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Bethe subalgebras of $U_q(\widehat{\mathfrak{gl}}_n)$ via shuffle algebras

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Abstract In this article, we construct certain commutative subalgebras of the *big* shuffle algebra of type $A_{n-1}^{(1)}$. This can be considered as a generalization of the similar construction for the *small* shuffle algebra, obtained in Feigin et al. (J Math Phys 50(9):42, 2009). We present a *Bethe algebra* realization of these subalgebras. The latter identifies them with the Bethe subalgebras of $U_q(\widehat{\mathfrak{gl}}_n)$.

Keywords Toroidal algebra · Shuffle algebra · Bethe algebra · Drinfeld double

Mathematics Subject Classification 16Gxx · 16Txx · 17Bxx

1 Introduction

Elliptic shuffle algebras were first introduced and studied by the first author and Odesskii, see [6–8]. In the *loc.cit.*, they were associated with an elliptic curve $\mathcal{E}$ endowed with two automorphisms τ_1, τ_2. A similar class of algebras, depending on two parameters (alternatively q_1, q_2, q_3 with $q_1q_2q_3 = 1$), became of interest in the recent years, due to their geometric interpretations and different algebraic incarnations (see [4,9,13,16] for the related results). We will refer to these algebras as the *small* shuffle algebras. In this paper, we study the higher-rank generalizations of those

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algebras, which we refer to as the *big* shuffle algebras (of $A^{(1)}_{n-1}$-type). These algebras were also recently considered in [14], where they were identified with the positive half of the quantum toroidal algebras $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

The aim of this paper is to study particular *large* commutative subalgebras of the *big* shuffle algebra S, similar to the one from [4]. We also establish a *Bethe algebra* realization of these subalgebras (which seems to be new even for the *small* shuffle algebras). In other words, we identify those commutative subalgebras with the standard Bethe subalgebras of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$, which is *horizontally* embedded into the quantum toroidal algebra.

The aforementioned commutative subalgebras of S admit a one-parameter deformation: the commutative subalgebras $\mathcal{A}(s_0, \ldots, s_{n-1}; t) \subset (S^{\geq})^{\wedge}$ (the algebra $S^{\geq}$ is a slight enhancement of S, see Sect. 4.3, while $^{\wedge}$ indicates the completion with respect to the natural $\mathbb{Z}$-grading). These algebras are closely related to the study of nonlocal integrals of motion for the deformed W-algebras $W_{q,t}(\widehat{\mathfrak{sl}}_n)$ from [5], as well as provide a framework for the generalization of the recent results from [3] to $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. This will be elaborated elsewhere.

This paper is organized as follows:

- In Sect. 2, we recall the definition and key results about the quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, $n \geq 3$. We also recall the notion of the *small* shuffle algebra S^{sm} and its commutative subalgebra $\mathcal{A}^{\mathrm{sm}}$, and introduce a higher-rank generalization, the *big* shuffle algebra S.
- In Sect. 3, we introduce a family of subspaces $\mathcal{A}(s_0, \ldots, s_{n-1}) \subset S$ depending on n parameters and generalizing the construction of $\mathcal{A}^{\mathrm{sm}} \subset S^{\mathrm{sm}}$. If $(\frac{1}{s_1 \ldots s_{n-1}}, s_1, \ldots, s_{n-1})$ is *generic* (see Sect. 3.2), then we prove that $\mathcal{A}(\frac{1}{s_1 \ldots s_{n-1}}, s_1, \ldots, s_{n-1})$ is a polynomial algebra on explicitly given generators; in particular, it is a commutative subalgebra of S.
- In Sect. 4, we use the universal R-matrix and vertex-type representations to establish an alternative viewpoint toward $\mathcal{A}(s_0, \ldots, s_{n-1})$. This allows us to identify them with the well-known Bethe subalgebras of the quantum affine $U_q(\widehat{\mathfrak{gl}}_n)$, horizontally embedded into $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.
- In Sect. 5, we discuss generalizations of the results from Sects. 2–4 to the cases $n = 1, 2$.

2 Basic definitions and constructions

2.1 Quantum toroidal algebras of $\mathfrak{sl}_n$ for $n \geq 3$

Let $q, d \in \mathbb{C}^*$ be two parameters. We set $[n] := \{0, 1, \ldots, n-1\}$, $[n]^{\times} := [n]\backslash\{0\}$, the former viewed as a set of mod n residues. Let $g_m(z) := \frac{q^m z-1}{z-q^m}$. Define $\{a_{i,j}, m_{i,j}\}_{i\in[n]}^{j\in[n]}$ by

$$a_{i,i} = 2,\ a_{i,i\pm 1} = -1,\ m_{i,i\pm 1} = \mp 1, \quad \text{and } a_{i,j} = m_{i,j} = 0 \quad \text{otherwise.}$$

The quantum toroidal algebra of $\mathfrak{sl}_n$, denoted by $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, is the unital associative algebra generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i\in[n]}^{k\in\mathbb{Z}}$ with the following defining relations:[1]

$$[\psi_i^{\pm}(z), \psi_j^{\pm}(w)] = 0, \ \gamma^{\pm 1/2} - \text{central}, \tag{T0.1}$$

$$\psi_{i,0}^{\pm 1} \cdot \psi_{i,0}^{\mp 1} = \gamma^{\pm 1/2} \cdot \gamma^{\mp 1/2} = q^{\pm d_1} \cdot q^{\mp d_1} = q^{\pm d_2} \cdot q^{\mp d_2} = 1, \tag{T0.2}$$

$$q^{d_1} e_i(z) q^{-d_1} = e_i(qz), \quad q^{d_1} f_i(z) q^{-d_1} = f_i(qz), \quad q^{d_1} \psi_i^{\pm}(z) q^{-d_1} = \psi_i^{\pm}(qz), \tag{T0.3}$$

$$q^{d_2} e_i(z) q^{-d_2} = q e_i(z), \quad q^{d_2} f_i(z) q^{-d_2} = q^{-1} f_i(z), \quad q^{d_2} \psi_i^{\pm}(z) q^{-d_2} = \psi_i^{\pm}(z), \tag{T0.4}$$

$$g_{a_{i,j}}(\gamma^{-1} d^{m_{i,j}} z/w) \psi_i^+(z) \psi_j^-(w) = g_{a_{i,j}}(\gamma d^{m_{i,j}} z/w) \psi_j^-(w) \psi_i^+(z), \tag{T1}$$

$$e_i(z) e_j(w) = g_{a_{i,j}}(d^{m_{i,j}} z/w) e_j(w) e_i(z), \tag{T2}$$

$$f_i(z) f_j(w) = g_{a_{i,j}}(d^{m_{i,j}} z/w)^{-1} f_j(w) f_i(z), \tag{T3}$$

$$(q - q^{-1})[e_i(z), f_j(w)] = \delta_{i,j} \left(\delta(\gamma w/z) \psi_i^+(\gamma^{1/2} w) - \delta(\gamma z/w) \psi_i^-(\gamma^{1/2} z) \right), \tag{T4}$$

$$\psi_i^{\pm}(z) e_j(w) = g_{a_{i,j}}(\gamma^{\pm 1/2} d^{m_{i,j}} z/w) e_j(w) \psi_i^{\pm}(z), \tag{T5}$$

$$\psi_i^{\pm}(z) f_j(w) = g_{a_{i,j}}(\gamma^{\mp 1/2} d^{m_{i,j}} z/w)^{-1} f_j(w) \psi_i^{\pm}(z), \tag{T6}$$

$$\operatorname*{Sym}_{z_1,z_2} [e_i(z_1), [e_i(z_2), e_{i\pm 1}(w)]_q]_{q^{-1}} = 0, \ [e_i(z), e_j(w)] = 0 \quad \text{for } j \neq i, i \pm 1, \tag{T7.1}$$

$$\operatorname*{Sym}_{z_1,z_2} [f_i(z_1), [f_i(z_2), f_{i\pm 1}(w)]_q]_{q^{-1}} = 0, \ [f_i(z), f_j(w)] = 0 \text{ for } j \neq i, i \pm 1, \tag{T7.2}$$

where we set $[a, b]_x := ab - x \cdot ba$ and define the generating series as follows:

$$e_i(z) := \sum_{k=-\infty}^{\infty} e_{i,k} z^{-k}, \quad f_i(z) := \sum_{k=-\infty}^{\infty} f_{i,k} z^{-k},$$

$$\psi_i^{\pm}(z) := \psi_{i,0}^{\pm 1} + \sum_{r>0} \psi_{i,\pm r} z^{\mp r}, \quad \delta(z) := \sum_{k=-\infty}^{\infty} z^k.$$

[1] Our notation are consistent with that of [17], but following [15] we add the elements $q^{\pm d_1}, q^{\pm d_2}$ satisfying (T0.3, T0.4). This update is essential for our discussion of the Drinfeld double and the universal R-matrix.

It will be convenient to use the generators $\{h_{i,k}\}_{k\neq 0}$ instead of $\{\psi_{i,k}\}_{k\neq 0}$, where $h_{i,\pm r} \in \mathbb{C}\left[\psi_{i,0}^{\mp 1}, \psi_{i,\pm 1}, \psi_{i,\pm 2}, \ldots\right]$ are defined by

$$\exp\left(\pm(q-q^{-1})\sum_{r>0} h_{i,\pm r} z^{\mp r}\right) = \bar{\psi}_i^{\pm}(z) := \psi_{i,0}^{\mp 1}\psi_i^{\pm}(z).$$

Then the relations (T5, T6) are equivalent to the following:

$$\psi_{i,0}e_{j,l} = q^{a_{i,j}} e_{j,l}\psi_{i,0},\ [h_{i,k}, e_{j,l}] = d^{-km_{i,j}}\gamma^{-|k|/2}\frac{[ka_{i,j}]_q}{k} e_{j,l+k}\ (k \neq 0), \quad \text{(T5}'\text{)}$$

$$\psi_{i,0}f_{j,l} = q^{-a_{i,j}} f_{j,l}\psi_{i,0},\ [h_{i,k}, f_{j,l}] = -d^{-km_{i,j}}\gamma^{|k|/2}\frac{[ka_{i,j}]_q}{k} f_{j,l+k}\ (k \neq 0), \quad \text{(T6}'\text{)}$$

where $[m]_q := \frac{q^m - q^{-m}}{q-q^{-1}}$. We also introduce $h_{i,0}, c, c'$ via $\psi_{i,0} = q^{h_{i,0}}, \gamma^{1/2} = q^c, c' = \sum_{i\in[n]} h_{i,0}$, so that c, c' are central.

Let $\ddot{U}^-$ and $\ddot{U}^+$ be the subalgebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}\}_{i\in[n]}^{k\in\mathbb{Z}}$ and $\{f_{i,k}\}_{i\in[n]}^{k\in\mathbb{Z}}$, respectively, while $\ddot{U}^0$ is generated by $\{\psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i\in[n]}^{k\in\mathbb{Z}}$.

Proposition 2.1 [10] (Triangular decomposition) *The multiplication map*

$$m : \ddot{U}^- \otimes \ddot{U}^0 \otimes \ddot{U}^+ \to \ddot{U}_{q,d}(\mathfrak{sl}_n)$$

is an isomorphism of vector spaces.

We equip the algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ with the $\mathbb{Z}^{[n]} \times \mathbb{Z}$-grading by assigning

$$\begin{gathered}\deg(e_{i,k}) := (1_i; k),\ \deg(f_{i,k}) := (-1_i; k),\ \deg(\psi_{i,k}) := (0; k),\\ \deg(x) := (0; 0) \text{ for } x = \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\ \ \forall\, i \in [n], k \in \mathbb{Z},\end{gathered}$$

where $1_j \in \mathbb{Z}^{[n]}$ is the vector with the jth coordinate 1 and all other coordinates zero.

2.2 Horizontal and vertical $U_q(\widehat{\mathfrak{gl}}_n)$

Following [17], we introduce the *vertical* and *horizontal* copies of the quantum affine algebra of $\mathfrak{sl}_n$, denoted by $U_q(\widehat{\mathfrak{sl}}_n)$, inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. Consider the subalgebra $\dot{U}^{\mathrm{v}}(\mathfrak{sl}_n)$ of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1} | i \in [n]^{\times}, k \in \mathbb{Z}\}$. This algebra is isomorphic to $U_q(\widehat{\mathfrak{sl}}_n)$, realized via the "new Drinfeld presentation". Let $\dot{U}^{\mathrm{h}}(\mathfrak{sl}_n)$ be the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,0}, f_{i,0}, \psi_{i,0}^{\pm 1}, q^{\pm d_2} | i \in [n]\}$. This algebra is also isomorphic to $U_q(\widehat{\mathfrak{sl}}_n)$, realized via the classical Drinfeld–Jimbo presentation.

Following [2], we recall a slight upgrade of this construction, which provides two copies of the quantum affine algebra of $\mathfrak{gl}_n$, rather than $\mathfrak{sl}_n$, inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. For every $r \neq 0$, choose $\{c_{i,r} | i \in [n]\}$ to be a nontrivial solution of the following system of linear equations:[2]

$$\sum_{i=0}^{n-1} c_{i,r} d^{-rm_{i,j}} [ra_{i,j}]_q = 0, \quad j \in [n]^\times.$$

Let $\mathfrak{h}^{\mathrm{v}}$ be the subspace of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ spanned by

$$h_r^{\mathrm{v}} = \begin{cases} \sum_{i\in[n]} c_{i,r} h_{i,r} & \text{if } r \neq 0 \\ \gamma^{1/2} & \text{if } r = 0 \end{cases}.$$

Note that $\mathfrak{h}^{\mathrm{v}}$ is well defined and commutes with $\dot{U}^{\mathrm{v}}(\mathfrak{sl}_n)$, due to (T5′, T6′). Moreover, $\mathfrak{h}^{\mathrm{v}}$ is isomorphic to the Heisenberg Lie algebra. Let $\dot{U}^{\mathrm{v}}(\mathfrak{gl}_n)$ be the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\dot{U}^{\mathrm{v}}(\mathfrak{sl}_n)$ and $\mathfrak{h}^{\mathrm{v}}$. The above discussions imply that $\dot{U}^{\mathrm{v}}(\mathfrak{gl}_n) \simeq U_q(\widehat{\mathfrak{gl}}_n)$, the quantum affine algebra of $\mathfrak{gl}_n$. We let $\dot{U}^{\mathrm{v}}(\mathfrak{gl}_1) \subset \dot{U}^{\mathrm{v}}(\mathfrak{gl}_n)$ be the subalgebra generated by $\mathfrak{h}^{\mathrm{v}}$.

Our next goal is to provide a *horizontal* copy of $U_q(\widehat{\mathfrak{gl}}_n)$, containing $\dot{U}^{\mathrm{h}}(\mathfrak{sl}_n)$, inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. The following approach was proposed in [2], and it is based on a beautiful result of Miki:

Theorem 2.2 [12] *There exists an automorphism π of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ such that*

$$\pi(\dot{U}^{\mathrm{v}}(\mathfrak{sl}_n)) = \dot{U}^{\mathrm{h}}(\mathfrak{sl}_n),\ \pi(\dot{U}^{\mathrm{h}}(\mathfrak{sl}_n)) = \dot{U}^{\mathrm{v}}(\mathfrak{sl}_n).$$

Moreover:

$$\pi(q^c) = q^{c'}, \quad \pi(q^{c'}) = q^{-c}.$$

Let us define $\mathfrak{h}^{\mathrm{h}} := \pi(\mathfrak{h}^{\mathrm{v}})$ and let $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_n)$ be the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\dot{U}^{\mathrm{h}}(\mathfrak{sl}_n)$ and $\mathfrak{h}^{\mathrm{h}}$. Then $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_n) = \pi(\dot{U}^{\mathrm{v}}(\mathfrak{gl}_n))$ and it is isomorphic to $U_q(\widehat{\mathfrak{gl}}_n)$. We also define $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_1) \subset \dot{U}^{\mathrm{h}}(\mathfrak{gl}_n)$ as the subalgebra generated by $\mathfrak{h}^{\mathrm{h}}$.

