st hamiltonian of the type G_2 quantum difference Toda lattice as defined in [6]:

$$
\begin{aligned}
\mathbf{D}_1 = 1 &+ (T_{2\varpi_1} + T_{-2\varpi_1} + T_{2\varpi_2} + T_{-2\varpi_2} + T_{2\varpi_1+2\varpi_2} + T_{-2\varpi_1-2\varpi_2}) - \\
& (v-v^{-1})^2 \cdot e^{-\alpha_1}(v^{-1}(v+v^{-1})T_{\varpi_1} + v(v+v^{-1})T_{-\varpi_1} + T_{\varpi_1+2\varpi_2} + T_{-\varpi_1-2\varpi_2}) - \\
& (v^3-v^{-3})^2 \cdot e^{-\alpha_2}(T_{\varpi_1+\varpi_2} + T_{-\varpi_1-\varpi_2}) + v^{-2}(1-v^{-1})^2(v^2-v^{-2})^2 \cdot e^{-2\alpha_1}. \qquad (3.33)
\end{aligned}
$$

Let $\mathcal{G}_2$ be the associative $\mathbb{C}(v)$-algebra generated by $\{\mathsf{w}_i^{\pm 1}, \mathsf{D}_i^{\pm 1}\}_{i=1}^2$ subject to

$$
\begin{aligned}
& [\mathsf{w}_1, \mathsf{w}_2] = [\mathsf{D}_1, \mathsf{D}_2] = 0,\ \mathsf{w}_i^{\pm 1}\mathsf{w}_i^{\mp 1} = \mathsf{D}_i^{\pm 1}\mathsf{D}_i^{\mp 1} = 1, \\
& \mathsf{D}_1\mathsf{w}_1 = v^2\mathsf{w}_1\mathsf{D}_1,\ \mathsf{D}_1\mathsf{w}_2 = v^{-1}\mathsf{w}_2\mathsf{D}_1,\ \mathsf{D}_2\mathsf{w}_1 = v^{-3}\mathsf{w}_1\mathsf{D}_2,\ \mathsf{D}_2\mathsf{w}_2 = v^3\mathsf{w}_2\mathsf{D}_2.
\end{aligned}
$$

Consider the anti-isomorphism from $\mathcal{G}_2$ to the algebra $\mathcal{D}_v(H_{\mathfrak{g}_2}^{\mathrm{ad}})$ of Section 2.5, sending $\mathsf{w}_i \mapsto T_{-\varpi_i}, \mathsf{D}_i \mapsto e^{-\alpha_i}$. Let $\mathsf{H} = \mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ be the element of $\mathcal{G}_2$ that corresponds to $\mathbf{D}_1$ under this anti-isomorphism. The following is the key property of $\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ in type G_2.

Proposition 3.38. $\mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ depends only on $\epsilon := \frac{\epsilon_{12}^{+}-\epsilon_{12}^{-}}{2} \in \{-1, 0, 1\}$, up to algebra automorphisms of $\mathcal{G}_2$.

The proof of Proposition 3.38 is completely analogous to that of Proposition 3.11 given in Appendix A; we leave the details to the interested reader. Proposition 3.38 implies that given two pairs of Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ and $(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$ with $\epsilon_{12}^{+} - \epsilon_{12}^{-} = \tilde{\epsilon}_{12}^{+} - \tilde{\epsilon}_{12}^{-}$, there exists an algebra automorphism of $\mathcal{D}_v(H_{\mathfrak{g}_2}^{\mathrm{ad}})$ that maps the 1st hamiltonian $\mathbf{D}_1(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathbf{D}_1(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$. As we will see in Appendix B, the same automorphism maps the modified quantum Toda system $\mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathcal{T}(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$.

Remark 3.39. For completeness of our list (3.22, 3.29, 3.30, 3.31), let us present the explicit formula for the 1st hamiltonian $\hat{\mathbf{D}}_1$ of the type $G_2^{(1)}$ quantum difference affine Toda system:

$$\begin{aligned}\hat{\mathbf{D}}_1 = 1 + (T_{2\varpi_1} + T_{-2\varpi_1} + T_{2\varpi_2} + T_{-2\varpi_2} + T_{2\varpi_1+2\varpi_2} + T_{-2\varpi_1-2\varpi_2}) - \\ (v - v^{-1})^2 e^{-\varpi_1} (v^{-1}(v + v^{-1}) T_{\varpi_1} + v(v + v^{-1}) T_{-\varpi_1} + T_{\varpi_1+2\varpi_2} + T_{-\varpi_1-2\varpi_2}) - \\ (v^3 - v^{-3})^2 e^{\varpi_1-\varpi_2} (T_{\varpi_1+\varpi_2} + T_{-\varpi_1-\varpi_2}) + v^{-2}(1 - v^{-1})^2 (v^2 - v^{-2})^2 \cdot e^{-2\varpi_1} - \\ \kappa v^{-12} (v^3 - v^{-3})^2 e^{\varpi_1+2\varpi_2} (T_{\varpi_1} + T_{-\varpi_1}) + \kappa v^{-12} (v - v^{-1})^2 (v^3 - v^{-3})^2 e^{2\varpi_2}. \quad (3.34)\end{aligned}$$

For $\kappa = 0$, this recovers $\mathbf{D}_1$ of (3.33).

4 Whittaker Vectors and Their Pairing

In this section, we study a pairing of two general Whittaker vectors (associated with a pair of Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$) in universal Verma modules, following [7]. We obtain a fermionic formula for the corresponding terms $\tilde{J}_{\beta}$. We show that their generating series is a natural solution of the modified quantum difference Toda system $\mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ of Section 2. This provides a natural generalization of [7, Section 3], where $\epsilon^{+} = \epsilon^{-}$ and $\mathsf{n}^{+} = \mathsf{n}^{-}$.

4.1 Whittaker vectors

Following the notations of [7], consider $U_v(\mathfrak{g})$ and $U_{v^{-1}}(\mathfrak{g})$, whose generators will be denoted by E_i, F_i, L_i and $\bar{E}_i, \bar{F}_i, \bar{L}_i$, respectively. In contrast to [7], we will work with universal Verma modules. Let $\{u_i\}_{i=1}^{n}$ be indeterminates and consider an extension $\mathsf{k} := \mathbb{C}(v^{1/\mathsf{N}}, u_1, \ldots, u_n)$ of $\mathbb{C}(v^{1/\mathsf{N}})$. Let $U_v(\mathfrak{g})^{\leq}$ be the subalgebra of $U_v(\mathfrak{g})$ generated by $\{L_i^{\pm 1}, F_i\}_{i=1}^{n}$ and consider its action on k with F_i acting trivially and L_i acting via multiplication by u_i. We define the universal Verma module $\mathcal{V}$ over $U_v(\mathfrak{g})$ as

$\mathcal{V} := U_v(\mathfrak{g}) \otimes_{U_v(\mathfrak{g})^{\leq}} \mathsf{k}$. It is generated by $\mathbf{1} \in \mathsf{k}$ such that $E_i(\mathbf{1}) = 0$ and $L_i(\mathbf{1}) = u_i \cdot \mathbf{1}$ for $1 \leq i \leq n$. We define the formal symbol $\lambda := \sum_{i=1}^n \frac{\log(u_i)}{\mathsf{d}_i \log(v)} \alpha_i$, which will appear only as an index or in the context of the homomorphism $v^{(\lambda,\cdot)} \colon P \to \mathsf{k}$ defined by $P \ni m_i\omega_i \mapsto v^{(\lambda, \sum_{i=1}^n m_i\omega_i)} = \prod_{i=1}^n u_i^{m_i}$, so that $K_\mu(\mathbf{1}) = v^{(\lambda,\mu)} \cdot \mathbf{1}$ for $\mu \in P$. In particular, $\mathcal{V}$ is graded by Q_+: $\mathcal{V} = \oplus_{\beta \in Q_+} \mathcal{V}_\beta$ with $\mathcal{V}_\beta = \{w \in \mathcal{V} | K_\mu(w) = v^{(\mu, \lambda-\beta)} w \ (\mu \in P)\}$. Similarly, let $\bar{\mathcal{V}}$ be the universal Verma module over $U_{v^{-1}}(\mathfrak{g})$ generated by the highest weight vector $\bar{\mathbf{1}}$ such that $\bar{E}_i(\bar{\mathbf{1}}) = 0$ and $\bar{L}_i(\bar{\mathbf{1}}) = u_i^{-1} \cdot \bar{\mathbf{1}}$ for $1 \leq i \leq n$. It is also Q_+-graded: $\bar{\mathcal{V}} = \oplus_{\beta \in Q_+} \bar{\mathcal{V}}_\beta$ with $\bar{\mathcal{V}}_\beta = \{w \in \bar{\mathcal{V}} | \bar{K}_\mu(w) = v^{-(\mu, \lambda-\beta)} w \ (\mu \in P)\}$.

Remark 4.2. One can alternatively work with the standard Verma modules $\mathcal{V}^\lambda$ and $\bar{\mathcal{V}}^\lambda$, $\lambda \in P$ (one should further require λ to be strictly antidominant for the existence of Whittaker vectors), so that $u_i = v^{(\lambda,\omega_i)} \in \mathbb{C}(v^{1/\mathsf{N}})$. This viewpoint is used in [7]. We prefer the current exposition as it is compatible with our discussion in Section 5. Nevertheless, motivated by the above standard set-up, we will freely use the above notation $v^{(\lambda,\mu)}$ for $\mu \in P$.

There is a unique nondegenerate k-bilinear pairing $(\cdot,\cdot) \colon \mathcal{V} \times \bar{\mathcal{V}} \to \mathsf{k}$ such that $(\mathbf{1}, \bar{\mathbf{1}}) = 1$ and $(xw, w') = (w, \sigma(x)w')$ for all $x \in U_v(\mathfrak{g}), w \in \mathcal{V}, w' \in \bar{\mathcal{V}}$, where the algebra anti-isomorphism $\sigma \colon U_v(\mathfrak{g}) \to U_{v^{-1}}(\mathfrak{g})$ is determined by $\sigma(E_i) = \bar{F}_i, \sigma(F_i) = \bar{E}_i, \sigma(L_i) = \bar{L}_i^{-1}$.

Remark 4.3. One can alternatively work with a single universal Verma module $\mathcal{V}$ over $U_v(\mathfrak{g})$ endowed with the Shapovalov form $(\cdot,\cdot) \colon \mathcal{V} \times \mathcal{V} \to \mathsf{k}$; see our discussion in Remark 5.12.

For the key definition of this section, consider the completions $\mathcal{V}^\wedge, \bar{\mathcal{V}}^\wedge$ of $\mathcal{V}, \bar{\mathcal{V}}$, defined via

$$\mathcal{V}^\wedge := \prod_{\beta \in Q_+} \mathcal{V}_\beta, \ \bar{\mathcal{V}}^\wedge := \prod_{\beta \in Q_+} \bar{\mathcal{V}}_\beta.$$

Given a pair of Sevostyanov triples $(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ (now $c_i^\pm \in \mathsf{k}^\times$), define $\nu_i^\pm \in P$ via $\nu_i^\pm := \sum_{k=1}^n \mathsf{n}_{ik}^\pm \omega_k$, so that $(\nu_i^\pm, \alpha_j) - (\nu_j^\pm, \alpha_i) = \epsilon_{i,j}^\pm b_{i,j}$. We have associated **Whittaker vectors**

$$\theta = \theta(\epsilon^+, \mathsf{n}^+, c^+) = \sum_{\beta \in Q_+} \theta_\beta \in \mathcal{V}^\wedge \quad (\text{with } \theta_\beta \in \mathcal{V}_\beta)$$

and

$$\bar{\theta} = \bar{\theta}(\epsilon^-, \mathsf{n}^-, c^-) = \sum_{\beta \in Q_+} \bar{\theta}_\beta \in \bar{\mathcal{V}}^\wedge \quad (\text{with } \bar{\theta}_\beta \in \bar{\mathcal{V}}_\beta),$$

which are uniquely determined by the following conditions:

$$\theta_0 = \mathbf{1},\ E_i K_{\nu_i^+}(\theta) = c_i^+ \cdot \theta \ \text{ and } \ \bar{\theta}_0 = \bar{\mathbf{1}},\ \bar{E}_i \bar{K}_{\nu_i^-}(\bar{\theta}) = c_i^- \cdot \bar{\theta}. \tag{4.1}$$

Remark 4.4. This is a direct generalization of the classical notion of Whittaker vectors for Lie algebras as defined by B. Kostant in his milestone work on the subject [14].

4.5. Pairing of Whittaker vectors

Set $(t;t)_r := \prod_{k=1}^{r}(1-t^k)$ for $r \in \mathbb{Z}_{>0}$, and $(t;t)_0 := 1$. Choose convex orderings $\prec_\pm$ on Δ_+ such that $\epsilon_{ij}^{\pm} = -1 \Rightarrow \alpha_i \prec_\pm \alpha_j$, cf. Section 3.4. Let $\alpha_{i_1^\pm} \prec_\pm \cdots \prec_\pm \alpha_{i_n^\pm}$ be the simple roots ordered with respect to $\prec_\pm$. For $1 \le i \neq j \le n$ we write $i \prec_\pm j$ if $\alpha_i \prec_\pm \alpha_j$. Define

$$J_\beta = J_\beta(\epsilon^\pm, \mathsf{n}^\pm, c^\pm) := \Big(\theta_\beta(\epsilon^+, \mathsf{n}^+, c^+), \bar{\theta}_\beta(\epsilon^-, \mathsf{n}^-, c^-)\Big). \tag{4.2}$$

Following [7, (3.11)] (we note that $\tilde{J}_\beta$ is denoted by J_β^λ in [7]) we also consider its slight modification

$$\tilde{J}_\beta = \tilde{J}_\beta(\epsilon^\pm, \mathsf{n}^\pm, c^\pm) := v^{-(\beta,\beta)/2+(\lambda,\beta)} \Big(\theta_\beta(\epsilon^+, \mathsf{n}^+, c^+), \bar{\theta}_\beta(\epsilon^-, \mathsf{n}^-, c^-)\Big). \tag{4.3}$$

For $\beta \notin Q_+$, we set $J_\beta := 0$ and $\tilde{J}_\beta := 0$. Our 1st result provides a recursive formula for $\tilde{J}_\beta$.

Theorem 4.6. We have

$$\tilde{J}_\beta = \sum_{0 \le \gamma \le \beta} \frac{1}{(v^2)_{\beta-\gamma}} v^{(\gamma,\gamma)-2(\lambda+\rho,\gamma)} c^{\beta-\gamma} v^{\tau_\lambda(\beta-\gamma,\beta)} \cdot \tilde{J}_\gamma, \tag{4.4}$$

where $(v^2)_\alpha := \prod_{i=1}^n (v_i^2; v_i^2)_{m_i}$, $c^\alpha := \prod_{i=1}^n (-c_i^+ c_i^- (v_i - v_i^{-1})^2)^{m_i}$, $\tau_\lambda(\alpha,\beta) := \sum_{i=1}^n m_i(\nu_i^- - \nu_i^+, \lambda - \beta) + \sum_{j \prec_+ i} m_i m_j (\nu_i^- - \nu_i^+, \alpha_j) + \sum_{i=1}^n \frac{m_i(m_i-1)}{2}(\nu_i^- - \nu_i^+, \alpha_i)$ for $\alpha = \sum_{i=1}^n m_i \alpha_i \in Q_+$.

