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The affine Yangian of $\mathfrak{gl}_1$ revisited

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ABSTRACT

The affine Yangian of $\mathfrak{gl}_1$ has recently appeared simultaneously in the work of Maulik–Okounkov [11] and Schiffmann–Vasserot [20] in connection with the Alday–Gaiotto–Tachikawa conjecture. While the presentation from [11] is purely geometric, the algebraic presentation in [20] is quite involved. In this article, we provide a simple loop realization of this algebra which can be viewed as an "additivization" of the quantum toroidal algebra of $\mathfrak{gl}_1$ in the same way as the Yangian $Y_h(\mathfrak{g})$ is an "additivization" of the quantum loop algebra $U_q(L\mathfrak{g})$ for a simple Lie algebra $\mathfrak{g}$. We also explain the similarity between the representation theories of the affine Yangian and the quantum toroidal algebras of $\mathfrak{gl}_1$ by generalizing the main result of [10] to the current settings.

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0. Introduction

The goal of this note is twofold. First, we provide an explicit loop type presentation of the affine Yangian of $\mathfrak{gl}_1$, which appeared simultaneously in [11,20], and discuss its representation theory. We also explain its relation to the quantum toroidal algebra of $\mathfrak{gl}_1$.

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The first half of the paper has an expository role. However, the author feels that it is worth recalling the geometric and representation theoretic aspects of the quantum toroidal algebra of $\mathfrak{gl}_1$, since in many cases they admit parallel counterparts for the case of the affine Yangian of $\mathfrak{gl}_1$.

The other source of motivation comes from the similarity between these theories for the $\mathfrak{gl}_1$ and $\mathfrak{sl}_n$ cases. In particular, most of the results from this paper admit the corresponding analogues for the quantum toroidal and affine Yangian algebras of $\mathfrak{sl}_n$. As such analogues are generally much more technical, they will be addressed separately in the forthcoming publications. Let us mention some new results of the current paper which admit $\mathfrak{sl}_n$-generalizations:

- We treat the $\mathfrak{sl}_n$-generalizations of Theorems 5.4, 5.5, 5.8, 5.9 in [22].
- We treat the $\mathfrak{sl}_n$-generalization of Proposition 8.2 in [21].
- We treat the $\mathfrak{sl}_n$-generalization of Proposition 8.3 in [9].
- We treat the $\mathfrak{sl}_n$-generalizations of Proposition 4.4, Theorem 6.6, and Corollary 6.7 in [23].
- We construct homomorphisms $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1) \to \widehat{\ddot{Y}}'_{h_1,h_2,h_3}(\mathfrak{sl}_n)$ generalizing Υ in [23].

This paper is organized as follows:

• In Section 1, we recall the definition and discuss some basic properties of the quantum toroidal algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. A similar class of algebras was first considered in [4]. This algebra was rediscovered later by different groups of people in [2,8,12] (see Remark 1.1).

We also introduce the key object of this paper, the affine Yangian $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$. This algebra was considered by the author and B. Feigin in an unpublished work, where it was viewed as a natural "additivization" of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ in the same way as $Y_h(\mathfrak{g})$ is an "additivization" of $U_q(L\mathfrak{g})$. The only non-trivial relations (Y4′) and (Y5′) were determined from the requirement that this algebra should naturally act on the sum of equivariant cohomology groups of the Hilbert schemes of points on $\mathbb{A}^2$. As pointed out by the referee, this algebra also appeared in [1], where the authors proved that it is isomorphic to the algebra $\mathbf{SH}^{\mathbf{c}}$ from [20].

• In Section 2, we recall the $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-action on the direct sum of equivariant K-groups of the Hilbert scheme of points on $\mathbb{A}^2$, discovered simultaneously in [8,19]. We also formulate a similar result about the $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action on an analogous sum of equivariant cohomologies, discovered by Schiffmann–Vasserot in [20] and Feigin–Tsymbaliuk (unpublished). We conclude that section by sketching our proof of this result, which is completely parallel to [8] with only one modification (Lemma 2.4) required.

• In Section 3, we present the generalizations of the results from Section 2 to the Gieseker moduli spaces $M(r,n)$ (also known as the instanton moduli spaces). In the $r=1$ case, we have $M(1,n) \simeq (\mathbb{A}^2)^{[n]}$ and we recover the actions from Section 2.

• In Section 4, we recall some series of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations discovered in [5,6]. Those are constructed from the simplest family of *vector representations* $V(u)$ by using the formal coproduct on $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. The simplest example of representations from

the category $\mathcal{O}$ are the *Fock representations* $F(u)$, whose basis is labeled by all Young diagrams.

We introduce the analogous *vector representations* ${}^aV(u)$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$, as well as the Fock representations ${}^aF(u)$, which are of particular interest for us. We also prove that the representations of geometric origin from Section 3 are the tensor products of the aforementioned Fock representations. We conclude that section by introducing appropriate categories $\mathcal{O}$ and generalizing the standard result of [3] to the current settings.

• In Section 5, we describe the limits of the appropriately renormalized algebras $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ as $q_3 \to 1$ and $h_3 \to 0$, respectively. The resulting limit algebras are closely related to central extensions of the algebras of *difference operators* on $\mathbb{C}^*$ and $\mathbb{C}$.

• In Section 6, we construct a homomorphism $\Upsilon : \ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1) \to \widehat{\ddot{Y}}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$. In the limit $h_3 \to 0$, this homomorphism is induced by a natural isomorphism between the completions of the aforementioned algebras of difference operators. This construction is motivated by the corresponding homomorphism $U_q(L\mathfrak{g}) \to \widehat{Y_h(\mathfrak{g})}$ from [10].

We also prove that the formal algebras $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ are flat deformations of their limits $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)/(q_3-1)$ and $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)/(h_3)$. In particular, this implies that Υ is injective. We also establish the faithfulness of the action of the two algebras in interest on the sum of the representations from Section 3.

• In Section 7, we recall the definition of the small shuffle algebra S^m and its commutative subalgebra $\mathcal{A}^m$, which played a crucial role in [8]. We introduce its additive analogue S^a and the corresponding commutative subalgebra $\mathcal{A}^a$.

• In Section 8, we discuss a *horizontal* realization of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, under which the Fock representations $F(u)$ correspond to the vertex type representations ρ_c from [7]. The representations ρ_c provide a new viewpoint towards the commutative algebra $\mathcal{A}^m$ (see [9] for more details). We conclude that section by introducing and discussing properties of the *Whittaker* vectors (see also [20] for the cohomological case).

• In Appendix, we present main computations.

1. Basic definitions

In this section, we introduce two associative algebras $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$, which are the key objects studied in this paper.

1.1. Quantum toroidal algebra of $\mathfrak{gl}_1$

Let $q_1, q_2, q_3 = q^2 \in \mathbb{C}^*$ satisfy $q_1q_2q_3 = 1$, while $q_i^m \neq 1$ for any $m \in \mathbb{N}, i \in \{1,2,3\}$. The quantum toroidal algebra of $\mathfrak{gl}_1$, denoted by $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, is the unital associative $\mathbb{C}$-algebra generated by $\{e_i, f_i, \psi_i, \psi_0^{-1} | i \in \mathbb{Z}\}$ with the following defining relations:

$$\psi_0 \cdot \psi_0^{-1} = \psi_0^{-1} \cdot \psi_0 = 1, \quad [\psi^{\pm}(z), \psi^{\pm}(w)] = 0, \quad [\psi^+(z), \psi^-(w)] = 0, \tag{T0}$$

$$e(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w) = -e(w)e(z)(w-q_1z)(w-q_2z)(w-q_3z), \quad \text{(T1)}$$

$$f(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z) = -f(w)f(z)(z-q_1w)(z-q_2w)(z-q_3w), \quad \text{(T2)}$$

$$[e(z), f(w)] = \frac{\delta(z/w)}{(1-q_1)(1-q_2)(1-q_3)}(\psi^+(w) - \psi^-(z)), \quad \text{(T3)}$$

$$\psi^{\pm}(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w) = -e(w)\psi^{\pm}(z)(w-q_1z)(w-q_2z)(w-q_3z), \quad \text{(T4)}$$

$$\psi^{\pm}(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z) = -f(w)\psi^{\pm}(z)(z-q_1w)(z-q_2w)(z-q_3w), \quad \text{(T5)}$$

$$\mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2+1}, e_{i_3-1}]] = 0, \quad \mathrm{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2+1}, f_{i_3-1}]] = 0, \quad \text{(T6)}$$

where these generating series are defined as follows:

$$e(z) := \sum_{i=-\infty}^{\infty} e_i z^{-i}, \ f(z) := \sum_{i=-\infty}^{\infty} f_i z^{-i}, \ \psi^{\pm}(z) := \psi_0^{\pm 1} + \sum_{j>0} \psi_{\pm j} z^{\mp j}, \ \delta(z) := \sum_{i=-\infty}^{\infty} z^i.$$

The relations (T0–T5) should be viewed as collections of termwise relations which can be recovered by evaluating the coefficients of $z^k w^l$ $(k, l \in \mathbb{Z})$ on both sides of the equalities.

Remark 1.1. The first reference of such algebras (but without the *Serre* relation (T6)) goes back to [4]. To distinguish, we refer to the algebra with the same collection of generators and the defining relations (T0–T5) as the *Ding–Iohara* algebra, see [8]. These algebras also appeared independently in the work of Burban and Schiffmann (see [2, Section 6]) as a direct specialization of Kapranov's theorem to the case of elliptic curves, and later with the Serre relation in the work of Schiffmann (see [18]) on the Drinfeld realization of elliptic Hall algebras. The algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ was studied in [5,6] under the name *quantum continuous* $\mathfrak{gl}_\infty$. We would like to thank the referee for pointing out that the algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ also appeared independently in the work of Miki on quantum deformations of $W_{1+\infty}$, see [12].

1.2. Some properties of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$

Let $\ddot{U}^0$ be the subalgebra of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by $\{\psi_i, \psi_0^{-1}\}_{i\in\mathbb{Z}}$. It is often more convenient to use the generators $\{\psi_0^{\pm 1}, t_i\}_{i\in\mathbb{Z}^*}$ of $\ddot{U}^0$ (here $\mathbb{Z}^* := \mathbb{Z}\backslash\{0\}$), defined via

$$\psi^{\pm}(z) = \psi_0^{\pm 1} \cdot \exp\left(\mp \sum_{m>0} \frac{\beta_m}{m} t_{\pm m} z^{\mp m}\right), \text{ where } \beta_m := (1-q_1^m)(1-q_2^m)(1-q_3^m).$$

This choice of t_i is motivated by the following two basic results.

Proposition 1.1. *The relations* (T4,T5) *are equivalent to*

$$[\psi_0, e_j] = 0,\ [t_i, e_j] = e_{i+j} \text{ for } i \in \mathbb{Z}^*,\ j \in \mathbb{Z}, \tag{T4t}$$

$$[\psi_0, f_j] = 0,\ [t_i, f_j] = -f_{i+j} \text{ for } i \in \mathbb{Z}^*,\ j \in \mathbb{Z}. \tag{T5t}$$

Proof. The proof of this proposition follows formally from the identity

$$\log\left(\frac{(z-q_1^{-1}w)(z-q_2^{-1}w)(z-q_3^{-1}w)}{(z-q_1w)(z-q_2w)(z-q_3w)}\right) = \sum_{m>0} -\frac{\beta_m}{m}\cdot\frac{w^m}{z^m},$$

where the left-hand side is expressed as a Taylor series in w/z. □

Proposition 1.2. *If the relations* (T4t,T5t) *hold, then* (T6) *is equivalent to*

$$[e_0, [e_1, e_{-1}]] = 0, \quad [f_0, [f_1, f_{-1}]] = 0. \tag{T6t}$$

Proof. Setting $i_1 = i_2 = i_3 = 0$ in (T6), we get (T6t). Let us now deduce (T4t), (T5t),(T6t)⇒(T6). We consider only the case with $\{e_i\}$ (the case with $\{f_i\}$ is completely analogous). For any $i \in \mathbb{Z}^*$, define the operator $T_i : \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \to \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ via $T_i : X \mapsto [t_i, X] = t_iX - Xt_i$. Combining the relation (T4t) with the algebraic equality

$$[t, [a, [b, c]]] = [[t, a], [b, c]] + [a, [[t, b], c]] + [a, [b, [t, c]]],$$

we get

$$T_{i_1}T_{i_2}T_{i_3} - T_{i_1+i_2}T_{i_3} - T_{i_1+i_3}T_{i_2} - T_{i_2+i_3}T_{i_1} + 2T_{i_1+i_2+i_3} : [e_0, [e_1, e_{-1}]]$$
$$\mapsto \mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2+1}, e_{i_3-1}]],$$

where we set $T_0 := 3\mathrm{Id}$. This completes our proof. □

Let $\ddot{U}^-$ and $\ddot{U}^+$ be the subalgebras of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by $\{f_i\}$ and $\{e_i\}$, respectively. We conclude this section by the standard result (see Appendix A for a proof):

Proposition 1.3. *(a) (Triangular decomposition for $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$) The multiplication map $m : \ddot{U}^- \otimes \ddot{U}^0 \otimes \ddot{U}^+ \to \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ is an isomorphism of vector spaces.*

(b) The subalgebras $\ddot{U}^-, \ddot{U}^+, \ddot{U}^0$ are generated by $\{f_i\}$, $\{e_i\}$, $\{\psi_i, \psi_0^{-1}\}$ with the defining relations (T2,T6), (T1,T6), and (T0), respectively.

1.3. Affine Yangian of $\mathfrak{gl}_1$

Let $h_1, h_2, h_3 \in \mathbb{C}$ satisfy $h_1 + h_2 + h_3 = 0$. The affine Yangian of $\mathfrak{gl}_1$, denoted by $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$, is the unital associative $\mathbb{C}$-algebra generated by $\{e_j, f_j, \psi_j | j \in \mathbb{Z}_+\}$ (here $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$) with the following defining relations:

$$[\psi_i, \psi_j] = 0, \tag{Y0}$$

$$[e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] + \sigma_2([e_{i+1}, e_j] - [e_i, e_{j+1}]) = \sigma_3\{e_i, e_j\}, \tag{Y1}$$

$$[f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] + \sigma_2([f_{i+1}, f_j] - [f_i, f_{j+1}]) = -\sigma_3\{f_i, f_j\}, \tag{Y2}$$

$$[e_i, f_j] = \psi_{i+j}, \tag{Y3}$$

$$[\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}] + \sigma_2([\psi_{i+1}, e_j] - [\psi_i, e_{j+1}]) = \sigma_3\{\psi_i, e_j\}, \tag{Y4}$$

$$[\psi_0, e_j] = 0, \quad [\psi_1, e_j] = 0, \quad [\psi_2, e_j] = 2e_j, \tag{Y4$'$}$$

$$[\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}] + \sigma_2([\psi_{i+1}, f_j] - [\psi_i, f_{j+1}]) = -\sigma_3\{\psi_i, f_j\}, \tag{Y5}$$

$$[\psi_0, f_j] = 0, \quad [\psi_1, f_j] = 0, \quad [\psi_2, f_j] = -2f_j, \tag{Y5$'$}$$

$$\mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = 0, \quad \mathrm{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2}, f_{i_3+1}]] = 0, \tag{Y6}$$

where $i, j \in \mathbb{Z}_+$ and we set $\{a, b\} := ab + ba$, $\sigma_2 := h_1h_2 + h_1h_3 + h_2h_3$, $\sigma_3 := h_1h_2h_3$.

Remark 1.2. This algebra can be considered as a natural "additivization" of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ in the same way as the classical Yangian $Y_h(\mathfrak{g})$ is an "additivization" of the quantum loop algebra $U_q(L\mathfrak{g})$. The relations (Y0–Y6) were obtained jointly with B. Feigin in an unpublished work. To the best of our knowledge, the first written reference goes back to [1, Section 3.2] (with the particular choice $\{h_1, h_2, h_3\} = \{1, -\kappa, \kappa - 1\}$), where it was shown to be isomorphic to the algebra $\mathbf{SH}^{\mathbf{c}}$ from [20]. The same algebra was also implicitly considered in [11].

1.4. Some properties of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$

Define the generating series

$$e(z) := \sum_{j \geq 0} e_j z^{-j-1}, \quad f(z) := \sum_{j \geq 0} f_j z^{-j-1}, \quad \psi(z) := 1 + \sigma_3 \sum_{j \geq 0} \psi_j z^{-j-1}.$$

Let $\ddot{Y}^-, \ddot{Y}^0, \ddot{Y}^+$ be the subalgebras of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ generated by $\{f_j\}$, $\{\psi_j\}$, and $\{e_j\}$, respectively. Let $\ddot{Y}^{\geq}$ and $\ddot{Y}^{\leq}$ be the subalgebras of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ generated by $\ddot{Y}^0, \ddot{Y}^+$ and $\ddot{Y}^0, \ddot{Y}^-$, respectively. The following is analogous to Proposition 1.3:

Proposition 1.4. *(a) $\ddot{Y}^0$ is a polynomial algebra in the generators $\{\psi_j\}$.*

(b) $\ddot{Y}^-$ and $\ddot{Y}^+$ are the algebras generated by $\{f_j\}$ and $\{e_j\}$ with the defining relations $(Y2, Y6)$ and $(Y1, Y6)$, respectively.

(c) $\ddot{Y}^{\leq}$ and $\ddot{Y}^{\geq}$ are the algebras generated by $\{\psi_j, f_j\}$ and $\{\psi_j, e_j\}$ with the defining relations $(Y0, Y2, Y5, Y5', Y6)$ and $(Y0, Y1, Y4, Y4', Y6)$, respectively.

(d) Multiplication induces an isomorphism of vector spaces

$$m : \ddot{Y}^- \otimes \ddot{Y}^0 \otimes \ddot{Y}^+ \xrightarrow{\sim} \ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1).$$

Let us consider the algebra homomorphisms

$$\sigma^+ : \ddot{Y}^{\geq} \to \ddot{Y}^{\geq} \text{ determined by } \psi_j \mapsto \psi_j,\ e_j \mapsto e_{j+1}$$

and

$$\sigma^- : \ddot{Y}^{\leq} \to \ddot{Y}^{\leq} \text{ determined by } \psi_j \mapsto \psi_j,\ f_j \mapsto f_{j+1}.$$

These are well defined due to Proposition 1.4. Let

$$\mu : \ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)^{\otimes 2} \to \ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$$

be the multiplication. The following result is straightforward:

Proposition 1.5. *Define $\sigma^{\pm}_{(1)}(a \otimes b) := \sigma^{\pm}(a) \otimes b$ and $\sigma^{\pm}_{(2)}(a \otimes b) := a \otimes \sigma^{\pm}(b)$. We also set $P(z, w) := (z - w - h_1)(z - w - h_2)(z - w - h_3)$. Then:*

(a) The relation (Y0) is equivalent to $[\psi(z), \psi(w)] = 0$.

(b) The relation (Y1) is equivalent to

$$\partial_z^3 \mu(P(z, \sigma^+_{(2)})e(z) \otimes e_j + P(\sigma^+_{(1)}, z)e_j \otimes e(z)) = 0 \quad \text{for } j \in \mathbb{Z}_+.$$

(c) The relation (Y2) is equivalent to

$$\partial_z^3 \mu(P(\sigma^-_{(2)}, z)f(z) \otimes f_j + P(z, \sigma^-_{(1)})f_j \otimes f(z)) = 0 \quad \text{for } j \in \mathbb{Z}_+.$$

(d) The relation (Y3) is equivalent to

$$\sigma_3 \cdot (w - z)[e(z), f(w)] = \psi(z) - \psi(w).$$

(e) The relations (Y4, Y4′) are equivalent to

$$P(z, \sigma^+)\psi(z)e_j + P(\sigma^+, z)e_j\psi(z) = 0 \quad \text{for } j \in \mathbb{Z}_+.$$

(f) The relations (Y5, Y5′) are equivalent to

$$P(\sigma^-, z)\psi(z)f_j + P(z, \sigma^-)f_j\psi(z) = 0 \quad \text{for } j \in \mathbb{Z}_+.$$

2. Representation theory via the Hilbert scheme

2.1. Correspondences and fixed points for $(\mathbb{A}^2)^{[n]}$

Throughout this section $X = \mathbb{A}^2$. Let $X^{[n]}$ be the Hilbert scheme of n points on X. Its $\mathbb{C}$-points are the codimension n ideals $J \subset \mathbb{C}[x, y]$. Let $P[i] \subset \coprod_n X^{[n]} \times X^{[n+i]}$ be the Nakajima–Grojnowski correspondence. For $i > 0$, the correspondence $P[i] \subset \coprod_n X^{[n]} \times X^{[n+i]}$ consists of all pairs of ideals (J_1, J_2) of $\mathbb{C}[x, y]$ of codimension $n,\ n+i$ respectively, such that $J_2 \subset J_1$ and the factor J_1/J_2 is supported at a single point. It is known that $P[1]$ is a smooth variety. Let L be the *tautological* line bundle on $P[1]$ whose fiber at a point $(J_1, J_2) \in P[1]$ equals J_1/J_2. There are natural projections $\mathbf{p}, \mathbf{q}$ from $P[1]$ to $X^{[n]}$ and $X^{[n+1]}$, correspondingly.

Consider a natural action of $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ on each $X^{[n]}$, induced from the one on X given by the formula $(t_1, t_2) \cdot (x, y) = (t_1 x, t_2 y)$. The set $(X^{[n]})^{\mathbb{T}}$ of $\mathbb{T}$-fixed points in $X^{[n]}$ is finite and is in bijection with size n Young diagrams. For such a Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$, the corresponding ideal $J_\lambda \in (X^{[n]})^{\mathbb{T}}$ is given by $J_\lambda = \mathbb{C}[x, y] \cdot (\mathbb{C}x^{\lambda_1}y^0 \oplus \cdots \oplus \mathbb{C}x^{\lambda_k}y^{k-1} \oplus \mathbb{C}y^k)$.

Notation. For a Young diagram λ, let λ^* be the conjugate diagram and define $|\lambda| := \sum \lambda_i$. For a box $\square$ with the coordinates (i, j), we define $a_\lambda(\square) := \lambda_j - i$, $l_\lambda(\square) := \lambda_i^* - j$. The diagram obtained from λ by adding a box to its jth row is denoted by $\lambda + \square_j$, or simply by $\lambda + j$.

2.2. Geometric $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-action I

Let $'M$ be the direct sum of equivariant (complexified) K-groups: $'M = \bigoplus_n K^{\mathbb{T}}(X^{[n]})$. It is a module over $K^{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$. Set $t_3 := t_1^{-1}t_2^{-1} \in \mathbb{C}(t_1, t_2) = \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt}))$ and consider a quadratic extension $\mathbb{F} = \mathbb{C}(t_1, t_2)[t]/(t^2 - t_3)$. We define

$$M := {}'M \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathbb{F}.$$

It has a natural grading: $M = \bigoplus_n M_n$, $M_n = K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathbb{F}$. According to the localization theorem, restriction to the $\mathbb{T}$-fixed point set induces an isomorphism $K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathbb{F} \xrightarrow{\sim} K^{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathbb{F}$. The structure sheaves $\{\lambda\}$ of the $\mathbb{T}$-fixed points J_λ (defined in Section 2.1) form a basis in $\bigoplus_n K^{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathbb{F}$. Since embedding of a point J_λ into $X^{[|\lambda|]}$ is a proper morphism, the direct image in the equivariant K-theory is well defined, and we denote by $[\lambda] \in M_{|\lambda|}$ the direct image of the structure sheaf $\{\lambda\}$. The set $\{[\lambda]\}$ forms a basis of M.