However, this construction is not very enlightening, as the images $\pi(h_r^{\mathrm{v}})$ are hardly computable. An alternative approach, based on the RTT realization of $U_q(\widehat{\mathfrak{gl}}_n)$, was proposed in [14]. We will discuss the related results in Sect. 3.3.

2.3 Hopf pairing, Drinfeld double and a universal *R*-matrix

We recall the general notion of a Hopf pairing, following [11, Chapter 3]. Given two Hopf algebras A and B with invertible antipodes S_A and S_B, the bilinear map

[2] It is easy to see that the space of solutions of this system is 1-dimensional if q is not a root of unity.

$$\varphi : A \times B \to \mathbb{C}$$

is called a *Hopf pairing* if it satisfies the following properties:

$$\begin{gathered}
\varphi(a, bb') = \varphi(a_1, b)\varphi(a_2, b') \quad \forall\, a \in A,\ b, b' \in B, \\
\varphi(aa', b) = \varphi(a, b_2)\varphi(a', b_1) \quad \forall\, a, a' \in A,\ b \in B, \\
\varphi(a, 1_B) = \epsilon_A(a) \quad \text{and } \varphi(1_A, b) = \epsilon_B(b) \quad \forall\, a \in A,\ b \in B, \\
\varphi(S_A(a), b) = \varphi(a, S_B^{-1}(b)) \quad \forall\, a \in A,\ b \in B,
\end{gathered}$$

where we use the Sweedler notation for the coproduct:

$$\Delta(x) = x_1 \otimes x_2.$$

For such a data, one can define the *generalized Drinfeld double* $D_\varphi(A, B)$ as follows:

Theorem 2.3 [11, Theorem 3.2] *There is a unique Hopf algebra $D_\varphi(A, B)$ satisfying the following properties:*

(i) *As coalgebras $D_\varphi(A, B) \simeq A \otimes B$.*
(ii) *Under the natural inclusions*

$$\begin{gathered}
A \hookrightarrow D_\varphi(A, B) \text{ given by } a \mapsto a \otimes 1_B, \\
B \hookrightarrow D_\varphi(A, B) \text{ given by } b \mapsto 1_A \otimes b,
\end{gathered}$$

A and B are Hopf subalgebras of $D_\varphi(A, B)$.
(iii) *For any $a \in A, b \in B$, we have*

$$(a \otimes 1_B) \cdot (1_A \otimes b) = a \otimes b$$

and

$$(1_A \otimes b) \cdot (a \otimes 1_B) = \varphi(S_A^{-1}(a_1), b_1)\varphi(a_3, b_3)a_2 \otimes b_2.$$

Remark 2.4 The notion of the *Drinfeld double* is reserved for the case $B = A^{\star,\mathrm{cop}}$ with φ being the natural pairing.

A Hopf algebra A is *quasitriangular* (*formally quasitriangular*) if there is an invertible element

$$R \in A \otimes A \ (\text{or } R \in A \widehat{\otimes} A)$$

satisfying the following properties:

$$\begin{gathered}
R\Delta(x) = \Delta^{\mathrm{op}}(x)R \quad \forall\, x \in A, \\
(\Delta \otimes \mathrm{Id})(R) = R^{13}R^{23}, \\
(\mathrm{Id} \otimes \Delta)(R) = R^{13}R^{12}.
\end{gathered}$$

Such an element R is called a *universal R-matrix* of A.
The fundamental property of Drinfeld doubles is their quasitriangularity:

Theorem 2.5 [11, Theorem 3.2] *For a nondegenerate Hopf pairing $\varphi : A \times B \to \mathbb{C}$, the generalized Drinfeld double $D_\varphi(A, B)$ is formally quasitriangular with the universal R-matrix*

$$R = \sum_i e_i \otimes e_i^*,$$

where $\{e_i\}$ is a basis of A and $\{e_i^*\}$ is the dual basis of B (with respect to φ).

2.4 Quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ as a Drinfeld double

In order to apply the constructions of the previous section to the quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ and its subalgebras, we need to endow the former with a Hopf algebra structure. This was first done (in a more general setup) in [1, Theorem 2.1]:

Theorem 2.6 *The formulas* (H1-H9) *endow $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ with a topological Hopf algebra structure:*

$$\Delta(e_i(z)) = e_i(z) \otimes 1 + \psi_i^-(\gamma_{(1)}^{1/2} z) \otimes e_i(\gamma_{(1)} z), \tag{H1}$$

$$\Delta(f_i(z)) = 1 \otimes f_i(z) + f_i(\gamma_{(2)} z) \otimes \psi_i^+(\gamma_{(2)}^{1/2} z), \tag{H2}$$

$$\Delta(\psi_i^\pm(z)) = \psi_i^\pm(\gamma_{(2)}^{\pm 1/2} z) \otimes \psi_i^\pm(\gamma_{(1)}^{\mp 1/2} z), \tag{H3}$$

$$\Delta(x) = x \otimes x \quad \text{for } x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}, \tag{H4}$$

$$\epsilon(e_i(z)) = \epsilon(f_i(z)) = 0, \ \epsilon(\psi_i^\pm(z)) = 1, \tag{H5}$$

$$\epsilon(x) = 1 \quad \text{for } x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}, \tag{H6}$$

$$S(e_i(z)) = -\psi_i^-(\gamma^{-1/2} z)^{-1} e_i(\gamma^{-1} z), \tag{H7}$$

$$S(f_i(z)) = -f_i(\gamma^{-1} z) \psi_i^+(\gamma^{-1/2} z)^{-1}, \tag{H8}$$

$$S(x) = x^{-1} \quad \text{for } x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}, \psi_i^\pm(z), \tag{H9}$$

where $\gamma_{(1)}^{1/2} := \gamma^{1/2} \otimes 1$ and $\gamma_{(2)}^{1/2} := 1 \otimes \gamma^{1/2}$.

Let $\ddot{U}^{\geq}$ be the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}, \psi_{i,l}, \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{k\in\mathbb{Z}}^{l\in -\mathbb{N}}$, and let $\ddot{U}^{\leq}$ be the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{f_{i,k}, \psi_{i,l}, \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{k\in\mathbb{Z}}^{l\in\mathbb{N}}$. Now we are ready to state the main result of this section (the proof is straightforward):

Theorem 2.7 *(a) There exists a unique Hopf algebra pairing $\varphi : \ddot{U}^{\geq} \times \ddot{U}^{\leq} \to \mathbb{C}$ satisfying*

$$\varphi(e_i(z), f_j(w)) = \frac{\delta_{i,j}}{q-q^{-1}} \cdot \delta\left(\frac{z}{w}\right), \quad \varphi(\psi_i^-(z), \psi_j^+(w)) = g_{a_{i,j}}(d^{m_{i,j}}z/w), \tag{P1}$$

$$\varphi(e_i(z), x^-) = \varphi(x^+, f_i(z)) = 0 \quad \text{for } x^{\pm} = \psi_j^{\mp}(w), \psi_{j,0}^{\pm 1}, \gamma^{1/2}, q^{d_1}, q^{d_2}, \tag{P2}$$

$$\varphi(\gamma^{1/2}, q^{d_1}) = \varphi(q^{d_1}, \gamma^{1/2}) = q^{-1/2}, \quad \varphi(\psi_i^-(z), q^{d_2}) = q^{-1}, \quad \varphi(q^{d_2}, \psi_i^+(z)) = q, \tag{P3}$$

$$\varphi(\psi_i^-(z), x) = \varphi(x, \psi_i^+(z)) = 1 \quad \text{for } x = \gamma^{1/2}, q^{d_1}, \tag{P4}$$

$$\varphi(\gamma^{1/2}, q^{d_2}) = \varphi(q^{d_2}, \gamma^{1/2}) = \varphi(q^{d_a}, q^{d_b}) = \varphi(\gamma^{1/2}, \gamma^{1/2}) = 1 \quad \text{for } 1 \leq a, b \leq 2. \tag{P5}$$

(b) The natural Hopf algebra homomorphism $D_\varphi(\ddot{U}^{\geq}, \ddot{U}^{\leq}) \to \ddot{U}_{q,d}(\mathfrak{sl}_n)$ induces the isomorphism

$$\Xi : D_\varphi(\ddot{U}^{\geq}, \ddot{U}^{\leq})/I \xrightarrow{\sim} \ddot{U}_{q,d}(\mathfrak{sl}_n)$$

with $I := (x \otimes 1 - 1 \otimes x | x = \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2})$.
(c) Consider a slight modification $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$, obtained from $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ by "throwing away" the generator $q^{\pm d_2}$ and taking the quotient by the central element c'. As in (b), this algebra admits the double Drinfeld realization via $D_{\varphi'}(\ddot{U}'^{\geq}, \ddot{U}'^{\leq})$, where $\ddot{U}'^{\leq}$ and $\ddot{U}'^{\geq}$ are obtained from $\ddot{U}^{\leq}$ and $\ddot{U}^{\geq}$ by "throwing away" $q^{\pm d_2}$ and taking the quotient by c', while φ' is induced by φ.
(d) The pairings φ and φ' are nondegenerate if and only if q, qd, qd^{-1} are not roots of unity.

2.5 Bethe subalgebras

Let us recall the standard way of constructing large commutative subalgebras of a (formally) quasitriangular Hopf algebra A. Fix a group-like element $x \in A$ (or in an appropriate completion $x \in A^\wedge$). For an A-representation V, we consider the *transfer matrix*

$$T_V(x) := (1 \otimes \mathrm{tr}_V)((1 \otimes x)R)$$

if the latter is well defined. The properties of the R-matrix imply

$$T_{V_1\oplus V_2}(x) = T_{V_1}(x) + T_{V_2}(x),$$
$$T_{V_1\otimes V_2}(x) = T_{V_2}(x)\cdot T_{V_1}(x).$$

In particular, we see that $T_{V_1}(x)\cdot T_{V_2}(x) = T_{V_2}(x)\cdot T_{V_1}(x)$.

To summarize, $\bullet \mapsto T_\bullet(x)$ is a ring homomorphism from the Grothendieck group of any suitable tensor category of A-modules to the suitable completion $A^\wedge$, with the image being a commutative subalgebra of that completion. The commutative subalgebras constructed in this way are sometimes called the *Bethe (sub)algebras*.

In Sect. 4, we will apply this construction to the following two cases:

∘ The formally quasitriangular algebra is $A = \ddot{U}'_{q,d}(\mathfrak{sl}_n)$, the corresponding group-like element is $x = q^{\lambda_1 h_{1,0}+\cdots+\lambda_{n-1}h_{n-1,0}+\lambda_n d_1}$, and we consider a tensor category of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-representations generated by vertex $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-representations $\rho_{p,\bar{c}}$ from Sect. 4.1.[3]

∘ The formally quasitriangular algebra is $A = U_q(L\mathfrak{gl}_n)$ (see Sect. 4.4), the corresponding group-like element is $x = q^{\lambda_1 h_{1,0}+\cdots+\lambda_{n-1}h_{n-1,0}}$ (the most generic element of the *finite Cartan* part), and we consider the tensor category of all finite-dimensional $U_q(L\mathfrak{gl}_n)$-representations.

2.6 Small shuffle algebra

As a motivating point for the current paper, we briefly recall the notion of the *small* shuffle algebra and its particular commutative subalgebra. Let $\mathbb{Z}_+ := \{n\in\mathbb{Z}|n\geq 0\} = \mathbb{N}\cup\{0\}$. Consider a $\mathbb{Z}_+$-graded $\mathbb{C}$-vector space $\mathbb{S}^{\mathrm{sm}} = \bigoplus_{n\geq 0}\mathbb{S}^{\mathrm{sm}}_n$, where $\mathbb{S}^{\mathrm{sm}}_n$ consists of rational functions $\frac{f(x_1,\ldots,x_n)}{\Delta(x_1,\ldots,x_n)}$ with $f\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]^{\mathfrak{S}_n}$ and $\Delta(x_1,\ldots,x_n) := \prod_{i\neq j}(x_i - x_j)$. Define the star product $\overset{\mathrm{sm}}{\star} : \mathbb{S}^{\mathrm{sm}}_k\times\mathbb{S}^{\mathrm{sm}}_l\to\mathbb{S}^{\mathrm{sm}}_{k+l}$ by

$$(F\overset{\mathrm{sm}}{\star}G)(x_1,\ldots,x_{k+l}) := \mathrm{Sym}_{\mathfrak{S}_{k+l}}\left(F(x_1,\ldots,x_k)G(x_{k+1},\ldots,x_{k+l})\prod_{i\leq k}^{j>k}\lambda(x_i/x_j)\right)$$

with

$$\lambda(x) := \frac{(q_1x-1)(q_2x-1)(q_3x-1)}{(x-1)^3},\quad \text{where } q_i\in\mathbb{C}\backslash\{0,1\}\ \text{ and } q_1q_2q_3 = 1.$$

This endows $\mathbb{S}^{\mathrm{sm}}$ with a structure of an associative unital $\mathbb{C}$-algebra with the unit $\mathbf{1}\in\mathbb{S}^{\mathrm{sm}}_0$.

[3] Actually, one can consider the whole category of highest weight $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-representations, see [12].

We say that an element $\frac{f(x_1,\ldots,x_n)}{\Delta(x_1,\ldots,x_n)} \in \mathbb{S}_n^{\mathrm{sm}}$ satisfies the *wheel conditions* if and only if

$$f(x_1,\ldots,x_n)=0 \quad \text{once } x_{i_1}/x_{i_2}=q_1 \quad \text{and } x_{i_2}/x_{i_3}=q_2 \quad \text{for some } 1\le i_1,i_2,i_3\le n.$$

Let $S^{\mathrm{sm}} \subset \mathbb{S}^{\mathrm{sm}}$ be a $\mathbb{Z}_+$-graded subspace, consisting of all such elements. The subspace S^{sm} is $\overset{\mathrm{sm}}{\star}$-closed (see [4, Proposition 2.10]).

Definition 2.8 The algebra $(S^{\mathrm{sm}}, \overset{\mathrm{sm}}{\star})$ is called the *small shuffle algebra*.

Following [4], we introduce an important $\mathbb{Z}_+$-graded subspace $\mathcal{A}^{\mathrm{sm}} = \bigoplus_n \mathcal{A}_n^{\mathrm{sm}}$ of S^{sm}. Its degree n component is defined by

$$\mathcal{A}_n^{\mathrm{sm}} := \{F \in S_n^{\mathrm{sm}} | \partial^{(0;k)}F, \partial^{(\infty;k)}F \text{ exist and } \partial^{(0;k)}F = \partial^{(\infty;k)}F \ \ \forall\, 0\le k\le n\},$$

where

$$\partial^{(0;k)}F := \lim_{\xi\to 0} F(x_1,\ldots,x_{n-k},\xi\cdot x_{n-k+1},\ldots,\xi\cdot x_n),$$
$$\partial^{(\infty;k)}F := \lim_{\xi\to \infty} F(x_1,\ldots,x_{n-k},\xi\cdot x_{n-k+1},\ldots,\xi\cdot x_n)$$

whenever the limits exist.