Proof. The proof is completely analogous to that of [7, Theorem 3.1] and is based on an evaluation of $(C(\theta_\beta), \bar{\theta}_\beta)$ in two different ways, where C is the Drinfeld Casimir element. ■

Solving the recursive relation (4.4), one obtains an explicit fermionic formula for $\tilde{J}_\beta$.

Theorem 4.7. We have

$$\tilde{J}_\beta = c^\beta \cdot \sum_{\underline{\beta} = \{\beta^{(t)}\}_{t=0}^{\infty} \in Q_+^{\mathbb{N}}}^{\sum_{t=0}^{\infty} \beta^{(t)} = \beta} \frac{v^{2B(\underline{\beta})+R(\underline{\beta})}}{\prod_{t=0}^{\infty} (v^2)_{\beta^{(t)}}}, \tag{4.5}$$

where we set $R(\underline{\beta}) := \sum_{t=0}^{\infty} \tau_\lambda \left(\beta^{(t)}, \beta - \sum_{s=0}^{t-1} \beta^{(s)}\right)$ and

$$B(\underline{\beta}) := \frac{1}{2} \sum_{t,t'=0}^{\infty} \min(t,t') \left(\beta^{(t)}, \beta^{(t')}\right) - \sum_{t=0}^{\infty} t \left(\lambda + \rho, \beta^{(t)}\right).$$

Proof. We will give a direct proof as the general machinery of fermionic formulas developed in [7, Section 2] does not apply to our set-up. The formula is obvious for $\beta = 0$. From now on, fix $\beta > 0$ (that is, $\beta \in Q_+ \backslash \{0\}$). Let us rewrite the equality (4.4) as

$$\left(1 - v^{(\beta,\beta)-2(\lambda+\rho,\beta)}\right) \tilde{J}_\beta = \sum_{0<\beta_1\leq\beta} \frac{1}{(v^2)_{\beta_1}} v^{(\beta-\beta_1,\beta-\beta_1)-2(\lambda+\rho,\beta-\beta_1)} c^{\beta_1} v^{\tau_\lambda(\beta_1,\beta)} \tilde{J}_{\beta-\beta_1}.$$

We apply the same formula for $\tilde{J}_{\beta-\beta_1}$ if $\beta_1 < \beta$. Proceeding in the same way, we finally obtain

$$\tilde{J}_\beta = \sum_{d\geq 1} \sum_{\beta_1,\ldots,\beta_d>0}^{\beta_1+\ldots+\beta_d=\beta} \frac{\prod_{e=1}^d c^{\beta_e} v^{\tau_\lambda(\beta_e,\beta_e+\ldots+\beta_d)} v^{(\beta_{e+1}+\ldots+\beta_d,\beta_{e+1}+\ldots+\beta_d)-2(\lambda+\rho,\beta_{e+1}+\ldots+\beta_d)}}{\prod_{e=1}^d (v^2)_{\beta_e} (1 - v^{(\beta_e+\ldots+\beta_d,\beta_e+\ldots+\beta_d)-2(\lambda+\rho,\beta_e+\ldots+\beta_d)})}. \quad (4.6)$$

On the other hand, the summation in the right-hand side of (4.5) is over all $\underline{\beta} = \{\beta^{(t)}\}_{t=0}^{\infty} \in Q_+^{\mathbb{N}}$ with $\sum_{t=0}^{\infty} \beta^{(t)} = \beta$. Such sequences are in bijection with tuples $\{d, \{\beta_e\}_{e=1}^d, \{t_e\}_{e=1}^d | d \geq 1, \beta_e > 0, t_e \in \mathbb{N}, \sum_{e=1}^d \beta_e = \beta\}$ via $\beta^{(t_1+\ldots+t_e+e-1)} = \beta_e$ $(1 \leq e \leq d)$ and $\beta^{(t)} = 0$ otherwise.

Hence, the right-hand side of (4.5) equals

$$\sum_{d\geq 1} \sum_{\beta_1,\ldots,\beta_d>0}^{\beta_1+\ldots+\beta_d=\beta} \left\{ \frac{c^\beta v^{\sum_{e=1}^d \tau_\lambda(\beta_e,\beta-\beta_1-\ldots-\beta_{e-1})}}{\prod_{e=1}^d (v^2)_{\beta_e}} \times \right.$$
$$\left. \sum_{t_1,\ldots,t_d\geq 0} v^{\sum_{e=1}^d (t_1+\ldots+t_e+e-1)(\beta_e,\beta_e)+2\sum_{e<e'}(t_1+\ldots+t_e+e-1)(\beta_e,\beta_{e'})-2(\lambda+\rho,\sum_{e=1}^d (t_1+\ldots+t_e+e-1)\beta_e)} \right\}. \quad (4.7)$$

It is straightforward to verify that the right-hand side of (4.6) coincides with (4.7). ■

4.8. J-functions: eigenfunctions of modified quantum difference Toda systems

Recall the elements $\omega_i^\vee = \omega_i/\mathsf{d}_i \in P \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $(\alpha_j, \omega_i^\vee) = \delta_{i,j}$. Consider a vector space N_λ that consists of all formal sums $\left\{\sum_{\beta\in Q} a_\beta \underline{y}^{\beta-\lambda} | a_\beta \in \mathsf{k}\right\}$ for which there exists $\beta_0 \in Q$ such that $a_\beta = 0$ unless $\beta - \beta_0 \in Q_+$, where $\underline{y}^{\beta-\lambda}$ is used to denote $\prod_{i=1}^n y_i^{(\beta-\lambda,\omega_i^\vee)}$. The vector space $N_{\lambda+\rho}$ is defined analogously with λ being replaced by $\lambda + \rho$. Consider the natural action of the algebra $\mathcal{D}_v(H^{\mathrm{ad}})$ of Section 2.5 on the vector space N_λ, determined by

$$T_\mu(\underline{y}^{\beta-\lambda}) = v^{-(\mu,\beta-\lambda)}\underline{y}^{\beta-\lambda},\ e^\alpha(\underline{y}^{\beta-\lambda}) = \underline{y}^{\beta-\alpha-\lambda} \text{ for } \alpha,\beta \in Q, \mu \in P. \tag{4.8}$$

The action of $\mathcal{D}_v(H^{\mathrm{ad}})$ on $N_{\lambda+\rho}$ is defined analogously.

Consider the following generating functions of the terms J_β defined in (4.2):

$$\begin{aligned}\tilde{J} = \tilde{J}(\{y_i\}_{i=1}^n) &:= \sum_{\beta\in Q_+} J_\beta \prod_{i=1}^n y_i^{(\beta-\lambda,\omega_i^\vee)} \in N_\lambda,\\ J = J(\{y_i\}_{i=1}^n) &:= \sum_{\beta\in Q_+} J_\beta \prod_{i=1}^n y_i^{(\beta-\lambda-\rho,\omega_i^\vee)} \in N_{\lambda+\rho}.\end{aligned} \tag{4.9}$$

Recall the difference operators $\tilde{\mathbf{D}}_V, \mathbf{D}_V \in \mathcal{D}_v(H^{\mathrm{ad}})$ of Section 2.5, associated with the pair of Sevostyanov triples $(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ and a finite-dimensional $U_v(\mathfrak{g})$-representation V. The following is the key result of this section.

Theorem 4.9. (a) We have $\tilde{\mathbf{D}}_V(\tilde{J}) = \mathrm{tr}_V\left(v^{2(\lambda+\rho)}\right)\cdot\tilde{J}$.
(b) We have $\mathbf{D}_V(J) = \mathrm{tr}_V\left(v^{2(\lambda+\rho)}\right)\cdot J$.

Proof. First, we note that part (a) implies part (b), due to $\mathbf{D}_V = e^\rho\tilde{\mathbf{D}}_V e^{-\rho}$ and $J = e^\rho(\tilde{J})$. The proof of part (a) is based on an evaluation of $\left(C_V(\theta_\beta), \bar{\theta}_\beta\right)$ in two different ways, where C_V is the central element of (2.2). On the one hand, C_V acts on $\mathcal{V}$ as a multiplication by $\mathrm{tr}_V(v^{2(\lambda+\rho)})$ (since C_V is central, $\mathcal{V}$ is generated by $\mathbf{1}$, and $C_V(\mathbf{1}) = \mathrm{tr}_V(v^{2(\lambda+\rho)})\cdot\mathbf{1}$), so that $\left(C_V(\theta_\beta), \bar{\theta}_\beta\right) = \mathrm{tr}_V(v^{2(\lambda+\rho)})\cdot J_\beta$. On the other hand, we can use the explicit formula for C_V.

Let $\{w_k\}_{k=1}^N$ be a weight basis of V, and $\mu_k \in P$ be the weight of w_k, cf. Section 2.5. Then we have

$$\begin{aligned}C_V(\theta_\beta) = &\sum_{\vec{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n}^{1\le k\le N} c(\vec{m})\cdot v^{(\lambda-\beta+2\rho,\mu_k)+(\lambda-\beta+\alpha(\vec{m}),\mu_k-\alpha(\vec{m}))}\times\\ &\left\langle w_k \middle| E_{\alpha_{i_1^-}}^{m_{i_1^-}}\cdots E_{\alpha_{i_n^-}}^{m_{i_n^-}}\cdot F_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots F_{\alpha_{i_n^+}}^{m_{i_n^+}} \middle| w_k\right\rangle\cdot F_{\alpha_{i_1^-}}^{m_{i_1^-}}\cdots F_{\alpha_{i_n^-}}^{m_{i_n^-}}\cdot E_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots E_{\alpha_{i_n^+}}^{m_{i_n^+}}(\theta_\beta),\end{aligned} \tag{4.10}$$

where $\langle w_k|x|w_k\rangle$ is the matrix coefficient of $x \in U_v(\mathfrak{g})$, $\alpha(\vec{m}) := \sum_{i=1}^n m_i\alpha_i \in Q_+$, and $c(\vec{m}) \in \mathbb{C}(v)$ are certain coefficients for which we currently do not need explicit formulas.

Using the defining property of $(\cdot,\cdot)$, we get

$$\left(F_{\alpha_{i_1^-}}^{m_{i_1^-}}\cdots F_{\alpha_{i_n^-}}^{m_{i_n^-}}\cdot E_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots E_{\alpha_{i_n^+}}^{m_{i_n^+}}(\theta_\beta),\bar{\theta}_\beta\right)=\left(E_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots E_{\alpha_{i_n^+}}^{m_{i_n^+}}(\theta_\beta),\bar{E}_{\alpha_{i_n^-}}^{m_{i_n^-}}\cdots\bar{E}_{\alpha_{i_1^-}}^{m_{i_1^-}}(\bar{\theta}_\beta)\right).$$

To evaluate the pairing in the right-hand side, note that the defining conditions (4.1) of the Whittaker vectors imply $E_iK_{\nu_i^+}(\theta_\gamma)=c_i^+\theta_{\gamma-\alpha_i}$ and $\bar{E}_i\bar{K}_{\nu_i^-}(\bar{\theta}_\gamma)=c_i^-\bar{\theta}_{\gamma-\alpha_i}$ for $\gamma\in Q$; hence, $E_i(\theta_\gamma)=c_i^+v^{-(\nu_i^+,\lambda-\gamma)}\theta_{\gamma-\alpha_i}$ and $\bar{E}_i(\bar{\theta}_\gamma)=c_i^-v^{(\nu_i^-,\lambda-\gamma)}\bar{\theta}_{\gamma-\alpha_i}$. Applying this iteratively, we find

$$E_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots E_{\alpha_{i_n^+}}^{m_{i_n^+}}(\theta_\beta)=\prod_{i=1}^n(c_i^+)^{m_i}v^{\tau_\lambda^+(\vec{m},\beta)}\cdot\theta_{\beta-\alpha(\vec{m})},$$
$$\bar{E}_{\alpha_{i_n^-}}^{m_{i_n^-}}\cdots\bar{E}_{\alpha_{i_1^-}}^{m_{i_1^-}}(\bar{\theta}_\beta)=\prod_{i=1}^n(c_i^-)^{m_i}v^{\tau_\lambda^-(\vec{m},\beta)}\cdot\bar{\theta}_{\beta-\alpha(\vec{m})}$$

with $\tau_\lambda^\pm(\vec{m},\beta)$ given by $\tau_\lambda^+(\vec{m},\beta):=-\sum_{k=1}^n\sum_{r=0}^{m_{i_k^+}-1}\left(\nu_{i_k^+}^+,\lambda-\beta+r\alpha_{i_k^+}+\sum_{s=k+1}^n m_{i_s^+}\alpha_{i_s^+}\right)$ and $\tau_\lambda^-(\vec{m},\beta):=\sum_{k=1}^n\sum_{r=0}^{m_{i_k^-}-1}\left(\nu_{i_k^-}^-,\lambda-\beta+r\alpha_{i_k^-}+\sum_{s=1}^{k-1}m_{i_s^-}\alpha_{i_s^-}\right)$.