Let $\mathfrak{F}$ be the *tautological* vector bundle on $X^{[n]}$ whose fiber at a point $J \in X^{[n]}$ equals the quotient $\mathbb{C}[x,y]/J$. Consider the following generating series $\mathbf{a}(z), \mathbf{c}(z) \in M(z)$:

$$\mathbf{a}(z) := \Lambda^{\bullet}_{-1/z}(\mathfrak{F}) = \sum_{i\geq 0} [\Lambda^i(\mathfrak{F})](-1/z)^i,$$
$$\mathbf{c}(z) := \mathbf{a}(zt_1)\mathbf{a}(zt_2)\mathbf{a}(zt_3)\mathbf{a}(zt_1^{-1})^{-1}\mathbf{a}(zt_2^{-1})^{-1}\mathbf{a}(zt_3^{-1})^{-1}.$$

Finally, we define the linear operators $e_i, f_i, \psi_i, \psi_0^{-1}$ $(i \in \mathbb{Z})$ acting on M:

$$e_i = \mathbf{q}_*(L^{\otimes i} \otimes \mathbf{p}^*) : \ M_n \to M_{n+1}, \tag{1}$$
$$f_i = -t^{-1} \cdot \mathbf{p}_*(L^{\otimes (i-1)} \otimes \mathbf{q}^*) : \ M_n \to M_{n-1}, \tag{2}$$
$$\psi^{\pm}(z) = \psi_0^{\pm 1} + \sum_{j=1}^{\infty} \psi_{\pm j} z^{\mp j} := \left(\frac{t^{-1} - tz^{-1}}{1 - z^{-1}} \mathbf{c}(z)\right)^{\pm} \in \prod_n \mathrm{End}(M_n)[[z^{\mp 1}]], \tag{3}$$

where $\gamma(z)^{\pm}$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

Theorem 2.1. *The operators $e_i, f_i, \psi_i, \psi_0^{-1}$, defined in (1–3), satisfy the relations (T0–T6) with the parameters $q_i = t_i$. This endows M with the structure of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representation.*

Remark 2.1. This theorem was proved simultaneously and independently in [8] and [19].

2.3. Geometric $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action I

Let $'V$ be the direct sum of equivariant (complexified) cohomology: $'V = \bigoplus_n \mathrm{H}^{\bullet}_{\mathbb{T}}(X^{[n]})$. It is a module over $\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}] = \mathbb{C}[s_1, s_2]$, where $\mathfrak{t}$ is the Lie algebra of $\mathbb{T}$. We define

$$V := {}'V \otimes_{\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt})) = {}'V \otimes_{\mathbb{C}[s_1,s_2]} \mathbb{C}(s_1, s_2).$$

It has a natural grading: $V = \bigoplus_n V_n$, $V_n = \mathrm{H}^{\bullet}_{\mathbb{T}}(X^{[n]}) \otimes_{\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt}))$. According to the localization theorem, restriction to the $\mathbb{T}$-fixed point set induces an isomorphism

$$\mathrm{H}^{\bullet}_{\mathbb{T}}(X^{[n]}) \otimes_{\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt})) \xrightarrow{\sim} \mathrm{H}^{\bullet}_{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt})).$$

The fundamental cycles of the $\mathbb{T}$-fixed points J_λ form a basis in $\bigoplus_n \mathrm{H}^{\bullet}_{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^{\bullet}_{\mathbb{T}}(\mathrm{pt}))$. Since embedding of a point J_λ into $X^{[|\lambda|]}$ is a proper morphism, the direct image in the equivariant cohomology is well defined, and we denote by $[\lambda] \in V_{|\lambda|}$ the direct image of the fundamental cycle of the point J_λ. The set $\{[\lambda]\}$ forms a basis of V.

Consider the following generating series $\mathbf{C}(z) \in V[[z^{-1}]]$:

$$\mathbf{C}(z) := \left(\frac{\mathrm{ch}(\mathfrak{F}t_1^{-1}, -z^{-1})\mathrm{ch}(\mathfrak{F}t_2^{-1}, -z^{-1})\mathrm{ch}(\mathfrak{F}t_3^{-1}, -z^{-1})}{\mathrm{ch}(\mathfrak{F}t_1, -z^{-1})\mathrm{ch}(\mathfrak{F}t_2, -z^{-1})\mathrm{ch}(\mathfrak{F}t_3, -z^{-1})}\right)^{+},$$

where $\mathrm{ch}(F, \bullet)$ denotes the Chern polynomial of F. We also set $s_3 := -s_1 - s_2$.

Finally, we define the linear operators e_j, f_j, ψ_j $(j \in \mathbb{Z}_+)$ acting on V:

$$e_j = \mathbf{q}_*(c_1(L)^j \cdot \mathbf{p}^*) : \ V_n \to V_{n+1}, \tag{1'}$$

$$f_j = \mathbf{p}_*(c_1(L)^j \cdot \mathbf{q}^*) : \ V_n \to V_{n-1}, \tag{2'}$$

$$\psi(z) = 1 + s_1 s_2 s_3 \sum_{j=0}^{\infty} \psi_j z^{-j-1} := (1 - s_3/z)\mathbf{C}(z) \in \prod_n \mathrm{End}(V_n)[[z^{-1}]]. \tag{3'}$$

Theorem 2.2. *The operators e_j, f_j, ψ_j, defined in (1′–3′), satisfy the relations (Y0–Y6) with the parameters $h_i = s_i$. This endows V with the structure of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representation.*

Remark 2.2. This result is a natural "cohomological" analogue of Theorem 2.1. It was obtained jointly with B. Feigin in an unpublished work. The first written reference goes back to [20] (to be precise, in [20] an action of an algebra $\mathbf{SH^c}$ on the space V was constructed, and in [1] it was shown that $\mathbf{SH^c}$ is isomorphic to the affine Yangian of $\mathfrak{gl}_1$).

In the remaining part of this section, we explain how the proof of Theorem 2.1 from [8] can be adapted almost automatically to the cohomological case of Theorem 2.2. We start with an explicit computation of the matrix coefficients of $e_p, f_p, \psi(z)$ in the fixed point basis $\{[\lambda]\}$ of V. For a linear operator $A \in \mathrm{End}(V)$, we use $A_{|[\lambda,\mu]}$ to denote the coefficient of $[\mu]$ in $A([\lambda])$. We also set $\chi(\square_{i,j}) := (i-1)s_1 + (j-1)s_2$ for a box $\square_{i,j}$ located in the jth row and ith column.

Lemma 2.3. *(a) The only nonzero matrix coefficients of the operators e_p, f_p are as follows:*

$$e_{p|[\lambda-i,\lambda]} = \frac{((\lambda_i - 1)s_1 + (i-1)s_2)^p}{(s_1+s_2)((\lambda_1 - \lambda_i + 1)s_1 + (1-i)s_2)} \cdot \prod_{j \geq 1} \frac{(\lambda_j - \lambda_i + 1)s_1 + (j-i+1)s_2}{(\lambda_{j+1} - \lambda_i + 1)s_1 + (j-i+1)s_2},$$

$$f_{p|[\lambda+i,\lambda]} = \frac{(\lambda_i s_1 + (i-1)s_2)^p((\lambda_i - \lambda_1 + 1)s_1 + is_2)}{s_1+s_2} \cdot \prod_{j \geq 1} \frac{(\lambda_i - \lambda_{j+1} + 1)s_1 + (i-j)s_2}{(\lambda_i - \lambda_j + 1)s_1 + (i-j)s_2}.$$

(b) $\psi(z)$ is diagonal in the fixed point basis and the eigenvalue of $\psi(z)$ on $[\lambda]$ equals

$$\psi(z)_{|\lambda} = \left(\left(1 - \frac{s_3}{z}\right) \prod_{\square \in \lambda} \frac{(z - \chi(\square) + s_1)(z - \chi(\square) + s_2)(z - \chi(\square) + s_3)}{(z - \chi(\square) - s_1)(z - \chi(\square) - s_2)(z - \chi(\square) - s_3)} \right)^+.$$

This is a "cohomological" analogue of [8, Lemma 3.1, Proposition 3.1]. Using this result, proof of Theorem 2.2 is reduced to a routine verification of the relations (Y0–Y6) in the fixed point basis. The only non-trivial relation is (Y3). A similar issue in the case of K-theory was resolved by [8, Lemma 4.1]. We conclude this section by proving an analogous result:

Lemma 2.4. *Consider the linear operator $\phi_{i,j} := [e_i, f_j]$ acting on V.*

(a) The operator $\phi_{i,j}$ is diagonal in the fixed point basis $\{[\lambda]\}$ of V.

(b) For any Young diagram λ, we have $\phi_{i,j}([\lambda]) = \gamma_{i+j|_\lambda} \cdot [\lambda]$, where

$$\gamma_{m|_\lambda} = s_1^{-2} \sum_{i=1}^{k} y_i^m \prod_{1\leq j\leq k}^{j\neq i} \frac{(y_i - y_j + s_2)(y_j - y_i + s_1 + s_2)}{(y_i - y_j)(y_j - y_i + s_1)} \cdot \frac{y_i + s_1 + (1-k)s_2}{-y_i + ks_2} -$$

$$s_1^{-2} \sum_{i=1}^{k} (y_i + s_1)^m \prod_{1\leq j\leq k}^{j\neq i} \frac{(y_j - y_i + s_2)(y_i - y_j + s_1 + s_2)}{(y_j - y_i)(y_i - y_j + s_1)} \cdot \frac{y_i + 2s_1 + (1-k)s_2}{-y_i - s_1 + ks_2}. \quad (\sharp)$$

Here $y_i := (\lambda_i - 1)s_1 + (i-1)s_2$ and k is a positive integer such that $\lambda_k = 0$.

(c) For any Young diagram λ, we have

$$\gamma_{0|_\lambda} = -1/s_1 s_2, \quad \gamma_{1|_\lambda} = 0, \quad \gamma_{2|_\lambda} = 2|\lambda|.$$

Proof. Parts (a) and (b) follow from Lemma 2.3(a) by straightforward calculations.

Let us now prove part (c). For $m \geq 0$, the expression in the right-hand side of $(\sharp)$ is a rational function in y_i with the only possible (simple) poles at $y_i = y_j$, $y_i = y_j + s_1$, $y_i = ks_2$, $y_i = -s_1 + ks_2$. A straightforward computation shows that the residues at these points are in fact zero. Therefore, $\gamma_{m|_\lambda} \in \mathbb{C}(s_1, s_2)[y_1, y_2, \ldots]$ for $m \geq 0$.

∘ *Case 1:* $m = 0$.

Since $\gamma_{0|_\lambda}$ is a polynomial in y_i of degree ≤ 0, it must be an element of $\mathbb{C}(s_1, s_2)$ independent of λ. Evaluating at the empty diagram, we find $\gamma_{0|_\lambda} = \gamma_{0|_\emptyset} = -1/s_1 s_2$.

∘ *Case 2:* $m = 1$.

The eigenvalue $\gamma_{1|_\lambda}$ is a polynomial in y_i of degree ≤ 1. However, the limit of the right-hand side of $(\sharp)$ with $m = 1$ as $y_{i_0} \to \infty$ while y_j are fixed for all $j \neq i_0$ is finite for any index i_0. Therefore, $\gamma_{1|_\lambda}$ must be a degree 0 polynomial. Hence, $\gamma_{1|_\lambda} = \gamma_{1|_\emptyset} = 0$.

∘ *Case 3:* $m = 2$.

Recall that $\gamma_{2|_\lambda}$ is a polynomial in y_i of degree ≤ 2. Arguments similar to the above show that $\gamma_{2|_\lambda}$ is actually a degree ≤ 1 polynomial in y_i. Let us now compute its principal linear part. The coefficient of y_{i_0} equals the limit $\lim\limits_{\xi\to\infty} \frac{1}{\xi}\gamma_{2|_\lambda}$ as y_j is fixed for $j \neq i_0$ and $y_{i_0} = \xi \to \infty$. By $(\sharp)$ this is just $\frac{2}{s_1}$. Therefore, there exists $F(s_1, s_2) \in \mathbb{C}(s_1, s_2)$ such that $\gamma_{2|_\lambda} = \frac{2}{s_1}(\widetilde{y}_1 + \widetilde{y}_2 + \ldots) + F(s_1, s_2) = 2|\lambda| + F(s_1, s_2)$, where $\widetilde{y}_i := y_i - ((i-1)s_2 - s_1)$ (the sequence $\{\widetilde{y}_i\}$ stabilizes to 0 as $i \to \infty$, unlike $\{y_i\}$). Evaluating at the empty Young diagram, we find $F(s_1, s_2) = \gamma_{2|_\emptyset} = 0$. The equality $\gamma_{2|_\lambda} = 2|\lambda|$ follows. □

Arguments similar to those from [8] imply $\gamma_{m|_\lambda} = \psi_{m|_\lambda}$ (see also Appendix B).

Remark 2.3. Due to Lemma 2.3(b), the next ψ-coefficient acts in the following way:

$$\psi_{3|_\lambda} = 6 \sum_{\square\in\lambda} \chi(\square) + 2(s_1 + s_2)|\lambda|.$$

In particular, $\frac{1}{6}(\psi_3 + s_3\psi_2)$ corresponds to the cup product with $c_1(\mathfrak{F})$. This operator was first studied by M. Lehn. It is also related to the Laplace–Beltrami operator (see [15, Section 4]).

3. Representation theory via the Gieseker space

The purpose of this section is to provide generalizations of the results from Section 2 to the higher rank r cases, that is, replacing $(\mathbb{A}^2)^{[n]}$ by the Gieseker moduli spaces $M(r,n)$.

3.1. Correspondences and fixed points for $M(r,n)$

We recall some basics on the Gieseker framed moduli spaces $M(r,n)$ of torsion free sheaves on $\mathbb{P}^2$ of rank r with $c_2 = n$. Its $\mathbb{C}$-points are the isomorphism classes of pairs $\{(E,\Phi)\}$, where E is a torsion free sheaf on $\mathbb{P}^2$ of rank r with $c_2(E) = n$, and which is locally free in a neighborhood of the line $l_\infty = \{(0 : z_1 : z_2)\} \subset \mathbb{P}^2$, while $\Phi : E_{|_{l_\infty}} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r}$ (called a *framing at infinity*).

This space has an alternative quiver description (see [14, Ch. 2] for details):

$$M(r,n) = \mathcal{M}(r,n)/GL_n(\mathbb{C}),\ \mathcal{M}(r,n) = \{(B_1,B_2,i,j)|[B_1,B_2]+ij=0\}^s,$$

where $B_1, B_2 \in \mathrm{End}(\mathbb{C}^n), i \in \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n), j \in \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)$, the $GL_n(\mathbb{C})$-action is given by $g\cdot(B_1,B_2,i,j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$, while the superscript s symbolizes the stability condition "there exists no subspace $S \subsetneq \mathbb{C}^n$ such that $B_\alpha(S) \subset S$ ($\alpha = 1,2$) and $\mathrm{Im}(i) \subset S$". Let $\mathfrak{F}_r$ be the *tautological* rank n vector bundle on $M(r,n)$.

Consider a natural action of $\mathbb{T}_r = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$ on $M(r,n)$, where $(\mathbb{C}^*)^2$ acts on $\mathbb{P}^2$ via $(t_1,t_2)\cdot([z_0 : z_1 : z_2]) = [z_0 : t_1z_1 : t_2z_2]$, while $(\mathbb{C}^*)^r$ acts by rescaling the framing isomorphism. The set $M(r,n)^{\mathbb{T}_r}$ of $\mathbb{T}_r$-fixed points in $M(r,n)$ is finite and is in bijection with r-tuples of Young diagrams $\bar{\lambda} = (\lambda^1,\ldots,\lambda^r)$ satisfying $|\bar{\lambda}| := |\lambda^1| + \ldots + |\lambda^r| = n$, denoted by $\bar{\lambda} \vdash n$ (see [16, Proposition 2.9]). For such $\bar{\lambda}$, the corresponding point $\xi_{\bar{\lambda}} \in M(r,n)^{\mathbb{T}_r}$ is represented by $(E_{\bar{\lambda}}, \Phi_{\bar{\lambda}})$, where $E_{\bar{\lambda}} = J_{\lambda^1} \oplus \cdots \oplus J_{\lambda^r}$ and $\Phi_{\bar{\lambda}}$ is a sum of natural inclusions $J_{\lambda^j|_{l_\infty}} \hookrightarrow \mathcal{O}_{l_\infty}$.

Following [13, Section 5], we recall the *Hecke correspondences*, which generalize $P[1]$ from Section 2.1 to the higher rank r cases. Consider $\mathcal{M}(r;n,n+1) \subset \mathcal{M}(r,n)\times\mathcal{M}(r,n+1)$ consisting of pairs of tuples $\{(B_1^{(k)}, B_2^{(k)}, i^{(k)}, j^{(k)})\}$ for $k = n, n+1$, such that there exists $\xi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ satisfying

$$\xi B_1^{(n+1)} = B_1^{(n)}\xi,\ \xi B_2^{(n+1)} = B_2^{(n)}\xi,\ \xi i^{(n+1)} = i^{(n)},\ j^{(n+1)} = j^{(n)}\xi.$$

The stability condition implies ξ is surjective. Therefore $S := \mathrm{Ker}\ \xi \subset \mathbb{C}^{n+1}$ is a 1-dimensional subspace of $\mathrm{Ker}\ j^{(n+1)}$ invariant with respect to $B_1^{(n+1)}, B_2^{(n+1)}$. This provides an identification of $\mathcal{M}(r;n,n+1)$ with pairs of $(B_1^{(n+1)}, B_2^{(n+1)}, i^{(n+1)}, j^{(n+1)}) \in \mathcal{M}(r,n+1)$ and a 1-dimensional subspace $S \subset \mathbb{C}^{n+1}$ satisfying the above conditions.

Taking the latter viewpoint, we define the Hecke correspondence $M(r;n,n+1)$ as the quotient $M(r;n,n+1)=\mathcal{M}(r;n,n+1)/GL_{n+1}(\mathbb{C})$. Let L_r be the *tautological* line bundle on $M(r;n,n+1)$, while $\mathbf{p}_r,\mathbf{q}_r$ be the natural projections from $M(r;n,n+1)$ to $M(r,n)$ and $M(r,n+1)$, correspondingly. The set $M(r;n,n+1)^{\mathbb{T}_r}$ of $\mathbb{T}_r$-fixed points in $M(r;n,n+1)$ is in bijection with pairs of r-tuples of Young diagrams $\bar{\lambda}\vdash n,\bar{\mu}\vdash n+1$ such that $\lambda^j\subseteq\mu^j$ for $1\leq j\leq r$; the corresponding fixed point will be denoted by $\xi_{\bar{\lambda},\bar{\mu}}$.

Our computations are based on the following well-known result (see [13,16]):

Proposition 3.1. *(a) The variety $M(r;n,n+1)$ is smooth of complex dimension $2rn+r+1$.*

(b) The $\mathbb{T}_r$-character of the tangent space to $M(r,n)$ at the $\mathbb{T}_r$-fixed point $\xi_{\bar{\lambda}}$ equals

$$T_{\bar{\lambda}}=\sum_{a,b=1}^{r}\frac{\chi_b}{\chi_a}\left(\sum_{\square\in\lambda^a}t_1^{-a_{\lambda^b}(\square)}t_2^{l_{\lambda^a}(\square)+1}+\sum_{\square\in\lambda^b}t_1^{a_{\lambda^a}(\square)+1}t_2^{-l_{\lambda^b}(\square)}\right).$$

(c) The map $(\mathbf{p}_r,\mathbf{q}_r):M(r;n,n+1)\to M(r,n)\times M(r,n+1)$ is a closed immersion and the $\mathbb{T}_r$-character of the fiber of the normal bundle of $M(r;n,n+1)$ at $\xi_{\bar{\lambda},\bar{\mu}}$ equals

$$N_{\bar{\lambda},\bar{\mu}}=-t_1t_2+\sum_{a,b=1}^{r}\frac{\chi_b}{\chi_a}\left(\sum_{\square\in\lambda^a}t_1^{-a_{\lambda^b}(\square)}t_2^{l_{\mu^a}(\square)+1}+\sum_{\square\in\mu^b}t_1^{a_{\mu^a}(\square)+1}t_2^{-l_{\lambda^b}(\square)}\right).$$

3.2. Geometric $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-action II

Let ${}'M^r$ be the direct sum of equivariant (complexified) K-groups: ${}'M^r=\bigoplus_n K^{\mathbb{T}_r}(M(r,n))$. It is a module over $K^{\mathbb{T}_r}(\mathrm{pt})=\mathbb{C}[\mathbb{T}_r]=\mathbb{C}[t_1^{\pm1},t_2^{\pm1},\chi_1^{\pm1},\dots,\chi_r^{\pm1}]$. Consider a quadratic extension $\mathbb{F}_r=\mathrm{Frac}(K^{\mathbb{T}_r}(\mathrm{pt}))[t]/(t^2-t_3)$, where $t_3:=t_1^{-1}t_2^{-1}\in\mathrm{Frac}(K^{\mathbb{T}_r}(\mathrm{pt}))$. We define

$$M^r:={}'M^r\otimes_{K^{\mathbb{T}_r}(\mathrm{pt})}\mathbb{F}_r.$$

It has a natural grading: $M^r=\bigoplus_n M^r_n$, $M^r_n=K^{\mathbb{T}_r}(M(r,n))\otimes_{K^{\mathbb{T}_r}(\mathrm{pt})}\mathbb{F}_r$. By the localization theorem, restriction to the $\mathbb{T}_r$-fixed point set induces an isomorphism $K^{\mathbb{T}_r}(M(r,n))\otimes_{K^{\mathbb{T}_r}(\mathrm{pt})}\mathbb{F}_r\xrightarrow{\sim}K^{\mathbb{T}_r}(M(r,n)^{\mathbb{T}_r})\otimes_{K^{\mathbb{T}_r}(\mathrm{pt})}\mathbb{F}_r$. The structure sheaves $\{\bar{\lambda}\}$ of the $\mathbb{T}_r$-fixed points $\xi_{\bar{\lambda}}$ form a basis in $\bigoplus_n K^{\mathbb{T}_r}(M(r,n)^{\mathbb{T}_r})\otimes_{K^{\mathbb{T}_r}(\mathrm{pt})}\mathbb{F}_r$. Since embedding of a point $\xi_{\bar{\lambda}}$ into $M(r,|\bar{\lambda}|)$ is a proper morphism, the direct image in the equivariant K-theory is well defined, and we denote by $[\bar{\lambda}]\in M^r_{|\bar{\lambda}|}$ the direct image of the structure sheaf $\{\bar{\lambda}\}$. The set $\{[\bar{\lambda}]\}$ forms a basis of M^r.