This subspace satisfies the following properties:

Theorem 2.9 [4, Section 2] *We have:*

(a) *Suppose* $F \in S_n^{\mathrm{sm}}$ *and* $\partial^{(\infty;k)}F$ *exist for all* $0\le k\le n$, *then* $F \in \mathcal{A}_n^{\mathrm{sm}}$.
(b) *The subspace* $\mathcal{A}^{\mathrm{sm}} \subset S^{\mathrm{sm}}$ *is* $\overset{\mathrm{sm}}{\star}$*-commutative.*
(c) $\mathcal{A}^{\mathrm{sm}}$ *is* $\overset{\mathrm{sm}}{\star}$*-closed and it is a polynomial algebra in* $\{K_j\}_{j\ge 1}$ *with* $K_j \in S_j^{\mathrm{sm}}$ *defined by:*

$$K_1(x_1) = x_1^0,\ \ K_m(x_1,\ldots,x_m) = \prod_{1\le i<j\le m} \frac{(x_i-q_1x_j)(x_j-q_1x_i)}{(x_i-x_j)^2}.$$

2.7 Big shuffle algebra

Consider a $\mathbb{Z}_+^{[n]}$-graded $\mathbb{C}$-vector space

$$\mathbb{S} = \bigoplus_{\bar{k}=(k_0,\ldots,k_{n-1})\in\mathbb{Z}_+^{[n]}} \mathbb{S}_{\bar{k}},$$

where $\mathbb{S}_{k_0,\ldots,k_{n-1}}$ consists of $\prod \mathfrak{S}_{k_i}$-symmetric rational functions in the variables $\{x_{i,j}\}_{i\in[n]}^{1\le j\le k_i}$. We also fix an $n\times n$ matrix of rational functions $\Omega = (\omega_{i,j}(z))_{i,j\in[n]} \in \mathrm{Mat}_{n\times n}(\mathbb{C}(z))$ by setting

$$\omega_{i,i}(z)=\frac{z-q^{-2}}{z-1},\quad \omega_{i,i+1}(z)=\frac{d^{-1}z-q}{z-1},$$
$$\omega_{i,i-1}(z)=\frac{z-qd^{-1}}{z-1},\quad \text{and}\quad \omega_{i,j}(z)=1\quad \text{otherwise.}$$

Let us now introduce the bilinear $\star$ product on $\mathbb{S}$: given $f\in\mathbb{S}_{\overline{k}}$, $g\in\mathbb{S}_{\overline{l}}$ define $f\star g\in\mathbb{S}_{\overline{k}+\overline{l}}$ by

$$(f\star g)(x_{0,1},\ldots,x_{0,k_0+l_0};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+l_{n-1}}):=$$
$$\mathrm{Sym}_{\prod\mathfrak{S}_{k_i+l_i}}\left(f(\{x_{i,j}\}_{i\in[n]}^{1\le j\le k_i})g(\{x_{i,j}\}_{i\in[n]}^{k_i<j\le k_i+l_i})\times\prod_{i\in[n]}^{i'\in[n]}\prod_{j\le k_i}^{j'>k_{i'}}\omega_{i,i'}(x_{i,j}/x_{i',j'})\right).$$

This endows $\mathbb{S}$ with a structure of an associative unital algebra with the unit $\mathbf{1}\in\mathbb{S}_{0,\ldots,0}$. We will be interested only in a certain subspace of $\mathbb{S}$, defined by the *pole* and *wheel conditions*:

- We say that $F\in\mathbb{S}_{\overline{k}}$ satisfies the *pole conditions* if and only if

$$F=\frac{f(x_{0,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i\in[n]}\prod_{j\le k_i}^{j'\le k_{i+1}}(x_{i,j}-x_{i+1,j'})},\quad \text{where } f\in(\mathbb{C}[x_{i,j}^{\pm1}]_{i\in[n]}^{1\le j\le k_i})^{\prod\mathfrak{S}_{k_i}}.$$

- We say that $F\in\mathbb{S}_{\overline{k}}$ satisfies the *wheel conditions* if and only if

$$F(x_{0,1},\ldots,x_{n-1,k_{n-1}})=0\quad\text{once } x_{i,j_1}/x_{i+\epsilon,l}=qd^{\epsilon}\quad\text{and}$$
$$x_{i+\epsilon,l}/x_{i,j_2}=qd^{-\epsilon}\quad\text{for some } i,\epsilon,j_1,j_2,l,$$

  where $\epsilon\in\{\pm1\}$, $i\in[n]$, $1\le j_1,j_2\le k_i$, $1\le l\le k_{i+\epsilon}$ and we use the cyclic notation $x_{n,l}:=x_{0,l}$, $k_n:=k_0$, $x_{-1,l}:=x_{n-1,l}$, $k_{-1}:=k_{n-1}$ as before.

Let $S_{\overline{k}}\subset\mathbb{S}_{\overline{k}}$ be the subspace of all elements F satisfying the above two conditions and set

$$S:=\bigoplus_{\overline{k}\in\mathbb{Z}_+^{[n]}}S_{\overline{k}}.$$

Further $S_{\overline{k}}=\oplus_{d\in\mathbb{Z}}S_{\overline{k},d}$ with $S_{\overline{k},d}:=\{F\in S_{\overline{k}}|\mathrm{tot.deg}(F)=d\}$. The following is straightforward:

Lemma 2.10 *The subspace $S\subset\mathbb{S}$ is $\star$-closed.*

Now we are ready to introduce the main algebra of this paper:

Definition 2.11 The algebra $(S,\star)$ is called the big shuffle algebra (of $A_{n-1}^{(1)}$-type).

2.8 Relation between S and $\ddot{U}^+$

Recall the subalgebra $\ddot{U}^+$ of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ from Sect. 2.1. By standard results, $\ddot{U}^+$ is generated by $\{e_{i,k}\}_{i\in[n]}^{k\in\mathbb{Z}}$ with the defining relations (T2, T7.1). The following is straightforward:

Proposition 2.12 *There exists a unique algebra homomorphism $\Psi : \ddot{U}^+ \to \mathbb{S}$ such that $\Psi(e_{i,k}) = x_{i,1}^k \ \forall\, i \in [n], k \in \mathbb{Z}$.*

As a consequence, $\mathrm{Im}(\Psi) \subset S$. The following beautiful result was recently proved by Negut:

Theorem 2.13 [14, Theorem 1.1] *The homomorphism $\Psi : \ddot{U}^+ \to S$ is an isomorphism of $\mathbb{Z}_+^{[n]} \times \mathbb{Z}$-graded algebras.*

Remark 2.14 In the *loc. cit.* $d = 1$, but the proof can be easily modified for any d. Note that the algebra $\mathcal{A}^+$ from [14] is isomorphic to our S with $d = 1$ via the map $S_{|d=1} \to \mathcal{A}^+$ given by

$$F(\{x_{i,j}\}_{i\in[n]}^{1\le j\le k_i}) \mapsto q^{\sum_{i=0}^{n-1}\frac{k_i(k_i-1)}{2}} F(\{z_{i,j}\}_{1\le i\le n}^{1\le j\le k_{i-1}}) \cdot \prod_{i=1}^{n}\prod_{j\ne j'} \frac{z_{i,j}-z_{i,j'}}{q^{-1}z_{i,j}-qz_{i,j'}} \cdot \prod_{i=1}^{n}\prod_{j,j'} \frac{z_{i,j}-z_{i+1,j'}}{z_{i,j}-qz_{i+1,j'}}.$$

3 Subalgebras $\mathcal{A}(s_0, \ldots, s_{n-1})$

3.1 Key constructions

In this section, we introduce the key objects of our paper, the commutative subalgebras of S, analogous to $\mathcal{A}^{\mathrm{sm}} \subset S^{\mathrm{sm}}$ from Sect. 2.6. The new feature of our setup (in comparison to the small shuffle algebras) is that we get an $(n-1)$-parameter family of those.

For any $0 \le \bar{l} \le \bar{k} \in \mathbb{Z}_+^{[n]}$, $\xi \in \mathbb{C}^*$ and $F \in S_{\bar{k}}$, we define $F_\xi^{\bar{l}} \in \mathbb{C}(x_{0,1}, \ldots, x_{n-1,k_{n-1}})$ by

$$F_\xi^{\bar{l}} := F(\xi\cdot x_{0,1}, \ldots, \xi\cdot x_{0,l_0}, x_{0,l_0+1}, \ldots, x_{0,k_0}; \ldots; \xi\cdot x_{n-1,1}, \ldots, \xi\cdot x_{n-1,l_{n-1}}, x_{n-1,l_{n-1}+1}, \ldots, x_{n-1,k_{n-1}}).$$

For any integer numbers $a \le b$, define the degree vector $\bar{l} := [a; b] \in \mathbb{Z}_+^{[n]}$ by

$$\bar{l} = (l_0, \ldots, l_{n-1}) \ \text{ with } \ l_i = \#\{c \in \mathbb{Z} | a \le c \le b \text{ and } c \equiv i \pmod n\}.$$

For such a choice of $\bar{l}$, we will denote $F_\xi^{\bar{l}}$ simply by $F_\xi^{(a,b)}$.

Definition 3.1 For any $\overline{s} = (s_0, \dots, s_{n-1}) \in (\mathbb{C}^*)^{[n]}$, consider a $\mathbb{Z}_+^{[n]}$-graded subspace $\mathcal{A}(\overline{s}) \subset S$ whose degree $\overline{k} = (k_0, \dots, k_{n-1})$ component is defined by

$$\mathcal{A}(\overline{s})_{\overline{k}} := \left\{ F \in S_{\overline{k},0} \mid \partial^{(\infty;a,b)} F = \prod_{i=a}^{b} s_i \cdot \partial^{(0;a,b)} F \right.$$
$$\left. \forall\, a, b \in \mathbb{Z} \text{ such that } a \le b \text{ and } [a;b] \le \overline{k} \right\},$$

where $\partial^{(\infty;a,b)} F := \lim_{\xi\to\infty} F_\xi^{(a,b)}$, $\partial^{(0;a,b)} F := \lim_{\xi\to 0} F_\xi^{(a,b)}$ whenever these limits exist, $s_i := s_{i \bmod n}$.

A certain class of such elements is provided by the following result:

Lemma 3.2 *For any $k \in \mathbb{N}$, $\mu \in \mathbb{C}$, and $\overline{s} \in (\mathbb{C}^*)^{[n]}$, define $F_k^\mu(\overline{s}) \in S_{k,\dots,k}$ by*

$$F_k^\mu(\overline{s}) := \frac{\prod_{i\in[n]} \prod_{1\le j\ne j'\le k} (x_{i,j} - q^{-2} x_{i,j'}) \cdot \prod_{i\in[n]} (s_0 \dots s_i \prod_{j=1}^k x_{i,j} - \mu \prod_{j=1}^k x_{i+1,j})}{\prod_{i\in[n]} \prod_{1\le j,j'\le k} (x_{i,j} - x_{i+1,j'})},$$

where we set $x_{n,j} := x_{0,j}$ as before. If $s_0 \dots s_{n-1} = 1$, then $F_k^\mu(\overline{s}) \in \mathcal{A}(\overline{s})$.

Proof Without loss of generality, we can assume $\mu \ne 0, a = 0, b = nr + c$, $0 \le r \le k-1, 0 \le c \le n-1$. Then $l_0 = \dots = l_c = r+1$ and $l_{c+1} = \dots = l_{n-1} = r$. As $\xi \to \infty$, the function $F_k^\mu(\overline{s})_\xi^{(a,b)}$ grows at the speed $\xi^{\sum_{i\in[n]} l_i(l_{i+1}-l_i-1) + \sum_{i\in[n]} \max\{l_i,l_{i+1}\}}$, while as $\xi \to 0$, the function $F_k^\mu(\overline{s})_\xi^{(a,b)}$ grows at the speed $\xi^{\sum_{i\in[n]} l_i(-l_{i+1}+l_i-1) + \sum_{i\in[n]} \min\{l_i,l_{i+1}\}}$. For the above values of l_i, both powers of ξ are zero and hence both limits $\partial^{(\infty;a,b)} F_k^\mu(\overline{s})$ and $\partial^{(0;a,b)} F_k^\mu(\overline{s})$ exist. Moreover, for α being 0 or ∞, we have $\partial^{(\alpha;a,b)} F_k^\mu(\overline{s}) = (-1)^{\sum_{i\in[n]} l_i(l_i-l_{i-1})} q^{-2\sum_{i\in[n]} l_i(k-l_i)} \cdot G \cdot \prod_{i\in[n]} G_{\alpha,i}$, where

$$G = \frac{\prod_{i\in[n]} \prod_{1\le j\ne j'\le l_i} (x_{i,j} - q^{-2} x_{i,j'}) \cdot \prod_{i\in[n]} \prod_{l_i<j\ne j'\le k} (x_{i,j} - q^{-2} x_{i,j'})}{\prod_{i\in[n]} \prod_{1\le j\le l_i}^{1\le j'\le l_{i+1}} (x_{i,j} - x_{i+1,j'}) \cdot \prod_{i\in[n]} \prod_{l_i<j\le k}^{l_{i+1}<j'\le k} (x_{i,j} - x_{i+1,j'})},$$

$$G_{\infty,i} = \prod_{j=1}^{l_i} x_{i,j}^{l_{i+1}+l_{i-1}-2l_i} \cdot \begin{cases} s_0 \dots s_i \prod_{j=1}^k x_{i,j} - \mu \prod_{j=1}^k x_{i+1,j} & \text{if } l_i = l_{i+1} \\ s_0 \dots s_i \prod_{j=1}^k x_{i,j} & \text{if } l_i > l_{i+1} \\ -\mu \prod_{j=1}^k x_{i+1,j} & \text{if } l_i < l_{i+1} \end{cases},$$

$$G_{0,i} = \prod_{j=l_i+1}^{k} x_{i,j}^{-l_{i+1}-l_{i-1}+2l_i} \cdot \begin{cases} s_0 \dots s_i \prod_{j=1}^k x_{i,j} - \mu \prod_{j=1}^k x_{i+1,j} & \text{if } l_i = l_{i+1} \\ -\mu \prod_{j=1}^k x_{i+1,j} & \text{if } l_i > l_{i+1} \\ s_0 \dots s_i \prod_{j=1}^k x_{i,j} & \text{if } l_i < l_{i+1} \end{cases}.$$

The equality $\partial^{(\infty;a,b)} F_k^\mu(\overline{s}) = \prod_{i=0}^c s_i \cdot \partial^{(0;a,b)} F_k^\mu(\overline{s})$ follows, while the total degree of $F_k^\mu(\overline{s})$ is zero. □

3.2 Main result

A collection $\{s_0, \ldots, s_{n-1}\} \subset \mathbb{C}^*$ satisfying $s_0 \ldots s_{n-1} = 1$ is called *generic* if and only if

$$s_0^{\alpha_0} \ldots s_{n-1}^{\alpha_{n-1}} \in q^{\mathbb{Z}} \cdot d^{\mathbb{Z}} \Rightarrow \alpha_0 = \ldots = \alpha_{n-1}.$$

The main result of this section describes $\mathcal{A}(s_0, \ldots, s_{n-1})$ for such generic n-tuples $\{s_0, \ldots, s_{n-1}\}$.

Theorem 3.3 *For a generic $\overline{s} = (s_0, \ldots, s_{n-1})$ satisfying $s_0 \ldots s_{n-1} = 1$, the space $\mathcal{A}(\overline{s})$ is shuffle-generated by $\{F_k^{\mu}(\overline{s}) | k \in \mathbb{N}, \mu \in \mathbb{C}\}$. Moreover, $\mathcal{A}(\overline{s})$ is a polynomial algebra in free generators $\{F_k^{\mu_l}(\overline{s}) | k \in \mathbb{N}, 1 \leq l \leq n\}$ for arbitrary pairwise distinct $\mu_1, \ldots, \mu_n \in \mathbb{C}$. In particular, $\mathcal{A}(\overline{s})$ is a commutative subalgebra of S.*

The proof of this theorem will proceed in several steps. First, we will use an analogue of the *Gordon filtration* from [4], further generalized in [14] to prove Theorem 2.13, in order to obtain the upper bound on dimensions of $\mathcal{A}(\overline{s})_{\overline{k}}$. Next, we will show that the subalgebra $\mathcal{A}'(\overline{s}) \subset S$, shuffle generated by all $F_k^{\mu}(\overline{s})$, belongs to $\mathcal{A}(\overline{s})$. We will use another filtration to argue that the dimension of $\mathcal{A}'(\overline{s})_{\overline{k}}$ is at least as big as the upper bound for the dimension of $\mathcal{A}(\overline{s})_{\overline{k}}$, implying $\mathcal{A}'(\overline{s}) = \mathcal{A}(\overline{s})$. Similar arguments will also imply the commutativity of $\mathcal{A}(\overline{s})$.