Summarizing all these calculations, we obtain the following equality:

$$\sum_{\vec{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n}^{1\le k\le N}c(\vec{m})\prod_{i=1}^n(c_i^+c_i^-)^{m_i}\cdot v^{(\lambda-\beta+2\rho,\mu_k)+(\lambda-\beta+\alpha(\vec{m}),\mu_k-\alpha(\vec{m}))+\tau_\lambda^+(\vec{m},\beta)+\tau_\lambda^-(\vec{m},\beta)}\times$$
$$\left\langle w_k\middle|E_{\alpha_{i_1^-}}^{m_{i_1^-}}\cdots E_{\alpha_{i_n^-}}^{m_{i_n^-}}\cdot F_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots F_{\alpha_{i_n^+}}^{m_{i_n^+}}\middle|w_k\right\rangle\cdot J_{\beta-\alpha(\vec{m})}=\mathrm{tr}_V\left(v^{2(\lambda+\rho)}\right)\cdot J_\beta.\qquad(4.11)$$

Let us now compute $\tilde{\mathbf{D}}_V(\tilde{J})$. First, we need to rewrite $F_{\alpha_{i_1^-}}^{m_{i_1^-}}\cdots F_{\alpha_{i_n^-}}^{m_{i_n^-}}$ and $E_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots E_{\alpha_{i_n^+}}^{m_{i_n^+}}$ in terms of the Sevostyanov generators e_i,f_i and Cartan terms, moving the latter to the right of f_i's and to the left of e_i's. We have

$$F_{\alpha_{i_1^-}}^{m_{i_1^-}}\cdots F_{\alpha_{i_n^-}}^{m_{i_n^-}}=v^{\tilde{\tau}_\lambda^-(\vec{m})}f_{\alpha_{i_1^-}}^{m_{i_1^-}}\cdots f_{\alpha_{i_n^-}}^{m_{i_n^-}}\cdot K_{\sum_{i=1}^n m_i\nu_i^-},$$
$$E_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots E_{\alpha_{i_n^+}}^{m_{i_n^+}}=v^{\tilde{\tau}_\lambda^+(\vec{m})}K_{-\sum_{i=1}^n m_i\nu_i^+}\cdot e_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots e_{\alpha_{i_n^+}}^{m_{i_n^+}},$$

where $\tilde{\tau}_\lambda^\pm(\vec{m})$ are given by $\tilde{\tau}_\lambda^+(\vec{m}):=\sum_{k=1}^n\sum_{r=1}^{m_{i_k^+}}\left(\nu_{i_k^+}^+,r\alpha_{i_k^+}+\sum_{s=1}^{k-1}m_{i_s^+}\alpha_{i_s^+}\right)$ and $\tilde{\tau}_\lambda^-(\vec{m}):=-\sum_{k=1}^n\sum_{r=1}^{m_{i_k^-}}\left(\nu_{i_k^-}^-,r\alpha_{i_k^-}+\sum_{s=k+1}^n m_{i_s^-}\alpha_{i_s^-}\right)$. Tracing back the definition of $\tilde{\mathbf{D}}_V$, we find

$$\tilde{\mathbf{D}}_V(\tilde{J}) = \tilde{\mathbf{D}}_V\left(\sum_{\tilde{\beta}\in Q_+} J_{\tilde{\beta}}\underline{v}^{\tilde{\beta}-\lambda}\right) = \sum_{\tilde{\beta}\in Q_+} \sum_{\vec{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n}^{1\le k\le N} c(\vec{m})\prod_{i=1}^{n}(c_i^+c_i^-)^{m_i}\times$$
$$v^{(2\rho,\mu_k)+\sum_{i=1}^n m_i(\nu_i^+-\nu_i^-,\tilde{\beta}-\lambda)-(2\mu_k-\alpha(\vec{m}),\tilde{\beta}-\lambda)-(\mu_k,\alpha(\vec{m}))+\tilde{\tau}_\lambda^+(\vec{m})+\tilde{\tau}_\lambda^-(\vec{m})}\times$$
$$\left\langle w_k\left|E_{\alpha_{i_1^-}}^{m_{i_1^-}}\cdots E_{\alpha_{i_n^-}}^{m_{i_n^-}}\cdot F_{\alpha_{i_1^+}}^{m_{i_1^+}}\cdots F_{\alpha_{i_n^+}}^{m_{i_n^+}}\right|w_k\right\rangle\cdot J_{\tilde{\beta}}\underline{v}^{\alpha(\vec{m})+\tilde{\beta}-\lambda}. \tag{4.12}$$

Due to the equalities $\tilde{\tau}_\lambda^+(\vec{m})+\tilde{\tau}_\lambda^-(\vec{m}) = \tau_\lambda^+(\vec{m},\beta)+\tau_\lambda^-(\vec{m},\beta)+\sum_{i=1}^n m_i(\nu_i^+-\nu_i^-,\alpha(\vec{m})+\lambda-\beta)$ and $-(\mu_k,\alpha(\vec{m}))-(2\mu_k-\alpha(\vec{m}),\beta-\lambda-\alpha(\vec{m})) = (\lambda-\beta,\mu_k)+(\lambda-\beta+\alpha(\vec{m}),\mu_k-\alpha(\vec{m}))$, the coefficient of $\underline{v}^{\beta-\lambda}$ in the right-hand side of (4.12) coincides with the left-hand side of (4.11).

The equality $\tilde{\mathbf{D}}_V(\tilde{J}) = \mathrm{tr}_V\left(v^{2(\lambda+\rho)}\right)\cdot\tilde{J}$ follows. ■

5 Geometric Realization of the Whittaker Vectors in Type A

In [3], A. Braverman and M. Finkelberg provided a geometric realization of the universal Verma module over $U_v(\mathfrak{sl}_n)$, the Shapovalov form on it, and two particular Whittaker vectors $\mathfrak{k},\mathfrak{w}$ of it via the Laumon based quasiflags' moduli spaces. The vectors $\mathfrak{k}$ and $\mathfrak{w}$ correspond to particular Sevostyanov triples (ϵ,n,c) with the corresponding orientations of the A_{n-1} Dynkin diagram being equioriented.x' In this section, we generalize their construction by providing a geometric interpretation of all Whittaker vectors and their pairing.

5.1 Laumon spaces

First, we recall the set-up of [9]. Let $\mathbf{C}$ be a smooth projective curve of genus zero. We fix a coordinate z on $\mathbf{C}$, and consider the action of $\mathbb{C}^\times$ on $\mathbf{C}$ such that $v(z) = v^{-2}z$. We have $\mathbf{C}^{\mathbb{C}^\times} = \{0,\infty\}$. We consider an n-dimensional vector space W with a basis $w_1,\ldots,w_n$. This defines a Cartan torus $T \subset G = \mathrm{SL}(n) \subset \mathrm{Aut}(W)$. We also consider its 2^{n-1}-fold cover, the bigger torus $\widetilde{T}$, acting on W as follows: for $\widetilde{T} \ni \underline{t} = (t_1,\ldots,t_n)$ we have $\underline{t}(w_k) = t_k^2 w_k$.

Given an $(n-1)$-tuple of nonnegative integers $\underline{d} = (d_1,\ldots,d_{n-1}) \in \mathbb{N}^{n-1}$, we consider the *Laumon's based quasiflags' space* $\mathfrak{Q}_{\underline{d}}$. It is the moduli space of flags of locally free subsheaves $0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$ such that $\mathrm{rk}(\mathcal{W}_k) = k, \deg(\mathcal{W}_k) = -d_k$, $\mathcal{W}_k \subset \mathcal{W}$ is a vector subbundle in a neighborhood of $\infty \in \mathbf{C}$, and the fiber of $\mathcal{W}_k$ at ∞ equals the span $\langle w_1,\ldots,w_k\rangle \subset W$. It is a smooth connected quasi-projective variety of dimension $\sum_{i=1}^{n-1} 2d_i$.

The group $\widetilde{T} \times \mathbb{C}^\times$ acts naturally on $\mathfrak{Q}_{\underline{d}}$. The set of fixed points of $\widetilde{T} \times \mathbb{C}^\times$ on $\mathfrak{Q}_{\underline{d}}$ is finite and is parametrized by collections $\widetilde{\underline{d}}$ of nonnegative integers $(d_{ij})_{1\le j\le i\le n-1}$ such that $d_i = \sum_{j=1}^{i} d_{ij}$ and $d_{kj} \ge d_{ij}$ for $i \ge k \ge j$; see [9, 2.11]. Given a collection $\widetilde{\underline{d}}$ as above, we will denote by $\widetilde{\underline{d}} + \delta_{ij}$ the collection $\widetilde{\underline{d}}'$, such that $d'_{ij} = d_{ij} + 1$, while $d'_{kl} = d_{kl}$ for $(k,l) \neq (i,j)$. By abuse of notation, we use $\widetilde{\underline{d}}$ to denote the corresponding $\widetilde{T} \times \mathbb{C}^\times$-fixed point in $\mathfrak{Q}_{\underline{d}}$.

For $i \in \{1,\ldots,n-1\}$ and $\underline{d} = (d_1,\ldots,d_{n-1})$, we set $\underline{d}+i := (d_1,\ldots,d_i+1,\ldots,d_{n-1})$. We have a correspondence $\mathfrak{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}} \times \mathfrak{Q}_{\underline{d}+i}$ formed by the pairs $(\mathcal{W}_\bullet, \mathcal{W}'_\bullet)$ such that $\mathcal{W}'_i \subset \mathcal{W}_i$ and $\mathcal{W}_j = \mathcal{W}'_j$ for $j \neq i$. It is a smooth quasi-projective variety of dimension $1 + \sum_{i=1}^{n-1} 2d_i$. We denote by $\mathbf{p}$ (resp. $\mathbf{q}$) the natural projection $\mathfrak{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}}$ (resp. $\mathfrak{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}+i}$). We also have a map $\mathbf{s}\colon \mathfrak{E}_{\underline{d},i} \to \mathbf{C}$, given by $(\mathcal{W}_\bullet, \mathcal{W}'_\bullet) \mapsto \mathrm{supp}(\mathcal{W}_i/\mathcal{W}'_i)$. The correspondence $\mathfrak{E}_{\underline{d},i}$ comes equipped with a natural line bundle $\mathcal{L}_i$ whose fiber at a point $(\mathcal{W}_\bullet, \mathcal{W}'_\bullet)$ equals $\Gamma(\mathbf{C}, \mathcal{W}_i/\mathcal{W}'_i)$.

We denote by $'M$ the direct sum of equivariant K-groups: $'M := \oplus_{\underline{d}} K^{\widetilde{T}\times\mathbb{C}^\times}(\mathfrak{Q}_{\underline{d}})$. It is a module over $K^{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[\widetilde{T} \times \mathbb{C}^\times] = \mathbb{C}[t_1^{\pm1},\ldots,t_n^{\pm1},v^{\pm1} : \ t_1\cdots t_n = 1]$. We define $M := {}'M \otimes_{K^{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})} \mathrm{Frac}(K^{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})) \otimes_{\mathbb{C}(v)} \mathbb{C}(v^{1/\mathsf{N}})$. It is naturally graded: $M = \oplus_{\underline{d}} M_{\underline{d}}$. According to the Thomason localization theorem, restriction to the $\widetilde{T} \times \mathbb{C}^\times$-fixed point set induces an isomorphism of localized K-groups . The classes of the structure sheaves $[\widetilde{\underline{d}}]$ of the $\widetilde{T} \times \mathbb{C}^\times$-fixed points $\widetilde{\underline{d}}$ form a basis in $\oplus_{\underline{d}} K^{\widetilde{T}\times\mathbb{C}^\times}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^\times})_{\mathrm{loc}}$. The embedding of a point $\widetilde{\underline{d}}$ into $\mathfrak{Q}_{\underline{d}}$ is a proper morphism, so the direct image in the equivariant K-theory is well-defined, and we will denote by $[\widetilde{\underline{d}}] \in M_{\underline{d}}$ the direct image of the structure sheaf of the point $\widetilde{\underline{d}}$. The set $\{[\widetilde{\underline{d}}]\}$ forms a basis of M.

5.2 $U_v(\mathfrak{sl}_n)$-action via Laumon spaces

Following Section 4.1, consider the universal Verma module $\mathcal{V}$ over $U_v(\mathfrak{sl}_n)$ with $u_i = v^{\frac{i(i-1)}{2}} t_1\cdots t_i$, that is, $L_i(\mathbf{1}) = v^{\frac{i(i-1)}{2}} t_1\cdots t_i \cdot \mathbf{1}$. We identify $\mathsf{k} \simeq \mathrm{Frac}(K^{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})) \otimes_{\mathbb{C}(v)} \mathbb{C}(v^{1/\mathsf{N}})$. Define the following operators on M:

$$E_i := t_{i+1}^{-1} v^{d_{i+1}-d_i+1-i} \mathbf{p}_* \mathbf{q}^* \colon M_{\underline{d}} \to M_{\underline{d}-i},$$
$$F_i := -t_i^{-1} v^{d_i-d_{i-1}+i} \mathbf{q}_*(\mathcal{L}_i \otimes \mathbf{p}^*) \colon M_{\underline{d}} \to M_{\underline{d}+i},$$
$$L_i := t_1\cdots t_i v^{-d_i+\frac{i(i-1)}{2}} \colon M_{\underline{d}} \to M_{\underline{d}},$$
$$K_i := L_{i-1}^{-1} L_i^2 L_{i+1}^{-1} = t_{i+1}^{-1} t_i v^{d_{i+1}-2d_i+d_{i-1}-1} \colon M_{\underline{d}} \to M_{\underline{d}}.$$

To each $\widetilde{\underline{d}}$, we also assign a collection of $\widetilde{T} \times \mathbb{C}^\times$-weights $s_{ij} := t_j^2 v^{-2d_{ij}}$.

The following result is due to [3] (though our formulas follow [10, 24]).

Theorem 5.3. (a) The operators $\{E_i, F_i, L_i^{\pm1}\}_{i=1}^{n-1}$ give rise to the action of $U_v(\mathfrak{sl}_n)$ on M.
(b) There is a unique $U_v(\mathfrak{sl}_n)$-module isomorphism $M \xrightarrow{\sim} \mathcal{V}$ taking $[\mathcal{O}_{\mathfrak{Q}_{\underline{0}}}]$ to $\mathbf{1}$.

(c) The action of L_i is diagonal in the basis $\{[\widetilde{\underline{d}}]\}$ and $L_i([\widetilde{\underline{d}}]) = t_1 \cdots t_i v^{-d_i+\frac{i(i-1)}{2}} \cdot [\widetilde{\underline{d}}]$.
(d) The matrix coefficients of F_i, E_i in the fixed point basis $\{[\widetilde{\underline{d}}]\}$ of M are as follows:

$$F_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = -(1-v^2)^{-1} t_i^{-1} v^{d_i - d_{i-1}+i} s_{ij} \prod_{j\neq k\leq i} (1 - s_{ij}/s_{ik})^{-1} \prod_{k\leq i-1} (1 - s_{ij}/s_{i-1,k})$$

if $\widetilde{\underline{d}}' = \widetilde{\underline{d}} + \delta_{ij}$ for certain $j \leq i$;

$$E_{i[\widetilde{\underline{d}},\widetilde{\underline{d}}']} = (1-v^2)^{-1} t_{i+1}^{-1} v^{d_{i+1}-d_i+1-i} \prod_{j\neq k\leq i} (1 - s_{ik}/s_{ij})^{-1} \prod_{k\leq i+1} (1 - s_{i+1,k}/s_{ij})$$

if $\widetilde{\underline{d}}' = \widetilde{\underline{d}} - \delta_{ij}$ for certain $j \leq i$. All the other matrix coefficients of F_i, E_i vanish.

5.4. Geometric realization of the Whittaker vectors

Choose a Sevostyanov triple $(\epsilon, \mathsf{n}, c)$ and let $e_i := E_i \prod_{p=1}^{n-1} L_p^{\mathsf{n}_{ip}} = E_i K_\nu, \nu := \sum_{p=1}^{n-1} \mathsf{n}_{ip}\omega_p$, be the corresponding Sevostyanov generators. Choose $\underline{a} = (a_1, \ldots, a_{n-1}) \in \{0,1\}^{n-1}$ so that $a_i = \frac{1+\epsilon_{i-1,i}}{2} = \frac{1-\mathsf{n}_{i-1,i}+\mathsf{n}_{i,i-1}}{2}$ for $1 < i \leq n-1$, while a_1 equals either 0 or 1.

Consider the line bundle $\mathcal{D}_i$ on $\mathfrak{Q}_{\underline{d}}$ whose fiber at the point $(\mathcal{W}_\bullet)$ equals $\det R\Gamma(\mathbf{C}, \mathcal{W}_i)$. We also define the line bundle $\mathcal{D}^{\underline{a}}$ on $\mathfrak{Q}_{\underline{d}}$ via $\mathcal{D}^{\underline{a}} := \otimes_{i=1}^{n-1} \mathcal{D}_i^{-a_i}$. Note that $\mathcal{D}_1$ is a pull-back of the 1st line bundle on the Drinfeld compactification and therefore is trivial, which explains the irrelevance of our choice of a_1. Finally, we introduce the constants

$$X(\underline{d}) := \prod_{i=1}^{n-1} ((1-v^2)c_i)^{d_i} \prod_{p=1}^{n-1} (t_1 \cdots t_p)^{-2a_p - \sum_{i=1}^{n-1} d_i \mathsf{n}_{ip}} \prod_{p=1}^{n-1} t_p^{d_{p-1}-2a_p d_p} \times$$
$$v^{\sum_{i=1}^{n-1}\left((\mathsf{n}_{ii}+i)d_i - 2a_{i+1}d_i d_{i+1} + \frac{d_i(d_i-1)}{2}(\mathsf{n}_{ii}+2a_i+1)\right) + \sum_{i<j} \mathsf{n}_{ji} d_i d_j - \sum_{i,p=1}^{n-1} \frac{p(p-1)}{2} d_i \mathsf{n}_{ip}}.$$

The following is the key result of this section.