Consider the following generating series $\mathbf{a}_r(z),\mathbf{c}_r(z)\in M^r(z)$:

$$\mathbf{a}_r(z):=\Lambda^\bullet_{-1/z}(\mathfrak{F}_r)=\sum_{i\geq0}[\Lambda^i(\mathfrak{F}_r)](-1/z)^i,$$

$$\mathbf{c}_r(z):=\mathbf{a}_r(zt_1)\mathbf{a}_r(zt_2)\mathbf{a}_r(zt_3)\mathbf{a}_r(zt_1^{-1})^{-1}\mathbf{a}_r(zt_2^{-1})^{-1}\mathbf{a}_r(zt_3^{-1})^{-1}.$$

Finally, we define the linear operators $e_i, f_i, \psi_i, \psi_0^{-1}$ $(i \in \mathbb{Z})$ acting on M^r:

$$e_i = \mathbf{q}_{r*}(L_r^{\otimes i} \otimes \mathbf{p}_r^*) : \ M_n^r \to M_{n+1}^r, \tag{4}$$

$$f_i = (-t)^{r-2}\chi_1^{-1} \cdots \chi_r^{-1} \cdot \mathbf{p}_{r*}(L_r^{\otimes (i-r)} \otimes \mathbf{q}_r^*) : \ M_n^r \to M_{n-1}^r, \tag{5}$$

$$\psi^{\pm}(z) = \psi_0^{\pm 1} + \sum_{j=1}^{\infty} \psi_{\pm j} z^{\mp j} := \left(\prod_{a=1}^{r} \frac{t^{-1} - t\chi_a^{-1}/z}{1 - \chi_a^{-1}/z} \mathbf{c}_r(z) \right)^{\pm} \in \prod_n \mathrm{End}(M_n^r)[[z^{\mp 1}]]. \tag{6}$$

Theorem 3.2. *The operators $e_i, f_i, \psi_i, \psi_0^{-1}$, defined in (4–6), satisfy the relations (T0–T6) with the parameters $q_i = t_i$. This endows M^r with the structure of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representation.*

Remark 3.1. This higher rank generalization of Theorem 2.1 first appeared in [19].

We refer the interested reader to Appendix B, where it is explained how the proof of Theorem 2.1 from [8] can be easily adapted to the general rank r case. We conclude this section by computing explicitly the matrix coefficients of $e_p, f_p, \psi^{\pm}(z)$ in the fixed point basis of M^r.

Lemma 3.3. *Consider the fixed point basis $\{[\bar{\lambda}]\}$ of M^r. Define $\chi_k^{(a)} := t_1^{\lambda_k^a - 1} t_2^{k-1} \chi_a^{-1}$.*

(a) The only nonzero matrix coefficients of the operators e_p, f_p are as follows:

$$e_{p|[\bar{\lambda} - \square_j^l, \bar{\lambda}]} = \frac{(\chi_j^{(l)})^p}{1 - t_1 t_2} \cdot \prod_{a=1}^{r} \prod_{k=1}^{\infty} \frac{1 - t_1 t_2 \chi_k^{(a)}/\chi_j^{(l)}}{1 - t_1 \chi_k^{(a)}/\chi_j^{(l)}},$$

$$f_{p|[\bar{\lambda} + \square_j^l, \bar{\lambda}]} = (-t)^{r-2}\chi_1^{-1} \cdots \chi_r^{-1} \cdot \frac{(t_1 \chi_j^{(l)})^{p-r}}{1 - t_1 t_2} \cdot \prod_{a=1}^{r} \prod_{k=1}^{\infty} \frac{1 - t_1 t_2 \chi_j^{(l)}/\chi_k^{(a)}}{1 - t_1 \chi_j^{(l)}/\chi_k^{(a)}},$$

where $\bar{\lambda} \pm \square_j^l$ denotes the r-tuple of diagrams $(\lambda^1, \ldots, \lambda^{l-1}, \lambda^l \pm j, \lambda^{l+1}, \ldots, \lambda^r)$.

(b) $\psi^{\pm}(z)$ is diagonal in the fixed point basis and the eigenvalue of $\psi^{\pm}(z)$ on $[\bar{\lambda}]$ equals

$$\psi^{\pm}(z)_{|_{\bar{\lambda}}} = \left(\prod_{a=1}^{r} \frac{t^{-1} - t\chi_a^{-1}/z}{1 - \chi_a^{-1}/z} \prod_{a=1}^{r} \prod_{\square \in \lambda^a} \frac{(1 - t_1^{-1}\chi(\square)/z)(1 - t_2^{-1}\chi(\square)/z)(1 - t_3^{-1}\chi(\square)/z)}{(1 - t_1\chi(\square)/z)(1 - t_2\chi(\square)/z)(1 - t_3\chi(\square)/z)} \right)^{\pm},$$

where $\chi(\square_{i,j}^a) := t_1^{i-1} t_2^{j-1} \chi_a^{-1}$ for a box $\square_{i,j}^a$ located in the jth row and ith column of λ^a.

3.3. Geometric $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action II

Let $'V^r$ be the direct sum of equivariant (complexified) cohomology: $'V^r = \bigoplus_n \mathrm{H}^{\bullet}_{\mathbb{T}_r}(M(r,n))$. It is a module over $\mathrm{H}^{\bullet}_{\mathbb{T}_r}(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}_r] = \mathbb{C}[s_1, s_2, x_1, \ldots, x_r]$, where

$\mathfrak{t}_r$ is the Lie algebra of $\mathbb{T}_r$. We define

$$V^r := {}'V^r \otimes_{\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt})) = {}'V^r \otimes_{\mathbb{C}[s_1,s_2,x_1,\ldots,x_r]} \mathbb{C}(s_1,s_2,x_1,\ldots,x_r).$$

It has a natural grading: $V^r = \bigoplus_n V^r_n$, $V^r_n = \mathrm{H}^\bullet_{\mathbb{T}_r}(M(r,n)) \otimes_{\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt}))$. According to the localization theorem, restriction to the $\mathbb{T}_r$-fixed point set induces an isomorphism

$$\mathrm{H}^\bullet_{\mathbb{T}_r}(M(r,n)) \otimes_{\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt})) \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathbb{T}_r}(M(r,n)^{\mathbb{T}_r}) \otimes_{\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt})).$$

The fundamental cycles of the $\mathbb{T}_r$-fixed points $\xi_{\bar\lambda}$ form a basis in $\bigoplus_n \mathrm{H}^\bullet_{\mathbb{T}_r}(M(r,n)^{\mathbb{T}_r}) \otimes_{\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(\mathrm{H}^\bullet_{\mathbb{T}_r}(\mathrm{pt}))$. Since embedding of a point $\xi_{\bar\lambda}$ into $M(r,|\bar\lambda|)$ is a proper morphism, the direct image in the equivariant cohomology is well defined, and we denote by $[\bar\lambda] \in V^r_{|\bar\lambda|}$ the direct image of the fundamental cycle of the point $\xi_{\bar\lambda}$. The set $\{[\bar\lambda]\}$ forms a basis of V^r.

Set $s_3 := -s_1 - s_2$. Consider the following generating series $\mathbf{C}_r(z) \in V^r[[z^{-1}]]$:

$$\mathbf{C}_r(z) := \left(\frac{\mathrm{ch}(\mathfrak{F}_r t_1^{-1}, -z^{-1})\mathrm{ch}(\mathfrak{F}_r t_2^{-1}, -z^{-1})\mathrm{ch}(\mathfrak{F}_r t_3^{-1}, -z^{-1})}{\mathrm{ch}(\mathfrak{F}_r t_1, -z^{-1})\mathrm{ch}(\mathfrak{F}_r t_2, -z^{-1})\mathrm{ch}(\mathfrak{F}_r t_3, -z^{-1})} \right)^+.$$

Finally, we define the linear operators e_j, f_j, ψ_j $(j \in \mathbb{Z}_+)$ acting on V^r:

$$e_j = \mathbf{q}_{r*}(c_1(L_r)^j \cdot \mathbf{p}_r^*) : \ V^r_n \to V^r_{n+1}, \tag{4'}$$

$$f_j = (-1)^{r-1} \mathbf{p}_{r*}(c_1(L_r)^j \cdot \mathbf{q}_r^*) : \ V^r_n \to V^r_{n-1}, \tag{5'}$$

$$\psi(z) = 1 + s_1 s_2 s_3 \sum_{j=0}^{\infty} \psi_j z^{-j-1} := \left(\prod_{a=1}^{r} \frac{z + x_a - s_3}{z + x_a} \right)^+ \cdot \mathbf{C}_r(z) \in \prod_n \mathrm{End}(V^r_n)[[z^{-1}]]. \tag{6'}$$

Theorem 3.4. *The operators e_j, f_j, ψ_j, defined in (4′–6′), satisfy the relations (Y0–Y6) with the parameters $h_i = s_i$. This endows V^r with the structure of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representation.*

Remark 3.2. This result is a natural "cohomological" analogue of Theorem 3.2. It was obtained jointly with B. Feigin in an unpublished work (motivated by its K-theoretical version from [19]): we sketch our proof in Appendix B. The first written reference goes back to [20].

We conclude this section by computing explicitly the matrix coefficients of $e_p, f_p, \psi(z)$ in the fixed point basis $\{[\bar\lambda]\}$ of V^r.

Lemma 3.5. *Define* $x_k^{(a)} := (\lambda_k^a - 1)s_1 + (k-1)s_2 - x_a$, $\chi(\square_{i,j}^a) := (i-1)s_1 + (j-1)s_2 - x_a$.

(a) The only nonzero matrix coefficients of the operators e_p, f_p *are as follows:*

$$e_{p|[\bar\lambda - \square_j^l, \bar\lambda]} = \frac{(x_j^{(l)})^p}{s_1+s_2} \cdot \prod_{a=1}^{r} \prod_{k=1}^{\infty} \frac{s_1+s_2+x_k^{(a)} - x_j^{(l)}}{s_1 + x_k^{(a)} - x_j^{(l)}},$$

$$f_{p|[\bar\lambda + \square_j^l, \bar\lambda]} = (-1)^{r-1} \frac{(s_1+x_j^{(l)})^p}{s_1+s_2} \cdot \prod_{a=1}^{r} \prod_{k=1}^{\infty} \frac{s_1+s_2+x_j^{(l)} - x_k^{(a)}}{s_1 + x_j^{(l)} - x_k^{(a)}}.$$

(b) $\psi(z)$ *is diagonal in the fixed point basis and the eigenvalue of* $\psi(z)$ *on* $[\bar\lambda]$ *equals*

$$\psi(z)_{|\bar\lambda} = \left(\prod_{a=1}^{r} \frac{z+x_a-s_3}{z+x_a} \prod_{a=1}^{r} \prod_{\square \in \lambda^a} \frac{(z-\chi(\square)+s_1)(z-\chi(\square)+s_2)(z-\chi(\square)+s_3)}{(z-\chi(\square)-s_1)(z-\chi(\square)-s_2)(z-\chi(\square)-s_3)} \right)^+.$$

Corollary 3.6. *We have*

$$\psi(z)_{|\bar\lambda} = 1 - \frac{rs_3}{z} + \frac{s_3 \sum x_a + \binom{r}{2} s_3^2}{z^2} + \frac{2\sigma_3|\bar\lambda| - s_3 \sum x_a^2 - (r-1)s_3^2 \sum x_a - \binom{r}{3} s_3^3}{z^3} + o(z^{-3}).$$

4. Some representations of $\ddot U_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot Y_{h_1,h_2,h_3}(\mathfrak{gl}_1)$

In this section, we recall several families of $\ddot U_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations from [5,6] and introduce their analogues for the case of $\ddot Y_{h_1,h_2,h_3}(\mathfrak{gl}_1)$. We also establish the relation to the representations from Sections 2–3 and introduce the appropriate categories $\mathcal{O}$.

4.1. Vector representations $V(u)$ *and* ${}^aV(u)$

The main 'building block' of all known $\ddot U_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations is the family of *vector representations* $\{V(u)\}_{u\in\mathbb{C}^*}$, whose basis is parametrized by $\mathbb{Z}$ (see [5, Proposition 3.1]).

Proposition 4.1 *(Vector representation of* $\ddot U_{q_1,q_2,q_3}(\mathfrak{gl}_1)$*). For* $u \in \mathbb{C}^*$*, let* $V(u)$ *be a* $\mathbb{C}$*-vector space with the basis* $\{[u]_j\}_{j\in\mathbb{Z}}$*. The following formulas define a* $\ddot U_{q_1,q_2,q_3}(\mathfrak{gl}_1)$*-action on* $V(u)$*:*

$$e(z)[u]_i = (1-q_1)^{-1} \delta(q_1^i u/z) \cdot [u]_{i+1},$$

$$f(z)[u]_i = (q_1^{-1}-1)^{-1} \delta(q_1^{i-1} u/z) \cdot [u]_{i-1},$$

$$\psi^{\pm}(z)[u]_i = \left(\frac{(z-q_1^i q_2 u)(z-q_1^i q_3 u)}{(z-q_1^i u)(z-q_1^{i-1} u)}\right)^{\pm} \cdot [u]_i.$$

Define $\delta^+(w) := 1 + w + w^2 + \ldots = (\frac{1}{1-w})^+$. Our next result provides an analogous construction of *vector representations* $\{{}^aV(u)\}_{u\in\mathbb{C}}$ for the case of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ (the proof is straightforward and is left to the interested reader):

Proposition 4.2 *(Vector representation of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$). For $u \in \mathbb{C}$, let ${}^aV(u)$ be a $\mathbb{C}$-vector space with the basis $\{[u]_j\}_{j\in\mathbb{Z}}$. The following formulas define a $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action on ${}^aV(u)$:*

$$e(z)[u]_i = \frac{1}{h_1 z}\delta^+((ih_1+u)/z)[u]_{i+1} = \left(\frac{1}{h_1(z-ih_1-u)}\right)^+ \cdot [u]_{i+1},$$

$$f(z)[u]_i = -\frac{1}{h_1 z}\delta^+(((i-1)h_1+u)/z)[u]_{i-1} = \left(\frac{-1}{h_1(z-(i-1)h_1-u)}\right)^+ \cdot [u]_{i-1},$$

$$\psi(z)[u]_i = \left(\frac{(z-(ih_1+h_2+u))(z-(ih_1+h_3+u))}{(z-(ih_1+u))(z-((i-1)h_1+u))}\right)^+ \cdot [u]_i.$$

4.2. Fock representations $F(u)$ and ${}^aF(u)$

A more interesting family of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations, whose basis is labeled by Young diagrams $\{\lambda\}$, was established in [5, Theorem 4.3].

Proposition 4.3 *(Fock representation of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$). For $u \in \mathbb{C}^*$, let $F(u)$ be a $\mathbb{C}$-vector space with the basis $\{|\lambda\rangle\}$. The following formulas define a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-action on $F(u)$:*

$$e(z)|\lambda\rangle = \sum_{i\geq 1}\prod_{j=1}^{i-1}\frac{(1-q_1^{\lambda_i-\lambda_j}q_2^{i-j-1})(1-q_1^{\lambda_i-\lambda_j+1}q_2^{i-j+1})}{(1-q_1^{\lambda_i-\lambda_j}q_2^{i-j})(1-q_1^{\lambda_i-\lambda_j+1}q_2^{i-j})} \cdot \frac{\delta(q_1^{\lambda_i}q_2^{i-1}u/z)}{1-q_1} \cdot |\lambda+i\rangle,$$

$$f(z)|\lambda\rangle = q^{-1}\sum_{i\geq 1}\prod_{j=i+1}^{\infty}\frac{(1-q_1^{\lambda_j-\lambda_i+1}q_2^{j-i+1})(1-q_1^{\lambda_{j+1}-\lambda_i}q_2^{j-i})}{(1-q_1^{\lambda_{j+1}-\lambda_i+1}q_2^{j-i+1})(1-q_1^{\lambda_j-\lambda_i}q_2^{j-i})}$$
$$\times \frac{1-q_1^{\lambda_{i+1}-\lambda_i}}{1-q_1^{\lambda_{i+1}-\lambda_i+1}q_2}\frac{\delta(q_1^{\lambda_i-1}q_2^{i-1}u/z)}{q_1^{-1}-1} \cdot |\lambda-i\rangle,$$

$$\psi^{\pm}(z)|\lambda\rangle = q^{-1}\cdot\left(\frac{z-q_1^{\lambda_1-1}q_2^{-1}u}{z-q_1^{\lambda_1}u}\prod_{i=1}^{\infty}\frac{(z-q_1^{\lambda_i}q_2^i u)(z-q_1^{\lambda_{i+1}-1}q_2^{i-1}u)}{(z-q_1^{\lambda_{i+1}}q_2^i u)(z-q_1^{\lambda_i-1}q_2^{i-1}u)}\right)^{\pm} \cdot |\lambda\rangle.$$

Remark 4.1. The Fock module $F(u)$ was originally constructed from $V(u)$ by using the semi-infinite wedge construction and the formal coproduct structure on $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ defined by

$$\Delta(e(z)) = e(z) \otimes 1 + \psi^-(z) \otimes e(z),\ \Delta(f(z)) = f(z) \otimes \psi^+(z) + 1 \otimes f(z),$$

$$\Delta(\psi^\pm(z)) = \psi^\pm(z) \otimes \psi^\pm(z).$$

Remark 4.2. (a) According to [5, Corollary 4.5], there exist constants $\{c_\lambda\}$ such that the map $[\lambda] \mapsto c_\lambda|\lambda\rangle$ induces an isomorphism $M \xrightarrow{\sim} F(1)$ of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations, where M is the representation from Theorem 2.1 and $q_i = t_i$.

(b) For $u \in \mathbb{C}^*$, let ϕ_u be the *shift automorphism* of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ defined on the generators by

$$\psi_0^{-1} \mapsto \psi_0^{-1},\ \ \psi_i \mapsto u^{-i} \cdot \psi_i,\ \ e_i \mapsto u^{-i} \cdot e_i,\ \ f_i \mapsto u^{-i} \cdot f_i \ \text{ for } i \in \mathbb{Z}.$$

Then the modules $F(u)$ and $V(u)$ are obtained from $F(1)$ and $V(1)$ via a twist by $\phi_{1/u}$.[1]

Let us analogously define *Fock representations* $\{{}^aF(u)\}_{u\in\mathbb{C}}$ for the case of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$.

Proposition 4.4 *(Fock representation of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$). For $u \in \mathbb{C}$, let ${}^aF(u)$ be a $\mathbb{C}$-vector space with the basis $\{|\lambda\rangle\}$. The following formulas define a $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action on ${}^aF(u)$:*

$$e(z)|\lambda\rangle = \frac{1}{h_1 z} \sum_{i\geq 1} \prod_{j=1}^{i-1} \frac{((\lambda_i-\lambda_j)h_1+(i-j-1)h_2)((\lambda_i-\lambda_j+1)h_1+(i-j+1)h_2)}{((\lambda_i-\lambda_j)h_1+(i-j)h_2)((\lambda_i-\lambda_j+1)h_1+(i-j)h_2)} \times$$
$$\delta^+\left(\frac{\lambda_i h_1+(i-1)h_2+u}{z}\right) \cdot |\lambda+i\rangle,$$

$$f(z)|\lambda\rangle = -\frac{1}{h_1 z} \sum_{i\geq 1} \prod_{j=i+1}^{\infty} \frac{((\lambda_j-\lambda_i+1)h_1+(j-i+1)h_2)((\lambda_{j+1}-\lambda_i)h_1+(j-i)h_2)}{((\lambda_{j+1}-\lambda_i+1)h_1+(j-i+1)h_2)((\lambda_j-\lambda_i)h_1+(j-i)h_2)} \times$$
$$\frac{(\lambda_{i+1}-\lambda_i)h_1}{(\lambda_{i+1}-\lambda_i+1)h_1+h_2} \delta^+\left(\frac{(\lambda_i-1)h_1+(i-1)h_2+u}{z}\right) \cdot |\lambda-i\rangle,$$

$$\psi(z)|\lambda\rangle = \left(\prod_{i=1}^{\infty} \frac{(z-(\lambda_i h_1+ih_2+u))(z-((\lambda_{i+1}-1)h_1+(i-1)h_2+u))}{(z-(\lambda_{i+1}h_1+ih_2+u))(z-((\lambda_i-1)h_1+(i-1)h_2+u))}\right)^+ \times$$
$$\left(\frac{z-((\lambda_1-1)h_1-h_2+u)}{z-(\lambda_1 h_1+u)}\right)^+ \cdot |\lambda\rangle.$$

The proof of this proposition follows from the following lemma:

[1] For an algebra A and an automorphism $\sigma \in \mathrm{Aut}(A)$, the twist of an A-representation $\rho : A \to \mathrm{End}(V)$ is $\rho^\sigma : A \to \mathrm{End}(V)$ with $\rho^\sigma(a) := \rho(\sigma(a))$.

Lemma 4.5. *(a) For $u \in \mathbb{C}$, there exists the* shift automorphism ϕ_u^a *of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ such that*

$$\phi_u^a: \ e(z) \mapsto e(z+u), \ \ f(z) \mapsto f(z+u), \ \ \psi(z) \mapsto \psi(z+u).$$

(b) The Fock representation ${}^aF(u)$ is obtained from ${}^aF(0)$ via a twist by ϕ_{-u}^a.

(c) There exist constants $\{c_\lambda^a\}$ such that the map $[\lambda] \mapsto c_\lambda^a|\lambda\rangle$ induces an isomorphism $V \xrightarrow{\sim} {}^aF(0)$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations, where V is the representation from Theorem 2.2 and $h_i = s_i$.

Proof. Parts (a) and (b) are straightforward.

We define c_λ^a by the following formula:

$$c_\lambda^a = \prod_{i\geq 1} \prod_{p=0}^{\lambda_i - 1} (-ph_1 + h_2) \cdot \prod_{i\geq 2} \prod_{j=1}^{i-1} \prod_{p=1}^{\lambda_i} \frac{(p-\lambda_j)h_1 + (i-j+1)h_2}{(p-\lambda_j)h_1 + (i-j)h_2}.$$

It is a routine verification to check that the map $[\lambda] \mapsto c_\lambda^a|\lambda\rangle$ intertwines the formulas for the matrix coefficients of e_k, f_k, ψ_k $(k \in \mathbb{Z}_+)$ from Lemma 2.3 and Proposition 4.4. □

Definition 4.1. We say that a $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representation U has a *central charge* $(c_0, c_1) \in \mathbb{C}^2$ if $\psi_{0|_U} = c_0 \cdot \mathrm{Id}_U$ and $\psi_{1|_U} = c_1 \cdot \mathrm{Id}_U$.

Thus ${}^aV(u)$ has a central charge $\left(0, \frac{1}{h_1}\right)$, while ${}^aF(u)$ has a central charge $\left(-\frac{1}{h_1 h_2}, -\frac{u}{h_1 h_2}\right)$.

4.3. Tensor products $\otimes F(u_i)$

In this section, we relate the representation M^r from Theorem 3.2 to the Fock modules $F(u)$. Let Δ be the *formal* comultiplication on $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ from Remark 4.1. This is not a comultiplication in the usual sense, since $\Delta(e_i)$ and $\Delta(f_i)$ contain infinite sums. However, for all modules of interest "with general spectral parameters" these formulas make sense (this is explained below in the particular case of Fock modules). The following theorem can be viewed as a higher rank generalization of Remark 4.2.