Lemma 3.4 *Consider the polynomial algebra $\mathcal{R} = \mathbb{C}[T_{i,m}]_{i \in [n]}^{m \geq 1}$ with $\deg(T_{i,m}) = m$. Then:*

(a) *For $\overline{k} = k\delta := (k, \ldots, k)$, we have $\dim \mathcal{A}(\overline{s})_{\overline{k}} \leq \dim \mathcal{R}_k$.*
(b) *For $\overline{k} \notin \{0, \delta, 2\delta, \ldots\}$, we have $\mathcal{A}(\overline{s})_{\overline{k}} = 0$.*

Proof An unordered set L of integer intervals $\{[a_1, b_1], \ldots, [a_r, b_r]\}$ is called a *partition* of $\overline{k} \in \mathbb{Z}_+^{[n]}$ (denoted by $L \vdash \overline{k}$) if $\overline{k} = [a_1; b_1] + \cdots + [a_r; b_r]$. We order the elements of L so that $b_1 - a_1 \geq b_2 - a_2 \geq \cdots \geq b_r - a_r$. The two sets L and L' as above are said to be equivalent if $|L| = |L'|$, and we can order their elements so that $b'_i - b_i = a'_i - a_i = nc_i$ for all i and some $c_i \in \mathbb{Z}$. Note that the collection of $L \vdash \overline{k}$, up to the above equivalence, is finite for any $\overline{k} \in \mathbb{Z}_+^{[n]}$. Finally, we say $L' > L$ if there exists s, such that $b'_s - a'_s > b_s - a_s$ and $b'_t - a'_t = b_t - a_t$ for $1 \leq t \leq s - 1$.

Any $L \vdash \overline{k}$ defines a linear map $\phi_L : \mathcal{A}(\overline{s})_{\overline{k}} \to \mathbb{C}[y_1^{\pm 1}, \ldots, y_r^{\pm 1}]$ as follows. Split the variables $\{x_{i,j}\}$ in r groups, each group corresponding to one of the intervals in L. Specialize the variables corresponding to the interval $[a_t, b_t]$ to $(qd)^{-a_t} \cdot y_t, \ldots, (qd)^{-b_t} \cdot y_t$ in the natural order. For

$$F = \frac{f(x_{0,1}, \ldots, x_{n-1,k_{n-1}})}{\prod_{i \in [n]} \prod_{1 \leq j \leq k_i}^{1 \leq j' \leq k_{i+1}} (x_{i,j} - x_{i+1,j'})} \in \mathcal{A}(\overline{s})_{\overline{k}},$$

define $\phi_L(F)$ as the corresponding specialization of f. The result is independent of our splitting of variables since f is symmetric. Finally, we define the filtration on $\mathcal{A}(\overline{s})_{\overline{k}}$ by

$$\mathcal{A}(\bar{s})^{L}_{\bar{k}} := \bigcap_{L'>L} \mathrm{Ker}(\phi_{L'}).$$

Let us now consider the images $\phi_L(\mathcal{A}(\bar{s})^{L}_{\bar{k}})$ for any $L \vdash \bar{k}$. For $F \in \mathcal{A}(\bar{s})^{L}_{\bar{k}}$, we have:

∘ The total degree $\mathrm{tot.deg}(\phi_L(F)) = \sum_{i\in[n]} k_i k_{i+1}$, since $\mathrm{tot.deg}(F) = 0$.

∘ For each $1 \le t \le r$, the degree of $\phi_L(F)$ with respect to y_t is bounded by

$$\deg_{y_t}(\phi_L(F)) \le \sum_{i\in[n]} (l^t_i(k_{i-1} + k_{i+1}) - l^t_i l^t_{i+1})$$

due to the existence of the limit $\partial^{(\infty;a_t,b_t)}F$ (here $\bar{l}^t := [a_t; b_t] \in \mathbb{Z}^{[n]}_+$ for $1 \le t \le r$).

On the other hand, the wheel conditions for F guarantee that $\phi_L(F)(y_1, \ldots, y_r)$ becomes zero under the following specializations:

(i) $(qd)^{-x'}y_v = (q/d)(qd)^{-x}y_u$ for any $1 \le u < v \le r,\ a_u \le x < b_u, a_v \le x' \le b_v,\ x' \equiv x + 1$,

(ii) $(qd)^{-x'}y_v = (d/q)(qd)^{-x}y_u$ for any $1 \le u < v \le r,\ a_u < x \le b_u, a_v \le x' \le b_v,\ x' \equiv x - 1$.

Finally, the conditions $\phi_{L'}(F) = 0$ for any $L' > L$ guarantee that $\phi_L(F)(y_1, \ldots, y_r)$ becomes zero under the following specializations:

(iii) $(qd)^{-x'}y_v = (qd)^{-b_u-1}y_u$ for any $1 \le u < v \le r,\ a_v \le x' \le b_v,\ x' \equiv b_u + 1$,

(iv) $(qd)^{-x'}y_v = (qd)^{-a_u+1}\cdot y_u$ for any $1 \le u < v \le r,\ a_v \le x' \le b_v,\ x' \equiv a_u - 1$.

In particular, we see that $\phi_L(F)$ is divisible by $Q_L \in \mathbb{C}[y_1, \ldots, y_r]$, defined as a product of the linear terms in y_t coming from (i)–(iv) (if some of these coincide, we still count them with the correct multiplicity). Note that

$$\mathrm{tot.deg}(Q_L) = \sum_{1\le u<v\le r}\sum_{i\in[n]} (l^u_i l^v_{i+1} + l^u_i l^v_{i-1}) = \sum_{i\in[n]} k_i k_{i+1} - \sum_{t=1}^{r}\sum_{i\in[n]} l^t_i l^t_{i+1},$$

while the degree with respect to each variable y_t $(1 \le t \le r)$ is given by

$$\deg_{y_t}(Q_L) = \sum_{i\in[n]} (l^t_i(k_{i-1} + k_{i+1}) - 2l^t_i l^t_{i+1}).$$

Define $r_L := \phi_L(F)/Q_L \in \mathbb{C}[y_1^{\pm1}, \ldots, y_r^{\pm1}]$. Then:

$$\mathrm{tot.deg}(r_L) = \sum_{t=1}^{r}\sum_{i\in[n]} l^t_i l^t_{i+1} \quad \text{and} \quad \deg_{y_t}(r_L) \le \sum_{i\in[n]} l^t_i l^t_{i+1}.$$

Hence, $r_L = \nu \cdot \prod_{t=1}^{r} y_t^{\sum_{i\in[n]} l^t_i l^t_{i+1}}$ for some $\nu \in \mathbb{C}$, so that

$$\phi_L(F) = \nu \cdot \prod_{t=1}^{r} y_t^{\sum_{i\in[n]} l^t_i l^t_{i+1}} \cdot Q_L.$$

On the other hand, applying the above specialization to the entire function F rather than f, we get $\phi_L(F)/Q$, where $Q \in \mathbb{C}[y_1,\ldots,y_r]$ is given by

$$Q = \nu' \cdot \prod_{t=1}^{r} y_t^{\sum_{i\in[n]} l_i^t l_{i+1}^t} \cdot \prod_{1\le u<v\le r} \prod_{a_u\le x\le b_u} \prod_{a_v\le x'\le b_v}^{x'\equiv x\pm 1} ((qd)^{-x}y_u - (qd)^{-x'}y_v)$$

for some $\nu' \in \mathbb{C}^*$.

The condition $F \in \mathcal{A}(\overline{s})$ implies

$$\lim_{\xi\to\infty}\left(\frac{\phi_L(F)}{Q}\right)_{|y_t\mapsto \xi\cdot y_t} = s_{a_t}\ldots s_{b_t}\cdot \lim_{\xi\to 0}\left(\frac{\phi_L(F)}{Q}\right)_{|y_t\mapsto \xi\cdot y_t} \quad \forall\, 1\le t\le r.$$

For $\nu \neq 0$, this equality enforces $s_{a_t}\ldots s_{b_t} \in q^{\mathbb{Z}}\cdot d^{\mathbb{Z}}$. Due to our condition on $\{s_i\}$, we get $b_t - a_t + 1 = nc_t$ for every $1\le t\le r$ and some $c_t \in \mathbb{N}$. The claim (ii) of the lemma is now obvious, while part (i) of the lemma follows from the inequality $\dim \mathcal{A}(\overline{s})_{\overline{k}} \le \sum \dim \phi_L(\mathcal{A}(\overline{s})^L_{\overline{k}})$, where the last sum is taken over all equivalence classes of $L \vdash \overline{k}$. □

Lemma 3.5 *Let $\mathcal{A}'(\overline{s})$ be the subalgebra of S generated by $\{F_k^{\mu}(\overline{s})|k\ge 1, \mu\in\mathbb{C}\}$. Then $\mathcal{A}'(\overline{s}) \subset \mathcal{A}(\overline{s})$.*

Proof It suffices to show $F_{k_1,\ldots,k_r}^{\mu_1,\ldots,\mu_r}(\overline{s}) := F_{k_1}^{\mu_1}(\overline{s}) \star \cdots \star F_{k_r}^{\mu_r}(\overline{s}) \in \mathcal{A}(\overline{s})$ for any r, $k_i \ge 1$, and $\mu_i \in \mathbb{C}^*$. The case of $r=1$ has been already treated in Lemma 3.2. The arguments for general r are similar. Choose any $a\le b$, such that $[a;b]\le k\delta$, where $k := k_1+\cdots+k_r$. We can further assume $a=0$. Let us consider any summand from the definition of $F_{k_1,\ldots,k_r}^{\mu_1,\ldots,\mu_r}(\overline{s})$ with $\bar{l} := [a;b]$ variables being multiplied by ξ. We will check that as ξ tends to ∞ or 0, both limits exist and differ by the constant $s_a\ldots s_b$.

For a fixed summand as above, define $\{\bar{l}^t\}_{t=1}^r \in \mathbb{Z}_+^{[n]}$ satisfying $\bar{l} = \bar{l}^1+\cdots+\bar{l}^r$ by considering those variables $x_{i,j}$ which are multiplied by ξ and get substituted into $F_{k_t}^{\mu_t}(\overline{s})$. Following the proof of Lemma 3.2, the function $F_{k_t}^{\mu_t}(\overline{s})_{\xi}^{\bar{l}^t}$ grows at the speed $\xi^{\sum_{i\in[n]} l_i^t(l_{i+1}^t - l_i^t - 1) + \sum_{i\in[n]}\max\{l_i^t, l_{i+1}^t\}}$ as $\xi\to\infty$ and at the speed $\xi^{\sum_{i\in[n]} l_i^t(-l_{i+1}^t + l_i^t - 1) + \sum_{i\in[n]}\min\{l_i^t, l_{i+1}^t\}}$ as $\xi \to 0$. To estimate these powers, we note that $(a-b)(a-b-1)\ge 0$ for any $a,b\in\mathbb{Z}$, implying

$$\min(a,b) + \frac{a^2+b^2-a-b}{2} \ge ab$$

with equality holding if and only if $a-b\in\{-1,0,1\}$. Therefore,

$$\sum_{i\in[n]} l_i^t(l_{i+1}^t - l_i^t - 1) + \sum_{i\in[n]}\max\{l_i^t, l_{i+1}^t\} \le 0,$$

$$0 \le \sum_{i\in[n]} l_i^t(-l_{i+1}^t + l_i^t - 1) + \sum_{i\in[n]}\min\{l_i^t, l_{i+1}^t\},$$

with equalities holding if and only if $l_i^t - l_{i+1}^t \in \{\pm 1, 0\}$ for any $i \in [n]$. Since the limits of

$$\omega_{i,j}(\xi \cdot x, y), \quad \omega_{i,j}(x, \xi \cdot y), \quad \omega_{i,j}(\xi \cdot x, \xi \cdot y) \quad \text{as } \xi \to 0, \infty \text{ exist} \quad \forall\, i, j \in [n],$$

the limits of the corresponding summands in the symmetrization are well defined as either $\xi \to 0, \infty$. Moreover, they are both zero if $|l_i^t - l_{i+1}^t| > 1$ for some $i \in [n]$, $1 \leq t \leq r$.

Assuming finally that $|l_i^t - l_{i+1}^t| \leq 1$ for any t, i, the formulas from the proof of Lemma 3.2 imply that the ratio of the limits as ξ goes to ∞ and 0 equals to

$$\prod_{t=1}^{r} \prod_{i \in [n]} \left(\frac{s_0 \dots s_i}{-\mu_t} \right)^{l_i^t - l_{i+1}^t} = \prod_{i \in [n]} (s_0 \dots s_i)^{l_i - l_{i+1}} = s_a \dots s_b.$$

The result follows. $\square$

Lemma 3.6 *For any $k \in \mathbb{N}$, we have* $\dim \mathcal{A}'(\overline{s})_{k\delta} \geq \dim \mathcal{R}_k$.

Proof Choose any pairwise distinct $\mu_1, \dots, \mu_n \in \mathbb{C}$ and consider a subspace $\mathcal{A}''(\overline{s})$ of $\mathcal{A}'(\overline{s})$ spanned by $F_{k_1,\dots,k_r}^{\mu_{i_1},\dots,\mu_{i_r}}(\overline{s})$ with $r \geq 0, k_1 \geq k_2 \geq \dots \geq k_r > 0$, and $1 \leq i_1, \dots, i_r \leq n$. It suffices to show

$$\dim \mathcal{A}''(\overline{s})_{k\delta} \geq \dim \mathcal{R}_k.$$

For a Young diagram λ, we introduce the specialization map

$$\varphi_\lambda : S_{|\lambda| \cdot \delta} \to \mathbb{C}(\{y_{i,j}\}_{i \in [n]}^{1 \leq j \leq l(\lambda)})$$

by specializing the variables $x_{i,j}$ as follows

$$x_{i, \lambda_1 + \dots + \lambda_{t-1} + j} \mapsto q^{2j} y_{i,t} \quad \text{for any} \quad 1 \leq t \leq l(\lambda), 1 \leq j \leq \lambda_t, i \in [n].$$

It is clear that for any $\overline{k} = (k_1, \dots, k_r)$ with $\sum_i k_i = k = |\lambda|$ and $\overline{k} > \lambda'$ (here $>$ denotes the lexicographic order on Young diagrams and λ' denotes the transposed to λ Young diagram), we have $\varphi_\lambda(F_{\overline{k}}^{\overline{\mu}}(\overline{s})) = 0$ for all $\overline{\mu} \in \mathbb{C}^r$. Therefore, it remains to prove

$$\sum_{\overline{k} \vdash k} \dim \varphi_{\overline{k}'}(\text{span}\{F_{\overline{k}}^{\overline{\mu}}(\overline{s}) | \overline{\mu} \in \mathbb{C}^r\}) \geq \dim \mathcal{R}_k.$$

Let us first consider the case $k_1 = \dots = k_r \Rightarrow k = rk_1$. Then

$$\varphi_{\overline{k}'}(F_{\overline{k}}^{\overline{\mu}}(\overline{s})) = Z \cdot \prod_{t=1}^{r} \prod_{i \in [n]} \left(s_0 \dots s_i \prod_{j=1}^{k_1} y_{i,j} - \mu_t \prod_{j=1}^{k_1} y_{i+1,j} \right)$$

for a certain nonzero common factor Z. Define $Y_i := y_{i,1} \cdots y_{i,k_1}$. Since the polynomials

$$f_t(Y_1,\ldots,Y_n) := \prod_{i\in[n]} (s_0\ldots s_i Y_i - \mu_t Y_{i+1}), \quad 1\le t\le n,$$

are algebraically independent, we immediately get the required dimension estimate for this particular $\overline{k}$. The general case follows immediately. □

By Lemmas 3.4–3.6, the subspace $\mathcal{A}(\overline{s})$ is generated by $F_k^{\mu}(\overline{s})$ and has the prescribed dimensions of each $\mathbb{Z}_+^{[n]}$-graded component.