Theorem 5.5. Define $\theta_{\underline{d}} := X(\underline{d}) \cdot [\mathcal{D}^{\underline{a}}] \in M_{\underline{d}}$ and set $\theta := \sum_{\underline{d}} \theta_{\underline{d}}$. Then $e_i(\theta) = c_i \cdot \theta$ for any $1 \leq i \leq n-1$.

Remark 5.6. (a) Due to Theorem 5.3, this provides a geometric realization of all Whittaker vectors (associated with Sevostyanov triples) of the universal Verma module $\mathcal{V}$ over $U_v(\mathfrak{sl}_n)$.
(b) It is straightforward to verify that θ does not depend on the choice of $a_1 \in \{0, 1\}$.

Proof. According to the Bott–Lefschetz formula, we have the following:

- $\theta = \sum_{\widetilde{\underline{d}}} a_{\widetilde{\underline{d}}} \cdot X(\underline{d}) \cdot (\mathcal{D}^{\underline{a}})_{|_{\widetilde{\underline{d}}}} \cdot [\widetilde{\underline{d}}]$, where $a_{\widetilde{\underline{d}}} = \prod_{w \in T_{\widetilde{\underline{d}}} \mathfrak{Q}_{\underline{d}}} (1-w)^{-1}$;
- $\frac{a_{\widetilde{\underline{d}}'}}{a_{\widetilde{\underline{d}}}} (\mathbf{p}_* \mathbf{q}^*)_{[\widetilde{\underline{d}}', \widetilde{\underline{d}}]} = (\mathbf{q}_* \mathbf{p}^*)_{[\widetilde{\underline{d}}, \widetilde{\underline{d}}']}$.

According to Theorem 5.3(c, d), we have

- $(\mathbf{q}_* \mathbf{p}^*)_{[\widetilde{\underline{d}}, \widetilde{\underline{d}}+\delta_{ij}]} = \frac{1}{1-v^2} \prod_{j \neq k \leq i} (1 - s_{ij}/s_{ik})^{-1} \prod_{k \leq i-1} (1 - s_{ij}/s_{i-1,k})$;
- $(\prod_{p=1}^{n-1} L_p^{\mathsf{n}_{ip}})[\widetilde{\underline{d}} + \delta_{ij}] = v^{-\mathsf{n}_{ii}} \prod_{p=1}^{n-1} \left(t_1 \cdots t_p v^{-d_p + \frac{p(p-1)}{2}} \right)^{\mathsf{n}_{ip}} \cdot [\widetilde{\underline{d}} + \delta_{ij}]$,

where $s_{ij} = t_j^2 v^{-2d_{ij}}$ as before. Finally, we also have

- $(\mathcal{D}^{\underline{a}})_{|_{\widetilde{\underline{d}}+\delta_{ij}}} / (\mathcal{D}^{\underline{a}})_{|_{\widetilde{\underline{d}}}} = s_{ij}^{a_i}$.

Therefore, it suffices to prove the following equality for any $\widetilde{\underline{d}}$ and $1 \leq i \leq n-1$:

$$\frac{X(\underline{d}+i)}{X(\underline{d})} \frac{t_{i+1}^{-1} v^{d_{i+1} - d_i - i}}{1 - v^2} v^{-\mathsf{n}_{ii}} \prod_{p=1}^{n-1} \left(t_1 \cdots t_p v^{-d_p + \frac{p(p-1)}{2}} \right)^{\mathsf{n}_{ip}} \cdot \sum_{j \leq i} s_{ij}^{a_i} \frac{\prod_{k \leq i-1} (1 - s_{ij}/s_{i-1,k})}{\prod_{k \leq i}^{k \neq j} (1 - s_{ij}/s_{ik})} = c_i.$$

Lemma 5.7. For any $\widetilde{\underline{d}}$ and $1 \leq i \leq n-1$, the following equality holds:

$$\sum_{j \leq i} s_{ij}^{a_i} \frac{\prod_{k \leq i-1} (1 - s_{ij}/s_{i-1,k})}{\prod_{k \leq i}^{k \neq j} (1 - s_{ij}/s_{ik})} = (t_i^2 v^{2d_{i-1} - 2d_i})^{a_i}. \tag{5.1}$$

Proof. First, let us rewrite the left-hand side of (5.1) as

$$\frac{s_{i1} \cdots s_{ii}}{s_{i-1,1} \cdots s_{i-1,i-1}} \cdot \sum_{j \leq i} s_{ij}^{a_i - 1} \frac{\prod_{k=1}^{i-1} (s_{i-1,k} - s_{ij})}{\prod_{k \leq i}^{k \neq j} (s_{ik} - s_{ij})}.$$

If $a_i = 1$, then the above sum $\sum_{j \leq i} s_{ij}^{a_i - 1} \frac{\prod_{k=1}^{i-1} (s_{i-1,k} - s_{ij})}{\prod_{k \leq i}^{k \neq j} (s_{ik} - s_{ij})}$ is a rational function in $\{s_{ij}\}_{j=1}^{i}$ of degree 0 and without poles, hence, a constant. To evaluate this constant, let $s_{ii} \to \infty$, in which case the 1st $i-1$ summands tend to zero, while the last one tends to 1. Hence, this constant is 1, and the left-hand side of (5.1) equals $\frac{s_{i1} \cdots s_{ii}}{s_{i-1,1} \cdots s_{i-1,i-1}} = t_i^2 v^{2d_{i-1} - 2d_i}$.

If $a_i = 0$, then $\sum_{j \leq i} s_{ij}^{-1} \frac{\prod_{k=1}^{i-1} (s_{i-1,k} - s_{ij})}{\prod_{k \leq i}^{k \neq j} (s_{ik} - s_{ij})} - \sum_{j \leq i} s_{ij}^{-1} \frac{\prod_{k=1}^{i-1} s_{i-1,k}}{\prod_{k \leq i}^{k \neq j} (s_{ik} - s_{ij})} = 0$ as the left-hand side is a rational function in $\{s_{ij}\}_{j=1}^{i}$ of degree -1 and without poles. Thus, the left-hand side of (5.1) equals $\sum_{j \leq i} \frac{\prod_{k \leq i}^{k \neq j} s_{ik}}{\prod_{k \leq i}^{k \neq j} (s_{ik} - s_{ij})}$. This is a rational function in $\{s_{ij}\}_{j=1}^{i}$ of degree 0 and without poles, hence, a constant. Specializing $s_{ii} \mapsto 0$, we see that this constant equals 1 (as the 1st $i-1$ summands specialize to 0, while the last one specializes to 1). ∎

Due to Lemma 5.7, it remains to verify

$$\frac{X(\underline{d}+i)}{X(\underline{d})} = (1-v^2)c_i t_{i+1} v^{d_i-d_{i+1}+i} v^{\mathsf{n}_{ii}} \prod_{p=1}^{n-1} \left(t_1\cdots t_p v^{-d_p+\frac{p(p-1)}{2}}\right)^{-\mathsf{n}_{ip}} (t_i^2 v^{2d_{i-1}-2d_i})^{-a_i},$$

which is straightforward. This completes our proof of Theorem 5.5. ■

Remark 5.8. Note that if $\epsilon_{i,i+1} = -1$ (resp. $\epsilon_{i,i+1} = 1$) for all i, then θ is a linear combination of $[\mathcal{O}_{\mathfrak{Q}_{\underline{d}}}]$ (resp. $[\mathcal{D}_{\underline{d}}^{-1}]$ with $\mathcal{D}_{\underline{d}} := \otimes_{i=1}^{n-1}\mathcal{D}_i$). These are exactly the two cases considered in [3].

5.9. Geometric realization of the J-function

Recall the Shapovalov form $(\cdot,\cdot)$ on the universal Verma module $\mathcal{V}$, which is a unique nondegenerate symmetric bilinear form on $\mathcal{V}$ with values in $\mathsf{k} \simeq \mathrm{Frac}(K^{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})) \otimes_{\mathbb{C}(v)} \mathbb{C}(v^{1/\mathsf{N}})$ characterized by $(\mathbf{1},\mathbf{1}) = 1$ and $(xw, w') = (w, \widetilde{\sigma}(x)w')$ for all $w, w' \in \mathcal{V}, x \in U_v(\mathfrak{sl}_n)$, where $\widetilde{\sigma}$ is the antiautomorphism of $U_v(\mathfrak{sl}_n)$ determined by $\widetilde{\sigma}(E_i) = F_i, \widetilde{\sigma}(F_i) = E_i, \widetilde{\sigma}(L_i) = L_i$.

Identifying $\mathcal{V} \cong M$ via Theorem 5.3(b), a geometric expression for the Shapovalov form was obtained in [3, Proposition 2.29] (note that our formula differs from the one of [3] as we use a slightly different action of $U_v(\mathfrak{sl}_n)$).

Proposition 5.10. If $\underline{d} \neq \underline{d}'$, then $M_{\underline{d}}$ is orthogonal to $M_{\underline{d}'}$. For $\mathcal{F}, \mathcal{F}' \in M_{\underline{d}}$, we have

$$(\mathcal{F},\mathcal{F}') = (-1)^{\sum_{i=1}^{n-1} d_i} v^{\sum_{i=1}^{n-1}(d_i d_{i+1} - d_i^2 + (1-2i)d_i)} \prod_{i=1}^{n} t_i^{d_i - d_{i-1}} \cdot [R\Gamma(\mathfrak{Q}_{\underline{d}}, \mathcal{F}\otimes\mathcal{F}'\otimes\mathcal{D}_{\underline{d}})], \tag{5.2}$$

where $\mathcal{D}_{\underline{d}} = \otimes_{i=1}^{n-1}\mathcal{D}_i$ as in Remark 5.8.

Given a pair of Sevostyanov triples $(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$, choose the corresponding $\underline{a}^\pm \in \{0,1\}^{n-1}$ and $X(\underline{d})^\pm \in \mathrm{Frac}(K^{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt}))$, and define vectors $\theta_{\underline{d}}^\pm := X(\underline{d})^\pm[\mathcal{D}^{\underline{a}^\pm}] \in M_{\underline{d}}$ as in Section 5.4. Consider the following generating function:

$$\mathfrak{J} = \mathfrak{J}(y_1,\ldots,y_{n-1}) := \prod_{i=1}^{n-1} y_i^{-\frac{\log(t_1\cdots t_i)}{\log(v)} - \frac{i(n-1)}{2}} \cdot \sum_{\underline{d}} \left(\theta_{\underline{d}}^+, \theta_{\underline{d}}^-\right) y_1^{d_1}\cdots y_{n-1}^{d_{n-1}}. \tag{5.3}$$

Due to (5.2), the coefficient $(\theta_{\underline{d}}^+, \theta_{\underline{d}}^-)$ equals

$$(-1)^{\sum_{i=1}^{n-1} d_i} v^{\sum_{i=1}^{n-1}(d_i d_{i+1} - d_i^2 + (1-2i)d_i)} X(\underline{d})^+ X(\underline{d})^- \prod_{i=1}^{n} t_i^{d_i-d_{i-1}} \cdot [R\Gamma(\mathfrak{Q}_{\underline{d}}, \otimes_{i=1}^{n-1}\mathcal{D}_i^{1-a_i^+-a_i^-})].$$

The following result is an immediate consequence of Theorems 4.9, 5.3, and Remark 5.6.

Theorem 5.11. $\mathfrak{J}$ is an eigenfunction of the type A_{n-1} modified quantum difference Toda system $\mathcal{T}(\pm\epsilon^{\pm}, \pm\mathsf{n}^{\pm}, c^{\pm})$. In particular, for $\mathbf{D}_1$ computed explicitly in (3.5), we have

$$\mathbf{D}_1(\mathfrak{J}) = \left(v^{n-1}\sum_{i=1}^{n} t_i^2\right)\cdot\mathfrak{J}. \tag{5.4}$$

Remark 5.12. (a) There is an algebra isomorphism $\varsigma\colon U_v(\mathfrak{sl}_n) \xrightarrow{\sim} U_{v^{-1}}(\mathfrak{sl}_n)$ determined by $E_i \mapsto \bar{E}_i, F_i \mapsto \bar{F}_i, L_i \mapsto \bar{L}_i^{-1}$. Note that $\sigma = \varsigma \circ \widetilde{\sigma}$ (with σ defined in Section 4.1) and the action of $U_v(\mathfrak{sl}_n)$ on $\bar{\mathcal{V}}$ (as a ς-pull-back of $U_{v^{-1}}(\mathfrak{sl}_n)$-action) is isomorphic to $\mathcal{V}$. This implies that the Shapovalov form on $\mathcal{V}$ is identified with the k-bilinear form on $\mathcal{V}\times\bar{\mathcal{V}}$ of Section 4.1.
(b) Under the identification of part (a), the Whittaker vector of $\bar{\mathcal{V}}$ associated with a Sevostyanov triple $(\epsilon^-, \mathsf{n}^-, c^-)$ becomes the Whittaker vector of $\mathcal{V}$ associated with the Sevostyanov triple $(-\epsilon^-, -\mathsf{n}^-, c^-)$. This explains the appearance of the sign '–' in front of ϵ^-, n^- in Theorem 5.11.

5.13. B. Feigin's viewpoint via $U_v(L\mathfrak{sl}_n)$-action

According to [24, Theorem 2.12] (see also [10, Theorem 12.7]), the action of $U_v(\mathfrak{sl}_n)$ on M can be extended to an action of the quantum loop algebra $U_v(L\mathfrak{sl}_n)$ on M (actually, this action factors through the one of $U_v(\mathfrak{gl}_n)$, extending the $U_v(\mathfrak{sl}_n)$-action from Theorem 5.3, via the evaluation homomorphism $\mathrm{ev}\colon U_v(L\mathfrak{sl}_n) \to U_v(\mathfrak{gl}_n)$). In particular, loop generators $\{e_{i,r}, f_{i,r}\}_{1\le i\le n-1}^{r\in\mathbb{Z}}$ (see [24, 2.10]) act via

$$e_{i,r} = t_{i+1}^{-1} v^{d_{i+1}-d_i+1-i}\mathbf{p}_*((v^i\mathcal{L}_i)^{\otimes r}\otimes\mathbf{q}^*)\colon M_{\underline{d}} \to M_{\underline{d}-i},$$

$$f_{i,r} = -t_i^{-1} v^{d_i-d_{i-1}+i}\mathbf{q}_*(\mathcal{L}_i\otimes(v^i\mathcal{L}_i)^{\otimes r}\otimes\mathbf{p}^*)\colon M_{\underline{d}} \to M_{\underline{d}+i}.$$

Note that $e_{i,0} = E_i$ and $f_{i,0} = F_i$. Following [3], define $\mathfrak{k} \in M^\wedge := \prod_{\underline{d}} M_{\underline{d}} \cong \mathcal{V}^\wedge$ via

$$\mathfrak{k} := \sum_{\underline{d}} \mathfrak{k}_{\underline{d}} \text{ with } \mathfrak{k}_{\underline{d}} := [\mathcal{O}_{\mathfrak{Q}_{\underline{d}}}] \in M_{\underline{d}}. \tag{5.5}$$

Proposition 5.14. (a) For any $1\le i\le n-1$, we have $e_{i,0}L_i^{-1}L_{i+1}(\mathfrak{k}) = \frac{v}{1-v^2}\mathfrak{k}$.
(b) For any $1\le i\le n-1$, we have $e_{i,1}L_{i-1}^2L_i^{-3}L_{i+1}(\mathfrak{k}) = \frac{v^{5-i}}{1-v^2}\mathfrak{k}$.