Theorem 4.6. *There exists a unique collection of constants $\{c_{\bar{\lambda}}\}$ from the field $\mathbb{F}_r$ with $c_{\bar{\emptyset}} = 1$ such that the map $[\bar{\lambda}] = [(\lambda^1, \dots, \lambda^r)] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ induces an isomorphism $M^r \xrightarrow{\sim} F(\chi_1^{-1}) \otimes \cdots \otimes F(\chi_r^{-1})$ of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations with $q_i = t_i$.*

Let us first make sense of the tensor product $F(\chi_1) \otimes F(\chi_2)$ (the case $r > 2$ is completely analogous). The action of $e(z)$ and $f(z)$ in the basis $\{|\lambda\rangle\}$ of $F(\chi)$ can be written as follows:

$$e(z)|\lambda\rangle = \sum_{\square} a_{\lambda,\square}\delta\left(\frac{\chi(\square)}{z}\right)|\lambda+\square\rangle,\ f(z)|\lambda\rangle = \sum_{\square} b_{\lambda,\square}\delta\left(\frac{\chi(\square)}{z}\right)|\lambda-\square\rangle,$$

where $a_{\lambda,\square}, b_{\lambda,\square} \in \mathbb{F}_r$, the first sum is over $\square \notin \lambda$ such that $\lambda+\square$ is a Young diagram, while the second sum is over $\square \in \lambda$ such that $\lambda-\square$ is a Young diagram.

According to the comultiplication formula, we have

$$\Delta(e(z))(|\lambda^1\rangle \otimes |\lambda^2\rangle) = e(z)(|\lambda^1\rangle) \otimes |\lambda^2\rangle + \psi^-(z)(|\lambda^1\rangle) \otimes e(z)(|\lambda^2\rangle).$$

The first summand is well defined. To make sense of the second summand, we use the formula

$$g(z)\delta(a/z) = g(a)\delta(a/z) \text{ for any rational function } g(z). \tag{*}$$

Recall that $\psi^{\pm}(z)(|\lambda\rangle) = \gamma_{\lambda}(z)^{\pm} \cdot |\lambda\rangle$, where $\gamma_{\lambda}(z)$ is a rational function in z depending on λ. Combining this with (*), we find

$$\psi^-(z)(|\lambda^1\rangle) \otimes e(z)(|\lambda^2\rangle) = \sum_{\square} a_{\lambda^2,\square}\gamma_{\lambda^1}(\chi(\square))\delta\left(\frac{\chi(\square)}{z}\right) \cdot |\lambda^1\rangle \otimes |\lambda^2+\square\rangle.$$

The action of f_i on $F(\chi_1)\otimes F(\chi_2)$ is defined analogously. Finally, the formula $\Delta(\psi^{\pm}(z)) = \psi^{\pm}(z) \otimes \psi^{\pm}(z)$ provides a well defined action of ψ_i on $F(\chi_1) \otimes F(\chi_2)$.

Proof of Theorem 4.6. Due to Remark 4.2, we can identify $F(\chi_k^{-1}) \simeq M^{\phi_{\chi_k}}$, the twist of M by ϕ_{χ_k}. For any r-tuple of Young diagrams $\bar{\lambda} = (\lambda^1, \dots, \lambda^r)$, Lemma 3.3(b) implies that the eigenvalue of $\psi^{\pm}(z)$ on $[\bar{\lambda}] \in M^r$ equals the eigenvalue of $\psi^{\pm}(z)$ on $|\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle \in F(\chi_1^{-1}) \otimes \cdots \otimes F(\chi_r^{-1})$. Hence, the map $[\bar{\lambda}] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ intertwines actions of ψ_i and ψ_0^{-1} for any constants $c_{\bar{\lambda}}$.

Consider constants $c_{\bar{\lambda}}$ such that $c_{\bar{\emptyset}} = 1$ and $c_{\bar{\lambda}+\square_i^k}/c_{\bar{\lambda}} = d_{\bar{\lambda},\square_i^k}$ with

$$d_{\bar{\lambda},\square_i^k} := \frac{t^{k-1}t_2(-t_1\chi_i^{(k)})^{-r}}{\chi_1 \cdots \chi_r} \cdot \prod_{(l,j)\prec(k,i)} \frac{1-t_1t_2\chi_i^{(k)}/\chi_j^{(l)}}{1-t_1\chi_i^{(k)}/\chi_j^{(l)}} \cdot \prod_{(l,j)\succ(k,i)} \frac{1-\chi_i^{(k)}/\chi_j^{(l)}}{1-t_2^{-1}\chi_i^{(k)}/\chi_j^{(l)}}, \tag{7}$$

where $\chi_p^{(m)} = t_1^{\lambda_p^m - 1}t_2^{p-1}\chi_m^{-1}$ as before and $(l,j) \prec (k,i)$ if and only if $l<k$ or $l=k$ and $j<i$. Since $\chi_{p+1}^{(m)} = t_2\chi_p^{(m)}$ for $p \geq |\lambda^m|$, it is easy to see that the infinite products in (7) are actually finite. The existence of $c_{\bar{\lambda}}$ satisfying $c_{\bar{\lambda}+\square_i^k}/c_{\bar{\lambda}} = d_{\bar{\lambda},\square_i^k}$ is equivalent to

$$d_{\bar{\lambda}+\square_i^k,\square_j^l} \cdot d_{\bar{\lambda},\square_i^k} = d_{\bar{\lambda}+\square_j^l,\square_i^k} \cdot d_{\bar{\lambda},\square_j^l} \text{ for all possible } \square_i^k, \square_j^l.$$

The proof of these equalities as well as verification that the map $[\bar{\lambda}] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ intertwines actions of e_i and f_i are left to the interested reader. The result follows. □

4.4. Tensor products $\otimes {}^aF(u_i)$

In this section, we relate the representation V^r from Theorem 3.4 to the Fock modules ${}^aF(u)$. First we need to define the tensor product $W_1 \otimes W_2$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations W_1, W_2.

Definition 4.2. A $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representation W is called *admissible* if there exists a basis $\{w_\alpha\}_{\alpha\in I}$ of W such that

∘ $e(z)(w_\alpha) = \sum_{\alpha'\in I} \frac{c_{\alpha,\alpha'}}{z}\delta^+(\lambda_{\alpha,\alpha'}/z)w_{\alpha'}$, $f(z)(w_\alpha) = \sum_{\alpha''\in I} \frac{d_{\alpha,\alpha''}}{z}\delta^+(\lambda_{\alpha'',\alpha}/z)w_{\alpha''}$ for some $c_{\alpha,\alpha'}, d_{\alpha,\alpha''}, \lambda_{\alpha,\alpha'} \in \mathbb{C}$, so that both sums are finite for every $\alpha \in I$.

∘ $\psi(z)(w_\alpha) = \gamma_W(\alpha,z)^+ \cdot w_\alpha$ for a rational function $\gamma_W(\alpha,z)$ defined by

$$\gamma_W(\alpha,z) = 1 + \sigma_3 \sum_{\alpha''\in I} \frac{d_{\alpha,\alpha''}c_{\alpha'',\alpha}}{z-\lambda_{\alpha'',\alpha}} - \sigma_3 \sum_{\alpha'\in I} \frac{c_{\alpha,\alpha'}d_{\alpha',\alpha}}{z-\lambda_{\alpha,\alpha'}}.$$

∘ For any $\alpha_1 \neq \alpha_2 \in I$, there is a bijection between $\{\alpha' \in I | c_{\alpha_1,\alpha'}d_{\alpha',\alpha_2} \neq 0\}$ and $\{\alpha'' \in I | d_{\alpha_1,\alpha''}c_{\alpha'',\alpha_2} \neq 0\}$ such that $\lambda_{\alpha_1,\alpha'} = \lambda_{\alpha'',\alpha_2}$ and $\lambda_{\alpha_2,\alpha'} = \lambda_{\alpha'',\alpha_1}$.

Example 4.1. The modules ${}^aV(u)$ and ${}^aF(u)$ are admissible.

Let W_1 and W_2 be admissible $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations with the corresponding bases $\{w^1_\alpha\}_{\alpha\in I}$ and $\{w^2_\beta\}_{\beta\in J}$. Define the operator series $\psi(z), e(z), f(z)$ from $\mathrm{End}(W_1 \otimes W_2)[[z^{-1}]]$ by

$$\begin{aligned}
\psi(z)(w^1_\alpha \otimes w^2_\beta) &= \psi(z)(w^1_\alpha) \otimes \psi(z)(w^2_\beta),\\
e(z)(w^1_\alpha \otimes w^2_\beta) &= \sum_{\alpha'\in I} \frac{c^1_{\alpha,\alpha'}}{z}\delta^+(\lambda^1_{\alpha,\alpha'}/z)w^1_{\alpha'} \otimes w^2_\beta\\
&\quad + \sum_{\beta'\in J} \frac{c^2_{\beta,\beta'}\gamma_{W_1}(\alpha,\lambda^2_{\beta,\beta'})}{z}\delta^+(\lambda^2_{\beta,\beta'}/z)w^1_\alpha \otimes w^2_{\beta'},\\
f(z)(w^1_\alpha \otimes w^2_\beta) &= \sum_{\beta'\in J} \frac{d^2_{\beta,\beta''}}{z}\delta^+(\lambda^2_{\beta'',\beta}/z)w^1_\alpha \otimes w^2_{\beta''}\\
&\quad + \sum_{\alpha''\in I} \frac{d^1_{\alpha,\alpha''}\gamma_{W_2}(\beta,\lambda^1_{\alpha'',\alpha})}{z}\delta^+(\lambda^1_{\alpha'',\alpha}/z)w^1_{\alpha''} \otimes w^2_\beta.
\end{aligned}$$

Remark 4.3. These formulas are well defined only if $\gamma_{W_1}(\alpha,z)$ is regular at $\{\lambda^2_{\beta,\beta'} | c^2_{\beta,\beta'} \neq 0\}$ and $\gamma_{W_2}(\beta,z)$ is regular at $\{\lambda^1_{\alpha'',\alpha} | d^1_{\alpha,\alpha''} \neq 0\}$ for any $\alpha \in I$, $\beta \in J$.

The following is straightforward:

Lemma 4.7. *If W_1 and W_2 are admissible $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations and the assumptions of Remark 4.3 hold, then the above formulas define an action of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ on $W_1 \otimes W_2$.*

Remark 4.4. We refer the interested reader to [23, Section 1] for an alternative viewpoint.

It might be still possible to define an action of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ on a submodule or a factor-module of $W_1 \otimes W_2$ if the assumptions of Remark 4.3 fail, due to our next result:

Lemma 4.8. *Let S be a subset of $I \times J$ such that $e(z)(w^1_\alpha \otimes w^2_\beta)$, $f(z)(w^1_\alpha \otimes w^2_\beta)$ are well defined (in the sense of Remark 4.3) for any $(\alpha, \beta) \in S$ and satisfy one of the following conditions:*

(a) For any $(\alpha, \beta) \in S$, $(\alpha', \beta') \notin S$, the matrix coefficients, $\langle w^1_{\alpha'} \otimes w^2_{\beta'} | e(z) | w^1_\alpha \otimes w^2_\beta \rangle$ and $\langle w^1_{\alpha'} \otimes w^2_{\beta'} | f(z) | w^1_\alpha \otimes w^2_\beta \rangle$ vanish.
(b) For any $(\alpha, \beta) \in S$, $(\alpha', \beta') \notin S$, the matrix coefficients, $\langle w^1_\alpha \otimes w^2_\beta | e(z) | w^1_{\alpha'} \otimes w^2_{\beta'} \rangle$ and $\langle w^1_\alpha \otimes w^2_\beta | f(z) | w^1_{\alpha'} \otimes w^2_{\beta'} \rangle$ vanish.

Then the above formulas define a $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action on span $\{w^1_\alpha \otimes w^2_\beta\}_{(\alpha,\beta)\in S}$.

The key result of this section is proved completely analogously to Theorem 4.6:

Theorem 4.9. *There exists a unique collection of constants $\{c^a_{\bar{\lambda}}\}$ from $\mathbb{C}(s_1, s_2, x_1, \dots, x_r)$ with $c^a_{\bar{\emptyset}} = 1$ such that the map $[\bar{\lambda}] = [(\lambda^1, \dots, \lambda^r)] \mapsto c^a_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ induces an isomorphism $V^r \xrightarrow{\sim} {}^aF(-x_1) \otimes \cdots \otimes {}^aF(-x_r)$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations with $h_i = s_i$.*

Remark 4.5. As ${}^aF(0) \simeq V$ by Lemma 4.5(c), we get $V^r \simeq V^{\phi^a_{x_1}} \otimes \cdots \otimes V^{\phi^a_{x_r}}$. In other words, the representation of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ on the sum of equivariant cohomology groups of $M(r, n)$ is a tensor product of r copies of such representations for $M(1, m)$.

4.5. Other families of representations

We recall some other series of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations from [5,6]. All of them admit straightforward generalizations for the case of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$. These have the same bases, while the matrix coefficients of $e(z), f(z), \psi(z)$ in these bases are modified as follows:

$$1 - q_1^i q_2^j q_3^k u/z \rightsquigarrow z - (ih_1 + jh_2 + kh_3 + u),$$
$$\delta(q_1^i q_2^j q_3^k u/z) \rightsquigarrow \pm \frac{1}{z} \delta^+((ih_1 + jh_2 + kh_3 + u)/z),$$

where the latter sign is "+" for $e(z)$ and "−" for $f(z)$.

• *Representations* $W^N(u)$.

Consider the tensor product $V^N(u) := V(u) \otimes V(uq_3^{-1}) \otimes V(uq_3^{-2}) \otimes \cdots \otimes V(uq_3^{1-N})$. Define $\mathcal{P}^N := \{\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N | \lambda_1 \geq \cdots \geq \lambda_N\}, \mathcal{P}^{N,+} := \{\lambda \in \mathcal{P}^N | \lambda_N \geq 0\}$. Let $W^N(u) \subset V^N(u)$ be the subspace spanned by $[u]_\lambda := [u]_{\lambda_1} \otimes [uq_3^{-1}]_{\lambda_2 - 1} \otimes \cdots \otimes [uq_3^{1-N}]_{\lambda_N - N + 1}$ for $\lambda \in \mathcal{P}^N$.

According to [5, Lemma 3.7], $W^N(u)$ is a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-submodule of $V^N(u)$. The subspace $W^{N,+}(u) \subset W^N(u)$ corresponding to $\mathcal{P}^{N,+}$ is not a submodule. However, its limit as $N \to \infty$ is well defined (after a renormalization) and coincides with $F(u)$.

• *Representations* $G^{k,r}_{\mathbf{a}}$.

Let q_1, q_2 be in the (k,r)-*resonance relation*: $q_1^a q_2^b = 1$ iff $a = (1-r)c, b = (k+1)c$ for some $c \in \mathbb{Z}$ (assume $k \geq 1, r \geq 2$). In this case the action of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ on $W^N(u)$ is ill defined. Consider the set of (k,r)-*admissible* partitions $S^{k,r,N} := \{\lambda \in \mathcal{P}^N | \lambda_i - \lambda_{i+k} \geq r \ \forall\ i \leq N-k\}$.

Let $W^{k,r,N}(u)$ be the subspace of $W^N(u)$ spanned by $[u]_\lambda$ for $\lambda \in S^{k,r,N}$. According to [5, Lemma 6.2], the comultiplication rule makes $W^{k,r,N}(u)$ into a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module. We think of it as "a submodule of $W^N(u)$ or $V^N(u)$" even though none of them has a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module structure (we use an analogue of Lemma 4.8 there).

Let us fix a sequence of non-negative integers $\mathbf{a} = (a_1, \ldots, a_k)$ satisfying $\sum_{i=1}^k a_i = r$. Define $\mathcal{P}^{k,r}_{\mathbf{a}} := \{(\lambda_1 \geq \lambda_2 \geq \cdots) | \lambda_j - \lambda_{j+k} \geq r \ \forall\ j \geq 1 \text{ and } \lambda_j = \lambda_j^0 \ \forall\ j \gg 0\}$, where we set $\lambda^0_{\nu k + i + 1} := -\nu r - \sum_{j=1}^i a_j$ for $\nu \in \mathbb{Z}_+, 0 \leq i \leq k-1$. One can define an action of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ on the $N \to \infty$ limit of $W^{k,r,N}(u)$, yielding an action of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ on the space $G^{k,r}_{\mathbf{a}}$ whose basis is labeled by $\lambda \in \mathcal{P}^{k,r}_{\mathbf{a}}$, see [5, Theorem 6.5].

• *Representations* $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$.

Consider the tensor product $F(u_1) \otimes \cdots \otimes F(u_n)$. It is well defined as a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module if $q_1, q_2, u_1, \ldots, u_n$ are *generic*, i.e., $q_1^a q_2^b u_1^{c_1} \cdots u_n^{c_n} = 1 \Leftrightarrow a = b = c_1 = \ldots = c_n = 0$.

Consider the *resonance case* $u_i = u_{i+1} q_1^{a_i+1} q_2^{b_i+1}$ for $1 \leq i \leq n-1$ and some $a_i, b_i \in \mathbb{Z}_+$. Set $u := u_1$. Let $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u) \subset F(u_1) \otimes \cdots \otimes F(u_n)$ be the subspace spanned by $|\lambda^1, \ldots, \lambda^n\rangle := [u_1]_{\lambda^1} \otimes \cdots \otimes [u_n]_{\lambda^n}$, where Young diagrams $\lambda^1, \ldots, \lambda^n$ satisfy $\lambda^i_s \geq \lambda^{i+1}_{s+b_i} - a_i$ for $1 \leq i \leq n-1, s \in \mathbb{N}$. According to [6, Proposition 3.3], the comultiplication rule makes $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ into a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module for *generic* q_1, q_2, u.

• *Representations* $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$.

Assume further that q_1, q_2 are not generic: there exist $p \neq p' \in \mathbb{N}$ such that $q_1^a q_2^b = 1$ iff $a = p'c$, $b = pc$ for some $c \in \mathbb{Z}$. We require that $a_n := p' - 1 - \sum_{i=1}^{n-1}(a_i + 1), b_n := p - 1 - \sum_{i=1}^{n-1}(b_i + 1)$ are non-negative. In this case, the action of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ on $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ is ill defined.

Consider a subspace $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u) \subset F(u_1) \otimes \cdots \otimes F(u_n)$ spanned by $|\lambda^1, \ldots, \lambda^n\rangle$ satisfying the same conditions $\lambda^i_s \geq \lambda^{i+1}_{s+b_i} - a_i$ but for $1 \leq i \leq n$, where we set $\lambda^{n+1} := \lambda^1$. The comultiplication rule makes it into a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module, due to [6, Proposition 3.7]. We think of $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ as "a subquotient of $F(u_1) \otimes \cdots \otimes F(u_n)$". Their characters coincide with the characters from the $\mathcal{W}_n$-minimal series, according to the main result of [6].

4.6. Categories $\mathcal{O}$

We conclude this section by introducing the appropriate categories $\mathcal{O}$ for both algebras.

• Category $\mathcal{O}$ for $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$.

We equip $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ with the $\mathbb{Z}$-grading by assigning $\deg(e_i) = -1$, $\deg(f_i) = 1$, $\deg(\psi_i) = 0$, $\deg(\psi_0^{-1}) = 0$.

Definition 4.3. We say that a $\mathbb{Z}$-graded $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module L is in the category $\mathcal{O}$ if

(i) for any $v \in L$ there exists $N \in \mathbb{Z}$ such that $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)_{\geq N}(v) = 0$,

(ii) L is of *finite type*, that is, all graded components $\{L_k\}_{k\in\mathbb{Z}}$ are finite dimensional.

We say that L is a *highest weight* module if there exists $v_0 \in L$ generating L and such that $f_i(v_0) = 0$, $\psi_i(v_0) = p_i \cdot v_0$ $\psi_0^{-1}(v_0) = p_0^{-1} \cdot v_0$ for all $i \in \mathbb{Z}$ and some $p_i \in \mathbb{C}$ with $p_0 \neq 0$. To such a collection $\{p_i\}$, we associate two series $p^{\pm}(z) := p_0^{\pm 1} + \sum_{m=1}^{\infty} p_{\pm m} z^{\mp m} \in \mathbb{C}[[z^{\mp 1}]]$. Given any two such series $p^{\pm}(z) \in \mathbb{C}[[z^{\mp 1}]]$, there is a universal highest weight representation M_{p^+,p^-}, which may be defined as the quotient of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ by the left ideal generated by $\{f_i, \psi_i - p_i, \psi_0^{-1} - p_0^{-1}\}_{i\in\mathbb{Z}}$. Standard arguments show that M_{p^+,p^-} has a unique irreducible quotient V_{p^+,p^-}, which obviously satisfies the condition (i) of Definition 4.3.

Our next result provides a criterion for V_{p^+,p^-} to satisfy the condition (ii), or equivalently to be in the category $\mathcal{O}$.

Proposition 4.10. *The module V_{p^+,p^-} is of finite type if and only if there exists a rational function $P(z)$ such that $p^{\pm}(z) = P(z)^{\pm}$ and $P(0)P(\infty) = 1$.*

Proof. Our proof is standard and is based on the arguments from [3]. Define constants $\{\bar{p}_i\}_{i\in\mathbb{Z}}$ as p_i (for $i > 0$), $-p_i$ (for $i < 0$), and $p_0 - p_0^{-1}$ (for $i = 0$). To prove the "only if" part, we choose indices $k \in \mathbb{Z}, l \in \mathbb{Z}_+$ such that $\{e_k(v_0), \ldots, e_{k+l}(v_0)\}$ span the degree -1 component $(V_{p^+,p^-})_{-1}$ and $a_0 e_k(v_0) + a_1 e_{k+1}(v_0) + \ldots + a_l e_{k+l}(v_0) = 0$ for some complex numbers $a_0, \ldots, a_l \in \mathbb{C}$ with $a_l \neq 0$. Applying f_{r-k} to the above equality and using the relation (T3) in the form $f_i e_j(v_0) = -\beta_1^{-1} \bar{p}_{i+j} \cdot v_0$, we get $a_0\bar{p}_r + a_1\bar{p}_{r+1} + \ldots + a_l\bar{p}_{r+l} = 0$ for all $r \in \mathbb{Z}$. Therefore, the collection $\{\bar{p}_i\}_{i\in\mathbb{Z}}$ satisfies a simple recurrence relation. Solving this recurrence relation and using the condition $\bar{p}_0 = p_0 - p_0^{-1}$, we immediately see that $p^{\pm}(z)$ are extensions in $z^{\mp 1}$ of the same rational function.

To prove the "if" direction, let us assume that $p^{\pm}(z) = P(z)^{\pm}$ for a rational function $P(z)$. Reversing the above arguments, we get $\dim(V_{p^+,p^-})_{-1} < \infty$. Combining this with (T1), a simple induction argument implies that $\dim(V_{p^+,p^-})_{-l} < \infty$ for all $l \in \mathbb{N}$. □

• Category $\mathcal{O}$ for $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$.

We equip $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ with the $\mathbb{Z}$-grading by assigning $\deg(e_j) = -1$, $\deg(f_j) = 1$, $\deg(\psi_j) = 0$.

Definition 4.4. We say that a $\mathbb{Z}$-graded $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-module L is in the category $\mathcal{O}$ if

(i) for any $v \in L$ there exists $N \in \mathbb{Z}$ such that $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)_{\geq N}(v) = 0$,

(ii) L is of *finite type*, that is, all graded components $\{L_k\}_{k\in\mathbb{Z}}$ are finite dimensional.