Lemma 3.7 *The algebra $\mathcal{A}(\overline{s})$ is commutative. Moreover, for any $\overline{\mu}=(\mu_1,\ldots,\mu_n)\in\mathbb{C}^n$ with $\mu_i\ne\mu_j$ for $i\ne j$, there is an isomorphism $\mathcal{R}\xrightarrow{\sim}\mathcal{A}(\overline{s})$ given by $T_{i,k}\mapsto F_k^{\mu_i}(\overline{s})$.*

Proof It suffices to prove $F_{m_1}^{\nu_1}(\overline{s})\star F_{m_2}^{\nu_2}(\overline{s}) = F_{m_2}^{\nu_2}(\overline{s})\star F_{m_1}^{\nu_1}(\overline{s})$ for any $m_1,m_2\in\mathbb{N}$ and $\nu_1,\nu_2\in\mathbb{C}$. Define $F := F_{m_1,m_2}^{\nu_1,\nu_2}(\overline{s}) - F_{m_2,m_1}^{\nu_2,\nu_1}(\overline{s})$. Due to previous lemmas, F can be written as a certain linear combination of $F_{\overline{k}}^{\overline{\mu}}(\overline{s})$ with $\overline{k}=(k_1\ge k_2\ge\cdots)$.

We claim that $\varphi_{(2,1^{m_1+m_2-2})}(F)=0$. Together with the properties of φ_λ discussed above, this equality implies $F=\sum_{r=1}^n \pi_r\cdot F_{m_1+m_2}^{\mu_r}(\overline{s})$ for some $\pi_r\in\mathbb{C}$. Let us multiply both sides of this equality by $\prod_{i\in[n]}\prod_{1\le j,j'\le m_1+m_2}(x_{i,j}-x_{i+1,j'})$ and consider a specialization $x_{i,j}\mapsto y_i\ \forall i,j$. The left-hand side will clearly specialize to 0, while the right-hand side will specialize to

$$\prod_{i\in[n]}((1-q^{-2})y_i)^{(m_1+m_2)(m_1+m_2-1)}\cdot\sum_{r=1}^n\left\{\pi_r\cdot\prod_{i\in[n]}(s_0\ldots s_i y_i^{m_1+m_2}-\mu_r y_{i+1}^{m_1+m_2})\right\}.$$

This expression vanishes if and only if $\pi_1=\cdots=\pi_n=0$, and so $F=0$ as required.

Finally, let us prove the equality $\varphi_{(2,1^{m_1+m_2-2})}(F)=0$. The statement is obvious when either m_1 or m_2 is zero. To prove for general $m_1,m_2>0$, we can assume by induction that

$$F_{m'_1}^{\nu'_1}(\overline{s}')\star F_{m'_2}^{\nu'_2}(\overline{s}') = F_{m'_2}^{\nu'_2}(\overline{s}')\star F_{m'_1}^{\nu'_1}(\overline{s}')$$

for any $m'_1<m_1$, $m'_2<m_2$, $\nu'_1,\nu'_2\in\mathbb{C}$, and $\prod_i s'_i=1$ (though $\{s'_i\}$ are not necessarily generic).

By straightforward computations, $\varphi_{(2,1^{m_1+m_2-2})}(F_{m_1,m_2}^{\nu_1,\nu_2}(\overline{s}))=\mathrm{Sym}(A_1\cdot B_1)$, where the symmetrization is taken with respect to all permutations of $\{y_{i,j}\}_{i\in[n]}^{2\le j\le m_1+m_2-1}$ preserving index i, $A_1\in\mathbb{C}(\{y_{i,j}\})$ is symmetric, while B_1 is given by the following explicit formula

$$B_1 = \frac{\prod_{i\in[n]}\prod_{2\le j\ne j'\le m_1}(y_{i,j}-q^{-2}y_{i,j'})\cdot\prod_{i\in[n]}\prod_{m_1<j\ne j'<m_1+m_2}(y_{i,j}-q^{-2}y_{i,j'})}{\prod_{i\in[n]}\prod_{2\le j\ne j'\le m_1}(y_{i,j}-y_{i+1,j'})\cdot\prod_{i\in[n]}\prod_{m_1<j\ne j'<m_1+m_2}(y_{i,j}-y_{i+1,j'})}$$
$$\times\prod_{i,i'\in[n]}\prod_{2\le j\le m_1}\prod_{m_1<j'<m_1+m_2}\omega_{i,i'}(y_{i,j}/y_{i',j'})$$

$$
\begin{aligned}
&\times \prod_{i\in[n]} \left(s_0\dots s_i y_{i,1} \prod_{j=2}^{m_1} y_{i,j} - \nu_1 y_{i+1,1} \prod_{j=2}^{m_1} y_{i+1,j} \right)\\
&\times \prod_{i\in[n]} \left(s_0\dots s_i y_{i,1} \prod_{j=m_1+1}^{m_1+m_2-1} y_{i,j} - \nu_2 y_{i+1,1} \prod_{j=m_1+1}^{m_1+m_2-1} y_{i+1,j} \right)\\
&\Rightarrow \mathrm{Sym}(B_1)\\
&= \kappa \cdot \left(F^{\nu_1'}_{m_1-1}(\bar{s}') \star F^{\nu_2'}_{m_2-1}(\bar{s}') \right) (y_{0,2},\dots,y_{0,m_1+m_2-1};\dots;y_{n-1,2},\dots,y_{n-1,m_1+m_2-1})
\end{aligned}
$$

with $\nu_1' := \nu_1 \cdot \frac{y_{0,1}}{y_{n-1,1}}$, $\nu_2' := \nu_2 \cdot \frac{y_{0,1}}{y_{n-1,1}}$, $s_i' := s_i \cdot \frac{y_{i,1}^2}{y_{i-1,1}y_{i+1,1}}$, $\kappa := \prod_{i\in[n]} \frac{y_{i,1}^2 y_{n-1,1}^2}{y_{0,1}^2}$.

Permuting $m_1 \leftrightarrow m_2$, $\nu_1 \leftrightarrow \nu_2$, we get

$$
\varphi_{(2,1^{m_1+m_2-2})}(F^{\nu_2,\nu_1}_{m_2,m_1}(\bar{s})) = \kappa \cdot A_1 \cdot \left(F^{\nu_2'}_{m_2-1}(\bar{s}') \star F^{\nu_1'}_{m_1-1}(\bar{s}') \right) (y_{0,2},\dots,y_{n-1,m_1+m_2-1}).
$$

Applying the induction assumption, we find

$$
\varphi_{(2,1^{m_1+m_2-2})}(F) = \kappa A_1 [F^{\nu_1'}_{m_1-1}(\bar{s}'), F^{\nu_2'}_{m_2-1}(\bar{s}')] = 0.
$$

This proves the inductive step and, hence, completes the proof of the claim. $\square$

The results of Theorem 3.3 follow immediately by combining the above four lemmas.

Remark 3.8 The proof of Lemma 3.4 implies $\mathcal{A}(\bar{s}) = \mathbb{C}$ for any $s_0,\dots,s_{n-1} \in \mathbb{C}^*$ such that $\prod_i s_i^{\alpha_i} \notin q^{\mathbb{Z}} \cdot d^{\mathbb{Z}}$ unless $\alpha_0 = \ldots = \alpha_{n-1} = 0$.

3.3 Shuffle realization of $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_n)^+$ and $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_1)^+$

In [14], the author introduced the notion of the *slope filtration* on S. For a zero slope, the corresponding subspace $A^0 \subset S$ is $\mathbb{Z}_+^{[n]}$-graded with the graded component $A^0_{\underline{k}}$ given by

$$
F \in A^0_{\underline{k}} \Longleftrightarrow F \in S_{\underline{k},0} \quad \text{and } \exists \lim_{\xi\to\infty} F^{\bar{l}}_{\xi} \quad \forall\, 0 \le \bar{l} \le \bar{k}.
$$

While proving Theorem 2.13, the author obtained the following description of A^0:

Proposition 3.9 [14, Lemma 4.4]

(a) *The isomorphism* $\Psi : \ddot{U}^+ \xrightarrow{\sim} S$ *identifies* $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_n)^+$ *with* A^0.
(b) *Under the isomorphism* $\Psi^{\mathrm{h}} : \dot{U}^{\mathrm{h}}(\mathfrak{gl}_n)^+ \xrightarrow{\sim} A^0$ *from (a), the image* $X_k := \Psi^{\mathrm{h}}(h^{\mathrm{h}}_k)$ *of the kth generator* $h^{\mathrm{h}}_k \in \dot{U}^{\mathrm{h}}(\mathfrak{gl}_1)^+ \subset \dot{U}^{\mathrm{h}}(\mathfrak{gl}_n)^+$ *is uniquely (up to a constant) characterized by*

$$
X_k \in S_{k\delta,0} \quad \text{and } \lim_{\xi\to\infty} (X_k)^{\bar{l}}_{\xi} = 0 \quad \forall\, 0 < \bar{l} < k\delta.
$$

This proposition provides a shuffle characterization of both $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_n)^+$ and $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_1)^+$. In particular, we immediately obtain the following result:

Theorem 3.10 *We have* $\Psi^{-1}(\mathcal{A}(\overline{s})) \subset \dot{U}^{\mathrm{h}}(\mathfrak{gl}_n)^+$ *for generic* $\{s_i\}$ *such that* $s_0 \dots s_{n-1} = 1$.

Proof By Theorem 3.3 and Proposition 3.9(a), it suffices to show that $F_k^{\mu}(\overline{s}) \in A^0$. The latter is equivalent to the existence of limits $\lim_{\xi\to\infty}(F_k^{\mu}(\overline{s}))_{\xi}^{\overline{l}}$ for all $0 \le \overline{l} \le k\delta$. As $\xi \to \infty$, the function $(F_k^{\mu}(\overline{s}))_{\xi}^{\overline{l}}$ grows at the speed $\xi^{\sum_{i\in[n]} l_i(l_{i+1}-l_i+1)-\sum_{i\in[n]}\min\{l_i,l_{i+1}\}}$ (see the proof of Lemma 3.2). Since $\sum_{i\in[n]} l_i(l_{i+1}-l_i+1) - \sum_{i\in[n]}\min\{l_i, l_{i+1}\} = \sum_{i\in[n]}(l_i l_{i+1} - \frac{l_i^2+l_{i+1}^2-l_i-l_{i+1}}{2} - \min\{l_i, l_{i+1}\})$ and each summand is nonpositive (see the proof of Lemma 3.5), the aforementioned power of ξ is nonpositive as well. Hence, the limit $\lim\limits_{\xi\to\infty}(F_k^{\mu}(\overline{s}))_{\xi}^{\overline{l}}$ does exist. This completes the proof. □

We complete this section by providing explicit formulas for the elements $X_k = \Psi^{\mathrm{h}}(h_k^{\mathrm{h}}) \in S$ (this answers one of the questions raised in [14, Section 5.6]). Consider the elements

$$F_0 := \mathbf{1},\ F_k := \frac{\prod_{i\in[n]}\prod_{1\le j\ne j'\le k}(q^{-1}x_{i,j} - qx_{i,j'}) \cdot \prod_{i\in[n]}\prod_{j=1}^{k} x_{i,j}}{\prod_{i\in[n]}\prod_{1\le j,j'\le k}(x_{i+1,j'} - x_{i,j})} \in S_{k\delta,0} \quad \text{for } k>0.$$

Note that $F_k = \frac{(-q^{k-1})^{nk}}{s_0^n s_1^{n-1}\dots s_{n-1}} \cdot F_k^0(\overline{s}) \in \mathcal{A}(\overline{s})$ for any $\{s_i\}$ such that $\prod_{i\in[n]} s_i = 1$. We also define $L_k \in S_{k\delta,0}$ via

$$\exp\left(\sum_{k=1}^{\infty} L_k t^k\right) = \sum_{k=0}^{\infty} F_k t^k.$$

The relevant properties of these elements are formulated in our next theorem:

Theorem 3.11 (a) *For* $\overline{l} \notin \{0, \delta, 2\delta, \dots, k\delta\}$, *we have* $\lim_{\xi\to\infty}(F_k)_{\xi}^{\overline{l}} = 0$.

(b) *For any* $0 \le l \le k$, *we have* $\lim_{\xi\to\infty}(F_k)_{\xi}^{l\delta} = F_l \cdot F_{k-l}$.

(c) *For any* $0 < \overline{l} < k\delta$, *we have* $\lim_{\xi\to\infty}(L_k)_{\xi}^{\overline{l}} = 0$.

Proof (a) For any $0 \le \overline{l} \le k\delta$, the function $(F_k)_{\xi}^{\overline{l}}$ grows at the speed $\xi^{\sum_{i\in[n]} l_i(l_{i+1}-l_i)}$ as $\xi \to \infty$. Note that $\sum_{i\in[n]} l_i(l_{i+1}-l_i) = -\frac{1}{2}\sum_{i\in[n]}(l_i - l_{i+1})^2 \le 0$. Moreover, the equality holds if and only if $l_0 = \dots = l_{n-1} \Leftrightarrow \overline{l} \in \{0, \delta, 2\delta, \dots\}$. Part (a) follows.

(b) Straightforward.

(c) Standard (it is actually equivalent to the general *exponential* relation between group-like elements and primitive elements; see [14, Section 4.3] for the related coproduct). □

Corollary 3.12 *Combining this result with Proposition 3.9(b), we see that* L_k *and* X_k *coincide up to a nonzero constant, and the isomorphism* Ψ^{h} *identifies* $\dot{U}^{\mathrm{h}}(\mathfrak{gl}_1)^+$ *with* $\mathbb{C}[F_1, F_2, \dots]$.

4 Bethe algebra realization of $\mathcal{A}(\bar{s})$

We provide an alternative viewpoint on the subspaces $\mathcal{A}(\bar{s})$ for generic $\{s_i\}$ with $\prod_{i\in[n]} s_i = 1$. Some of the results from this section (the computation of $\phi^{\bar{u},t}_{p,\bar{c}}, \Gamma^{\bar{u},t}_{p,\bar{c}}, X^{\bar{u},t}_{p,\bar{c}}$) are not essential for the rest of this paper, but will be used in the forthcoming publications in order to formulate Bethe ansatz for $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ as well as establish connections with the results of [5].

4.1 Vertex representations $\rho_{p,\bar{c}}$

Recall the algebra $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ introduced in Theorem 2.7(c). We start by recalling the construction of vertex $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-representations from [15], which generalize the classical Frenkel–Jing construction. Let S_n be the *generalized* Heisenberg algebra generated by $\{H_{i,k}|i \in [n], k \in \mathbb{Z}\backslash\{0\}\}$ and a central element H_0 with the defining relations

$$[H_{i,k}, H_{j,l}] = d^{-km_{i,j}} \frac{[k]_q \cdot [ka_{i,j}]_q}{k} \delta_{k,-l} \cdot H_0.$$

Let S_n^+ be the Lie subalgebra generated by $\{H_{i,k}\}^{k>0}_{i\in[n]} \sqcup \{H_0\}$, and let $\mathbb{C}v_0$ be the S_n^+-representation with $H_{i,k}$ acting trivially and H_0 acting via the identity operator. The induced representation $F_n := \mathrm{Ind}^{S_n}_{S_n^+} \mathbb{C}v_0$ is called the *Fock representation* of S_n.