Proof. Part (a) follows from Theorem 5.5 (see also [10, Proposition 12.21]). The proof of part (b) is completely analogous to our proof of Theorem 5.5 (see also [10, Remark 12.22(b)]). ■

Let us now explain the relation between Proposition 5.14 regarding the "eigen-property" of the (geometrically) simplest Whittaker vector $\mathfrak{k}$ and the geometric description of the general Whittaker vectors from Theorem 5.5. In what follows, we will view the line bundle $\mathcal{D}_i^{\pm 1}$ as an endomorphism of M given by the multiplication by $[\mathcal{D}_i^{\pm 1}]$.

Proposition 5.15. We have the following equalities in $\mathrm{End}(M)$:
(a) $\mathcal{D}_i e_{j,0} \mathcal{D}_i^{-1} = e_{j,0}$ for $j \neq i$,
(b) $\mathcal{D}_i e_{i,0} \mathcal{D}_i^{-1} = v^{-i} e_{i,1}$.

Proof. According to [8, Corollary 6.5(a)], the operator $\mathcal{D}_i$ is diagonal in the fixed point basis $\{[\widetilde{\underline{d}}]\}$, and the eigenvalue at $[\widetilde{\underline{d}}]$ is equal to $\prod_{k=1}^{i} t_k^{2(1-d_{ik})} v^{d_{ik}(d_{ik}-1)}$. Part (a) follows as $e_{j,0}\colon M_{\underline{d}} \to M_{\underline{d}-j}$. Likewise, the only nonzero matrix coefficients of $\mathcal{D}_i e_{i,0} \mathcal{D}_i^{-1}$ are given by $\mathcal{D}_i e_{i,0} \mathcal{D}_i^{-1}{}_{[\widetilde{\underline{d}}, \widetilde{\underline{d}}-\delta_{ij}]} = t_j^2 v^{-2(d_{ij}-1)} \cdot e_{i,0[\widetilde{\underline{d}}, \widetilde{\underline{d}}-\delta_{ij}]} = v^2 s_{ij} \cdot e_{i,0[\widetilde{\underline{d}}, \widetilde{\underline{d}}-\delta_{ij}]}$, where $s_{ij} = t_j^2 v^{-2d_{ij}}$ as before. According to [24, Proposition 2.15], the only non-zero matrix coefficients of $e_{i,1}$ in the fixed point basis are given by $e_{i,1[\widetilde{\underline{d}}, \widetilde{\underline{d}}-\delta_{ij}]} = v^{i+2} s_{ij} \cdot e_{i,0[\widetilde{\underline{d}}, \widetilde{\underline{d}}-\delta_{ij}]}$. Part (b) follows. ■

Corollary 5.16. For any $\underline{a} = (a_1, \ldots, a_{n-1}) \in \{0,1\}^{n-1}$, the following holds in $\mathrm{End}(M)$:

$$(\mathcal{D}^{\underline{a}})^{-1} e_{i,0} \mathcal{D}^{\underline{a}} = \begin{cases} e_{i,0}, & \text{if } a_i = 0, \\ v^{-i} e_{i,1}, & \text{if } a_i = 1. \end{cases} \tag{5.6}$$

For $\underline{a} \in \{0,1\}^{n-1}$, define $\mathfrak{k}^{\underline{a}} \in M^\wedge$ via

$$\mathfrak{k}^{\underline{a}} := \sum_{\underline{d}} \mathfrak{k}^{\underline{a}}_{\underline{d}} \quad \text{with} \quad \mathfrak{k}^{\underline{a}}_{\underline{d}} := \prod_{i=1}^{n-1} (t_1 \cdots t_i)^{-2a_i} \cdot [\mathcal{D}^{\underline{a}}] \in M_{\underline{d}}. \tag{5.7}$$

Note that $\mathfrak{k}^{\underline{0}} = \mathfrak{k}$. The special case of Theorem 5.5 follows immediately from Proposition 5.14.

Proposition 5.17. $\mathfrak{k}^{\underline{a}} \in M^\wedge \cong \mathcal{V}^\wedge$ is the Whittaker vector corresponding to the Sevostyanov triple $(\epsilon, \mathsf{n}, c)$ with $\epsilon_{i,i+1} = 2a_{i+1} - 1$, $\mathsf{n}_{ij} = \delta_{j,i+1} - (1+2a_i)\delta_{j,i} + 2a_i \delta_{j,i-1}$, $c_i = \frac{v^{1+a_i(4-2i)}}{1-v^2}$.

Proof. Since $\mathfrak{k}^{\underline{a}}_{\underline{0}} = [\mathcal{O}_{\mathfrak{Q}_{\underline{0}}}]$, it remains to verify the following equality for any $1 \leq i \leq n-1$:

$$e_{i,0} L_{i-1}^{2a_i} L_i^{-1-2a_i} L_{i+1}([\mathcal{D}^{\underline{a}}]) = \frac{v^{1+a_i(4-2i)}}{1-v^2} \cdot [\mathcal{D}^{\underline{a}}]. \tag{5.8}$$

This follows by combining formula (5.6) with Proposition 5.14. ■

Remark 5.18. We note that the above operator of multiplication by $[\mathcal{D}_k]$ can be interpreted entirely algebraically as a product of the Drinfeld Casimir element of the subalgebra $U_v(\mathfrak{sl}_k) \subset U_v(\mathfrak{sl}_n)$ and a certain Cartan element, due to [8, Corollary 6.5(b)]. In *loc.cit.*, the authors choose to work with the action of $U_v(\mathfrak{gl}_n)$ instead of $U_v(\mathfrak{sl}_n)$, which results in shorter formulas.

We conclude this section by generalizing the construction of [4, Theorem 6.12]. In *loc.cit.*, the authors established an edge-weight path model for the type A_{n-1} Whittaker vector associated with a particular Sevostyanov triple $(\epsilon, \mathsf{n}, c)$ with $\epsilon_{i,i+1} = 1$ $(1 \leq i \leq n-2)$ and $\mathsf{n}_{ij} = (i-1)(\delta_{j,i+1} - 2\delta_{j,i} + \delta_{j,i-1})$. More generally, their construction can be applied to Whittaker vectors associated with $(\epsilon, \mathsf{n}, c)$ satisfying $\epsilon_{1,2} = \epsilon_{2,3} = \ldots = \epsilon_{n-2,n-1}$ (corresponding to an equioriented A_{n-1} Dynkin diagram). In particular, identifying $\mathcal{V} \cong M$, we obtain the following edge-weight path model for the Whittaker vector $\mathfrak{k} \in M^\wedge$ of (5.5).

Proposition 5.19. The following equality holds:

$$\mathfrak{k} = \sum_{\beta \in Q_+} \left(\frac{v}{1-v^2}\right)^{|\beta|} \sum_{\mathbf{P} \in \mathcal{P}_\beta} \gamma(\mathbf{P}) \cdot |\mathbf{P}\rangle, \tag{5.9}$$

where we use the following notations:

- $|\beta| := \sum_{i=1}^{n-1} m_i$ for $\beta = \sum_{i=1}^{n-1} m_i \alpha_i \in Q_+$,
- the set $\mathcal{P}_\beta$ consists of all paths $\mathbf{P} = (p_0, \ldots, p_N)$ such that $p_0 = 0$, $p_N = \beta$, and $p_k - p_{k-1} = \alpha_{i_k} (1 \leq i_k \leq n-1)$ for all $1 \leq k \leq N$,
- for $\mathbf{P} = (p_0, \ldots, p_N) \in \mathcal{P}_\beta$, the vector $|\mathbf{P}\rangle \in M$ is defined as $|\mathbf{P}\rangle := f_{i_N} f_{i_{N-1}} \cdots f_{i_1}([\mathcal{O}_{\mathfrak{Q}_{\underline{0}}}])$ with $f_i := L_i L_{i+1}^{-1} F_i$,
- for $\mathbf{P} = (p_0, \ldots, p_N) \in \mathcal{P}_\beta$ with $p_k - p_{k-1} = \alpha_{i_k}$, the coefficient $\gamma(\mathbf{P})$ edge-factorizes as $\gamma(\mathbf{P}) = \prod_{k=1}^{N} \frac{1}{\mathfrak{v}^{(i_k)}(p_k)}$, where $\mathfrak{v}^{(i)}(\gamma) = v^{\tau_i(\gamma)} \mathfrak{v}(\gamma)$ with $\tau_i(\gamma) = (\lambda + \rho - \gamma, \omega_{i-1} - \omega_{i+1})$ and $\mathfrak{v}(\gamma) = (v - v^{-1})^{-2} \sum_{i=0}^{n-1} \left(v^{2(\lambda+\rho, \omega_{i+1} - \omega_i)} - v^{2(\lambda+\rho-\gamma, \omega_{i+1}-\omega_i)}\right)$ for $1 \leq i \leq n-1$ and $\gamma \in Q_+$. Here ω_i is the i-th fundamental weight of $\mathfrak{sl}_n$ as before, and we set $\omega_0 := 0$, $\omega_n := 0$.

Noteworthy, there seems to be no such straightforward edge-weight path model for a general type A Whittaker vector. Nevertheless, one can fix this by changing the above definition of $|\mathbf{P}\rangle$ with the help of the quantum loop algebra $U_v(L\mathfrak{sl}_n)$ in spirit of Proposition 5.15 and Corollary 5.16. This is based on the following result.

Proposition 5.20. For any $\underline{a} = (a_1, \dots, a_{n-1}) \in \{0,1\}^{n-1}$, the following holds in $\mathrm{End}(M)$:

$$\mathcal{D}^{\underline{a}} f_{i,0} (\mathcal{D}^{\underline{a}})^{-1} = \begin{cases} f_{i,0}, & \text{if } a_i = 0, \\ v^{-i} f_{i,1}, & \text{if } a_i = 1, \end{cases} \tag{5.10}$$

where $\mathcal{D}^{\underline{a}}$ denotes an endomorphism of M given by the multiplication by $[\mathcal{D}^{\underline{a}}]$.

Proof. The proof is completely analogous to that of Proposition 5.15. ■

Recall the element $\mathfrak{k}^{\underline{a}} \in M^{\wedge}$ of (5.7). Since $[\mathcal{D}^{\underline{a}}] = \mathcal{D}^{\underline{a}}(\mathfrak{k})$, we obtain the following edge-weight path model for $\mathfrak{k}^{\underline{a}}$.

Proposition 5.21. The following equality holds:

$$\mathfrak{k}^{\underline{a}} = \prod_{i=1}^{n-1} (t_1 \cdots t_i)^{-2a_i} \cdot \sum_{\beta \in Q_+} \left(\frac{v}{1-v^2} \right)^{|\beta|} \sum_{\mathbf{P} \in \mathcal{P}_\beta} y(\mathbf{P}) \cdot |\mathbf{P}\rangle^{\underline{a}}, \tag{5.11}$$

where for a path $\mathbf{P} = (p_0, \dots, p_N) \in \mathcal{P}_\beta$ with $p_k - p_{k-1} = \alpha_{i_k}$ we set

$$|\mathbf{P}\rangle^{\underline{a}} := f^{\underline{a}}_{i_N} f^{\underline{a}}_{i_{N-1}} \cdots f^{\underline{a}}_{i_1} ([\mathcal{O}_{\mathfrak{Q}_{\underline{0}}}]) \text{ with } f^{\underline{a}}_i := \begin{cases} L_i L_{i+1}^{-1} f_{i,0}, & \text{if } a_i = 0, \\ v^{-i} L_i L_{i+1}^{-1} f_{i,1}, & \text{if } a_i = 1. \end{cases}$$

Proof. Follows by combining Propositions 5.19 and 5.20. ■

According to Proposition 5.17, $\mathfrak{k}^{\underline{a}}$ is the Whittaker vector corresponding to the Sevostyanov triple $(\epsilon, \mathsf{n}, c)$ with $\epsilon_{i,i+1} = 2a_{i+1} - 1, \mathsf{n}_{ij} = \delta_{j,i+1} - (1+2a_i)\delta_{j,i} + 2a_i \delta_{j,i-1}, c_i = \frac{v^{1+a_i(4-2i)}}{1-v^2}$. As $\underline{a} \in \{0,1\}^{n-1}$ varies, we get all possible orientations Or of $\mathrm{Dyn}(\mathfrak{sl}_n) = A_{n-1}$ (here Or is determined by ϵ). Since it is clear how the edge-weight path model gets modified once we change n, c (while ϵ is kept fixed), cf. [7, (3.8, 3.9)], Proposition 5.21 provides an edge-weight path model for a general type A Whittaker vector.

A Proof of Proposition 3.11

Given two pairs of type A_{n-1} Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ and $(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$ such that $\epsilon^+_{i,i+1} - \epsilon^-_{i,i+1} = \tilde{\epsilon}^+_{i,i+1} - \tilde{\epsilon}^-_{i,i+1}$ for $1 \le i \le n-2$, we will prove that there exist constants $\{r_{ij}, r_i\}_{1 \le i \le j \le n}$ such that the function

$$F = F(\mathsf{w}_1, \dots, \mathsf{w}_n) := \exp\left(\sum_{1 \le i \le j \le n} r_{ij} \log(\mathsf{w}_i) \log(\mathsf{w}_j) + \sum_{1 \le i \le n} r_i \log(\mathsf{w}_i) \right) \tag{A1}$$

satisfies the equality

$$F^{-1}\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F = \mathsf{H}(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm). \tag{A2}$$

We will view this as an equality in $\mathcal{A}_n$ (rather than $\bar{\mathcal{A}}_n$), treating H of (3.8) as elements of $\mathcal{A}_n$. This will immediately imply the result of Proposition 3.11. Set $\hbar := \log(v)$.

• First, we note that the terms without D_i's are the same (and equal to $\sum_{j=1}^n \mathsf{w}_j^{-2}$) both in $F^{-1}\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F$ and $\mathsf{H}(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$, independently of our choice of constants $\{r_{ij}, r_i\}$.