We say that L is a *highest weight* module if there exists $v_0 \in L$ generating L and such that $f_j(v_0) = 0,\ \psi_j(v_0) = p_j \cdot v_0$ for all $j \in \mathbb{Z}_+$ and some $p_j \in \mathbb{C}$. Set $p(z) := 1 + \sum_{j\geq 0} p_j z^{-j-1} \in \mathbb{C}[[z^{-1}]]$. For any $\{p_j\}$, there is a universal highest weight representation M_p, which may be defined as the quotient of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ by the left ideal generated by $\{f_j, \psi_j - p_j\}_{j\in\mathbb{Z}_+}$. It has a unique irreducible quotient V_p.

Module V_p obviously satisfies the condition (i) of Definition 4.4. The following criterion for V_p to be in the category $\mathcal{O}$ is completely analogous to the one for V_{p^+,p^-}:

Proposition 4.11. *The module V_p is in the category $\mathcal{O}$ if and only if there exists a rational function $P(z)$ such that $p(z) = P(z)^+$ and $P(\infty) = 1$.*

5. Limit algebras

The goal of this section is to relate certain limits of our two algebras of interest to the well-known algebras of difference operators on $\mathbb{C}$ and $\mathbb{C}^*$. Throughout this section, h is a formal variable.

5.1. Difference operators on $\mathbb{C}^*$

Set $q = \exp(h) \in \mathbb{C}[[h]]$. The algebra of *$q$-difference operators on $\mathbb{C}^*$*, denoted by $\mathfrak{d}_q$, is the unital associative $\mathbb{C}[[h]]$-algebra topologically generated by $Z^{\pm 1}, D^{\pm 1}$ subject to the relations:

$$Z \cdot Z^{-1} = Z^{-1} \cdot Z = 1,\ D \cdot D^{-1} = D^{-1} \cdot D = 1,\ D \cdot Z = qZ \cdot D.$$

We will view $\mathfrak{d}_q$ as a Lie algebra with the natural commutator-Lie bracket $[\cdot,\cdot]$. It is easy to check that the following formula defines a 2-cocycle $\phi_{\mathfrak{d}} \in C^2(\mathfrak{d}_q, \mathbb{C}[[h]])$:

$$\phi_{\mathfrak{d}}(Z^a D^j, Z^b D^{-j'}) = \begin{cases} \sum_{i=-j}^{-1} q^{ai+b(i+j)} & \text{if } \ j = j' > 0 \\ -\sum_{i=j}^{-1} q^{bi+a(i-j)} & \text{if } \ j = j' < 0 \\ 0 & \text{otherwise} \end{cases}.$$

This endows $\bar{\mathfrak{d}}_q = \mathfrak{d}_q \oplus \mathbb{C}[[h]] \cdot c_{\mathfrak{d}}$ with the Lie algebra structure.

5.2. Difference operators on $\mathbb{C}$

The algebra of *h-difference operators on $\mathbb{C}$*, denoted by $\mathfrak{D}_h$, is the unital associative $\mathbb{C}[[h]]$-algebra topologically generated by $x, \partial^{\pm 1}$ subject to the following defining relations:

$$\partial \cdot \partial^{-1} = \partial^{-1} \cdot \partial = 1,\ \partial \cdot x = (x+h) \cdot \partial.$$

We will view $\mathfrak{D}_h$ as a Lie algebra with the natural commutator-Lie bracket $[\cdot,\cdot]$. It is easy to check that the following formula defines a 2-cocycle $\phi_{\mathfrak{D}} \in C^2(\mathfrak{D}_h, \mathbb{C}[[h]])$:

$$\phi_{\mathfrak{D}}(f(x)\partial^r, g(x)\partial^{-s}) = \begin{cases} \sum_{l=-r}^{-1} f(lh)g((l+r)h) & \text{if } r = s > 0 \\ -\sum_{l=r}^{-1} g(lh)f((l-r)h) & \text{if } r = s < 0 \\ 0 & \text{otherwise} \end{cases}.$$

This endows $\bar{\mathfrak{D}}_h = \mathfrak{D}_h \oplus \mathbb{C}[[h]] \cdot c_{\mathfrak{D}}$ with the Lie algebra structure.

5.3. *Isomorphism* Υ_0

Let us introduce the appropriate completions of the algebras $\bar{\mathfrak{d}}_q$ and $\bar{\mathfrak{D}}_h$:

- $\widehat{\bar{\mathfrak{d}}_q}$ is the completion of $\bar{\mathfrak{d}}_q$ with respect to the powers of the two-sided ideal $J_{\mathfrak{d}} = (Z-1, q-1)$;
- $\widehat{\bar{\mathfrak{D}}_h}$ is the completion of $\bar{\mathfrak{D}}_h$ with respect to the powers of the two-sided ideal $J_{\mathfrak{D}} = (x, h)$.

In other words, we have:

$$\widehat{\bar{\mathfrak{d}}_q} := \varprojlim \bar{\mathfrak{d}}_q / \bar{\mathfrak{d}}_q \cdot (Z-1, q-1)^j,\ \widehat{\bar{\mathfrak{D}}_h} := \varprojlim \bar{\mathfrak{D}}_h / \bar{\mathfrak{D}}_h \cdot (x,h)^j.$$

Remark 5.1. Taking completions of $\mathfrak{d}_q$ and $\mathfrak{D}_h$ with respect to the ideals $J_{\mathfrak{d}}$ and $J_{\mathfrak{D}}$ commutes with taking central extensions with respect to the 2-cocycles $\phi_{\mathfrak{d}}$ and $\phi_{\mathfrak{D}}$.

The following result is straightforward:

Proposition 5.1. *The assignment*

$$D^{\pm 1} \mapsto \partial^{\pm 1},\ Z^{\pm 1} \mapsto e^{\pm x},\ c_{\mathfrak{d}} \mapsto c_{\mathfrak{D}}$$

extends to an isomorphism $\Upsilon_0 : \widehat{\bar{\mathfrak{d}}_q} \xrightarrow{\sim} \widehat{\bar{\mathfrak{D}}_h}$ *of* $\mathbb{C}[[h]]$*-algebras.*

Remark 5.2. Specializing h to a complex parameter $h_0 \in \mathbb{C}$, we get the classical $\mathbb{C}$-algebras of difference operators $\mathfrak{d}_q$ and $\mathfrak{D}_{h_0}$ as well as their one-dimensional central extensions $\bar{\mathfrak{d}}_q$ and $\bar{\mathfrak{D}}_{h_0}$, where $q = \exp(h_0) \in \mathbb{C}^*$. In other words, we consider the $\mathbb{C}$-algebras given by the same collections of the generators and the defining relations. However, one can not define their completions as above. This explains our preference to work over $\mathbb{C}[[h]]$.

5.4. Algebras $\ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ and $\ddot{U}'_h(\mathfrak{gl}_1)$

Throughout this section, we let h_2, h_3 be formal variables and set $h_1 := -h_2 - h_3$. We define $q_i := \exp(h_i) \in \mathbb{C}[[h_2, h_3]]$ for $i = 1, 2, 3$. In order to consider a *formal version* of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, that is, the $\mathbb{C}[[h_2, h_3]]$-algebra with the same generators and defining relations, we need to modify (T3) in an appropriate way. First, we *renormalize* (T3) to the following form:

$$[e(z), f(w)] = \delta(z/w)(\psi^+(w) - \psi^-(z))/(1 - q_3). \tag{T3$'$}$$

In the case of specialized values $q_i \in \mathbb{C}\backslash\{0, 1\}$, this corresponds to *rescaling* e_i, f_i by $1 - q_1, 1 - q_2$. Next, we present $\psi^{\pm}(z)$ in the form

$$\psi^{\pm}(z) = \exp\left(\mp\frac{h_3}{2}H_0\right) \cdot \exp\left(\pm(1 - q_3)\sum_{m>0} H_{\pm m} z^{\mp m}\right).$$

Switching from the generators $\{\psi_i, \psi_0^{-1}\}$ to $\{H_i\}$, the relations (T4,T5) get modified to:

$$[H_0, e_j] = 0, \quad [H_i, e_j] = -\frac{(1 - q_1^i)(1 - q_2^i)(1 - q_3^i)}{i(1 - q_3)} e_{i+j} \text{ for } i \in \mathbb{Z}^*, j \in \mathbb{Z}, \tag{T4H}$$

$$[H_0, f_j] = 0, \quad [H_i, f_i] = \frac{(1 - q_1^i)(1 - q_2^i)(1 - q_3^i)}{i(1 - q_3)} f_{i+j} \text{ for } i \in \mathbb{Z}^*, j \in \mathbb{Z}. \tag{T5H}$$

These relations are well defined in the formal setting since $\frac{(1-q_1^i)(1-q_2^i)(1-q_3^i)}{1-q_3} \in \mathbb{C}[[h_2, h_3]]$. Note that the right-hand side of (T3$'$) is also a polynomial in H_i with coefficients in $\mathbb{C}[[h_2, h_3]]$.

Definition 5.1. $\ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is the unital associative $\mathbb{C}[[h_2, h_3]]$-algebra topologically generated by $\{e_i, f_i, H_i | i \in \mathbb{Z}\}$ with the defining relations (T0,T1,T2,T3$'$,T4H,T5H,T6).

Finally, we define $\ddot{U}'_h(\mathfrak{gl}_1)$ by

$$\ddot{U}'_h(\mathfrak{gl}_1) := \ddot{U}'_{-h-h_3,h,h_3}(\mathfrak{gl}_1)/(h_3).$$

It is a unital associative $\mathbb{C}[[h]]$-algebra topologically generated by $\{e_i, f_i, H_i | i \in \mathbb{Z}\}$ subject to the relations (T1,T2,T6) and

$$[H_i, H_j] = 0, \tag{T0L}$$

$$[e_i, f_j] = H_{i+j}, \tag{T3L}$$

$$[H_i, e_j] = -(1 - q^i)(1 - q^{-i}) e_{i+j}, \tag{T4tL}$$

$$[H_i, f_j] = (1-q^i)(1-q^{-i})f_{i+j}, \tag{T5tL}$$

where $i, j \in \mathbb{Z}$ and $q = \exp(h) \in \mathbb{C}[[h]]$.

Remark 5.3. For $h_0 \in \mathbb{C}$, define $\ddot{U}_{h_0}(\mathfrak{gl}_1)$ as the $\mathbb{C}$-algebra generated by $\{e_i, f_i, H_i | i \in \mathbb{Z}\}$ with the same defining relations (T0L,T1,T2,T3L,T4tL,T5tL,T6), where $q = e^{h_0} \in \mathbb{C}^*$.

The following result is straightforward:

Proposition 5.2. *The assignment*

$$e_i \mapsto Z^i D, \ \ f_i \mapsto -D^{-1}Z^i, \ \ H_i \mapsto -(1-q^{-i})Z^i - q^{-i}c_{\mathfrak{d}}$$

extends to a homomorphism $\theta : \ddot{U}'_h(\mathfrak{gl}_1) \to U(\bar{\mathfrak{d}}_q)$.

Proof. It suffices to show that all the defining relations of $\ddot{U}'_h(\mathfrak{gl}_1)$ are preserved under the above assignment. This is a simple exercise, which we leave to the interested reader. □

Let $\bar{\mathfrak{d}}^0_q \subset \bar{\mathfrak{d}}_q$ be the free $\mathbb{C}[[h]]$-submodule spanned by

$$\{c_{\mathfrak{d}}, hZ^k D^0, h^{j-1} Z^i D^{\pm j} | i \in \mathbb{Z}, k \in \mathbb{Z}^*, j \in \mathbb{N}\}.$$

Lemma 5.3. $\bar{\mathfrak{d}}^0_q$ *is a Lie subalgebra of* $\bar{\mathfrak{d}}_q$ *and* $\mathrm{Im}(\theta) \subset U(\bar{\mathfrak{d}}^0_q)$.

In fact, we have the following result:

Theorem 5.4. *The homomorphism* θ *provides an isomorphism* $\theta : \ddot{U}'_h(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathfrak{d}}^0_q)$.

Note that all the defining relations of $\ddot{U}'_h(\mathfrak{gl}_1)$ are of Lie-type. Hence, $\ddot{U}'_h(\mathfrak{gl}_1)$ is an enveloping algebra of the Lie algebra generated by e_i, f_i, H_i with the aforementioned defining relations. Thus, Theorem 5.4 provides a presentation of the Lie algebra $\bar{\mathfrak{d}}^0_q$ by generators and relations.

Actually, we will prove a more general result in Appendix C:

Theorem 5.5. *If* $h_0 \in \mathbb{C} \backslash \{\mathbb{Q} \cdot \pi\sqrt{-1}\}$, *then* θ *induces an isomorphism of the* $\mathbb{C}$*-algebras:* $\theta : \ddot{U}_{h_0}(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathfrak{d}}^0_q)$, *where* $\bar{\mathfrak{d}}^0_q = \mathrm{span}_{\mathbb{C}}\{c_{\mathfrak{d}}, Z^i D^j\}_{(i,j)\neq(0,0)}$ *is a* $\mathbb{C}$*-Lie subalgebra of* $\bar{\mathfrak{d}}_q$.

5.5. *Algebras* $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ *and* $\ddot{Y}'_h(\mathfrak{gl}_1)$

Analogously to the previous section, let h_2, h_3 be formal variables and set $h_1 := -h_2 - h_3$. We view $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ as a *formal version* of the corresponding algebra introduced in Section 1.3. In other words, $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is the unital associative $\mathbb{C}[[h_2, h_3]]$-algebra topologically generated by $\{e_j, f_j, \psi_j | j \in \mathbb{Z}_+\}$ subject to the relations (Y0–Y6).

Let us renormalize the relations (Y4′) and (Y5′) to make them homogeneous:

$$[\psi_0, e_j] = 0, \quad [\psi_1, e_j] = 0, \quad [\psi_2, e_j] = -2h_1h_2e_j, \tag{Y4′H}$$

$$[\psi_0, f_j] = 0, \quad [\psi_1, f_j] = 0, \quad [\psi_2, f_j] = 2h_1h_2f_j. \tag{Y5′H}$$

Definition 5.2. $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is the unital associative $\mathbb{C}[[h_2, h_3]]$-algebra topologically generated by $\{e_j, f_j, \psi_j | j \in \mathbb{Z}_+\}$ subject to the relations (Y0–Y3,Y4,Y4′H,Y5,Y5′H,Y6).

We equip the algebra $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ with the $\mathbb{Z}_+$-grading by assigning

$$\deg(e_j) := j, \ \deg(f_j) := j, \ \deg(\psi_j) := j, \ \deg(h_k) := 1 \ \text{ for } j \in \mathbb{Z}_+, k \in \{1, 2, 3\}.$$

Finally, we define $\ddot{Y}'_h(\mathfrak{gl}_1)$ by

$$\ddot{Y}'_h(\mathfrak{gl}_1) := \ddot{Y}'_{-h-h_3,h,h_3}(\mathfrak{gl}_1)/(h_3).$$

It is an associative algebra over $\mathbb{C}[[h]]$. Its specialization at $h_0 \in \mathbb{C}$ is denoted by $\ddot{Y}_{h_0}(\mathfrak{gl}_1)$.

The following result is straightforward:

Proposition 5.6. *The assignment*

$$e_j \mapsto x^j\partial, \ f_j \mapsto -\partial^{-1}x^j, \ \psi_j \mapsto (x-h)^j - x^j - (-h)^j c_{\mathfrak{D}}$$

extends to an algebra homomorphism $\vartheta : \ddot{Y}'_h(\mathfrak{gl}_1) \to U(\bar{\mathfrak{D}}_h)$.

Let $\bar{\mathfrak{D}}^0_h \subset \bar{\mathfrak{D}}_h$ be the free $\mathbb{C}[[h]]$-submodule spanned by $\{c_{\mathfrak{D}}, hx^i\partial^0, h^{j-1}x^i\partial^{\pm j}\}_{i\in\mathbb{Z}_+}^{j\in\mathbb{N}}$.

Lemma 5.7. $\bar{\mathfrak{D}}^0_h$ *is a Lie subalgebra of* $\bar{\mathfrak{D}}_h$ *and* $\mathrm{Im}(\vartheta) \subset U(\bar{\mathfrak{D}}^0_h)$.

In fact, we have the following result:

Theorem 5.8. *The homomorphism* ϑ *provides an isomorphism* $\vartheta : \ddot{Y}'_h(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathfrak{D}}^0_h)$.

Note that all the defining relations of $\ddot{Y}'_h(\mathfrak{gl}_1)$ are of Lie-type. Hence, $\ddot{Y}'_h(\mathfrak{gl}_1)$ is an enveloping algebra of the Lie algebra generated by e_j, f_j, ψ_j with the aforementioned defining relations. Thus, Theorem 5.8 provides a presentation of the Lie algebra $\bar{\mathfrak{D}}^0_h$ by generators and relations.

Actually, we will prove a more general result in Appendix C:

Theorem 5.9. *For* $h_0 \neq 0$, ϑ *induces an isomorphism of* $\mathbb{C}$*-algebras* $\vartheta : \ddot{Y}_{h_0}(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathfrak{D}}_{h_0})$.

6. Key homomorphism and flatness of deformations

Following [10], we construct an algebra homomorphism $\Upsilon : \ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1) \to \widehat{\ddot{Y}}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ and establish a *compatible* isomorphism of their faithful representations. We also prove the flatness result (Theorem 6.6) for both $\ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ and $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$.

6.1. Homomorphism Υ

Let $\widehat{\ddot{Y}}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ be the completion of $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ with respect to the $\mathbb{Z}_+$-grading from Section 5.5. To state our main result, we introduce the following notation (see [10]):

- Define $\psi(z)$ as in Section 1.4: $\psi(z) := 1 - h_3 \sum_{j\geq 0} \psi_j z^{-j-1} \in \ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)[[z^{-1}]]$.
- Define $k_j \in \mathbb{C}[\psi_0, \psi_1, \psi_2, \ldots]$ via $\sum_{j\geq 0} k_j z^{-j-1} = k(z) := \log(\psi(z))$.
- Define the *inverse Borel transform*:

$$B : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[w]] \text{ by } \sum_{j=0}^{\infty} \frac{a_j}{z^{j+1}} \mapsto \sum_{j=0}^{\infty} \frac{a_j}{j!} w^j.$$

- Define $B(w) \in h_3 \ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)[[w]]$ to be the inverse Borel transform of $k(z)$.
- Define a function $G(v) := \log\left(\frac{v}{e^{v/2}-e^{-v/2}}\right) \in v\mathbb{Q}[[v]]$.
- Define $\gamma(v) := -B(-\partial_v)G'(v) \in \widehat{\ddot{Y}}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)[[v]]$.
- Define $\mathrm{g}(v) = \sum_{j\geq 0} \mathrm{g}_j v^j \in \widehat{\ddot{Y}}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)[[v]]$ by $\mathrm{g}(v) := \left(\frac{h_3}{q_3-1}\right)^{1/2} \cdot \exp\left(\frac{\gamma(v)}{2}\right)$.

The identity $B(\log(1-s/z)) = (1-e^{sw})/w$ immediately implies the following result:

Corollary 6.1. *The equalities from Proposition 1.5(e,f) are equivalent to*

$$[B(w), e_j] = \frac{\sum_{i=1}^{3}(e^{h_i w} - e^{-h_i w})}{w}\ e^{w\sigma^+} e_j,\ \ [B(w), f_j] = \frac{\sum_{i=1}^{3}(e^{-h_i w} - e^{h_i w})}{w}\ e^{w\sigma^-} f_j.$$

Now we are ready to state the main result of this section.

Theorem 6.2. *The assignment*

$$H_0 \mapsto \psi_0,\ H_m \mapsto \frac{B(m)}{1-q_3},\ e_k \mapsto e^{k\sigma^+}\mathrm{g}(\sigma^+)e_0,\ f_k \mapsto e^{k\sigma^-}\mathrm{g}(\sigma^-)f_0 \quad \text{for } k \in \mathbb{Z}, m \in \mathbb{Z}^* \quad (\ddagger)$$

extends to an algebra homomorphism

$$\Upsilon : \ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1) \to \widehat{\ddot{Y}}'_{h_1,h_2,h_3}(\mathfrak{gl}_1).$$

The proof of this theorem is presented in Section 6.6.

6.2. Limit of Υ

Recall the isomorphisms from Theorems 5.4 and 5.8:

$$\theta : \ddot{U}'_{-h-h_3,h,h_3}(\mathfrak{gl}_1)/(h_3) \xrightarrow{\sim} U(\bar{\mathfrak{d}}^0_q),$$

$$\vartheta : \ddot{Y}'_{-h-h_3,h,h_3}(\mathfrak{gl}_1)/(h_3) \xrightarrow{\sim} U(\bar{\mathfrak{D}}^0_h).$$

Note that Υ factors through the factors by (h_3), inducing $\Upsilon_{|_{h_3=0}} : U(\bar{\mathfrak{d}}^0_q) \to \widehat{U(\bar{\mathfrak{D}}^0_h)}$.

Proposition 6.3. *The limit homomorphism $\Upsilon_{|_{h_3=0}}$ is induced by Υ_0.*

Proof. It suffices to compute explicitly the images of the generators under $\Upsilon_{|_{h_3=0}}$:

- $\Upsilon_{|_{h_3=0}}(c_{\mathfrak{d}}) = c_{\mathfrak{D}}$.
- $\Upsilon_{|_{h_3=0}}((q^{-i}-1)Z^i - q^{-i}c_{\mathfrak{d}}) = \sum_{k\geq 0}((x-h)^k - x^k - (-h)^k c_{\mathfrak{D}})\frac{i^k}{k!} = (q^{-i}-1)e^{ix} - q^{-i}c_{\mathfrak{D}}$.
- $\Upsilon_{|_{h_3=0}}(Z^i D) = \sum_{k\geq 0}\frac{i^k}{k!}\cdot x^k\partial = e^{ix}\partial$.
- $\Upsilon_{|_{h_3=0}}(-D^{-1}Z^i) = -\sum_{k\geq 0}\frac{i^k}{k!}\partial^{-1}\cdot x^k = -\partial^{-1}e^{ix}$. $\square$

6.3. Elliptic Hall algebra

We recall the elliptic Hall algebra studied in [2]. First, we introduce the following notation:

- Define $(\mathbb{Z}^2)^* := \mathbb{Z}^2\backslash\{(0,0)\}$, $(\mathbb{Z}^2)^{\pm} := \{(a,b) | \pm a > 0 \text{ or } a = 0, \pm b > 0\}$.
- For $\mathbf{x} = (a,b) \in (\mathbb{Z}^2)^*$, we define $\deg(\mathbf{x}) := \gcd(a,b) \in \mathbb{N}$.
- For $\mathbf{x} \in (\mathbb{Z}^2)^*$, we define $\epsilon_{\mathbf{x}} := 1$ if $\mathbf{x} \in (\mathbb{Z}^2)^+$ and $\epsilon_{\mathbf{x}} := -1$ if $\mathbf{x} \in (\mathbb{Z}^2)^-$.
- For non-collinear $\mathbf{x}, \mathbf{y} \in (\mathbb{Z}^2)^*$, we define $\epsilon_{\mathbf{x},\mathbf{y}} := \mathrm{sign}(\det(\mathbf{x},\mathbf{y})) \in \{\pm 1\}$.
- For non-collinear $\mathbf{x}, \mathbf{y} \in (\mathbb{Z}^2)^*$, we denote the triangle with vertices $\{(0,0), \mathbf{x}, \mathbf{x}+\mathbf{y}\}$ by $\triangle_{\mathbf{x},\mathbf{y}}$.
- We say that $\triangle_{\mathbf{x},\mathbf{y}}$ is *empty* if there are no lattice points inside this triangle.
- For $n \in \mathbb{Z}^*$, we define $\alpha_n := -\frac{\beta_n}{n} = \frac{(1-q_1^{-n})(1-q_2^{-n})(1-q_3^{-n})}{n}$.