We denote by $\{\bar{\alpha}_i\}_{i=1}^{n-1}$ the simple roots of $\mathfrak{sl}_n$, by $\{\bar{\Lambda}_i\}_{i=1}^{n-1}$ the fundamental weights of $\mathfrak{sl}_n$, by $\{\bar{h}_i\}_{i=1}^{n-1}$ the simple coroots of $\mathfrak{sl}_n$. Let $\bar{Q} := \bigoplus_{i=1}^{n-1} \mathbb{Z}\bar{\alpha}_i$ be the root lattice of $\mathfrak{sl}_n$, $\bar{P} := \bigoplus_{i=1}^{n-1} \mathbb{Z}\bar{\Lambda}_i = \bigoplus_{i=2}^{n-1} \mathbb{Z}\bar{\alpha}_i \oplus \mathbb{Z}\bar{\Lambda}_{n-1}$ be the weight lattice of $\mathfrak{sl}_n$. We also set

$$\bar{\alpha}_0 := -\sum_{i=1}^{n-1} \bar{\alpha}_i \in \bar{Q}, \quad \bar{\Lambda}_0 := 0 \in \bar{P}, \quad \bar{h}_0 := -\sum_{i=1}^{n-1} \bar{h}_i.$$

Let $\mathbb{C}\{\bar{P}\}$ be the $\mathbb{C}$-algebra generated by $e^{\bar{\alpha}_2}, \ldots, e^{\bar{\alpha}_{n-1}}, e^{\bar{\Lambda}_{n-1}}$ with the defining relations:

$$e^{\bar{\alpha}_i} \cdot e^{\bar{\alpha}_j} = (-1)^{\langle \bar{h}_i, \bar{\alpha}_j\rangle} e^{\bar{\alpha}_j} \cdot e^{\bar{\alpha}_i}, \quad e^{\bar{\alpha}_i} \cdot e^{\bar{\Lambda}_{n-1}} = (-1)^{\delta_{i,n-1}} e^{\bar{\Lambda}_{n-1}} \cdot e^{\bar{\alpha}_i}.$$

For $\alpha = \sum_{i=2}^{n-1} m_i \bar{\alpha}_i + m_n \bar{\Lambda}_{n-1}$, we define $e^{\bar{\alpha}} \in \mathbb{C}\{\bar{P}\}$ via

$$e^{\bar{\alpha}} := (e^{\bar{\alpha}_2})^{m_2} \cdots (e^{\bar{\alpha}_{n-1}})^{m_{n-1}} (e^{\bar{\Lambda}_{n-1}})^{m_n}.$$

Let $\mathbb{C}\{\bar{Q}\}$ be the subalgebra of $\mathbb{C}\{\bar{P}\}$ generated by $\{e^{\bar{\alpha}_i}\}_{i=1}^{n-1}$.

For every $0 \le p \le n-1$, define the space

$$W(p)_n := F_n \otimes \mathbb{C}\{\bar{Q}\} e^{\bar{\Lambda}_p}.$$

Consider the operators $H_{i,l}$, $e^{\bar{\alpha}}$, $\partial_{\bar{\alpha}_i}$, $z^{H_{i,0}}$, d acting on $W(p)_n$, which assign to every element

$$v \otimes e^{\bar{\beta}} = (H_{i_1,-k_1} \cdots H_{i_N,-k_N} v_0) \otimes e^{\sum_{j=1}^{n-1} m_j \bar{\alpha}_j + \bar{\Lambda}_p} \in W(p)_n$$

the following values:

$$\begin{aligned} H_{i,l}(v \otimes e^{\bar{\beta}}) &:= (H_{i,l} v) \otimes e^{\bar{\beta}}, \\ e^{\bar{\alpha}}(v \otimes e^{\bar{\beta}}) &:= v \otimes e^{\bar{\alpha}} e^{\bar{\beta}}, \\ \partial_{\bar{\alpha}_i}(v \otimes e^{\bar{\beta}}) &:= \langle \bar{h}_i, \bar{\beta} \rangle v \otimes e^{\bar{\beta}}, \\ z^{H_{i,0}}(v \otimes e^{\bar{\beta}}) &:= z^{\langle \bar{h}_i, \bar{\beta} \rangle} d^{\frac{1}{2} \sum_{j=1}^{n-1} \langle \bar{h}_i, m_j \bar{\alpha}_j \rangle m_{i,j}} v \otimes e^{\bar{\beta}}, \\ \mathrm{d}(v \otimes e^{\bar{\beta}}) &:= -\left(\sum k_i + ((\bar{\beta}, \bar{\beta}) - (\bar{\Lambda}_p, \bar{\Lambda}_p))/2 \right) v \otimes e^{\bar{\beta}}. \end{aligned}$$

The following result provides a natural structure of an $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-module on $W(p)_n$.

Proposition 4.1 [15, Proposition 3.2.2] *For any $\bar{c} = (c_0, \ldots, c_{n-1}) \in (\mathbb{C}^*)^n$ and $0 \le p \le n-1$, the following formulas define an action of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ on $W(p)_n$ (which does not depend on $\bar{c}$):*

$$\begin{aligned} \rho_{p,\bar{c}}(e_i(z)) &= c_i \cdot \exp\left(\sum_{k>0} \frac{q^{-k/2}}{[k]_q} H_{i,-k} z^k \right) \\ &\quad \cdot \exp\left(-\sum_{k>0} \frac{q^{-k/2}}{[k]_q} H_{i,k} z^{-k} \right) \cdot e^{\bar{\alpha}_i} z^{H_{i,0}+1}, \\ \rho_{p,\bar{c}}(f_i(z)) &= c_i^{-1} \cdot \exp\left(-\sum_{k>0} \frac{q^{k/2}}{[k]_q} H_{i,-k} z^k \right) \\ &\quad \cdot \exp\left(\sum_{k>0} \frac{q^{k/2}}{[k]_q} H_{i,k} z^{-k} \right) \cdot e^{-\bar{\alpha}_i} z^{-H_{i,0}+1}, \\ \rho_{p,\bar{c}}(\psi_i^{\pm}(z)) &= \exp\left(\pm(q - q^{-1}) \sum_{k>0} H_{i,\pm k} z^{\mp k} \right) \cdot q^{\pm \partial_{\bar{\alpha}_i}}, \\ \rho_{p,\bar{c}}(\gamma^{\pm 1/2}) &= q^{\pm 1/2}, \quad \rho_{p,\bar{c}}(q^{\pm d_1}) = q^{\pm \mathrm{d}}. \end{aligned}$$

4.2 Functionals $\phi^0_{p,\bar{c}}$, $\phi^{\bar{u}}_{p,\bar{c}}$, $\phi^{\bar{u},t}_{p,\bar{c}}$ on $\ddot{U}'^{\le}$

In this subsection, we introduce and "explicitly compute" three functionals on $\ddot{U}'^{\le}$.

- *Top matrix coefficient.*

  Consider the functional

$$\phi^0_{p,\bar{c}} : \ddot{U}'^{\le} \longrightarrow \mathbb{C} \ \text{ defined by } \ \phi^0_{p,\bar{c}}(A) := \langle v_0 \otimes e^{\bar{\Lambda}_p} | \rho_{p,\bar{c}}(A) | v_0 \otimes e^{\bar{\Lambda}_p} \rangle.$$

Since $h_{i,j}(v_0 \otimes e^{\bar{\Lambda}_p}) = 0$ for $j > 0$, it remains to compute the values of $\phi^0_{p,\bar{c}}$ evaluated at

$$f_{i_1,j_1} f_{i_2,j_2} \cdots f_{i_m,j_m} \psi^{r_0}_{0,0} \cdots \psi^{r_{n-1}}_{n-1,0} \cdot (\gamma^{1/2})^a (q^{d_1})^b$$

with $a, b \in \mathbb{Z}, \bar{r} := (r_0, \ldots, r_{n-1}) \in \mathbb{Z}^{[n]}$ and $\sum_{s=1}^m \bar{\alpha}_{i_s} = 0 \in \bar{Q}$. The latter condition means that the multiset $\{i_1, \ldots, i_m\}$ contains an equal number of each of the indices $\{0, \ldots, n-1\}$. Due to the defining quadratic relation (T3) of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$, it suffices to compute the series

$$\begin{aligned} &\phi^0_{p,\bar{c};N,\bar{r},a,b}(z_{0,1}, \ldots, z_{n-1,N}) \\ &\quad := \phi^0_{p,\bar{c}} \left(\prod_{j=1}^N (f_0(z_{0,j}) \cdots f_{n-1}(z_{n-1,j})) \cdot \prod_{i \in [n]} \psi^{r_i}_{i,0} \cdot \gamma^{a/2} q^{bd_1} \right). \end{aligned}$$

In this expression, we order the z-variables as follows:

$$z_{0,1}, \ldots, z_{n-1,1}, z_{0,2}, \ldots, z_{n-1,2}, \ldots, z_{0,N}, \ldots, z_{n-1,N}.$$

Normally ordering the product $\prod_{j=1}^N (f_0(z_{0,j}) \cdots f_{n-1}(z_{n-1,j}))$, we get the following result:

Proposition 4.2 *For $n \geq 3$, we have:*

$$\begin{aligned} &\phi^0_{p,\bar{c};N,\bar{r},a,b}(z_{0,1}, \ldots, z_{n-1,N}) = (c_0 \ldots c_{n-1})^{-N} q^{a/2 + r_p - r_0} d^{\frac{N(n-2)}{2}} \cdot \prod_{j=1}^N \frac{z_{0,j}}{z_{p,j}} \\ &\times \frac{\prod_{i \in [n]} \prod_{1 \leq j < j' \leq N} (z_{i,j} - z_{i,j'})(z_{i,j} - q^2 z_{i,j'}) \cdot \prod_{i \in [n]} \prod_{j=1}^N z_{i,j}}{\prod_{i \in [n]} \prod_{1 \leq j \leq j' \leq N} (z_{i,j} - q d z_{i+1,j'}) \cdot \prod_{i \in [n]} \prod_{1 \leq j < j' \leq N} (z_{i,j} - q d^{-1} z_{i-1,j'})}. \end{aligned}$$

- *Top level graded trace.*

Recall the operator d acting diagonally in the natural basis of $W(p)_n$. Clearly all its eigenvalues are in $-\mathbb{Z}_+$. Let $M(p)_n := \mathrm{Ker}(\mathrm{d}) \subset W(p)_n$ be its kernel. The following is obvious:

Lemma 4.3 (a) *The subspace $M(p)_n$ is $U_q(\mathfrak{sl}_n)$-invariant and is isomorphic to the irreducible highest weight $U_q(\mathfrak{sl}_n)$-module $L_q(\bar{\Lambda}_p)$.*
(b) *For any $\bar{\sigma} = \{1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_p \leq n\}$, let $\bar{\Lambda}^{\bar{\sigma}}_p$ be the $\mathfrak{sl}_n$-weight having entries $1 - \frac{p}{n}$ at the places $\{\sigma_i\}_{i=1}^p$ and $-\frac{p}{n}$ elsewhere. Then $\{v_0 \otimes e^{\bar{\Lambda}^{\bar{\sigma}}_p}\}_{\bar{\sigma}}$ form a basis of $M(p)_n$.*

Define the degree operators $\mathrm{d}_1, \ldots, \mathrm{d}_{n-1}$ acting on $W(p)_n$ by

$$\mathrm{d}_r(v \otimes e^{\sum_{j=1}^{n-1} m_j \bar{\alpha}_j + \bar{\Lambda}_p}) = -m_r \cdot v \otimes e^{\sum_{j=1}^{n-1} m_j \bar{\alpha}_j + \bar{\Lambda}_p} \quad \forall\, v \in F_n.$$

For any $\bar{u} = (u_1, \dots, u_{n-1}) \in (\mathbb{C}^*)^{n-1}$, consider the functional

$$\phi^{\bar{u}}_{p,\bar{c}} : \ddot{U}'^{\leq} \longrightarrow \mathbb{C} \ \textit{defined by}$$
$$\phi^{\bar{u}}_{p,\bar{c}}(A) := \sum_{\bar{\sigma}} \left\langle v_0 \otimes e^{\bar{\Lambda}^{\bar{\sigma}}_p} | \rho_{p,\bar{c}}(A) u_1^{\mathrm{d}_1} \cdots u_{n-1}^{\mathrm{d}_{n-1}} | v_0 \otimes e^{\bar{\Lambda}^{\bar{\sigma}}_p} \right\rangle,$$

computing the $\bar{Q}$-*graded* trace of the A-action on the subspace $M(p)_n$ (here $u_i^{\mathrm{d}_i}$ makes sense as d_i acts with integer eigenvalues). Since $h_{i,j}(v_0 \otimes e^{\bar{\Lambda}^{\bar{\sigma}}_p}) = 0$ for $j > 0$, it suffices to compute the generating series

$$\phi^{\bar{u}}_{p,\bar{c};N,\bar{r},a,b}(z_{0,1}, \dots, z_{n-1,N})$$
$$:= \phi^{\bar{u}}_{p,\bar{c}} \left(\prod_{j=1}^{N} (f_0(z_{0,j}) \cdots f_{n-1}(z_{n-1,j})) \cdot \prod_{i\in[n]} \psi_{i,0}^{r_i} \cdot \gamma^{a/2} q^{bd_1} \right).$$

Normally ordering the product $\prod_{j=1}^{N} (f_0(z_{0,j}) \cdots f_{n-1}(z_{n-1,j}))$, we get the following result:

Proposition 4.4 *For $n \geq 3$, we have:*

$$\phi^{\bar{u}}_{p,\bar{c};N,\bar{r},a,b}(z_{0,1}, \dots, z_{n-1,N}) = (c_0 \dots c_{n-1})^{-N} q^{a/2} d^{\frac{N(n-2)}{2}}$$
$$\times \frac{\prod_{i\in[n]} \prod_{1\leq j<j'\leq N} (z_{i,j} - z_{i,j'})(z_{i,j} - q^2 z_{i,j'})}{\prod_{i\in[n]} \prod_{1\leq j\leq j'\leq N} (z_{i,j} - qdz_{i+1,j'}) \cdot \prod_{i\in[n]} \prod_{1\leq j<j'\leq N} (z_{i,j} - qd^{-1} z_{i-1,j'})}$$
$$\times (-1)^p \prod_{j=1}^{p} \frac{1}{u_1 \dots u_{j-1}} \cdot [\mu^p] \left\{ \prod_{i\in[n]} \left(\prod_{j=1}^{N} z_{i+1,j} - \mu u_1 \dots u_i q^{r_{i+1}-r_i} \prod_{j=1}^{N} z_{i,j} \right) \right\},$$

where $[\mu^p]\{\cdots\}$ denotes the coefficient of μ^p in $\{\cdots\}$.

- *Full graded trace.*

Finally, we introduce the most general functional

$$\phi^{\bar{u},t}_{p,\bar{c}} : \ddot{U}'^{\leq} \longrightarrow \mathbb{C}[[t]] \ \textit{defined by} \ \phi^{\bar{u},t}_{p,\bar{c}}(A) := \mathrm{tr}_{W(p)_n}(\rho_{p,\bar{c}}(A) u_1^{\mathrm{d}_1} \dots u_{n-1}^{\mathrm{d}_{n-1}} t^{-\mathrm{d}}),$$

computing the $\bar{Q} \times \mathbb{Z}_+$-*graded trace* of the A-action on the representation $W(p)_n$. Due to the quadratic relations and the $\bar{Q}$-grading, it suffices to compute the following generating series:

$$\phi^{\bar{u},t}_{p,\bar{c};N,\bar{k},\bar{r},a,b}(z_{0,1}, \dots, z_{n-1,N}; w_{0,1}, \dots, w_{0,k_0}, \dots, w_{n-1,1}, \dots, w_{n-1,k_{n-1}})$$
$$:= \phi^{\bar{u},t}_{p,\bar{c}} \left(\prod_{j=1}^{N} (f_0(z_{0,j}) \cdots f_{n-1}(z_{n-1,j})) \cdot \prod_{i\in[n]} \prod_{j=1}^{k_i} \bar{\psi}_i^+(w_{i,j}) \cdot \prod_{i\in[n]} \psi_{i,0}^{r_i} \cdot \gamma^{a/2} q^{bd_1} \right).$$

In what follows, $(z;t)_\infty$ is defined by $(z;t)_\infty := \prod_{a=0}^{\infty}(1-t^a z)$.