• Second, we will match the terms with $\{\frac{\mathsf{D}_i}{\mathsf{D}_{i+1}}\}_{i=1}^{n-1}$ appearing in $F^{-1}\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F$ and $\mathsf{H}(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$. Their equality is equivalent to the following system of equations on $\{r_{ij}\}$:

$$\frac{\mathsf{m}_{ij} - \tilde{\mathsf{m}}_{ij}}{\hbar} = \begin{cases} r_{ji} - r_{j,i+1}, & \text{if } 1 \le j < i \\ r_{ij} - r_{i+1,j}, & \text{if } i+2 \le j \le n \\ 2r_{ii} - r_{i,i+1}, & \text{if } j = i \\ r_{i,i+1} - 2r_{i+1,i+1}, & \text{if } j = i+1 \end{cases} \tag{A3}$$

and the following system of equations on $\{r_i\}$:

$$r_i - r_{i+1} = \hbar(r_{i,i+1} - r_{ii} - r_{i+1,i+1}) + \hbar^{-1}\log(\tilde{b}_i/b_i) + \sum_{k=1}^{n}(n-k+1/2)(\mathsf{m}_{ik} - \tilde{\mathsf{m}}_{ik}), \tag{A4}$$

where the coefficients $\mathsf{m}_{ij}, \tilde{\mathsf{m}}_{ij}, b_i, \tilde{b}_i$ are defined as in Section 3.9 via

$\mathsf{m}_{ij} := \sum_{p=j}^{n-1}(\mathsf{n}^-_{ip} - \mathsf{n}^+_{ip})$,
$\tilde{\mathsf{m}}_{ij} := \sum_{p=j}^{n-1}(\tilde{\mathsf{n}}^-_{ip} - \tilde{\mathsf{n}}^+_{ip})$,
$b_i := (v - v^{-1})^2 v^{\mathsf{n}^+_{ii} - \mathsf{n}^-_{ii}} c_i^+ c_i^-$,
$\tilde{b}_i := (v - v^{-1})^2 v^{\tilde{\mathsf{n}}^+_{ii} - \tilde{\mathsf{n}}^-_{ii}} \tilde{c}_i^+ \tilde{c}_i^-$.

It suffices to show that (A3) admits a solution, since (A4) obviously admits a solution in terms of r_i (unique up to a common constant). Pick any r_{11}. Using the last two cases of (A3), we determine uniquely $\{r_{i,i+1}, r_{i+1,i+1}\}_{i=1}^{n-1}$. Using the 1st case of (A3), we determine uniquely r_{ij} for $j > i+1$. The resulting collection $\{r_{ij}\}_{1 \le i \le j \le n}$ satisfies the 1st, 3rd, and 4th cases of (A3). It remains to verify that it also satisfies the 2nd case of (A3). We prove this by induction in $j - i \ge 2$.

(a) If $j = i+2$, then $r_{i,i+2} - r_{i+1,i+2} = (r_{i,i+1} - 2r_{i+1,i+1}) + (2r_{i+1,i+1} - r_{i+1,i+2}) - (r_{i,i+1} - r_{i,i+2}) = \hbar^{-1}(\mathsf{m}_{i,i+1} + \mathsf{m}_{i+1,i+1} - \mathsf{m}_{i+1,i} - \tilde{\mathsf{m}}_{i,i+1} - \tilde{\mathsf{m}}_{i+1,i+1} + \tilde{\mathsf{m}}_{i+1,i})$. Hence, it remains to prove $\mathsf{m}_{i,i+1} + \mathsf{m}_{i+1,i+1} - \mathsf{m}_{i+1,i} - \mathsf{m}_{i,i+2} = \tilde{\mathsf{m}}_{i,i+1} + \tilde{\mathsf{m}}_{i+1,i+1} - \tilde{\mathsf{m}}_{i+1,i} - \tilde{\mathsf{m}}_{i,i+2}$. Since $\mathsf{m}_{st} - \mathsf{m}_{s,t+1} = \mathsf{n}^-_{st} - \mathsf{n}^+_{st}$, this is reduced to $\mathsf{n}^-_{i,i+1} - \mathsf{n}^+_{i,i+1} - \mathsf{n}^-_{i+1,i} + \mathsf{n}^+_{i+1,i} = \tilde{\mathsf{n}}^-_{i,i+1} - \tilde{\mathsf{n}}^+_{i,i+1} - \tilde{\mathsf{n}}^-_{i+1,i} + \tilde{\mathsf{n}}^+_{i+1,i}$. The latter equality follows from $\mathsf{n}^\pm_{st} - \mathsf{n}^\pm_{ts} = \epsilon^\pm_{st} b_{st}$ and our assumption on the triples.

(b) If $j > i+2$, then $r_{ij} - r_{i+1,j} = (r_{i,j-1} - r_{i+1,j-1}) + (r_{i+1,j-1} - r_{i+1,j}) - (r_{i,j-1} - r_{ij}) = \hbar^{-1}(\mathsf{m}_{i,j-1} + \mathsf{m}_{j-1,i+1} - \mathsf{m}_{j-1,i} - \tilde{\mathsf{m}}_{i,j-1} - \tilde{\mathsf{m}}_{j-1,i+1} + \tilde{\mathsf{m}}_{j-1,i})$. Hence, it remains to prove $(\mathsf{m}_{i,j-1} - \mathsf{m}_{ij}) - (\mathsf{m}_{j-1,i} - \mathsf{m}_{j-1,i+1}) = (\tilde{\mathsf{m}}_{i,j-1} - \tilde{\mathsf{m}}_{ij}) - (\tilde{\mathsf{m}}_{j-1,i} - \tilde{\mathsf{m}}_{j-1,i+1})$. Similarly to (a), this is reduced to the proof of $b_{i,j-1}(\epsilon^-_{i,j-1} - \epsilon^+_{i,j-1}) = b_{i,j-1}(\tilde{\epsilon}^-_{i,j-1} - \tilde{\epsilon}^+_{i,j-1})$. The latter follows immediately from the equality $b_{i,j-1} = 0$.

Thus, we have determined a collection of constants $\{r_{ij}, r_i\}_{1\le i\le j\le n}$ satisfying (A3, A4) (this collection is uniquely determined by a choice of r_{11}, r_1; however, we note that the image of F defined via (A1) in $\bar{\mathcal{A}}_n$ is independent of this choice).

• Finally, it remains to verify that for F of (A1) with the constants r_{ij}, r_i chosen as above, the terms with $\frac{\mathsf{D}_i}{\mathsf{D}_j}$ $(j > i+1)$ in $F^{-1}\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F$ and $\mathsf{H}(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ do coincide. First, we note that the conditions $\epsilon^\pm_{i,i+1} = \ldots = \epsilon^\pm_{j-2,j-1} = \pm 1$ and $\tilde{\epsilon}^\pm_{i,i+1} = \ldots = \tilde{\epsilon}^\pm_{j-2,j-1} = \pm 1$ are equivalent under our assumption on the triples. Pick $j > i+1$ such that either of these equivalent conditions is satisfied. Then the compatibility of the terms with $\frac{\mathsf{D}_i}{\mathsf{D}_j}$ is equivalent to the following equality:

$$\frac{F(\mathsf{w}_1, \ldots, v\mathsf{w}_i, \ldots, v^{-1}\mathsf{w}_j, \ldots, \mathsf{w}_n)}{F(\mathsf{w}_1, \ldots, \mathsf{w}_n)} = \\ \prod_{k=1}^{n} \mathsf{w}_k^{\sum_{s=i}^{j-1}(\mathsf{m}_{sk} - \tilde{\mathsf{m}}_{sk})} \cdot \prod_{s=i}^{j-1} \frac{\tilde{b}_s}{b_s} \cdot v^{\sum_{i\le a<b\le j-1}(\mathsf{n}^-_{ab} - \mathsf{n}^+_{ab} - \tilde{\mathsf{n}}^-_{ab} + \tilde{\mathsf{n}}^+_{ab}) + \sum_{k=1}^{n}\sum_{s=i}^{j-1}\frac{n+1-2k}{2}(\mathsf{m}_{sk} - \tilde{\mathsf{m}}_{sk})}. \tag{A5}$$

We prove this by induction in $j-i$. Note that the $j = i+1$ counterpart of (A5) is just the compatibility of the terms with $\frac{\mathsf{D}_i}{\mathsf{D}_{i+1}}$, established in the previous step. Writing the left-hand side of (A5) as a product

$$\frac{F(\mathsf{w}_1, \ldots, v\mathsf{w}_i, \ldots, v^{-1}\cdot v\mathsf{w}_{j-1}, v^{-1}\mathsf{w}_j, \ldots, \mathsf{w}_n)}{F(\mathsf{w}_1, \ldots, v\mathsf{w}_{j-1}, v^{-1}\mathsf{w}_j, \ldots, \mathsf{w}_n)} \cdot \frac{F(\mathsf{w}_1, \ldots, v\mathsf{w}_{j-1}, v^{-1}\mathsf{w}_j, \ldots, \mathsf{w}_n)}{F(\mathsf{w}_1, \ldots, \mathsf{w}_n)} \tag{A6}$$

and applying the induction assumption to both fractions of (A6), it is straightforward to see that we obtain the right-hand side of (A5).

Thus, the function F defined via (A1) with the constants $\{r_{ij}, r_i\}_{1\le i\le j\le n}$ determined in our 2nd step satisfies the equality (A2). This completes our proof of Proposition 3.11. ∎

Remark A.1. (a) The proofs of Propositions 3.14, 3.17, and 3.20 are analogous to the above proof of Proposition 3.11. In each case, there exists a unique collection of constants $\{r_{ij}, r_i\}_{1\le i\le j\le n}$ such that the function F defined via (A1) satisfies the corresponding equality (A2). The way we choose such constants closely follows the above 2nd step in our proof of Proposition 3.11 and is determined by matching up the coefficients of

- $\{\mathsf{D}_i/\mathsf{D}_{i+1}\}_{i=1}^{n-1}$ and D_n^2 for the type C_n,
- $\{\mathsf{D}_i/\mathsf{D}_{i+1}\}_{i=1}^{n-1}$ and $\mathsf{D}_{n-1}\mathsf{D}_n$ for the type D_n,
- $\{\mathsf{D}_i/\mathsf{D}_{i+1}\}_{i=1}^{n-1}$ and D_n for the type B_n.

Finally, it remains to check that the function F defined via (A1) with thus determined $\{r_{ij}, r_i\}_{1\le i\le j\le n}$ conjugates each of the remaining terms appearing in $\mathsf{H}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ into the one of $\mathsf{H}(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$. This is verified by induction similarly to the above last step in our proof of Proposition 3.11.

(b) The proof of Proposition 3.38 is also analogous, but the constants r_{ij}, r_i are determined by matching the terms with D_1 and D_2.

B Proof of Theorem 3.2

Assume that $\mathfrak{g}$ is either of the classical type or G_2. Given two pairs of Sevostyanov triples $(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ and $(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ with $\vec{\epsilon} = \vec{\tilde{\epsilon}}$ (defined right before Theorem 3.2), we need to show that there exists an automorphism of $\mathcal{D}_v(H^{\mathrm{ad}})$ that maps $\mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ to $\mathcal{T}(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$.

According to our proof of Proposition 3.11 and Remark A.1 (which states that the same argument applies to all classical types and G_2), there exists a "formal function" F of the shift operators T_μ such that conjugation by F is a well-defined automorphism of $\mathcal{D}_v(H^{\mathrm{ad}})$ satisfying $F\mathbf{D}_1(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F^{-1} = \mathbf{D}_1(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$. It remains to prove the following result.

Proposition B.1. For any $1 \le i \le n$, we have $F\mathbf{D}_i(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F^{-1} = \mathbf{D}_i(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$.

Proof. Recall that $\mathbf{D}_i \in \mathcal{D}_v^{\le}(H^{\mathrm{ad}})$, where $\mathcal{D}_v^{\le}(H^{\mathrm{ad}})$ is the subalgebra of $\mathcal{D}_v(H^{\mathrm{ad}})$ generated by $\{e^{-\alpha_i}, T_\mu | 1 \le i \le n, \mu \in P\}$. Let us extend the field $\mathbb{C}(v^{1/\mathsf{N}})$ to k and recall the vector space N_λ of Section 4.8, which was equipped with a natural $\mathcal{D}_v(H^{\mathrm{ad}})$-action. In particular, the subspace W_λ of N_λ formed by the formal sums $\left\{\sum_{\beta\in Q_+} a_\beta \underline{y}^{\beta-\lambda} | a_\beta \in \mathsf{k}\right\}$ is $\mathcal{D}_v^{\le}(H^{\mathrm{ad}})$-stable. Moreover, $X(\underline{y}^{\beta-\lambda})$ contains only $\underline{y}^{\gamma-\lambda}$ with $\gamma \ge \beta$ for any $X \in \mathcal{D}_v^{\le}(H^{\mathrm{ad}})$, which we refer to as the "upper-triangular" property of the $\mathcal{D}_v^{\le}(H^{\mathrm{ad}})$-action. In particular, we have

$$\mathbf{D}_i(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)(\underline{y}^{\beta-\lambda}), \mathbf{D}_i(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)(\underline{y}^{\beta-\lambda}) \in \left(\sum_{k=1}^{N_i} v^{2(\mu_k^{(i)}, \lambda-\beta)}\right) \cdot \underline{y}^{\beta-\lambda} \oplus \bigoplus_{\gamma>\beta} \mathsf{k}\underline{y}^{\gamma-\lambda}, \tag{B1}$$

where N_i is the dimension and $\{\mu_k^{(i)}\}_{k=1}^{N_i}$ are the weights (counted with multiplicities) of the i-th fundamental $U_v(\mathfrak{g})$-representation V_i, while $v^{(\nu,\lambda)}$ ($\nu \in P$) is defined as in Section 4.1.

Therefore, the action of $\mathbf{D}_1(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ on W_λ is upper-triangular with pairwise distinct diagonal matrix coefficients; hence, it is diagonalizable with a simple spectrum. Moreover, the eigenvalues are exactly $\left\{\sum_{k=1}^{N_1} v^{2(\mu_k^{(1)},\lambda-\beta)} | \beta \in Q_+\right\}$.

Remark B.2. Due to Theorem 4.9, the corresponding eigenbasis consists of the J-functions $J(\{y_i\}_{i=1}^n)$ associated with $\{\lambda - \rho - \beta | \beta \in Q_+\}$, cf. (4.9).

Since $[\mathbf{D}_i(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm), \mathbf{D}_1(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)] = 0$, the action of $\mathbf{D}_i(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ on W_λ is diagonal in a $\mathbf{D}_1(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$-eigenbasis with the corresponding eigenvalues given by $\sum_{k=1}^{N_i} v^{2(\mu_k^{(i)},\lambda-\beta)}$ (this also follows from Remark B.2 and Theorem 4.9). On the other hand, the action of $F\mathbf{D}_i(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F^{-1}$ on W_λ is also upper-triangular with the same diagonal matrix coefficients and commutes with $\mathbf{D}_1(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ (since $F\mathbf{D}_1(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F^{-1} = \mathbf{D}_1(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ and $[\mathbf{D}_1(\epsilon^\pm, \mathsf{n}^\pm, c^\pm), \mathbf{D}_i(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)] = 0$). Thus, both $F\mathbf{D}_i(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F^{-1}$ and $\mathbf{D}_i(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ act diagonally in a $\mathbf{D}_1(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$-eigenbasis and have the same corresponding eigenvalues. ■

The equality $F\mathbf{D}_i(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)F^{-1} = \mathbf{D}_i(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$ follows.