Definition 6.1 *([2])*. The (central extension of) elliptic Hall algebra $\widetilde{\mathcal{E}}$ is the unital associative algebra generated by $\{u_{\mathbf{x}}, \kappa_{\mathbf{y}} | \mathbf{x} \in (\mathbb{Z}^2)^*, \mathbf{y} \in \mathbb{Z}^2\}$ with the following defining relations:

$$\kappa_{\mathbf{x}}\kappa_{\mathbf{y}} = \kappa_{\mathbf{x}+\mathbf{y}}, \ \kappa_{0,0} = 1, \tag{E0}$$

$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \delta_{\mathbf{x},-\mathbf{y}} \cdot \frac{\kappa_{\mathbf{x}} - \kappa_{\mathbf{x}}^{-1}}{\alpha_{\deg(\mathbf{x})}} \text{ if } \mathbf{x}, \mathbf{y} \text{ are collinear}, \tag{E1}$$

$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \epsilon_{\mathbf{x},\mathbf{y}}\kappa_{\alpha(\mathbf{x},\mathbf{y})}\frac{\theta_{\mathbf{x}+\mathbf{y}}}{\alpha_1} \text{ if } \triangle_{\mathbf{x},\mathbf{y}} \text{ is empty and } \deg(\mathbf{x}) = 1, \tag{E2}$$

where the elements $\{\theta_{\mathbf{x}}|\mathbf{x}\in(\mathbb{Z}^2)^*\}$ are determined from the equality

$$\sum_{n\geq 0}\theta_{n\mathbf{x}_0}x^n=\exp\left(\sum_{r>0}\alpha_r u_{r\mathbf{x}_0}x^r\right)\text{ for }\mathbf{x}_0\in(\mathbb{Z}^2)^*\text{ with }\deg(\mathbf{x}_0)=1, \tag{E3}$$

while $\alpha(\mathbf{x},\mathbf{y})$ is defined by

$$\alpha(\mathbf{x},\mathbf{y})=\begin{cases}\epsilon_{\mathbf{x}}(\epsilon_{\mathbf{x}}\mathbf{x}+\epsilon_{\mathbf{y}}\mathbf{y}-\epsilon_{\mathbf{x}+\mathbf{y}}(\mathbf{x}+\mathbf{y}))/2 & \text{if }\epsilon_{\mathbf{x},\mathbf{y}}=1\\ \epsilon_{\mathbf{y}}(\epsilon_{\mathbf{x}}\mathbf{x}+\epsilon_{\mathbf{y}}\mathbf{y}-\epsilon_{\mathbf{x}+\mathbf{y}}(\mathbf{x}+\mathbf{y}))/2 & \text{if }\epsilon_{\mathbf{x},\mathbf{y}}=-1\end{cases}. \tag{E4}$$

The relation to the quantum toroidal of $\mathfrak{gl}_1$ is given in the following theorem:

Theorem 6.4. *[18] The assignment*

$$u_{1,i}\mapsto e_i,\ u_{-1,i}\mapsto f_i,\ \theta_{0,\pm j}\mapsto\psi_{\pm j}\cdot\psi_0^{\mp 1},\ \kappa_{a,b}\mapsto\psi_0^a\quad \textit{for } i,a,b\in\mathbb{Z},j\in\mathbb{N}$$

extends to an isomorphism of algebras $\Xi:\widetilde{\mathcal{E}}/(\kappa_{0,1}-1)\xrightarrow{\sim}\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$.

Remark 6.1. This theorem was proved in [18] only for $\mathcal{E}:=\widetilde{\mathcal{E}}/(\kappa_{\mathbf{y}}-1)_{\mathbf{y}\in\mathbb{Z}^2}$, but the above generalization is straightforward. According to [2], $\mathcal{E}$ is also isomorphic to the Drinfeld double of the spherical Hall algebra of an elliptic curve over a finite field.

This result provides distinguished elements $\{u_{\mathbf{x}}|\mathbf{x}\in(\mathbb{Z}^2)^*\}$ of $\ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$. Their images in $\ddot{U}'_{h_2}(\mathfrak{gl}_1)=\ddot{U}'_{-h_2-h_3,h_2,h_3}(\mathfrak{gl}_1)/(h_3)$ will be denoted by $\bar{u}_{\mathbf{x}}$.

Lemma 6.5. *The isomorphism θ maps these elements $\bar{u}_{\mathbf{x}}$ as follows:*

$$\bar{u}_{0,r}\mapsto\operatorname{sign}(r)\frac{(1-q_2^{-1})(1-q_2)}{1-q_2^r}\left(Z^r-\frac{1}{1-q_2^r}c_{\mathfrak{d}}\right)\ \textit{for } r\in\mathbb{Z}^*, \tag{8}$$

$$\bar{u}_{\pm k,\pm l}\mapsto\pm q_2^{\pm f(k,l)}\frac{1-q_2^{\mp 1}}{1-q_2^{\mp d}}(1-q_2^{\pm 1})^{k-1}Z^{\pm l}D^{\pm k}q_2^{\frac{kl}{2}\mp\frac{kl}{2}}\ \textit{for } k\in\mathbb{N},l\in\mathbb{Z}, \tag{9}$$

where $d:=\gcd(k,l)\in\mathbb{N}$ and $f(k,l):=\frac{kl-k-l-d+2}{2}\in\mathbb{Z}$ (note that $f(k,l)$ equals the number of lattice points inside the triangle with vertices $\{(0,0),(0,l),(k,l)\}$ if $k,l>0$).

Proof. Considering the "$h_3\to 0$ limit" of the relation (E2), we find

$$[\bar{u}_{\mathbf{y}},\bar{u}_{\mathbf{x}}]=\epsilon_{\mathbf{x},\mathbf{y}}\frac{\alpha_{\deg(\mathbf{x}+\mathbf{y})}}{\alpha_1}\bar{u}_{\mathbf{x}+\mathbf{y}}\text{ if }\triangle_{\mathbf{x},\mathbf{y}}\text{ is empty and }\deg(\mathbf{x})=1. \tag{E2$'$}$$

In particular, we get $\bar{u}_{0,r}=\operatorname{sign}(r)\frac{\alpha_1}{\alpha_r}[\bar{u}_{-1,0},\bar{u}_{1,r}]$. Applying θ to both sides of this equality, we recover (8).

We prove (9) by an induction on k; we will consider only the sign "+" case. Case $k=1$ is trivial. Given $(k,l)\in\mathbb{Z}_{>1}\times\mathbb{Z}$, choose unique $\mathbf{x}=(k_1,l_1),\mathbf{y}=(k_2,l_2),0<k_1,k_2<k$,

such that $\mathbf{x}+\mathbf{y}=(k,l)$, $\epsilon_{\mathbf{x},\mathbf{y}}=1$, $\deg(\mathbf{x})=\deg(\mathbf{y})=1$, and $\triangle_{\mathbf{x},\mathbf{y}}$ is empty. Combining the formula (E2′) with the induction assumption on $\theta(\bar{u}_{\mathbf{x}})$ and $\theta(\bar{u}_{\mathbf{y}})$, we find

$$\theta(\bar{u}_{k,l})=\frac{(1-q_2)(1-q_2^{-1})}{(1-q_2^d)(1-q_2^{-d})}q_2^{f(k_1,l_1)+f(k_2,l_2)}(q_2^{k_2l_1}-q_2^{k_1l_2})(1-q_2)^{k_1+k_2-2}Z^{l_1+l_2}D^{k_1+k_2}.$$

Our choice of $\mathbf{x},\mathbf{y}$ and the Pick's formula imply that $k_1l_2-k_2l_1=d$. As a result, we have $q_2^{k_2l_1}-q_2^{k_1l_2}=q_2^{k_2l_1}(1-q_2^d)$ and $f(k_1,l_1)+f(k_2,l_2)+k_2l_1=f(k_1+k_2,l_1+l_2)=f(k,l)$. This completes the induction step. □

6.4. Flatness and faithfulness

The main result of this section is:

Theorem 6.6. *(a) The algebra $\ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is a flat $\mathbb{C}[[h_3]]$-deformation of $\ddot{U}'_{h_2}(\mathfrak{gl}_1)\simeq U(\bar{\mathfrak{d}}^0_{q_2})$.*

(b) The algebra $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is a flat $\mathbb{C}[[h_3]]$-deformation of $\ddot{Y}'_{h_2}(\mathfrak{gl}_1)\simeq U(\bar{\mathfrak{D}}^0_{h_2})$.

Proof. To prove Theorem 6.6, it suffices to provide a faithful $U(\bar{\mathfrak{d}}^0_{q_2})$-representation (respectively $U(\bar{\mathfrak{D}}^0_{h_2})$-representation) which admits a flat deformation to a representation of $\ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ (respectively $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$). Let R be a localization of $\mathbb{C}[[h_2,h_3]]$ by the multiplicative set $\{(h_2-\nu_1h_3)\cdots(h_2-\nu_sh_3)|s\in\mathbb{N},\nu_j\in\mathbb{C}\}$. Note that $\bar{R}:=R/(h_3)\simeq\mathbb{C}((h_2))$. The ring R is needed to make use of the representations from Sections 2–4, therefore, we define

$$\ddot{U}'_R(\mathfrak{gl}_1):=\ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)\otimes_{\mathbb{C}[[h_2,h_3]]}R,\quad \ddot{Y}'_R(\mathfrak{gl}_1):=\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)\otimes_{\mathbb{C}[[h_2,h_3]]}R.$$

Consider the Lie algebra

$$\mathfrak{gl}_\infty=\left\{\sum_{i,j\in\mathbb{Z}}a_{i,j}E_{i,j}|a_{i,j}\in\mathbb{C}[[h_2]]\text{ and }a_{i,j}=0\text{ for }|i-j|\gg 0\right\}.$$

Let $\bar{\mathfrak{gl}}_\infty=\mathfrak{gl}_\infty\oplus\mathbb{C}[[h_2]]\cdot\kappa$ be the central extension of this Lie algebra via the 2-cocycle

$$\phi_{\mathfrak{gl}}\left(\sum a_{i,j}E_{i,j},\sum b_{i,j}E_{i,j}\right)=\sum_{i\le 0<j}a_{i,j}b_{j,i}-\sum_{j\le 0<i}a_{i,j}b_{j,i}.$$

For $u\in 1+h_2\mathbb{C}[[h_2]]$, consider the homomorphism $\tau_u:U_{\bar{R}}(\bar{\mathfrak{d}}^0_{q_2})\to U_{\bar{R}}(\bar{\mathfrak{gl}}_\infty)$ induced by

$$c_{\mathfrak{d}}\mapsto-\kappa\quad\text{and}\quad Z^kD^l\mapsto\sum_{i\in\mathbb{Z}}u^kq_2^{(1-i)k}E_{i,i-l}-\delta_{0,l}\frac{1-q_2^ku^k}{1-q_2^k}\kappa\quad\text{for}\quad(k,l)\in(\mathbb{Z}^2)^*.$$

Let $\varpi_u:\ddot{U}'_{\bar{R}}(\mathfrak{gl}_1)\to U_{\bar{R}}(\bar{\mathfrak{gl}}_\infty)$ be the composition of $\theta:\ddot{U}'_{\bar{R}}(\mathfrak{gl}_1)\to U_{\bar{R}}(\bar{\mathfrak{d}}^0_{q_2})$ and τ_u. Then

$$\varpi_u(e(z)) = \sum_{i\in\mathbb{Z}} \delta(q_2^{-i}u/z)E_{i+1,i},\ \ \varpi_u(f(z)) = -\sum_{i\in\mathbb{Z}} \delta(q_2^{-i}u/z)E_{i,i+1}.$$

Let F_∞ be the fundamental representation of $\bar{\mathfrak{gl}}_\infty$. It is realized on $\wedge^{\infty/2}\mathbb{C}^\infty$ with the highest weight vector $w_0\wedge w_{-1}\wedge w_{-2}\wedge\cdots$ (here $\mathbb{C}^\infty$ is a $\mathbb{C}$-vector spaces with the basis $\{w_i\}_{i\in\mathbb{Z}}$). Comparing the formulas for the Fock $\ddot{U}'_R(\mathfrak{gl}_1)$-module $F'_R(u)$ with those for the $\bar{\mathfrak{gl}}_\infty$-action on F_∞, we see that $F'_R(u)$ degenerates to $\varpi_u^*(F_\infty)$ (the intertwining linear map is given by $|\lambda\rangle \mapsto w_{\lambda_1}\wedge w_{\lambda_2-1}\wedge w_{\lambda_3-2}\wedge\cdots$). Moreover, it is easy to see that any legible finite tensor product $F'_R(u_1)\otimes\cdots\otimes F'_R(u_n)$ degenerates to $\varpi_{u_1}^*(F_\infty)\otimes\cdots\otimes\varpi_{u_n}^*(F_\infty)$.

It remains to prove that the module $\bigoplus_n \bigoplus_{u_1,\ldots,u_n} \tau_{u_1}^*(F_\infty)\otimes\cdots\otimes\tau_{u_n}^*(F_\infty)$ is a faithful representation of $U(\bar{\mathfrak{d}}^0_{q_2})$, where the sum is over all collections $u_1,\ldots,u_n \in 1+h_2\mathbb{C}[[h_2]]$ which are not in resonance. This follows from the faithfulness of the $\bar{\mathfrak{d}}^0_{q_2}$-action on each $\tau_u^*(F_\infty)$ for any $u\in 1+h_2\mathbb{C}[[h_2]]$, which is a simple exercise left to the interested reader.

In part (b), for any $v\in h_2\mathbb{C}[[h_2]]$, we consider the homomorphism $\varsigma_v : U_{\bar{R}}(\bar{\mathfrak{D}}^0_{h_2}) \to U_{\bar{R}}(\bar{\mathfrak{gl}}_\infty)$ induced by

$$c_{\mathfrak{D}} \mapsto -\kappa \ \text{ and } \ x^n\partial^l \mapsto \sum_{i\in\mathbb{Z}} (v+(1-i)h_2)^n E_{i,i-l} + \delta_{0,l}c_n\kappa, \ \text{ for } n\in\mathbb{Z}_+,\ l\in\mathbb{Z}$$

where $c_n\in\bar{R}$ are determined recursively from

$$\binom{n}{1}h_2c_{n-1} - \binom{n}{2}h_2^2c_{n-2} + \ldots + (-1)^{n+1}h_2^nc_0 - (-h_2)^n + v^n = 0.$$

The rest of the arguments are the same. This completes our proof of Theorem 6.6. □

Corollary 6.7. *(a) The following is a faithful $\ddot{U}'_R(\mathfrak{gl}_1)$-representation:*

$$\mathbf{F}'_R := \bigoplus_{n\in\mathbb{N}} \bigoplus_{u_1,\ldots,u_n\in 1+h_2\mathbb{C}[[h_2]]-\text{not in resonance}} F'_R(u_1)\otimes\cdots\otimes F'_R(u_n).$$

(b) The following is a faithful $\ddot{Y}'_R(\mathfrak{gl}_1)$-representation:

$${}^a\mathbf{F}'_R := \bigoplus_{n\in\mathbb{N}} \bigoplus_{v_1,\ldots,v_n\in h_2\mathbb{C}[[h_2]]-\text{not in resonance}} {}^aF'_R(v_1)\otimes\cdots\otimes {}^aF'_R(v_n).$$

As another consequence of Theorem 6.6 and Proposition 6.3, we have:

Corollary 6.8. *The homomorphism Υ is injective.*

Remark 6.2. In contrast to [10], Υ does not induce an isomorphism of appropriate completions, since the homomorphism $\Upsilon_{|_{h_3=0}}$ does not induce an isomorphism of $\widehat{\bar{\mathfrak{d}}^0_{q_2}}$ and $\widehat{\bar{\mathfrak{D}}^0_{h_2}}$.

6.5. Compatible isomorphisms of representations

Given $n \in \mathbb{N}$, consider two n-tuples

$$\mathbf{v} = (v_1, \dots, v_n) \in ((h_2, h_3)\mathbb{C}[[h_2, h_3]])^n, \ \mathbf{u} = (u_1, \dots, u_n) \in (1 + (h_2, h_3)\mathbb{C}[[h_2, h_3]])^n.$$

Associated to this data, we have Fock $\ddot{U}'_R(\mathfrak{gl}_1)$-representations $\{F'_R(u_i)\}_{i=1}^n$ and Fock $\ddot{Y}'_R(\mathfrak{gl}_1)$-representations $\{{}^aF'_R(v_i)\}_{i=1}^n$. Consider the tensor products

$$F'_R(\mathbf{u}) := F'_R(u_1) \otimes F'_R(u_2) \otimes \dots \otimes F'_R(u_n) - \text{representation of } \ddot{U}'_R(\mathfrak{gl}_1),$$

$${}^aF'_R(\mathbf{v}) := {}^aF'_R(v_1) \otimes {}^aF'_R(v_2) \otimes \dots \otimes {}^aF'_R(v_n) - \text{representation of } \ddot{Y}'_R(\mathfrak{gl}_1),$$

whenever these representations are well defined, i.e., $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ are *not in resonance*. Both of these tensor products have natural bases $\{|\bar{\lambda}\rangle\}$ labeled by n-tuples of Young diagrams $\bar{\lambda}$. Our key result establishes an isomorphism of these tensor products compatible with Υ.

Theorem 6.9. *For any* $\mathbf{v}$ *as above, define* $u_i := e^{v_i}$. *There exists a unique collection of constants* $\mathbf{c}_{\bar{\lambda}} \in R$ *such that* $\mathbf{c}_{\bar{\emptyset}} = 1$ *and the corresponding* R*-linear isomorphism of vector spaces*

$$I_{\mathbf{v}} : F'_R(\mathbf{u}) \xrightarrow{\sim} {}^aF'_R(\mathbf{v}) \text{ given by } |\bar{\lambda}\rangle \mapsto \mathbf{c}_{\bar{\lambda}} \cdot |\bar{\lambda}\rangle$$

satisfies the property

$$I_{\mathbf{v}}(X(w)) = \Upsilon(X)(I_{\mathbf{v}}(w)) \text{ for all } w \in F'_R(\mathbf{u}) \text{ and } X \in \{H_i, e_i, f_i\}_{i \in \mathbb{Z}}. \tag{$\dagger$}$$

We say that $I_{\mathbf{v}}$ *is compatible with* Υ *if* ($\dagger$) *holds.*

Proof. By straightforward computation, one can see that ($\dagger$) holds for all $w = |\bar{\lambda}\rangle$, $X = H_i$ and an arbitrary choice of $\mathbf{c}_{\bar{\lambda}}$. On the other hand, the equalities

$$I_{\mathbf{v}}(e_i(|\bar{\lambda}\rangle)) = \Upsilon(e_i)(I_{\mathbf{v}}(|\bar{\lambda}\rangle)) \text{ and } I_{\mathbf{v}}(f_i(|\bar{\lambda}\rangle)) = \Upsilon(f_i)(I_{\mathbf{v}}(|\bar{\lambda}\rangle)) \text{ for all } \bar{\lambda}, i$$

are both equivalent to

$$\frac{\mathbf{c}_{\bar{\lambda}+\square_i^k}}{\mathbf{c}_{\bar{\lambda}}} = \mathbf{d}_{\bar{\lambda},\square_i^k}, \tag{10}$$

where

$$\mathbf{d}_{\bar{\lambda},\square_i^k} := q^{k-1-n/2} \cdot \left(\frac{h_1}{1-q_1^{-1}} \cdot \frac{q_2-1}{h_2}\right)^{1/2} \times$$

$$\prod_{(a,j)\neq(k,i)} \left\{ \left(\frac{(\chi_i^{(k)} - q_2\chi_j^{(a)})(\chi_i^{(k)} - q_3\chi_j^{(a)})}{(\chi_i^{(k)} - \chi_j^{(a)})(\chi_i^{(k)} - q_1^{-1}\chi_j^{(a)})} \right) \left(\frac{(x_i^{(k)} - x_j^{(a)})(x_i^{(k)} - x_j^{(a)} + h_1)}{(x_i^{(k)} - x_j^{(a)} - h_2)(x_i^{(k)} - x_j^{(a)} - h_3)} \right) \right\}^{\epsilon_{(k,i)}^{(a,j)}}. \tag{11}$$

In this formula, we use the following notation:

$$x_j^{(a)} := (\lambda_j^a - 1)h_1 + (j-1)h_2 + v_a,\ \chi_j^{(a)} := \exp(x_j^{(a)}),\ \epsilon_{(k,i)}^{(a,j)} := \begin{cases} 1/2 & \text{if } (a,j) \succ (k,i) \\ -1/2 & \text{if } (a,j) \prec (k,i) \end{cases}.$$

The uniqueness of $\mathbf{c}_{\bar{\lambda}} \in R$ satisfying the relation (10) with the initial condition $\mathbf{c}_{\bar{\emptyset}} = 1$ is obvious. The existence of such $\mathbf{c}_{\bar{\lambda}}$ is equivalent to

$$\mathbf{d}_{\bar{\lambda}+\square_i^k,\square_j^l} \cdot \mathbf{d}_{\bar{\lambda},\square_i^k} = \mathbf{d}_{\bar{\lambda}+\square_j^l,\square_i^k} \cdot \mathbf{d}_{\bar{\lambda},\square_j^l}$$

for all possible $\bar{\lambda}, \square_i^k, \square_j^l$. The verification of this identity is straightforward. □

6.6. Proof of Theorem 6.2

Recall the faithful $\ddot{U}'_R(\mathfrak{gl}_1)$-representation $\mathbf{F}'_R$ and $\ddot{Y}'_R(\mathfrak{gl}_1)$-representation ${}^a\mathbf{F}'_R$ from Corollary 6.7. Note that $\exp : (h_2, h_3)\mathbb{C}[[h_2, h_3]] \to 1 + (h_2, h_3)\mathbb{C}[[h_2, h_3]]$ is a bijective map and $\{v_i\}_{i=1}^n$ are not in resonance if and only if $\{e^{v_i}\}_{i=1}^n$ are not in resonance.

According to Theorem 6.9, we have an R-linear isomorphism $\mathbf{I} : \mathbf{F}'_R \xrightarrow{\sim} {}^a\mathbf{F}'_R$ compatible with Υ in the following sense:

$$\mathbf{I}(X(w)) = \Upsilon(X)(\mathbf{I}(w)) \text{ for all } w \in \mathbf{F}'_R \text{ and } X \in \{H_i, e_i, f_i\}_{i\in\mathbb{Z}}.$$

For any $X \in \{H_i, e_i, f_i\}_{i\in\mathbb{Z}}$, consider the assignment $X \mapsto \Upsilon(X)$ with the right-hand side defined via (‡). Then Theorem 6.2 is equivalent to saying that this assignment preserves all the defining relations of $\ddot{U}'_R(\mathfrak{gl}_1)$. The latter follows immediately from the faithfulness of ${}^a\mathbf{F}'_R$ combined with an existence of the compatible isomorphism $\mathbf{I}$.