Theorem 4.5 *For $n \geq 3$, we have:*

$$\phi^{\bar{u},t}_{p,\bar{c};N,\bar{k},\bar{r},a,b}(z_{0,1},\ldots,z_{n-1,N};w_{0,1},\ldots,w_{n-1,k_{n-1}}) = (c_0\ldots c_{n-1})^{-N} q^{a/2} d^{\frac{N(n-2)}{2}} \tag{$\diamond$}$$

$$\times \frac{\prod_{i\in[n]}\prod_{1\leq j<j'\leq N}(z_{i,j}-z_{i,j'})(z_{i,j}-q^2 z_{i,j'})\cdot\prod_{i\in[n]}\prod_{j=1}^{N} z_{i,j}}{\prod_{i\in[n]}\prod_{1\leq j\leq j'\leq N}(z_{i,j}-qdz_{i+1,j'})\cdot\prod_{i\in[n]}\prod_{1\leq j<j'\leq N}(z_{i,j}-qd^{-1}z_{i-1,j'})}$$

$$\times q^{r_p-r_0}\cdot\prod_{j=1}^{N}\frac{z_{0,j}}{z_{p,j}}\cdot\theta(\vec{y};\bar{\Omega})$$

$$\times \frac{1}{(T;T)^n_\infty}\cdot\prod_{i\in[n]}\prod_{a,b=1}^{N}\frac{(T\cdot\frac{z_{i,a}}{z_{i,b}};T)_\infty\cdot(Tq^2\frac{z_{i,a}}{z_{i,b}};T)_\infty}{(Tqd\frac{z_{i+1,a}}{z_{i,b}};T)_\infty\cdot(Tqd^{-1}\frac{z_{i-1,a}}{z_{i,b}};T)_\infty}$$

$$\times\prod_{i\in[n]}\prod_{a=1}^{N}\prod_{b=1}^{k_i}\frac{(Tq^2\frac{q^{1/2}z_{i,a}}{w_{i,b}};T)_\infty\cdot(Tq^{-1}d\frac{q^{1/2}z_{i+1,a}}{w_{i,b}};T)_\infty\cdot(Tq^{-1}d^{-1}\frac{q^{1/2}z_{i-1,a}}{w_{i,b}};T)_\infty}{(Tq^{-2}\frac{q^{1/2}z_{i,a}}{w_{i,b}};T)_\infty\cdot(Tqd\frac{q^{1/2}z_{i+1,a}}{w_{i,b}};T)_\infty\cdot(Tqd^{-1}\frac{q^{1/2}z_{i-1,a}}{w_{i,b}};T)_\infty},$$

where $T := \frac{t}{q^b}$ and $\theta(\vec{y},\bar{\Omega}) := \sum_{\vec{n}\in\mathbb{Z}^{n-1}}\exp(2\pi\sqrt{-1}(\frac{1}{2}\vec{n}\bar{\Omega}\vec{n}'+\vec{n}\vec{y}'))$ is the classical Riemann theta function with $\bar{\Omega} = \frac{1}{2\pi\sqrt{-1}}\cdot(a_{i,j}\ln(T))_{i,j=1}^{n-1}$ and

$$\vec{y} = (y_1,\ldots,y_{n-1}) \text{ with } y_i = \frac{1}{2\pi\sqrt{-1}}\ln\left(u_i^{-1}T^{\delta_{p,i}}q^{2r_i-r_{i-1}-r_{i+1}}\prod_{j=1}^{N}\frac{z_{i-1,j}z_{i+1,j}}{z_{i,j}^2}\right).$$

We start with the following two auxiliary results:

Lemma 4.6 *The matrix $\left(\frac{d^{-km_{i,j}}[k]_q[ka_{i,j}]_q}{k}\right)_{i\in[n]}^{j\in[n]}$ is nondegenerate if and only if $q^{2k}, q^k d^{\pm k} \neq 1$.*

Therefore if $q^2, dq, d^{-1}q$ are not roots of unity, we can choose a new basis $\{\widetilde{H}_{i,-k}\}_{i\in[n]}$ of the space $\mathrm{span}_{\mathbb{C}}\{H_{0,-k},\ldots,H_{n-1,-k}\}$, such that $[H_{i,k},\widetilde{H}_{j,-l}] = \delta_{i,j}\delta_{k,l}H_0$ for any $i,j\in[n]$, $k,l\in\mathbb{N}$. In particular, the elements $\{H_{i,k},\widetilde{H}_{i,-k},H_0\}_{k>0}$ form a Heisenberg Lie algebra $\mathfrak{h}_i$ for any $i\in[n]$, and $\mathfrak{h}_i$ commutes with $\mathfrak{h}_j$ for any $i\neq j\in[n]$.

Lemma 4.7 *Let $\mathfrak{a}$ be a Heisenberg Lie algebra with the basis $\{a_k\}_{k\in\mathbb{Z}}$ and the commutator relation $[a_k,a_l] = \delta_{k,-l}\lambda_k a_0$. Consider the Fock $\mathfrak{a}$-representation $F := \mathrm{Ind}_{\mathfrak{a}_+}^{\mathfrak{a}}\mathbb{C}v_0$ with the central charge $a_0 = 1$ and the degree operator $d\in\mathrm{End}(F)$ satisfying $[d,a_k] = ka_k$ and $d(v_0) = 0$. Then:*

$$\mathrm{tr}_F\left\{\exp\left(\sum_{j=1}^{\infty}x_j a_{-j}\right)\cdot\exp\left(\sum_{j=1}^{\infty}y_j a_j\right)\cdot t^{-d}\right\} = \frac{1}{(t;t)_\infty}\cdot\exp\left(\sum_{j=1}^{\infty}\frac{x_j y_j\lambda_j t^j}{1-t^j}\right)\ \forall\, x_j,y_j\in\mathbb{C}.$$

Proof Applying the formula $\langle a_{-j}^l v_0 | a_{-j}^k a_j^k | a_{-j}^l v_0 \rangle = l(l-1)\cdots(l-k+1)\lambda_j^k$, we get

$$\begin{aligned}
&\mathrm{tr}_F \left\{ \exp\left(\sum_{j=1}^{\infty} x_j a_{-j}\right) \exp\left(\sum_{j=1}^{\infty} y_j a_j\right) \cdot t^{-d} \right\} \\
&= \sum_{k_1,k_2,\ldots \geq 0} \mathrm{tr}_F \left(\prod_{j=1}^{\infty} \frac{(x_j y_j)^{k_j}}{(k_j!)^2} a_{-j}^{k_j} a_j^{k_j} \cdot t^{-d} \right) \\
&= \prod_{j=1}^{\infty} \left\{ \sum_{k_j=0}^{\infty} \sum_{l_j=k_j}^{\infty} \frac{(x_j y_j)^{k_j}}{(k_j!)^2} \cdot \frac{l_j! \cdot \lambda_j^{k_j}}{(l_j-k_j)!} \cdot t^{j l_j} \right\} \\
&= \prod_{j=1}^{\infty} \left\{ \sum_{k_j=0}^{\infty} \frac{(x_j y_j \lambda_j t^j)^{k_j}}{k_j!} \cdot \frac{1}{(1-t^j)^{k_j+1}} \right\}.
\end{aligned}$$

The result follows. □

Proof of Theorem 4.5 Reordering the factors of

$$\prod_{j=1}^{N} (f_0(z_{0,j}) \cdots f_{n-1}(z_{n-1,j})) \cdot \prod_{i\in[n]} \prod_{j=1}^{k_i} \bar{\psi}_i^{+}(w_{i,j}) \cdot \prod_{i\in[n]} \psi_{i,0}^{r_i}$$

in the normal order, we gain the product of factors from the first two lines of (◇). The $\bar{Q} \times \mathbb{Z}_{+}$-graded trace of the normally ordered product splits as $\mathrm{tr}_1 \cdot \mathrm{tr}_2$, where

$$\mathrm{tr}_1 = \mathrm{tr}_{\mathbb{C}\{\bar{Q}\}e^{\bar{\Lambda}_p}} \left(q^{\sum_{i\in[n]} r_i \partial_{\bar{\alpha}_i}} \cdot \prod_{i\in[n]} \prod_{j=1}^{N} z_{i,j}^{-H_{i,0}} \cdot \prod_{i=1}^{n-1} u_i^{d_i} \cdot (t/q^b)^{\mathrm{d}^{(2)}} \right),$$

$$\mathrm{tr}_2 = \mathrm{tr}_{F_n} \left(\exp\left(\sum_{i\in[n]} \sum_{k>0} u_{i,k} H_{i,-k} \right) \cdot \exp\left(\sum_{i\in[n]} \sum_{k>0} (v_{i,k}^{(1)} + v_{i,k}^{(2)}) H_{i,k} \right) \cdot (t/q^b)^{\mathrm{d}^{(1)}} \right)$$

with

$$u_{i,k} := \frac{-q^{k/2}}{[k]_q} \sum_{j=1}^{N} z_{i,j}^{k}, \quad v_{i,k}^{(1)} := \frac{q^{k/2}}{[k]_q} \sum_{j=1}^{N} z_{i,j}^{-k}, \quad v_{i,k}^{(2)} := (q-q^{-1}) \sum_{j=1}^{k_i} w_{i,j}^{-k}$$

and the operators $\mathrm{d}^{(1)} \in \mathrm{End}(F_n)$, $\mathrm{d}^{(2)} \in \mathrm{End}(\mathbb{C}\{\bar{Q}\}e^{\bar{\Lambda}_p})$ defined by

$$\mathrm{d}^{(1)}(H_{i_1,-k_1} \cdots H_{i_l,-k_l} v_0) = \sum_{i=1}^{l} k_i \cdot H_{i_1,-k_1} \cdots H_{i_l,-k_l} v_0,$$

$$\mathrm{d}^{(2)}(e^{\bar{\beta}}) = \frac{(\bar{\beta},\bar{\beta}) - (\bar{\Lambda}_p,\bar{\Lambda}_p)}{2} \cdot e^{\bar{\beta}}.$$

The computation of tr_1 is straightforward, and we get exactly the expression from the third line of ($\diamond$). To evaluate tr_2, we rewrite $\sum_{i\in[n]}\sum_{k>0} u_{i,k}H_{i,-k} = \sum_{i\in[n]}\sum_{k>0}\widetilde{u}_{i,k}\widetilde{H}_{i,-k}$ with $\widetilde{H}_{i,-k}$ defined right after Lemma 4.6 and $\widetilde{u}_{i,k} = \sum_{i'\in[n]} d^{-km_{i,i'}}\frac{[k]_q[ka_{i,i'}]_q}{k}u_{i',k}$. The commutativity of $\mathfrak{h}_i$ and $\mathfrak{h}_j$ for $i \neq j$ allows us to rewrite tr_2 as a product of the corresponding traces over the $\mathfrak{h}_i$-Fock modules. Applying Lemma 4.7, we see (after routine computations) that tr_2 is equal to the product of the factors from the last two lines in ($\diamond$). □

4.3 Functionals via pairing

Recall the Hopf algebra pairing $\varphi' : \ddot{U}'^{\geq} \times \ddot{U}'^{\leq} \to \mathbb{C}$ from Theorem 2.7. As φ' is nondegenerate, there exist unique elements $X^0_{p,\bar{c}}, X^{\bar{u}}_{p,\bar{c}} \in \ddot{U}'^{\geq,\wedge}$ and $X^{\bar{u},t}_{p,\bar{c}} \in \ddot{U}'^{\geq,\wedge}[[t]]$ such that

$$\phi^0_{p,\bar{c}}(X) = \varphi'(X^0_{p,\bar{c}}, X),\ \phi^{\bar{u}}_{p,\bar{c}}(X)=\varphi'(X^{\bar{u}}_{p,\bar{c}}, X), \phi^{\bar{u},t}_{p,\bar{c}}(X) = \varphi'(X^{\bar{u},t}_{p,\bar{c}}, X)\,\forall\, X \in \ddot{U}'^{\leq}.$$

The goal of this section is to find these elements explicitly.

We will actually compute these elements in the shuffle presentation. In order to do this, we first extend the isomorphism Ψ from Theorem 2.13 to the isomorphism

$$\Psi^{\geq} : \ddot{U}'^{\geq} \xrightarrow{\sim} S^{\geq}.$$

Here $S^{\geq}$ is generated by S and the formal generators $\psi_{i,k}(k<0)$, $\psi^{\pm 1}_{i,0}$, $\gamma^{\pm 1/2}$, $q^{\pm d_1}$ with the defining relations compatible with those for $\ddot{U}'^{\geq}$. In particular, for $F \in S_{\bar{k},d}$ we have

$$q^{d_1}Fq^{-d_1} = q^{-d}\cdot F.$$

We define $\Gamma^0_{p,\bar{c}}, \Gamma^{\bar{u}}_{p,\bar{c}}, \Gamma^{\bar{u},t}_{p,\bar{c}}$ as the images of $X^0_{p,\bar{c}}, X^{\bar{u}}_{p,\bar{c}}, X^{\bar{u},t}_{p,\bar{c}}$ under the isomorphism $\Psi^{\geq}$, respectively. Now we are ready to state the main result of this section:

Theorem 4.8 *We have the following formulas:*

(a) $\Gamma^0_{p,\bar{c}} = \sum_{N=0}^{\infty}(c_0\dots c_{n-1})^{-N}\cdot\Gamma^0_{p;N}\cdot q^{\bar{\Lambda}_p}q^{-d_1}$ *with* $\Gamma^0_{p;N} \in S_{N\delta}$ *given by*

$$\Gamma^0_{p;N} = (1-q^{-2})^{nN}(-q^n d^{-n/2})^{N^2}\cdot\prod_{j=1}^{N}\frac{x_{0,j}}{x_{p,j}}$$
$$\times\frac{\prod_{i\in[n]}\prod_{j\neq j'}(x_{i,j}-q^{-2}x_{i,j'})\cdot\prod_{i\in[n]}\prod_{j=1}^{N}x_{i,j}}{\prod_{i\in[n]}\prod_{j,j'}(x_{i,j}-x_{i+1,j'})}.$$

(b) $\Gamma^{\bar{u}}_{p,\bar{c}} = \sum_{N\geq 0}(c_0\dots c_{n-1})^{-N}\cdot\Gamma^{\bar{u}}_{p;N}\cdot q^{-d_1}$ *with* $\Gamma^{\bar{u}}_{p;N}\in S^{\geq}_{N\delta}$ *given by*

$$\Gamma^{\bar{u}}_{p;N} = (1-q^{-2})^{nN}(-q^n d^{-n/2})^{N^2}\prod_{j=1}^{p}\frac{1}{u_1\dots u_{j-1}}\cdot\frac{\prod_{i\in[n]}\prod_{j\neq j'}(x_{i,j}-q^{-2}x_{i,j'})}{\prod_{i\in[n]}\prod_{j,j'}(x_{i,j}-x_{i+1,j'})}$$
$$\times(-1)^p[\mu^p]\left\{\prod_{i\in[n]}\left(\prod_{j=1}^{N}x_{i+1,j}-\mu u_1\dots u_i\prod_{j=1}^{N}x_{i,j}\cdot q^{\bar{\Lambda}_{i+1}-\bar{\Lambda}_i}\right)\right\},$$

where in the last product we take all $x_{i,j}$ *to the left and all* $q^{\bar{\Lambda}_{i+1}-\bar{\Lambda}_i}$ *to the right.*