Thus, conjugation by F maps $\mathcal{T}(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ to $\mathcal{T}(\tilde{\epsilon}^\pm, \tilde{\mathsf{n}}^\pm, \tilde{c}^\pm)$. Theorem 3.2 follows. ■

C Proof of Theorem 3.3

Following the discussion in Appendix B, consider a basis of W_λ in which all $\mathbf{D}_i$ act simultaneously diagonally with the corresponding eigenvalues given by $\sum_{k=1}^{N_i} v^{2(\mu_k^{(i)},\lambda-\beta)}$. The latter can be viewed as characters χ_i of the fundamental representations evaluated at $v^{2(\lambda-\beta)}$.

Since the point $v^{2\lambda} \in H(\mathsf{k})$ is general and the characters $\{\chi_i\}_{i=1}^n$ are known to be algebraically independent, we immediately obtain part (a) of Theorem 3.3.

Part (c) of Theorem 3.3 follows from part (b) as $\mathbf{D}_V(\epsilon^\pm, \mathsf{n}^\pm, c^\pm) \in \mathcal{D}_v^{\leq}(H^{\mathrm{ad}})$ commutes with $\mathbf{D}_1(\epsilon^\pm, \mathsf{n}^\pm, c^\pm)$ for any finite-dimensional $U_v(\mathfrak{g})$-representation V, due to Lemma 2.8(c).

It remains to prove part (b) of Theorem 3.3. The algebra $\mathcal{D}_v^{\leq}(H^{\mathrm{ad}})$ is $\mathbb{Z}$-graded via $\deg(T_\mu) = 0$ and $\deg(e^{-\alpha_i}) = -1$, so that the degree zero component $\mathcal{D}_v^{\leq}(H^{\mathrm{ad}})^0$ has a basis $\{T_\mu | \mu \in P\}$. Note that the degree zero component $\mathbf{D}_i^{(0)}$ of $\mathbf{D}_i$ equals $\mathbf{D}_i^{(0)} = \sum_{k=1}^{N_i} T_{2\mu_k^{(i)}}$. Let $\mathcal{D}_v^{\leq}(H^{\mathrm{ad}})^0_{ev}$ be the subspace spanned by $\{T_{2\mu} | \mu \in P\}$. We also consider a natural action of the Weyl group W both on $\mathcal{D}_v^{\leq}(H^{\mathrm{ad}})^0$ and $\mathcal{D}_v^{\leq}(H^{\mathrm{ad}})^0_{ev}$ via $w(T_\mu) = T_{w\mu}$ for $\mu \in P, w \in W$. Given $\mathbf{D} \in \mathcal{D}_v^{\leq}(H^{\mathrm{ad}})$ that commutes with $\mathbf{D}_1$, let $\mathbf{D}^{(0)}$ denote its degree zero component.

Proposition C.1. We have $\mathbf{D}^{(0)} \in (\mathcal{D}_v^{\leq}(H^{\mathrm{ad}})^0_{ev})^W$.

The proof of Proposition C.1 is based on the rank 1 case, for which we prove a slightly more general result. In type A_1, the modified quantum difference Toda systems are conjugate to the q-Toda of [6] with the 1st hamiltonian $\mathbf{D}_1 = T_{2\varpi_1} + T_{-2\varpi_1} - (v - v^{-1})^2 e^{-\alpha} T_0$; see (3.6).

Lemma C.2. In type A_1, given $\mathbf{D} \in \mathcal{D}_{\bar{v}}^{\leq}(H^{\mathrm{ad}})$ that commutes with $\mathbf{D}' = a_r \mathbf{D}_1^r + a_{r-1}\mathbf{D}_1^{r-1} + \ldots + a_0$ for some $a_0, \ldots, a_r \in \mathbb{Q}(v^{1/\mathsf{N}})$ with $a_r \neq 0, r > 0$, $\mathbf{D}$ must be a polynomial in $\mathbf{D}_1$.

Proof. Let $\mathbf{D} = \mathbf{D}^{(0)} + e^{-\alpha}\mathbf{D}^{(-1)} + \ldots + e^{-s\alpha}\mathbf{D}^{(-s)}$ with $\mathbf{D}^{(0)}, \ldots, \mathbf{D}^{(-s)} \in \mathcal{D}_{\bar{v}}^{\leq}(H^{\mathrm{ad}})^0$ and $\mathbf{D}^{(-s)} \neq 0$. We prove the claim by induction in s. Comparing the degree $-r-s$ terms in $\mathbf{D}\mathbf{D}' = \mathbf{D}'\mathbf{D}$, we immediately get $\mathbf{D}^{(-s)} = c_s T_0$ for some constant c_s. Replacing $\mathbf{D}$ by $\mathbf{D} - c_s(-(v - v^{-1})^{-2})^s \mathbf{D}_1^s$, we obtain another element of $\mathcal{D}_{\bar{v}}^{\leq}(H^{\mathrm{ad}})$ that commutes with $\mathbf{D}'$ and has a smaller value of s, hence, is a polynomial in $\mathbf{D}_1$ by the induction assumption. Therefore, $\mathbf{D}$ is also a polynomial in $\mathbf{D}_1$. ■

Proof of Proposition C.1. The result of Proposition C.1 follows immediately from Lemma C.2. Indeed, it suffices to verify the following two claims for any $1 \leq i \leq n$:

(I) the operator $\mathbf{D}^{(0)}$ is invariant with respect to the simple reflection s_i,

(II) every μ appearing in $\mathbf{D}^{(0)}$ satisfies $(\mu, \alpha_i) \in 2\mathsf{d}_i\mathbb{Z}$.

To prove this, consider a subspace W'_λ of W_λ that consists of $\left\{\sum_{\beta \in Q_+ \setminus \mathbb{Z}\alpha_i} a_\beta \underline{Y}^{\beta-\lambda} | a_\beta \in \mathsf{k}\right\}$. It is stable under the action of $\mathcal{D}_{\bar{v}}^{\leq}(H^{\mathrm{ad}})$, hence, we obtain the action of $\mathcal{D}_{\bar{v}}^{\leq}(H^{\mathrm{ad}})$ on the quotient $\bar{W}_\lambda := W_\lambda / W'_\lambda$. We also specialize $u_j \mapsto 1$ for $j \neq i$ (recall that u_j were used in our definition of λ). As a result, summands with $e^{-\alpha_j}$ $(j \neq i)$ in $\mathbf{D}, \mathbf{D}_1$ act by zero on $\bar{W}_\lambda$, while T_{ω_j} $(j \neq i)$ act by the identity operator. Identifying further $\bar{W}_\lambda$ with the space $W_{\lambda'}^{(\mathfrak{sl}_2)}$ constructed for $\mathfrak{sl}_2$ instead of $\mathfrak{g}$ (hence, the superscript in our notations), $\mathbf{D}_1$ gives rise to the operator $\mathbf{D}_V^{(\mathfrak{sl}_2)}$ with V being the restriction of the 1st fundamental $U_v(\mathfrak{g})$-representation V_1 to the subalgebra generated by $E_i, F_i, L_i^{\pm 1}$, which is isomorphic to $U_{v_i}(\mathfrak{sl}_2)$. As V is not a trivial $U_{v_i}(\mathfrak{sl}_2)$-module, $\mathbf{D}_V^{(\mathfrak{sl}_2)}$ is a nonconstant polynomial in the 1st hamiltonian $\mathbf{D}_1^{(\mathfrak{sl}_2)}$. Hence, Lemma C.2 can be applied with $\mathbf{D}' = \mathbf{D}_V^{(\mathfrak{sl}_2)}$ and $\mathbf{D}$ denoting the image of $\mathbf{D}$ acting on $\bar{W}_\lambda \simeq W_{\lambda'}^{(\mathfrak{sl}_2)}$ by abuse of notation. Therefore, both claims (I) and (II) follow. ■

It is clear that $(\mathcal{D}_{\bar{v}}^{\leq}(H^{\mathrm{ad}})_{ev}^0)^W$ is generated by $\{\mathbf{D}_i^{(0)}\}_{i=1}^n$. Hence, due to Proposition C.1, there exists a polynomial P in n variables such that $\mathbf{D}' := \mathbf{D} - P(\mathbf{D}_1, \ldots, \mathbf{D}_n)$ is of strictly negative degree. Thus, the action of $\mathbf{D}'$ on W_λ is upper-triangular with zeros on the diagonal. As $[\mathbf{D}', \mathbf{D}_1] = 0$ and $\mathbf{D}_1$ acts on W_λ with a simple spectrum, we immediately get $\mathbf{D}' = 0$.

This completes our proof of Theorem 3.3. ■

D Proof of Theorem 3.1

Given a rank n simple Lie algebra $\mathfrak{g}$, fix an arbitrary orientation of the edges of $\mathrm{Dyn}(\mathfrak{g})$ as well as their labeling by numbers from 1 up to $n-1$. For an edge $1 \le e \le n-1$, the vertices $t(e), h(e)$ will denote the *tail* and the *head* of that edge, respectively. To every pair of Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$, we associate an invariant $\vec{\epsilon} = (\epsilon_{n-1}, \ldots, \epsilon_1) \in \{-1,0,1\}^{n-1}$ via $\epsilon_e := \frac{\epsilon^{+}_{t(e),h(e)} - \epsilon^{-}_{t(e),h(e)}}{2} \in \{-1,0,1\}$ for $1 \le e \le n-1$.

To prove Theorem 3.1, it suffices to verify that given two pairs of Sevostyanov triples $(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ and $(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$ satisfying $\vec{\epsilon} = \vec{\tilde{\epsilon}}$, there exists an automorphism of $\mathcal{D}_v(H^{\mathrm{ad}})$ that maps $\mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathcal{T}(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$. Our proof is similar to that of Theorem 3.2 presented in Appendix B, but is crucially based on the fermionic formula of Theorem 4.7 for $\tilde{J}_\beta$ instead of Propositions 3.11, 3.14, 3.17, 3.20, and 3.38 (we owe this observation to A. Braverman).

Following Appendix B, consider the action of $\mathcal{D}_v^{\le}(H^{\mathrm{ad}})$ on W_λ. Due to Remark B.2, the action of pairwise commuting operators $\mathbf{D}_i(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ (resp. $\mathbf{D}_i(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm})$) is simultaneously diagonalizable in the basis of J-functions $\{J^\Lambda(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm}; \{y_i\}) | \Lambda = \lambda - \rho - \beta, \beta \in Q_+\}$ (resp. $\{J^\Lambda(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm}; \{y_i\}) | \Lambda = \lambda - \rho - \beta, \beta \in Q_+\}$) (as Λ varies, we will use the notations $J^\Lambda(\{y_i\}), J^\Lambda_\beta$ instead of $J(\{y_i\}), J_\beta$ used in Section 4). We note that both $J^\Lambda(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm}; \{y_i\}) - \underline{y}^{\beta-\lambda}$ and $J^\Lambda(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm}; \{y_i\}) - \underline{y}^{\beta-\lambda}$ contain only $\{\underline{y}^{\gamma-\lambda}\}_{\gamma > \beta}$.

The following is the key observation.

Proposition D.1. If $\vec{\epsilon} = \vec{\tilde{\epsilon}}$, there exists a difference operator $\mathfrak{D}$ that acts on W_λ and maps $J^\Lambda(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm}; \{y_i\})$ to a nonzero multiple of $J^\Lambda(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm}; \{y_i\})$ for any $\Lambda \in \lambda - \rho - Q_+$.

Proof. Define $\nu_i^{\pm}, \tilde{\nu}_i^{\pm} \in P$ via $\nu_i^{\pm} := \sum_{k=1}^n \mathsf{n}^{\pm}_{ik}\omega_k$ and $\tilde{\nu}_i^{\pm} := \sum_{k=1}^n \tilde{\mathsf{n}}^{\pm}_{ik}\omega_k$. Due to Theorem 4.7, the pairing $J^\Lambda_\beta(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ depends only on $\{\nu_i^+ - \nu_i^-\}_{i=1}^n$ and $\{c_i^+ c_i^-\}_{i=1}^n$ for any fixed Λ, β. Hence, as $\vec{\epsilon} = \vec{\tilde{\epsilon}}$, we may assume $\tilde{\epsilon}^{\pm} = \epsilon^{\pm}, \tilde{\mathsf{n}}^- = \mathsf{n}^-, \tilde{c}^- = c^-$, while $\gamma_i := \tilde{\nu}_i^+ - \nu_i^+$ satisfy

$$(\alpha_i, \gamma_j) = (\alpha_j, \gamma_i) \text{ for any } 1 \le i, j \le n. \tag{D1}$$

In this set-up, we have the following.

Lemma D.2. There exist constants $\{s_i\}_{i=1}^n$ such that

$$J^\Lambda_\beta(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm}) = v^{\sum_{i=1}^n s_i(\beta,\omega_i) + \frac{1}{2}\sum_{i=1}^n (\beta,\omega_i^\vee)(\beta - 2\Lambda, \gamma_i)} \cdot J^\Lambda_\beta(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$$

for any $\Lambda \in \lambda - \rho - Q_+, \beta \in Q_+$.

This essentially follows from [7, (3.8, 3.9)], but let us provide a complete argument.

Proof. Since $\tilde{\epsilon}^{\pm} = \epsilon^{\pm}, \tilde{\mathsf{n}}^{-} = \mathsf{n}^{-}, \tilde{c}^{-} = c^{-}$ and $J^{\Lambda}_{\beta}(\bullet, \bullet, \bullet)$ is defined via (4.2), it suffices to prove the following equality:

$$\theta^{\Lambda}_{\beta}(\epsilon^{+}, \tilde{\mathsf{n}}^{+}, \tilde{c}^{+}) = \theta^{\Lambda}_{\beta}(\epsilon^{+}, \mathsf{n}^{+}, c^{+}) \cdot a_{\Lambda,\beta}, \tag{D2}$$

where

$$a_{\Lambda,\beta} := v^{\frac{1}{2}\sum_{i=1}^{n}(\beta,\omega_i^{\vee})(\beta-2\Lambda,\gamma_i)} \cdot \prod_{i=1}^{n}(\tilde{c}_i^{+} v^{\frac{1}{2}(\gamma_i,\alpha_i)}/c_i^{+})^{(\beta,\omega_i^{\vee})}. \tag{D3}$$

Let $\widetilde{\theta}^{\Lambda}_{\beta}$ denote the right-hand side of (D2). To prove (D2), it suffices to verify that $\sum_{\beta\in Q_+}\widetilde{\theta}^{\Lambda}_{\beta}$ satisfies the defining conditions (4.1) of the Whittaker vector associated with $(\epsilon^{+}, \tilde{\mathsf{n}}^{+}, \tilde{c}^{+})$.

First, the equality $a_{\Lambda,0} = 1$ implies $\widetilde{\theta}^{\Lambda}_{0} = \theta^{\Lambda}_{0}(\epsilon^{+}, \mathsf{n}^{+}, c^{+}) = \mathbf{1}$.