Remark 6.3. One can directly check that the aforementioned assignment given by (‡) preserves all the defining relations of $\ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$, except for the Serre relations (compare with [10]). In particular, the compatibility with the relations (T4H,T5H) follows from Corollary 6.1. Actually, we used this approach to determine the formulas in (‡). However, the arguments of [10] on the compatibility with the Serre relations are not applicable in our settings. Instead, one can prove this compatibility by utilizing the shuffle approach from the next section. We refer the interested reader to [23], where we discuss this in the greater generality.

7. Shuffle algebras S^m and S^a

We introduce the *small multiplicative* and *small additive shuffle algebras*. Their relation to $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is recalled. We also discuss their commutative subalgebras.

7.1. Small multiplicative shuffle algebra S^m

Consider a $\mathbb{Z}_+$-graded $\mathbb{C}$-vector space $\mathbb{S}^m = \bigoplus_{n\geq 0} \mathbb{S}^m_n$, where $\mathbb{S}^m_n$ consists of rational functions $\frac{f(x_1,\dots,x_n)}{\Delta(x_1,\dots,x_n)}$ with $f \in \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]^{\mathfrak{S}_n}$ and $\Delta(x_1,\dots,x_n) := \prod_{i\neq j}(x_i - x_j)$.

Define the star product $\overset{m}{\star} : \mathbb{S}^m_k \times \mathbb{S}^m_l \to \mathbb{S}^m_{k+l}$ by

$$(F \overset{m}{\star} G)(x_1,\dots,x_{k+l}) := \mathrm{Sym}_{\mathfrak{S}_{k+l}} \left(F(x_1,\dots,x_k) G(x_{k+1},\dots,x_{k+l}) \prod_{i\leq k}^{j>k} \omega^m(x_j,x_i) \right)$$

with

$$\omega^m(x,y) := \frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(x-y)^3}.$$

This endows $\mathbb{S}^m$ with a structure of an associative unital $\mathbb{C}$-algebra with the unit $\mathbf{1} \in \mathbb{S}^m_0$.

We say that an element $\frac{f(x_1,\dots,x_n)}{\Delta(x_1,\dots,x_n)} \in \mathbb{S}^m_n$ satisfies the *wheel conditions* if $f(x_1,\dots,x_n) = 0$ once $x_{i_1}/x_{i_2} = q_1$ and $x_{i_2}/x_{i_3} = q_2$ for some $1 \leq i_1, i_2, i_3 \leq n$. Let $S^m \subset \mathbb{S}^m$ be a $\mathbb{Z}_+$-graded subspace consisting of all such elements. The subspace S^m is closed with respect to $\overset{m}{\star}$.

Definition 7.1. The algebra $(S^m, \overset{m}{\star})$ is called the *small multiplicative shuffle algebra*.

Recall that q_1, q_2, q_3 are called *generic* if $q_1^a q_2^b q_3^c = 1 \iff a = b = c$. We have the following result:

Theorem 7.1. *[17, Proposition 3.5] The algebra S^m is generated by S^m_1 for generic q_1, q_2, q_3.*

The connection between the algebras S^m and $\widetilde{\mathcal{E}}$ was established in [19]:

Proposition 7.2. *[19] The map $u_{1,i} \mapsto x_1^i$ extends to an injective homomorphism $\widetilde{\mathcal{E}}^+ \to S^m$, where $\widetilde{\mathcal{E}}^+$ is the subalgebra of $\widetilde{\mathcal{E}}$ generated by $\{u_{i,j} | i \in \mathbb{N}, j \in \mathbb{Z}\}$.*

Combining this result with Theorems 7.1 and 6.4, we get:

Theorem 7.3. *The algebras $\widetilde{\mathcal{E}}^+, \ddot{U}^+, S^m$ are isomorphic.*

7.2. Commutative subalgebra $\mathcal{A}^m \subset S^m$

Following [7], we introduce an important $\mathbb{Z}_+$-graded subspace $\mathcal{A}^m = \bigoplus_{n\geq 0} \mathcal{A}^m_n$ of S^m. Its degree n component is defined by $\mathcal{A}^m_n = \{F \in S^m_n | \partial^{(0,k)} F = \partial^{(\infty,k)} F \ \ \forall\ 1 \leq k \leq n\}$, where

$$\partial^{(0,k)}F := \lim_{\xi\to 0} F(x_1,\dots,x_{n-k},\xi\cdot x_{n-k+1},\dots,\xi\cdot x_n),$$
$$\partial^{(\infty,k)}F := \lim_{\xi\to \infty} F(x_1,\dots,x_{n-k},\xi\cdot x_{n-k+1},\dots,\xi\cdot x_n).$$

This subspace satisfies the following properties:

Theorem 7.4. *[7, Section 2] We have:*
(a) Suppose $F\in S^m_n$ and $\partial^{(\infty,k)}F$ exist for all $1\le k\le n$, then $F\in \mathcal{A}^m_n$.
(b) The subspace $\mathcal{A}^m\subset S^m$ is $\overset{m}{\star}$-commutative and $\overset{m}{\star}$-closed.
(c) $\mathcal{A}^m$ is a polynomial algebra in $\{K^m_n\}_{n\in\mathbb{N}}$ with $K^m_n\in S^m_n$ defined by

$$K^m_1(x_1)=x^0_1 \text{ and } K^m_n(x_1,\dots,x_n)=\prod_{1\le i<j\le n}\frac{(x_i-q_1x_j)(x_j-q_1x_i)}{(x_i-x_j)^2} \text{ for } n\ge 2.$$

Remark 7.1. These elements $\{K^m_n\}_{n\in\mathbb{N}}$ played a crucial role in [8], where they were used to construct an action of the Heisenberg algebra on the vector space M from Section 2.2.

Our next result provides an alternative choice of generators for the algebra $\mathcal{A}^m$ explicitly expressed via S^m_1. We use the following notation: $[P,Q]_m = P\overset{m}{\star}Q - Q\overset{m}{\star}P$ for $P,Q\in S^m$.

Proposition 7.5. *The algebra $\mathcal{A}^m$ is a polynomial algebra in $\{L^m_n\}_{n\in\mathbb{N}}$ defined by*

$$L^m_1(x_1)=x^0_1 \quad \text{and} \quad L^m_n=\underbrace{[x^1,[x^0,[x^0,\dots,[x^0,x^{-1}]_m\dots]_m]_m]_m}_{n \text{ factors}}\in S^m_n \quad \text{for } n\ge 2.$$

We refer the reader to Appendix D for the proof of this result.

7.3. Small additive shuffle algebra S^a

Consider a $\mathbb{Z}_+$-graded $\mathbb{C}$-vector space $\mathbb{S}^a=\bigoplus_{n\ge 0}\mathbb{S}^a_n$, where $\mathbb{S}^a_n$ consists of rational functions $\frac{f(x_1,\dots,x_n)}{\Delta(x_1,\dots,x_n)}$ with $f\in\mathbb{C}[x_1,\dots,x_n]^{\mathfrak{S}_n}$. Define the star product $\overset{a}{\star}:\mathbb{S}^a_k\times\mathbb{S}^a_l\to\mathbb{S}^a_{k+l}$ by

$$(F\overset{a}{\star}G)(x_1,\dots,x_{k+l}):=\mathrm{Sym}_{\mathfrak{S}_{k+l}}\left(F(x_1,\dots,x_k)G(x_{k+1},\dots,x_{k+l})\prod_{i\le k}^{j>k}\omega^a(x_j,x_i)\right)$$

with

$$\omega^a(x,y):=\frac{(x-y-h_1)(x-y-h_2)(x-y-h_3)}{(x-y)^3}.$$

This endows $\mathbb{S}^a$ with a structure of an associative unital $\mathbb{C}$-algebra with the unit $\mathbf{1}\in\mathbb{S}^a_0$.

We say that an element $\frac{f(x_1,\ldots,x_n)}{\Delta(x_1,\ldots,x_n)} \in \mathbb{S}_n^a$ satisfies the *wheel conditions* if $f(x_1,\ldots,x_n) = 0$ once $x_{i_1} - x_{i_2} = h_1$ and $x_{i_2} - x_{i_3} = h_2$ for some $1 \leq i_1, i_2, i_3 \leq n$. Let $S^a \subset \mathbb{S}^a$ be a $\mathbb{Z}_+$-graded subspace consisting of all such elements. The subspace S^a is closed with respect to $\overset{a}{\star}$.

Definition 7.2. The algebra $(S^a, \overset{a}{\star})$ is called the *small additive shuffle algebra*.

The following result is proved analogously to Theorem 7.1:

Theorem 7.6. *For generic h_1, h_2, h_3 (that is $ah_1 + bh_2 + ch_3 = 0$ for $a, b, c \in \mathbb{Z} \iff a = b = c$), the map $e_j \mapsto x_1^j$ extends to an isomorphism $\ddot{Y}^+ \xrightarrow{\sim} S^a$. In particular, S^a is generated by S_1^a.*

7.4. Commutative subalgebra $\mathcal{A}^a \subset S^a$

Let us introduce an *additive version* of $\mathcal{A}^m$: a $\mathbb{Z}_+$-graded subspace $\mathcal{A}^a = \bigoplus_{n\geq 0} \mathcal{A}_n^a$ of S^a. Its degree n component $\mathcal{A}_n^a$ consists of those $F \in S_n^a$ such that the limits

$$\partial^{(\infty,k)} F := \lim_{\xi\to\infty} F(x_1, \ldots, x_{n-k}, x_{n-k+1} + \xi, \ldots, x_n + \xi)$$

exist for all $1 \leq k \leq n$. The following is an *additive* counterpart of Theorem 7.4:

Theorem 7.7. *We have:*
(a) The subspace $\mathcal{A}^a \subset S^a$ is $\overset{a}{\star}$-commutative and $\overset{a}{\star}$-closed.
(b) $\mathcal{A}^a$ is a polynomial algebra in $\{K_n^a\}_{n\in\mathbb{N}}$ with $K_n^a \in S_n^a$ defined by

$$K_1^a(x_1) = x_1^0 \text{ and } K_n^a(x_1, \ldots, x_n) = \prod_{1\leq i<j\leq n} \frac{(x_i - x_j - h_1)(x_j - x_i - h_1)}{(x_i - x_j)^2} \text{ for } n \geq 2.$$

Analogously to Proposition 7.5, the commutative subalgebra $\mathcal{A}^a$ admits an alternative set of generators explicitly expressed via S_1^a. Define $[P, Q]_a := P \overset{a}{\star} Q - Q \overset{a}{\star} P$ for $P, Q \in S^a$.

Proposition 7.8. *The algebra $\mathcal{A}^a$ is a polynomial algebra in $\{L_n^a\}_{n\in\mathbb{N}}$ defined by*

$$L_1^a(x_1) = x_1^0 \quad \text{and} \quad L_n^a = \underbrace{[x^0, [x^0, \ldots, [x^0, x^{n-1}]_a \ldots]_a]_a}_{n \text{ factors}} \in S_n^a \quad \text{for } n \geq 2.$$

Remark 7.2. The commutativity of $\{L_n^m\}$ and $\{L_n^a\}$ was shown in [19], [20, Appendix E].

8. Horizontal realization and Whittaker vectors

8.1. Horizontal realization of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$

Recall the distinguished collection of elements $\{u_{\mathbf{x}}, \kappa_{\mathbf{y}}\} \subset \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ from Theorem 6.4. Note that there is a natural $\mathrm{SL}_2(\mathbb{Z})$-action on $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)/(\psi_0^{\pm 1} - 1) \simeq \widetilde{\mathcal{E}}/(\kappa_{\mathbf{y}} - 1)_{\mathbf{y}\in\mathbb{Z}^2}$. In particular, we have a natural automorphism of $\widetilde{\mathcal{E}}/(\kappa_{\mathbf{y}} - 1)_{\mathbf{y}\in\mathbb{Z}^2}$ induced by $u_{k,l} \mapsto u_{-l,k}$. Although there is no such automorphism for $\widetilde{\mathcal{E}}/(\kappa_{0,1} - 1)$, but we still have a nice presentation of this algebra in terms of the generators $\{u_{i,\pm 1}, u_{j,0}, \kappa_{\pm 1,0}\}_{i\in\mathbb{Z}}^{j\in\mathbb{Z}^*}$ rather than $\{u_{\pm 1,i}, u_{0,j}, \kappa_{\pm 1,0}\}_{i\in\mathbb{Z}}^{j\in\mathbb{Z}^*}$.

To formulate the main result, we need to introduce a modification of the algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. The algebra $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$ is the unital associative $\mathbb{C}$-algebra generated by $\{\widetilde{e}_i, \widetilde{f}_i, \widetilde{\psi}_j, \gamma^{\pm 1/2}\}_{i\in\mathbb{Z}}^{j\in\mathbb{Z}^*}$ with the following defining relations:

$$\widetilde{\psi}^{\pm}(z)\widetilde{\psi}^{\pm}(w) = \widetilde{\psi}^{\pm}(w)\widetilde{\psi}^{\pm}(z),\ g(\gamma^{-1}z/w)\widetilde{\psi}^{+}(z)\widetilde{\psi}^{-}(w) = g(\gamma z/w)\widetilde{\psi}^{-}(w)\widetilde{\psi}^{+}(z), \quad \text{(TT0)}$$

$$\widetilde{e}(z)\widetilde{e}(w) = g(z/w)\widetilde{e}(w)\widetilde{e}(z), \quad \text{(TT1)}$$

$$\widetilde{f}(z)\widetilde{f}(w) = g(w/z)\widetilde{f}(w)\widetilde{f}(z), \quad \text{(TT2)}$$

$$\beta_1 \cdot [\widetilde{e}(z), \widetilde{f}(w)] = \delta(\gamma w/z)\widetilde{\psi}^{+}(\gamma^{1/2}w) - \delta(\gamma z/w)\widetilde{\psi}^{-}(\gamma^{1/2}z), \quad \text{(TT3)}$$

$$\widetilde{\psi}^{\pm}(z)\widetilde{e}(w) = g(\gamma^{\pm 1/2}z/w)\widetilde{e}(w)\widetilde{\psi}^{\pm}(z), \quad \text{(TT4)}$$

$$\widetilde{\psi}^{\pm}(z)\widetilde{f}(w) = g(\gamma^{\pm 1/2}w/z)\widetilde{f}(w)\widetilde{\psi}^{\pm}(z), \quad \text{(TT5)}$$

$$\mathrm{Sym}_{\mathfrak{S}_3}[\widetilde{e}_{i_1}, [\widetilde{e}_{i_2+1}, \widetilde{e}_{i_3-1}]] = 0,\ \ \mathrm{Sym}_{\mathfrak{S}_3}[\widetilde{f}_{i_1}, [\widetilde{f}_{i_2+1}, \widetilde{f}_{i_3-1}]] = 0, \quad \text{(TT6)}$$

where these generating series are defined as follows:

$$\widetilde{e}(z) := \sum_{i=-\infty}^{\infty} \widetilde{e}_i z^{-i},\ \widetilde{f}(z) := \sum_{i=-\infty}^{\infty} \widetilde{f}_i z^{-i},\ \widetilde{\psi}^{\pm}(z) := 1 + \sum_{j>0} \widetilde{\psi}_{\pm j} z^{\mp j},$$

while $g(y) := \frac{(1-q_1y)(1-q_2y)(1-q_3y)}{(1-q_1^{-1}y)(1-q_2^{-1}y)(1-q_3^{-1}y)}$. Note that $g(y) = g(y^{-1})^{-1}$.

The following result is analogous to Theorem 6.4:

Theorem 8.1. *The assignment*

$$\kappa_{1,0}^{\pm 1/2} \mapsto \gamma^{\mp 1/2},\ \theta_{j,0} \mapsto \widetilde{\psi}_{-j},\ u_{-i,1} \mapsto \gamma^{-|i|/2}\widetilde{e}_i,\ u_{-i,-1} \mapsto \gamma^{|i|/2}\widetilde{f}_i \ \ \textit{for } i \in \mathbb{Z}, j \in \mathbb{Z}^*$$

extends to an isomorphism of algebras $\Xi_h : \widetilde{\mathcal{E}}[\kappa_{1,0}^{\pm 1/2}]/(\kappa_{0,1} - 1) \xrightarrow{\sim} \ddot{\mathbf{U}}_{q_1,q_2,q_3}$.

Following [4], we equip the algebra $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$ with a formal coproduct Δ_h defined by

$$\Delta_h(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2},\ \Delta_h(\widetilde{\psi}^{\pm}(z)) = \widetilde{\psi}^{\pm}(\gamma_{(2)}^{\pm 1/2} z) \otimes \widetilde{\psi}^{\pm}(\gamma_{(1)}^{\mp 1/2} z),$$

$$\Delta_h(\widetilde{e}(z)) = \widetilde{e}(z) \otimes 1 + \widetilde{\psi}^{-}(\gamma_{(1)}^{1/2} z) \otimes \widetilde{e}(\gamma_{(1)} z),$$

$$\Delta_h(\widetilde{f}(z)) = 1 \otimes \widetilde{f}(z) + \widetilde{f}(\gamma_{(2)} z) \otimes \widetilde{\psi}^{+}(\gamma_{(2)}^{1/2} z),$$

where $\gamma_{(1)}^{\pm 1/2} := \gamma^{\pm 1/2} \otimes 1$ and $\gamma_{(2)}^{\pm 1/2} := 1 \otimes \gamma^{\pm 1/2}$.

Remark 8.1. According to Theorems 6.4 and 8.1, the algebras $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)[\psi_0^{\pm 1/2}]$ and $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$ are isomorphic. This allows to view Δ_h as a *horizontal coproduct* on $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)[\psi_0^{\pm 1/2}]$, providing a monoidal structure on the category $\mathcal{O}$ from Section 4.6 (the category $\mathcal{O}$ for the bigger algebra with the central elements $\psi_0^{\pm 1/2}$ added is defined as before). For two $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)[\psi_0^{\pm 1/2}]$-modules L_1, L_2, we denote the corresponding tensor product by $L_1 \underset{h}{\otimes} L_2$.

8.2. Horizontal realization of $F(u), V(u)$

Identifying $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$ with $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)[\psi_0^{\pm 1/2}]$, let us describe how $\widetilde{e}(z), \widetilde{f}(z), \widetilde{\psi}^{\pm}(z)$ act on the Fock module $F(u)$. Consider the Heisenberg algebra $\mathfrak{h}$ generated by $\{a_n\}_{n \in \mathbb{Z}^*}$ with the defining relation

$$[a_m, a_n] = m(1 - q_1^{|m|})/(1 - q_2^{-|m|})\delta_{m,-n}.$$

Let $\mathfrak{h}^+$ be the subalgebra generated by $\{a_n\}_{n \in \mathbb{N}}$ and $\mathcal{F} := \mathrm{Ind}_{\mathfrak{h}^+}^{\mathfrak{h}} \mathbb{C}\mathbf{1}$ be the Fock $\mathfrak{h}$-representation. On the other hand, note that $\{u_{j,0}\}_{j \in \mathbb{Z}^*} \subset \widetilde{\mathcal{E}}$ also form a Heisenberg algebra and the highest weight vector $|\emptyset\rangle \in F(u)$ is annihilated by $\{u_{j,0}\}_{j<0}$. The following is straightforward:

Proposition 8.2. *There exists a unique isomorphism $F(u) \xrightarrow{\sim} \mathcal{F}$ such that $|\emptyset\rangle \mapsto \mathbf{1}$ and action of $\widetilde{e}(z), \widetilde{f}(z), \widetilde{\psi}^{\pm}(z), \gamma^{\pm 1/2}$ gets intertwined with $\rho_c(\widetilde{e}(z)), \rho_c(\widetilde{f}(z)), \rho_c(\widetilde{\psi}^{\pm}(z))$, $\rho_c(\gamma^{\pm 1/2})$ acting on $\mathcal{F}$ in the following way:*

$$\rho_c(\gamma^{\pm 1/2}) = q_3^{\pm 1/4},\ \rho_c(\widetilde{\psi}^{\pm}(z)) = \exp\left(\mp \sum_{n>0} \frac{1 - q_2^{\mp n}}{n}(1 - q_3^n) q_3^{-n/4} a_{\pm n} z^{\mp n}\right),$$

$$\rho_c(\widetilde{e}(z)) = c \exp\left(\sum_{n>0} \frac{1 - q_2^n}{n} a_{-n} z^n\right) \exp\left(-\sum_{n>0} \frac{1 - q_2^{-n}}{n} a_n z^{-n}\right),$$

$$\rho_c(\widetilde{f}(z)) = \frac{-q_1q_2c^{-1}}{(1-q_1)^2(1-q_2)^2}\exp\left(-\sum_{n>0}\frac{1-q_2^n}{n}q_3^{n/2}a_{-n}z^n\right)$$
$$\times\exp\left(\sum_{n>0}\frac{1-q_2^{-n}}{n}q_3^{n/2}a_nz^{-n}\right),$$

where $c=-u/(1-q_1)(1-q_2)$.

These $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$-representations $\{\rho_c\}$ (acting on $\mathcal{F}$) were constructed in [7, Proposition A.6]. Likewise, computing the action of $\widetilde{e}(z), \widetilde{f}(z), \widetilde{\psi}^{\pm}(z)$ on the vector representation $V(u)$, one recovers the formulas for the $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$-representations π_c (acting on $\mathbb{C}[x,x^{-1}]$) with $c=-u/(1-q_1)(1-q_2)$ considered in [7, Proposition A.5].

8.3. Correlation functions

For a $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$-module L and a pair $v\in L, w\in L^*$, we define the correlation function

$$m_{w,v}(z_1,\dots,z_n) := \langle w|\widetilde{e}(z_1)\cdots\widetilde{e}(z_n)|v\rangle\cdot\prod_{i<j}\omega^m(z_i,z_j).$$

The relation (TT1) implies that $m_{w,v}(z_1,\dots,z_n)$ is $\mathfrak{S}_n$-symmetric.

Proposition 8.3. *For* $L=\rho_c$ *and* $v=\mathbf{1}\in L$, $w=\mathbf{1}^*\in L^*$*–the dual of* $\mathbf{1}$*, we have*

$$m_{\mathbf{1}^*,\mathbf{1}}(z_1,\dots,z_n) = (-q_3)^{-n(n-1)/2}c^n\prod_{i<j}\frac{(z_i-q_3z_j)(z_j-q_3z_i)}{(z_i-z_j)^2}.$$

Proof. For $n>0$, we have

$$\exp(u\cdot a_n)\exp(v\cdot a_{-n}) = \exp(v\cdot a_{-n})\exp(u\cdot a_n)\exp(uv\cdot n(1-q_1^n)/(1-q_2^{-n})).$$

Therefore

$$\rho_c(\widetilde{e}(z_i))\rho_c(\widetilde{e}(z_j)) = :\rho_c(\widetilde{e}(z_i))\rho_c(\widetilde{e}(z_j)):\cdot\prod_{n>0}\exp\left(-\frac{(1-q_1^n)(1-q_2^n)}{n}(z_j/z_i)^n\right).$$

It remains to use the equality $\prod_{n>0}\exp\left(-\frac{(1-q_1^n)(1-q_2^n)}{n}(z_j/z_i)^n\right)=\frac{(z_i-z_j)(z_i-q_1q_2z_j)}{(z_i-q_1z_j)(z_i-q_2z_j)}$. □

In the more general case of $\rho_{c_1}\underset{h}{\otimes}\cdots\underset{h}{\otimes}\rho_{c_k}$, we have the following result:

Proposition 8.4. *For* $L=\rho_{c_1}\underset{h}{\otimes}\cdots\underset{h}{\otimes}\rho_{c_k}$ *consider* $\bar{\mathbf{1}}:=\mathbf{1}\otimes\cdots\otimes\mathbf{1}\in L$ *and its dual* $\bar{\mathbf{1}}^*\in L^*$*. Then* $m_{\bar{\mathbf{1}}^*,\bar{\mathbf{1}}}(z_1,\dots,z_n)\in\mathcal{A}^m$.