(c) $\Gamma^{\bar{u},t}_{p,\bar{c}} = \sum_{N\geq 0}(c_0\dots c_{n-1})^{-N}\cdot\Gamma^{\bar{u},t}_{p;N}\cdot q^{\bar{\Lambda}_p}q^{-d_1}$ *with* $\Gamma^{\bar{u},t}_{p;N}\in S^{\geq}_{N\delta}$ *given by*

$$\Gamma^{\bar{u},t}_{p;N} = (1-q^{-2})^{nN}(-q^n d^{-n/2})^{N^2}$$
$$\times\frac{\prod_{i\in[n]}\prod_{j\neq j'}(x_{i,j}-q^{-2}x_{i,j'})\cdot\prod_{i\in[n]}\prod_{j=1}^{N}x_{i,j}}{\prod_{i\in[n]}\prod_{j,j'}(x_{i,j}-x_{i+1,j'})}\cdot\prod_{j=1}^{N}\frac{x_{0,j}}{x_{p,j}}\cdot\theta(\vec{x},\widetilde{\Omega})$$
$$\times\frac{1}{(\bar{t};\bar{t})^n_\infty}\cdot\prod_{i\in[n]}\prod_{a,b=1}^{N}\frac{(\bar{t}\frac{x_{i,a}}{x_{i,b}};\bar{t})_\infty\cdot(\bar{t}q^2\frac{x_{i,a}}{x_{i,b}};\bar{t})_\infty}{(\bar{t}qd\frac{x_{i+1,a}}{x_{i,b}};\bar{t})_\infty\cdot(\bar{t}qd^{-1}\frac{x_{i-1,a}}{x_{i,b}};\bar{t})_\infty}$$
$$\times\prod_{k>0}\prod_{i\in[n]}\prod_{a=1}^{N}\bar{\psi}^-_i(\bar{t}^k q^{1/2}x_{i,a}),$$

where $\bar{t}=t\gamma,\ \widetilde{\Omega}=\frac{1}{2\pi\sqrt{-1}}\cdot(a_{i,j}\ln(\bar{t}))^{n-1}_{i,j=1}$, *and*

$$\vec{x}=(x_1,\dots,x_{n-1})\text{ with }x_i=\frac{1}{2\pi\sqrt{-1}}\ln\left(u_i^{-1}\bar{t}^{\delta_{p,i}}\psi_{i,0}\prod_{j=1}^{N}\frac{x_{i-1,j}x_{i+1,j}}{x^2_{i,j}}\right).$$

In the above products, we take all $x_{i,j}$ *to the left and all* $\psi_{i,j}$ *to the right.*

The proof of this theorem follows by combining Proposition 4.2, Proposition 4.4 and Theorem 4.5 with the following technical lemma:

Lemma 4.9 *(a) For any elements* $a\in\ddot{U}^+,\ a'\in\ddot{U}^{\geq}\cap\ddot{U}^0,\ b\in\ddot{U}^-,\ b'\in\ddot{U}^{\leq}\cap\ddot{U}^0$, *we have*

$$\varphi(aa',bb')=\varphi(a,b)\cdot\varphi(a',b').$$

(b) For any $k_i,k'_i\in\mathbb{Z}_+$ *and* $A,B,C,A',B',C',a_i,b_i\in\mathbb{Z}$, *we have*

$$\varphi\left(\prod_{i\in[n]}\prod_{a=1}^{k_i}\bar{\psi}^-_i(z_{i,a})\prod_{i\in[n]}\psi^{a_i}_{i,0}\gamma^{A/2}q^{Bd_1}q^{Cd_2},\ \prod_{j\in[n]}\prod_{b=1}^{k'_j}\bar{\psi}^+_j(w_{j,b})\prod_{j\in[n]}\psi^{a'_j}_{j,0}\gamma^{A'/2}q^{B'd_1}q^{C'd_2}\right)$$

$$= q^{-\frac{1}{2}A'B-\frac{1}{2}AB'+C'\sum a_i+C\sum a'_i+\sum_{i,j} a_i a'_j a_{i,j}} \cdot \prod_{i\in[n]}^{j\in[n]} \prod_{a=1}^{k_i} \prod_{b=1}^{k'_j} \frac{w_{j,b} - q^{a_{i,j}} d^{m_{i,j}} z_{i,a}}{w_{j,b} - q^{-a_{i,j}} d^{m_{i,j}} z_{i,a}}.$$

(c) For $\bar{r} = (r_0, \ldots, r_{n-1}), \bar{s} = (s_0, \ldots, s_{n-1}) \in \mathbb{Z}_+^{[n]}$ and elements $X \in \ddot{U}^+, Y \in \ddot{U}^-$ of the form

$$X = e_{0,a_1^0} \cdots e_{0,a_{r_0}^0} \cdots e_{n-1,a_1^{n-1}} \cdots e_{n-1,a_{r_{n-1}}^{n-1}},$$
$$Y = f_{0,b_1^0} \cdots f_{0,b_{s_0}^0} \cdots f_{n-1,b_1^{n-1}} \cdots f_{n-1,b_{s_{n-1}}^{n-1}},$$

the pairing $\varphi(X, Y)$ is expressed by an integral formula similar to [14, Proposition 3.10]:

$$\varphi(X,Y) = \delta_{\bar{r},\bar{s}}$$
$$\times \int \frac{(q-q^{-1})^{-\sum r_i} u_{0,1}^{b_1^0} \ldots u_{n-1,s_{n-1}}^{b_{s_{n-1}}^{n-1}} \Psi(X)(u_{0,1},\ldots,u_{n-1,r_{n-1}})}{\prod_i \prod_{j<j'} \omega_{i,i}(u_{i,j}/u_{i,j'}) \cdot \prod_{i<i'} \prod_{j,j'} \omega_{i,i'}(u_{i,j}/u_{i',j'})} \prod_{i\in[n]} \prod_{j=1}^{s_i} \frac{du_{i,j}}{2\pi\sqrt{-1}u_{i,j}}.$$

4.4 Bethe incarnation of $\mathcal{A}(\bar{s})$

Recalling the notion of a transfer matrix from Sect. 2.5, it is easy to see that

$$X_{p,\bar{c}}^{\bar{u},t} = T_{\rho_{p,\bar{c}}}\left(u_1^{-\bar{\Lambda}_1} \ldots u_{n-1}^{-\bar{\Lambda}_{n-1}} t^{-d_1}\right) \cdot \prod_{j=1}^{n-1} u_j^{\langle \bar{\Lambda}_j, \bar{\Lambda}_p \rangle},$$

which provides a more elegant definition of $X_{p,\bar{c}}^{\bar{u},t}$. Moreover, the elements $X_{p,\bar{c}}^{\bar{u}}$ can be thought of as certain *truncations* of $X_{p,\bar{c}}^{\bar{u},t}$ obtained by setting $t \to 0$, while $X_{p,\bar{c}}^0$ are obtained by setting further $u_1, \ldots, u_{n-1} \to 0$.

The commutativity of the Bethe subalgebras implies the commutativity of $\{\Gamma_{p,\bar{c}}^{\bar{u},t} | p, \bar{c}\}$ and hence of $\{\Gamma_{p;N}^{\bar{u},t}\}_{0\le p\le n-1}^{N\ge 1}$. As a result, we get the commutativity of the families $\{\Gamma_{p;N}^0\}_{0\le p\le n-1}^{N\ge 1}$ and $\{\Gamma_{p;N}^{\bar{u}}\}_{0\le p\le n-1}^{N\ge 1}$. Due to Theorem 4.8(b), the elements $\Gamma_{p;N}^{\bar{u}}$ have the same form as the generators of the subalgebra $\mathcal{A}(s_0, \ldots, s_{n-1})$ from Sect. 3 with $s_i \in \mathbb{C}^* \cdot e^{\bar{P}}$ given by

$$s_i := u_i \cdot q^{\bar{\Lambda}_{i+1} - 2\bar{\Lambda}_i + \bar{\Lambda}_{i-1}} \quad \text{for all } i \in [n], \quad \text{where } u_0 := 1/(u_1 \ldots u_{n-1}).$$

Since e^h ($h \in \bar{P}$) commute with $\oplus_k S_{k\delta}$, we see that those $\{s_i\}$ can be treated as formal parameters with $s_0 \ldots s_{n-1} = 1$ and $\{s_i\}$ being generic for any choice of $\{u_i\}$.

Finally, let us notice that while $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ contained the horizontal copy of $U_q(\widehat{\mathfrak{gl}}_n)$, the algebra $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ contains a horizontal copy of $U_q(L\mathfrak{gl}_n)$ (that is no $q^{\pm d_2}$ and with trivial central charge $c' = 0$). The subspace $M(p)_n$ is $U_q(L\mathfrak{gl}_n)$-invariant and is just the pth fundamental representation. By standard results, $U_q(L\mathfrak{gl}_n)$ admits a double construction similar to the one for $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$. Combining all the previous discussions with the construction of the universal R-matrices for $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ and $U_q(L\mathfrak{gl}_n)$, we get the following result:

Theorem 4.10 *The Bethe subalgebra of $U_q(L\mathfrak{gl}_n)$, corresponding to the group-like element $x = u_1^{-\bar{\Lambda}_1}\dots u_{n-1}^{-\bar{\Lambda}_{n-1}}$ and the category of finite-dimensional $U_q(L\mathfrak{gl}_n)$-representations, can be identified with $\mathcal{A}(\{u_i\cdot q^{\bar{\Lambda}_{i+1}-2\bar{\Lambda}_i+\bar{\Lambda}_{i-1}}\}_{i\in[n]})$, where $u_0 := 1/(u_1\dots u_{n-1})$.*

Remark 4.11 (a) The commutativity of $\{\Gamma^0_{p;N}\}^{N\geq 1}_{0\leq p\leq n-1}$ implies that the family

$$\left\{\prod_{j=1}^N \frac{x_{0,j}}{x_{p,j}}\cdot\frac{\prod_{i\in[n]}\prod_{j\neq j'}(x_{i,j}-q^{-2}x_{i,j'})\cdot\prod_{i\in[n]}\prod_{j=1}^N x_{i,j}}{\prod_{i\in[n]}\prod_{j,j'}(x_{i,j}-x_{i+1,j'})}\right\}^{N\geq 1}_{0\leq p\leq n-1}$$

of elements from S is commutative. It is easy to see that the subalgebra they generate is the limit algebra of $\mathcal{A}(s_0,s_1,\dots,s_{n-1})$ as $s_1,\dots,s_{n-1}\to 0,\ s_0 = 1/(s_1\dots s_{n-1}),$ and $\{s_i\}$ stay generic.

(b) The commutative algebras generated by $\{\Gamma^{\bar{u},t}_{p;N}\}^{N\geq 1}_{0\leq p\leq n-1}$ can be viewed as one-parameter deformations of the algebras $\mathcal{A}(\bar{s})$. They play a crucial role in the Bethe ansatz for $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

5 Generalizations to $n = 1$ and $n = 2$

It turns out that all the previous results of this paper can be actually generalized to the $n = 1, 2$ cases. The goal of this last section is to explain the required slight modifications.

5.1 $n = 1$ case

The quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{gl}_1)$ has been extensively studied in the last few years. Roughly speaking, one just needs to modify the quadratic relations from Sect. 2.1 by replacing

$$g_{a_{i,j}}(t)\rightsquigarrow\frac{(q_1t-1)(q_2t-1)(q_3t-1)}{(t-q_1)(t-q_2)(t-q_3)},\ \text{where } q_1 := q^2,\ q_2 := q^{-1}d,\ q_3 := q^{-1}d^{-1},$$

and by replacing the Serre relations (T7.1, T7.2) by

$$\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\frac{z_2}{z_3}\cdot[e(z_1),[e(z_2),e(z_3)]] = 0,$$

$$\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\frac{z_2}{z_3}\cdot[f(z_1),[f(z_2),f(z_3)]] = 0.$$

Analogously to the $n \geq 3$ case, the map $e_i \mapsto x^i$ extends to the isomorphism

$$\ddot{U}_{q,d}(\mathfrak{gl}_1)^+ \xrightarrow{\sim} S^{\mathrm{sm}}.$$

The results of Sect. 3 recover the same commutative algebra $\mathcal{A}^{\mathrm{sm}}$ we started from. On the other hand, we can apply the constructions of Sect. 4 to the *Fock* $\ddot{U}'_{q,d}(\mathfrak{gl}_1)$-representations $\{F_c\}_{c\in\mathbb{C}^*}$ (see [4, Proposition A.6]). As a result, we will get:

∘ The elements Γ_c^0 (corresponding to the top matrix coefficient functional ϕ_c^0) are given by

$$\Gamma_c^0 = \sum_{N=0}^{\infty} c^{-N} q^{-N(N-1)} \cdot K_N(x_1, \ldots, x_N) \cdot q^{-d_1}.$$

∘ The elements Γ_c^t (corresponding to the full graded trace functional ϕ_c^t) are given by

$$\Gamma_c^t = \sum_{N=0}^{\infty} \frac{c^{-N} q^{-N(N-1)}}{(\bar{t};\bar{t})_\infty} \cdot K_N \cdot \prod_{a,b=1}^{N} \frac{(\bar{t}\frac{x_a}{x_b};\bar{t})_\infty \cdot (\bar{t}q^2\frac{x_a}{x_b};\bar{t})_\infty}{(\bar{t}qd^{-1}\frac{x_a}{x_b};\bar{t})_\infty \cdot (\bar{t}qd\frac{x_a}{x_b};\bar{t})_\infty}$$
$$\times \prod_{k>0}\prod_{a=1}^{N} \bar{\psi}^-(\bar{t}^k q^{1/2} x_a) \cdot q^{-d_1}.$$

5.2 $n = 2$ case

For $n = 2$, we need first to redefine both the quantum toroidal and the shuffle algebras.

∘ *Quantum toroidal algebra of* $\mathfrak{sl}_2$.

One needs to slightly modify the defining relations (T0.1–T7.2) of $\ddot{U}_{q,d}(\mathfrak{sl}_2)$ (see [2]). The function $g_{a_{i,j}}(z)$ from the relations (T1, T2, T3, T5, T6) should be changed as follows:

$$g_{a_{i,i}}(z) \rightsquigarrow \frac{q^2 z - 1}{z - q^2}, \quad g_{a_{i,i+1}}(z) \rightsquigarrow \frac{(dz-q)(d^{-1}z-q)}{(qz-d)(qz-d^{-1})},$$

while the cubic Serre relations (T7.1, T7.2) should be replaced with quartic Serre relations

$$\underset{z_1,z_2,z_3}{\mathrm{Sym}}\ [e_i(z_1),[e_i(z_2),[e_i(z_3),e_{i+1}(w)]_{q^2}]]_{q^{-2}} = 0,$$
$$\underset{z_1,z_2,z_3}{\mathrm{Sym}}\ [f_i(z_1),[f_i(z_2),[f_i(z_3),f_{i+1}(w)]_{q^2}]]_{q^{-2}} = 0.$$

∘ *Big shuffle algebra of type* $A_1^{(1)}$.

One needs to modify the matrix Ω used to define the $\star$ product as follows:

$$\omega_{i,i}(z) = \frac{z - q^{-2}}{z - 1}, \quad \omega_{i,i+1}(z) = \frac{(z-qd)(z-qd^{-1})}{(z-1)^2}.$$

∘ *Vertex representations* $\rho_{p,\bar{c}}$.

Finally, we need to slightly modify the formulas of $\rho_{p,\bar{c}}$ from Proposition 4.1:

(i) We redefine the commutator relations of the Heisenberg algebra S_n as follows:

$$[H_{i,k}, H_{i,l}] = \frac{[k]_q \cdot [2k]_q}{k} \delta_{k,-l} \cdot H_0,$$
$$[H_{i,k}, H_{i+1,l}] = -(d^k + d^{-k}) \frac{[k]_q \cdot [k]_q}{k} \delta_{k,-l} \cdot H_0.$$

(ii) We also redefine the operator $z^{H_{i,0}}$ via

$$z^{H_{i,0}}(v \otimes e^{\bar{\beta}}) := z^{\langle \bar{h}_i, \bar{\beta} \rangle} v \otimes e^{\bar{\beta}}.$$

Once the above modifications are made, all the results from Sects. 3 and 4 still hold.

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