Second, we note that the equality $E_i K_{\nu_i^{+}}(\theta^{\Lambda}_{\beta+\alpha_i}(\epsilon^{+}, \mathsf{n}^{+}, c^{+})) = c_i^{+} \cdot \theta^{\Lambda}_{\beta}(\epsilon^{+}, \mathsf{n}^{+}, c^{+})$ implies $E_i K_{\nu_i^{+}+\gamma_i}(\widetilde{\theta}^{\Lambda}_{\beta+\alpha_i}) = c_i^{+} v^{(\gamma_i,\Lambda-\beta-\alpha_i)} \frac{a_{\Lambda,\beta+\alpha_i}}{a_{\Lambda,\beta}} \cdot \widetilde{\theta}^{\Lambda}_{\beta}$. Therefore, it remains to verify

$$c_i^{+} v^{(\gamma_i,\Lambda-\beta-\alpha_i)} \frac{a_{\Lambda,\beta+\alpha_i}}{a_{\Lambda,\beta}} = \tilde{c}_i^{+}. \tag{D4}$$

Recalling the definition of $a_{\Lambda,\beta}$ of (D3), we find

$$\frac{a_{\Lambda,\beta+\alpha_i}}{a_{\Lambda,\beta}} = \frac{\tilde{c}_i^{+}}{c_i^{+}} v^{\frac{1}{2}(\gamma_i,\alpha_i)} \cdot v^{\frac{1}{2}\sum_{j=1}^{n}\{(\beta+\alpha_i,\omega_j^{\vee})(\beta-2\Lambda+\alpha_i,\gamma_j)-(\beta,\omega_j^{\vee})(\beta-2\Lambda,\gamma_j)\}} =$$

$$\frac{\tilde{c}_i^{+}}{c_i^{+}} v^{\frac{1}{2}(\gamma_i,\alpha_i)} \cdot v^{\frac{1}{2}(\beta-2\Lambda+\alpha_i,\gamma_i)+\frac{1}{2}\sum_j(\beta,\omega_j^{\vee})(\alpha_i,\gamma_j)} = \frac{\tilde{c}_i^{+}}{c_i^{+}} v^{(\beta-\Lambda+\alpha_i,\gamma_i)},$$

where we used (D1) to evaluate $\sum_j(\beta,\omega_j^{\vee})(\alpha_i,\gamma_j) = \sum_j(\beta,\omega_j^{\vee})(\alpha_j,\gamma_i) = (\beta,\gamma_i)$.

This implies (D4), which completes our proof of Lemma D.2. ■

Set

$$\mathfrak{D} := \exp\left(\sum_{i=1}^{n}\frac{\log(T_{\omega_i})\log(T_{\gamma_i})}{2\mathsf{d}_i\log(v)} - \sum_{i=1}^{n}s_i\log(T_{\omega_i})\right).$$

This definition is motivated by the following result.

Lemma D.3. $\mathfrak{D}(J^{\Lambda}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm}; \{y_i\}))$ is a nonzero multiple of $J^{\Lambda}(\tilde{\epsilon}^{\pm}, \tilde{\mathsf{n}}^{\pm}, \tilde{c}^{\pm}; \{y_i\})$ for any $\Lambda \in \lambda - \rho - Q_+$.

Proof. Evoking formula (4.8), we get

$$\mathfrak{D}(\underline{y}^{\beta-\Lambda}) = \underline{y}^{\beta-\Lambda} \cdot v^{\frac{1}{2}\sum_{i=1}^{n}(\omega_i^{\vee},\beta-\Lambda)(\gamma_i,\beta-\Lambda)+\sum_i s_i(\omega_i,\beta-\Lambda)}.$$

Combining this with Lemma D.2, the statement reduces to the β-independence of

$$\frac{1}{2}\sum_{i=1}^{n}(\omega_i^\vee,\beta-\Lambda)(\gamma_i,\beta-\Lambda)+\sum_i s_i(\omega_i,\beta-\Lambda)-\frac{1}{2}\sum_{i=1}^{n}\frac{(\beta,\omega_i^\vee)(\beta-2\Lambda,\gamma_i)}{2}-\sum_{i=1}^{n}s_i(\beta,\omega_i)=$$

$$\sum_{i=1}^{n}\frac{(\omega_i^\vee,\beta)(\gamma_i,\Lambda)-(\omega_i^\vee,\Lambda)(\gamma_i,\beta-\Lambda)}{2}-\sum_i s_i(\omega_i,\Lambda).$$

The latter follows from $\sum_{i=1}^{n}(\omega_i^\vee,\beta)(\gamma_i,\Lambda)=\sum_{i=1}^{n}(\omega_i^\vee,\Lambda)(\gamma_i,\beta)$, due to (D1). ■

This completes our proof of Proposition D.1. ■

Due to Proposition D.1, $\mathfrak{D}\mathbf{D}_i(\epsilon^\pm,\mathsf{n}^\pm,c^\pm)\mathfrak{D}^{-1}$ and $\mathbf{D}_i(\tilde{\epsilon}^\pm,\tilde{\mathsf{n}}^\pm,\tilde{c}^\pm)$ act diagonally in the basis $\{J^\Lambda(\tilde{\epsilon}^\pm,\tilde{\mathsf{n}}^\pm,\tilde{c}^\pm;\{y_i\})|\Lambda\in\lambda-\rho-Q_+\}$ of W_λ with the same eigenvalues, hence, they coincide for every $1\le i\le n$. Therefore, conjugation by $\mathfrak{D}$ is a well-defined automorphism of $\mathcal{D}_v(H^{\mathrm{ad}})$ that maps $\mathcal{T}(\epsilon^\pm,\mathsf{n}^\pm,c^\pm)$ to $\mathcal{T}(\tilde{\epsilon}^\pm,\tilde{\mathsf{n}}^\pm,\tilde{c}^\pm)$.

This completes our proof of Theorem 3.1. ■

E Proof of Theorem 3.24

The proof of Theorem 3.24 is similar to the one of Proposition 3.11 given in Appendix A and of Theorem 3.2 given in Appendix B, but we provide details as the formulas are different.

Proof of part (a).

Given a pair of type A_{n-1} Sevostyanov triples $(\epsilon^\pm,\mathsf{n}^\pm,c^\pm)$ and $\vec{k}=(k_n,\ldots,k_1)\in\{-1,0,1\}^n$ satisfying $k_{i+1}=\frac{\epsilon^+_{i,i+1}-\epsilon^-_{i,i+1}}{2}$ for $1\le i\le n-2$, we will prove that there exist constants $\{r_{ij},r_i\}_{1\le i\le j\le n}$ such that the function F defined in (A1) satisfies the equality

$$F^{-1}\mathsf{H}(\epsilon^\pm,\mathsf{n}^\pm,c^\pm)F=\mathsf{H}_2^{\vec{k}}. \tag{E1}$$

We will view this as an equality in $\mathcal{A}_n$, treating $\mathsf{H}(\epsilon^\pm,\mathsf{n}^\pm,c^\pm)$ as an element of $\mathcal{A}_n$.

- First, we note that the terms without D_i's are the same (and equal to $\sum_{j=1}^{n}\mathsf{w}_j^{-2}$) both in $F^{-1}\mathsf{H}(\epsilon^\pm,\mathsf{n}^\pm,c^\pm)F$ and $\mathsf{H}_2^{\vec{k}}$, independently of our choice of constants $\{r_{ij},r_i\}$.
- Second, we will match the terms with $\{\frac{\mathsf{D}_i}{\mathsf{D}_{i+1}}\}_{i=1}^{n-1}$ appearing in $F^{-1}\mathsf{H}(\epsilon^\pm,\mathsf{n}^\pm,c^\pm)F$ and $\mathsf{H}_2^{\vec{k}}$. Their equality is equivalent to the following system of equations on $\{r_{ij}\}$:

$$\frac{\mathsf{m}_{ij}-\delta_{j,i}k_i-\delta_{j,i+1}k_{i+1}}{\hbar}=\begin{cases} r_{ji}-r_{j,i+1}, & \text{if } 1\le j<i\\ r_{ij}-r_{i+1,j}, & \text{if } i+2\le j\le n\\ 2r_{ii}-r_{i,i+1}, & \text{if } j=i\\ r_{i,i+1}-2r_{i+1,i+1}, & \text{if } j=i+1\end{cases} \tag{E2}$$

and the following system of equations on $\{r_i\}$:

$$r_i - r_{i+1} = \hbar(r_{i,i+1} - r_{ii} - r_{i+1,i+1}) - \hbar^{-1}\log(b_i) + \sum_{k=1}^{n}(n-k+1/2)\mathsf{m}_{ik}. \tag{E3}$$

Pick any r_{11}, r_1. It suffices to show that (E2) admits a solution since (E3) obviously admits a unique solution with a given r_1 for any choice of r_{ij}. For a fixed r_{11}, the constants $\{r_{ij}\}_{1\le i\le j\le n}$ satisfying the 1st, 3rd, and 4th cases of (E2) are determined uniquely. It remains to verify that they also satisfy the 2nd case of (E2). We prove this by induction in $j-i\ge 2$.

(a) If $j=i+2$, then $r_{i,i+2}-r_{i+1,i+2} = (r_{i,i+1}-2r_{i+1,i+1}) + (2r_{i+1,i+1}-r_{i+1,i+2}) - (r_{i,i+1}-r_{i,i+2}) = \hbar^{-1}(\mathsf{m}_{i,i+1}-k_{i+1}+\mathsf{m}_{i+1,i+1}-k_{i+1}-\mathsf{m}_{i+1,i})$. Hence, it remains to prove $(\mathsf{m}_{i,i+1}-\mathsf{m}_{i,i+2})-(\mathsf{m}_{i+1,i}-\mathsf{m}_{i+1,i+1}) = 2k_{i+1}$. The left-hand side is equal to $\mathsf{n}^-_{i,i+1}-\mathsf{n}^+_{i,i+1}-\mathsf{n}^-_{i+1,i}+\mathsf{n}^+_{i+1,i} = b_{i,i+1}(\epsilon^-_{i,i+1}-\epsilon^+_{i,i+1}) = \epsilon^+_{i,i+1}-\epsilon^-_{i,i+1} = 2k_{i+1}$, due to the choice of k_{i+1}.

(b) If $j>i+2$, then $r_{ij}-r_{i+1,j} = (r_{i,j-1}-r_{i+1,j-1})+(r_{i+1,j-1}-r_{i+1,j})-(r_{i,j-1}-r_{ij}) = \hbar^{-1}(\mathsf{m}_{i,j-1}+\mathsf{m}_{j-1,i+1}-\mathsf{m}_{j-1,i})$. Hence, it remains to prove $(\mathsf{m}_{i,j-1}-\mathsf{m}_{ij})-(\mathsf{m}_{j-1,i}-\mathsf{m}_{j-1,i+1}) = 0$. The left-hand side is equal to $b_{i,j-1}(\epsilon^-_{i,j-1}-\epsilon^+_{i,j-1}) = 0$ as $b_{i,j-1}=0$.

Thus, we have determined a collection of constants $\{r_{ij}, r_i\}_{1\le i\le j\le n}$ satisfying (E2, E3).

• Finally, it remains to verify that for F of (A1) with the constants r_{ij}, r_i chosen as above the terms with $\frac{\mathsf{D}_i}{\mathsf{D}_j}$ $(j>i+1)$ in $F^{-1}\mathsf{H}(\epsilon^\pm,\mathsf{n}^\pm,c^\pm)F$ and $\mathsf{H}_2^{\vec{k}}$ do coincide. First, we note that the conditions $\epsilon^\pm_{i,i+1} = \ldots = \epsilon^\pm_{j-2,j-1} = \pm 1$ and $k_{i+1} = \ldots = k_{j-1} = 1$ are equivalent under our assumption. Pick $j>i+1$ such that either of these equivalent conditions is satisfied. Then the compatibility of the terms with $\frac{\mathsf{D}_i}{\mathsf{D}_j}$ is equivalent to the following equality:

$$\frac{F(\mathsf{w}_1,\ldots,v\mathsf{w}_i,\ldots,v^{-1}\mathsf{w}_j,\ldots,\mathsf{w}_n)}{F(\mathsf{w}_1,\ldots,\mathsf{w}_n)} = \prod_{k=1}^{n}\mathsf{w}_k^{\sum_{s=i}^{j-1}\mathsf{m}_{sk}+\delta_{k,i}+\delta_{k,j}} \cdot \prod_{p=i}^{j}\mathsf{w}_p^{-k_p-1} \cdot \prod_{s=i}^{j-1} b_s^{-1} \cdot v^{i+1-j+\sum_{i\le a<b\le j-1}(\mathsf{n}^-_{ab}-\mathsf{n}^+_{ab})+\sum_{k=1}^{n}\sum_{s=i}^{j-1}\frac{n+1-2k}{2}\mathsf{m}_{sk}}. \tag{E4}$$

This equality is proved by induction in $j-i$, factoring the left-hand side as in (A6) and noticing that the $j=i+1$ counterpart of (E4) is just the compatibility of the terms with $\frac{\mathsf{D}_i}{\mathsf{D}_{i+1}}$, established in the previous step.

Thus, the function F defined via (A1) with the constants $\{r_{ij}, r_i\}_{1\le i\le j\le n}$ determined in our 2nd step satisfies the equality (E1). This completes our proof of Theorem 3.24(a).

Proof of part (b).

Let us write

$$\mathsf{T}^{v}_{\vec{k}}(z)_{11} = (-1)^n \mathsf{w}_1 \cdots \mathsf{w}_n z^s \left(1 - \mathsf{H}^{\vec{k}}_2 z + \mathsf{H}^{\vec{k}}_3 z^2 - \ldots + (-1)^n \mathsf{H}^{\vec{k}}_{n+1} z^n\right),$$

where $s = \sum_{j=1}^{n} \frac{k_j - 1}{2}$. For $1 \le r \le n$, let $\bar{\mathsf{H}}^{\vec{k}}_{r+1} \in \mathcal{T}^{\vec{k}}$ be the image of $\mathsf{H}^{\vec{k}}_{r+1}$ in $\bar{\mathcal{A}}_n$. Consider the summands in $\bar{\mathsf{H}}^{\vec{k}}_{r+1}$ without D_i's and let $\bar{\mathsf{H}}^{\vec{k};0}_{r+1}$ denote their sum. Tracing back the definition of $\mathsf{T}^{v}_{\vec{k}}(z)$, we get $\bar{\mathsf{H}}^{\vec{k};0}_{r+1} = \sigma_r(\{\mathsf{w}_j^{-2}\})$: the r-th elementary symmetric polynomial of $\{\mathsf{w}_j^{-2}\}_{j=1}^{n}$.

Thus, the image of $\bar{\mathsf{H}}^{\vec{k}}_{r+1}$ under the anti-isomorphism $\bar{\mathcal{A}}_n \to \mathcal{D}_v(H^{\mathrm{ad}}_{\mathfrak{sl}_n})$ of Section 3.9 is an element of $\mathcal{D}^{\le}_v(H^{\mathrm{ad}}_{\mathfrak{sl}_n})$ whose action on W_λ (see Appendix B) is upper-triangular with the same diagonal matrix coefficients as in the action of $\mathbf{D}_r \in \mathcal{T}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$. Thus, the argument of Proposition B.1 can be applied to show that the function F of part (a), which conjugates $\mathsf{H}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ into $\bar{\mathsf{H}}^{\vec{k}}_2$ also conjugates the preimage of $\mathbf{D}_r$ in $\bar{\mathcal{A}}_n$ into $\bar{\mathsf{H}}^{\vec{k}}_{r+1}$ for all $1 \le r \le n$. Therefore, conjugation with F is an automorphism of $\bar{\mathcal{A}}_n$ that maps $\widetilde{\mathcal{T}}(\epsilon^{\pm}, \mathsf{n}^{\pm}, c^{\pm})$ to $\mathcal{T}^{\vec{k}}$.

This completes our proof of Theorem 3.24. ■

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