Proof. Combining the formulas from Proposition 8.2 with the formulas for Δ_h, we find

$$m_{\bar{\mathbf{1}}^*,\bar{\mathbf{1}}}(z_1,\dots,z_n)=\sum_f c_{f(1)}\cdots c_{f(n)}\prod_{i<j}W_f(z_i,z_j)\prod_{i<j}\omega^m(z_i,z_j),$$

where the sum is over all maps $f:\{1,\dots,n\}\to\{1,\dots,k\}$ and $W_f(z_i,z_j)$ equals 1 (if $f(i)>f(j)$), $\frac{(z_i-z_j)(z_i-q_1q_2z_j)}{(z_i-q_1z_j)(z_i-q_2z_j)}$ (if $f(i)=f(j)$), and $g(z_i/z_j)$ (if $f(i)<f(j)$). The claim follows. □

Remark 8.2. (a) This approach provides a *Bethe algebra realization* of $\mathcal{A}^m$ (see [9]).

(b) According to [7, Proposition A.10], the correlation functions of $\pi_{c_1}\underset{h}{\otimes}\cdots\underset{h}{\otimes}\pi_{c_k}$ are identified with the classical Macdonald difference operators.

8.4. Whittaker vector in K-theory

Consider the *Whittaker vector* $v_r^K:=\sum_{n\geq 0}\left[\mathcal{O}_{M(r,n)}\right]\in\widehat{M}^r$, where $\widehat{M}^r:=\prod_{n=1}^{\infty}M_n^r$. To state our main result, we introduce a family of the elements $\{K_n^{(m;j)}\}_{n\in\mathbb{N}}^{j\in\mathbb{Z}}$ of S^m defined by

$$\begin{aligned}K_n^{(m;j)}(x_1,\dots,x_n)&:=K_n^m(x_1,\dots,x_n)x_1^j\cdots x_n^j\\&=\prod_{1\leq a<b\leq n}\frac{(x_a-q_1x_b)(x_b-q_1x_a)}{(x_a-x_b)^2}\prod_{1\leq s\leq n}x_s^j.\end{aligned}$$

Let $\{K_{-n}^{(m;j)}\}_{n\in\mathbb{N}}^{j\in\mathbb{Z}}$ be analogous elements in the opposite algebra $(S^m)^{\mathrm{opp}}$. The name *Whittaker* is motivated by the following result (see Appendix E for a proof):

Theorem 8.5. *The vector v_r^K is an eigenvector with respect to $\{K_{-n}^{(m;j)}|0\leq j\leq r,n>0\}$. More precisely: $K_{-n}^{(m;j)}(v_r^K)=C_{j,-n}\cdot v_r^K$, where*

$$C_{0,-n}=\frac{(-1)^nt^{(r-2)n}(-t_1)^{n(n-1)/2}}{(1-t_1)^n(1-t_2)(1-t_2^2)\cdots(1-t_2^n)},\ C_{1,-n}=\ldots=C_{r-1,-n}=0,$$

$$C_{r,-n}=\frac{(-t)^{(r-2)n}(-t_1t_2)^{n(n-1)/2}}{(1-t_1)^n(1-t_2)(1-t_2^2)\cdots(1-t_2^n)\cdot(\chi_1\cdots\chi_r)^n}.$$

Remark 8.3. The subalgebra of $(S^m)^{\mathrm{opp}}$ generated by $\{K_{-n}^{(m;j)}\}_{n>0}^{0\leq j\leq r}$ corresponds to the subalgebra of $\ddot{U}^-$ generated by $\{f_j,[f_{j+1},f_{j-1}],[f_{j+1},[f_j,f_{j-1}]],\cdots\}_{j=0}^r$ under the identification $(S^m)^{\mathrm{opp}}\simeq\ddot{U}^-$, due to Proposition 7.5.

8.5. Whittaker vector in cohomology

Consider the *Whittaker vector* $v_r^H := \sum_{n\geq 0}[M(r,n)] \in \widehat{V}^r$, where $\widehat{V}^r := \prod_{n=1}^{\infty} V_n^r$. To state our main result, we introduce a family of the elements $\{K_n^{(a;j)}\}_{n\in\mathbb{N}}^{j\in\mathbb{Z}_+}$ of S^a defined by

$$K_n^{(a;j)}(x_1,\dots,x_n) := K_n^a(x_1,\dots,x_n)x_1^j\cdots x_n^j$$
$$= \prod_{1\leq a<b\leq n} \frac{(x_a-x_b-h_1)(x_b-x_a-h_1)}{(x_a-x_b)^2} \prod_{1\leq s\leq n} x_s^j.$$

Let $\{K_{-n}^{(a;j)}\}_{n\in\mathbb{N}}^{j\in\mathbb{Z}_+}$ be analogous elements in the opposite algebra $(S^a)^{\mathrm{opp}}$. The following result has been already proved in [20] (we refer the interested reader to Appendix E for an alternative proof).

Theorem 8.6. *The vector v_r^H is an eigenvector with respect to $\{K_{-n}^{(a;j)}|0\leq j\leq r, n>0\}$. More precisely: $K_{-n}^{(a;j)}(v_r^H) = D_{j,-n}\cdot v_r^H$, where*

$$D_{0,-n} = \ldots = D_{r-2,-n} = 0,\ D_{r-1,-n} = \frac{(-1)^{n(n-1)/2}}{n!s_1^n s_2^n},\ D_{r,-1} = \frac{-1}{s_1 s_2}\sum_{a=1}^{r} x_a$$

and $D_{r,-n}$ is a degree n polynomial in x_a.

Remark 8.4. The subalgebra of $(S^a)^{\mathrm{opp}}$ generated by $\{K_{-n}^{(a;j)}\}_{n>0}^{0\leq j\leq r}$ corresponds to the subalgebra of $\ddot{Y}^-$ generated by $\{f_j, [f_j,f_{j+1}], [f_j,[f_j,f_{j+2}]],\cdots\}_{j=0}^r$ under the identification $(S^a)^{\mathrm{opp}} \simeq \ddot{Y}^-$, due to Proposition 7.8.

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Appendix A. Proof of Proposition 1.3

We follow a standard argument. Consider a unital associative $\mathbb{C}$-algebra $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by $\{e_i, f_i, \psi_i, \psi_0^{-1} | i \in \mathbb{Z}\}$ subject to the relations (T0,T3,T4,T5). Let

$\ddot{V}^-, \ddot{V}^0, \ddot{V}^+$ be the subalgebras of $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by $\{f_i\}$, $\{\psi_i, \psi_0^{-1}\}$, and $\{e_i\}$, respectively. Let I^+ and I^- be the two-sided ideals of $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by the quadratic and cubic relations in e_i and f_i, respectively, arising from the relations (T1,T2,T6). Explicitly, I^+ is generated by

$$A_{i,j} = e_{i+3}e_j - \varsigma_1 e_{i+2}e_{j+1} + \varsigma_2 e_{i+1}e_{j+2} - e_i e_{j+3} \\ - e_j e_{i+3} + \varsigma_2 e_{j+1}e_{i+2} - \varsigma_1 e_{j+2}e_{i+1} + e_{j+3}e_i,$$

$$B_{i_1,i_2,i_3} = \mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2+1}, e_{i_3-1}]],$$

where $\varsigma_1 := q_1 + q_2 + q_3$, $\varsigma_2 := q_1^{-1} + q_2^{-1} + q_3^{-1}$ and $i, j, i_1, i_2, i_3 \in \mathbb{Z}$. Let $J^\pm$ stay for the corresponding two-sided ideals of $\ddot{V}^\pm$. Our next result implies Proposition 1.3.

Lemma A.1. *(a) Multiplication induces an isomorphism of vector spaces*

$$m : \ddot{V}^- \otimes \ddot{V}^0 \otimes \ddot{V}^+ \xrightarrow{\sim} \ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1).$$

(b) $\ddot{V}^-$ and $\ddot{V}^+$ are free associative algebras in $\{f_i\}$ and $\{e_i\}$, respectively, while $\ddot{V}^0$ is the algebra generated by $\{\psi_i, \psi_0^{-1}\}$ with the defining relations (T0).

(c) We have $I^+ = m(\ddot{V}^- \otimes \ddot{V}^0 \otimes J^+)$ and $I^- = m(J^- \otimes \ddot{V}^0 \otimes \ddot{V}^+)$.

Proof of Lemma A.1. Parts (a) and (b) are standard.

The first equality in part (c) is equivalent to $\ddot{V}^-\ddot{V}^0 J^+$ being a two-sided ideal of $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. To prove this, it suffices to show $[A_{i,j}, t_r], [B_{i_1,i_2,i_3}, t_r], [A_{i,j}, f_r]$, $[B_{i_1,i_2,i_3}, f_r] \in \ddot{V}^0 J^+$. The first two commutators are just the linear combinations of $A_{i',j'}$ and $B_{i'_1,i'_2,i'_3}$, respectively, due to (T4t). Also $[A_{i,j}, f_r] = 0$, due to (T3) and (T4).

To prove $[B_{i_1,i_2,i_3}, f_r] \in \ddot{V}^0 J^+$, we work with the generating series. The relation (T3) implies

$$\beta_1 \cdot [e(z_1)e(z_2)e(z_3), f(w)] = \\ \delta\left(\frac{z_1}{w}\right)\psi(z_1)e(z_2)e(z_3) + \delta\left(\frac{z_2}{w}\right)\psi(z_2)e(z_1)e(z_3)g\left(\frac{z_1}{z_2}\right) + \\ \delta\left(\frac{z_3}{w}\right)\psi(z_3)e(z_1)e(z_2)g\left(\frac{z_1}{z_3}\right)g\left(\frac{z_2}{z_3}\right),$$

where $\psi(z) = \psi^+(z) - \psi^-(z)$. Hence, we have

$$\left[\mathrm{Sym}_{\mathfrak{S}_3}\left\{\left(\frac{z_2}{z_1} + \frac{z_2}{z_3} - \frac{z_1}{z_2} - \frac{z_3}{z_2}\right)e(z_1)e(z_2)e(z_3)\right\}, f(w)\right] = \\ \beta_1^{-1}\left(\delta(z_1/w)\psi(z_1)C_1(z_2,z_3) + \delta(z_2/w)\psi(z_2)C_2(z_3,z_1) + \delta(z_3/w)\psi(z_3)C_3(z_1,z_2)\right),$$

where $C_1(z_2,z_3)=e(z_2)e(z_3)C_{123}+e(z_3)e(z_2)C_{132}$ and

$$C_{123}=\left(\frac{z_2}{z_1}+\frac{z_2}{z_3}-\frac{z_1}{z_2}-\frac{z_3}{z_2}\right)+g(z_2/z_1)\left(\frac{z_1}{z_2}+\frac{z_1}{z_3}-\frac{z_2}{z_1}-\frac{z_3}{z_1}\right)+$$
$$g(z_2/z_1)g(z_3/z_1)\left(\frac{z_3}{z_1}+\frac{z_3}{z_2}-\frac{z_1}{z_3}-\frac{z_2}{z_3}\right),$$
$$C_{132}=\left(\frac{z_3}{z_1}+\frac{z_3}{z_2}-\frac{z_1}{z_3}-\frac{z_2}{z_3}\right)+g(z_3/z_1)\left(\frac{z_1}{z_2}+\frac{z_1}{z_3}-\frac{z_2}{z_1}-\frac{z_3}{z_1}\right)+$$
$$g(z_2/z_1)g(z_3/z_1)\left(\frac{z_2}{z_1}+\frac{z_2}{z_3}-\frac{z_1}{z_2}-\frac{z_3}{z_2}\right).$$

The equality $C_{132}=-g(z_2/z_3)C_{123}$ implies that $C_1(z_2,z_3)$ is proportional to the generating function of $A_{i,j}$. The same holds for $C_2(z_3,z_1), C_3(z_1,z_2)$. This yields $[B_{i_1,i_2,i_3},f_r]\in \dot{V}^0J^+$ for any $i_1,i_2,i_3,r\in\mathbb{Z}$.

The second equality in part (c) is proved analogously. □

Appendix B. Proofs of Theorems 3.2, 3.4

B.1. Sketch of the proof of Theorem 3.2

We generalize the key technical result of [8], required to prove Theorem 3.2 (all other arguments stay the same). Verification of the relations (T0,T1,T2,T6t) is straightforward and is based on Lemma 3.3(a). Likewise, it is easy to check that the operators $[e_i,f_j]$ are diagonal in the fixed point basis and depend on $i+j$ only: $[e_i,f_j](|\bar{\lambda}]) = \gamma_{i+j|_{\bar{\lambda}}}\cdot|\bar{\lambda}]$.

Lemma B.1. *We have*

$$\gamma_{0|_{\bar{\lambda}}}=\frac{t^{-r}-t^r}{(1-t_1)(1-t_2)(1-t_3)},$$
$$\gamma_{1|_{\bar{\lambda}}}=t^{-r}\left(\frac{1}{(1-t_1)(1-t_2)}\sum_{a=1}^{r}\chi_a^{-1}-\sum_{a=1}^{r}\sum_{\square\in\lambda^a}\chi(\square)\right).$$

Proof. Fix positive integers $L_a>(\lambda^{a*})_1$ for $1\le a\le r$. Applying Lemma 3.3(a), we find

$$\gamma_{s|_{\bar{\lambda}}}=$$
$$\sum_{l=1}^{r}\sum_{j=1}^{L_l}\frac{T}{(1-t_1)^2}(\chi_j^{(l)})^{s-r}\cdot\frac{\chi_j^{(l)}(1-\chi_j^{(l)}t_1t_2^{1-L_l}\chi_l)}{\chi_j^{(l)}-t_2^{L_l}\chi_l^{-1}}\cdot\prod_{k\ne j}^{k\le L_l}\frac{(\chi_j^{(l)}-t_1t_2\chi_k^{(l)})(\chi_k^{(l)}-t_2\chi_j^{(l)})}{(\chi_j^{(l)}-t_1\chi_k^{(l)})(\chi_k^{(l)}-\chi_j^{(l)})}\times$$
$$\prod_{a\ne l}\left(\frac{\chi_j^{(l)}(1-\chi_j^{(l)}t_1t_2^{1-L_a}\chi_a)}{\chi_j^{(l)}-t_2^{L_a}\chi_a^{-1}}\cdot\prod_{k=1}^{L_a}\frac{(\chi_j^{(l)}-t_1t_2\chi_k^{(a)})(\chi_k^{(a)}-t_2\chi_j^{(l)})}{(\chi_j^{(l)}-t_1\chi_k^{(a)})(\chi_k^{(a)}-\chi_j^{(l)})}\right)-$$

$$\sum_{l=1}^{r}\sum_{j=1}^{L_l}\frac{T}{(1-t_1)^2}(t_1\chi_j^{(l)})^{s-r}\cdot\frac{\chi_j^{(l)}(1-\chi_j^{(l)}t_1^2t_2^{1-L_l}\chi_l)}{\chi_j^{(l)}-t_1^{-1}t_2^{L_l}\chi_l^{-1}}\cdot\prod_{k\neq j}^{k\leq L_l}\frac{(\chi_k^{(l)}-t_1t_2\chi_j^{(l)})(\chi_j^{(l)}-t_2\chi_k^{(l)})}{(\chi_k^{(l)}-t_1\chi_j^{(l)})(\chi_j^{(l)}-\chi_k^{(l)})}\times$$
$$\prod_{a\neq l}\left(\frac{\chi_j^{(l)}(1-\chi_j^{(l)}t_1^2t_2^{1-L_a}\chi_a)}{\chi_j^{(l)}-t_1^{-1}t_2^{L_a}\chi_a^{-1}}\cdot\prod_{k=1}^{L_a}\frac{(\chi_k^{(a)}-t_1t_2\chi_j^{(l)})(\chi_j^{(l)}-t_2\chi_k^{(a)})}{(\chi_k^{(a)}-t_1\chi_j^{(l)})(\chi_j^{(l)}-\chi_k^{(a)})}\right),\quad(\heartsuit)$$

where $T=(-t)^{r-2}\chi_1^{-1}\cdots\chi_r^{-1}$ and $\chi_k^{(a)}=t_1^{\lambda_k^a-1}t_2^{k-1}\chi_a^{-1}$ as before.

(i) For $s=0$, the right-hand side of $(\heartsuit)$ is a degree 0 rational function in the variables $\chi_k^{(a)}$. It is easy to check that it has no poles, in fact. Therefore, it is an element of $\mathbb{F}_r$ independent of $\bar{\lambda}$. It suffices to compute its value at $\bar{\emptyset}$, the r-tuple of empty Young diagrams. For $\bar{\lambda}=\bar{\emptyset}$, we can choose $L_1=\ldots=L_r=1$, while $\chi_k^{(a)}=t_1^{-1}t_2^{k-1}\chi_a^{-1}$. Applying $(\heartsuit)$, we get

$$\begin{aligned}\gamma_{0|_{\bar{\lambda}}}&=\gamma_{0|_{\bar{\emptyset}}}\\&=-\frac{T}{(1-t_1)^2}\sum_{l=1}^{r}t_1^{-r}\frac{1-t_1}{t_1^{-1}\chi_l^{-1}-t_2t_1^{-1}\chi_l^{-1}}\prod_{a\neq l}\frac{(t_1^{-1}\chi_a^{-1}-t_2\chi_l^{-1})(1-t_1\chi_a\chi_l^{-1})}{(t_1^{-1}\chi_l^{-1}-t_1^{-1}\chi_a^{-1})(t_1^{-1}\chi_a^{-1}-\chi_l^{-1})}\\&=\frac{t^{r-2}}{(1-t_1)(1-t_2)}\sum_{l=1}^{r}\prod_{a\neq l}\frac{\chi_l-t_1t_2\chi_a}{\chi_l-\chi_a}=\frac{t^{r-2}}{(1-t_1)(1-t_2)}\cdot\frac{1-t_1^rt_2^r}{1-t_1t_2}\\&=\frac{t^{-r}-t^r}{(1-t_1)(1-t_2)(1-t_3)},\end{aligned}$$

where we used the identity $\sum_{l=1}^{r}\prod_{a\neq l}\frac{\chi_l-u\chi_a}{\chi_l-\chi_a}=\frac{1-u^r}{1-u}$. The first equality follows.

(ii) For $s=1$, the right-hand side of $(\heartsuit)$ is a degree 1 rational function in the variables $\chi_k^{(a)}$. It is easy to check that it has no poles. Therefore, it is a linear function with the leading linear part $T\cdot(-1)^r\chi_1\ldots\chi_r\frac{t_1^rt_2^{r-1}}{1-t_1}\cdot\sum_{l=1}^{r}\sum_{j=1}^{L_l}\chi_j^{(l)}$. Hence, we have

$$\gamma_{1|_{\bar{\lambda}}}=\frac{t^{-r}t_1}{1-t_1}\cdot\sum_{l=1}^{r}\sum_{j=1}^{\infty}\widetilde{\chi}_j^{(l)}+C$$

for a constant $C\in\mathbb{F}_r$ independent of $\bar{\lambda}$, where $\widetilde{\chi}_j^{(l)}:=\chi_j^{(l)}-t_1^{-1}t_2^{j-1}\chi_l^{-1}$. Note that

$$\sum_{a=1}^{r}\sum_{\square\in\lambda^a}\chi(\square)=\sum_{a=1}^{r}\sum_{j=1}^{(\lambda^{a*})_1}t_2^{j-1}(1+t_1+\ldots+t_1^{\lambda_j^a-1})\chi_a^{-1}=\sum_{l=1}^{r}\sum_{j}\frac{-t_1}{1-t_1}\widetilde{\chi}_j^{(l)}.$$

On the other hand, we have $C=\gamma_{1|_{\bar{\emptyset}}}$. Applying $(\heartsuit)$, we find

$$\begin{aligned}C&=\gamma_{1|_{\bar{\emptyset}}}\\&=-\frac{T}{(1-t_1)^2}\sum_{l=1}^{r}t_1^{-r}\frac{\chi_l^{-1}(1-t_1)}{t_1^{-1}\chi_l^{-1}-t_2t_1^{-1}\chi_l^{-1}}\prod_{a\neq l}\frac{(t_1^{-1}\chi_a^{-1}-t_2\chi_l^{-1})(1-t_1\chi_a\chi_l^{-1})}{(t_1^{-1}\chi_l^{-1}-t_1^{-1}\chi_a^{-1})(t_1^{-1}\chi_a^{-1}-\chi_l^{-1})}\end{aligned}$$

$$= \frac{t^{r-2}}{(1-t_1)(1-t_2)} \cdot \sum_{l=1}^{r} \frac{1}{\chi_l} \prod_{a \neq l} \frac{\chi_l - t_1 t_2 \chi_a}{\chi_l - \chi_a} = \frac{t^{r-2} t_1^{r-1} t_2^{r-1}}{(1-t_1)(1-t_2)} \sum_{l=1}^{r} \chi_l^{-1}$$

$$= \frac{t^{-r}}{(1-t_1)(1-t_2)} \sum_{l=1}^{r} \chi_l^{-1},$$

where we used $\sum_{l=1}^{r} \frac{1}{\chi_l} \prod_{a \neq l} \frac{\chi_l - u\chi_a}{\chi_l - \chi_a} = u^{r-1} \sum_{l=1}^{r} \chi_l^{-1}$. The second equality follows. □

Next, we introduce the operator series $\phi^{\pm}(z) = \sum_{j=0}^{\infty} \phi_j^{\pm} z^{\mp j} \in \mathrm{End}(M^r)[[z^{\mp 1}]]$, diagonal in the fixed point basis with the eigenvalues given by

$$\phi^{+}_{0|_{\bar{\lambda}}} = t^{-r},\ \phi^{-}_{0|_{\bar{\lambda}}} = t^{r},\ \phi^{\pm}_{j|_{\bar{\lambda}}} = \pm(1-t_1)(1-t_2)(1-t_3)\gamma_{\pm j|_{\bar{\lambda}}} \text{ for } j \in \mathbb{N}.$$

The following is a consequence of Lemma B.1:

Corollary B.2. *We have*

$$[e(z), f(w)] = \frac{\delta(z/w)}{(1-t_1)(1-t_2)(1-t_3)} (\phi^{+}(w) - \phi^{-}(z)).$$

Our next result follows from the explicit computations in the fixed point basis.

Lemma B.3. *The series $\phi^{\pm}(z)$ satisfy the following relations:*

$$\phi^{\pm}(z)e(w)(z-t_1w)(z-t_2w)(z-t_3w) = -e(w)\phi^{\pm}(z)(w-t_1z)(w-t_2z)(w-t_3z), \tag{12}$$

$$\phi^{\pm}(z)f(w)(w-t_1z)(w-t_2z)(w-t_3z) = -f(w)\phi^{\pm}(z)(z-t_1w)(z-t_2w)(z-t_3w). \tag{13}$$

Relation (12) implies the following identity:

$$\phi^{+}(z)_{|_{\bar{\lambda}+\square_j^l}} = \phi^{+}(z)_{|_{\bar{\lambda}}} \cdot \frac{(1-t_1^{-1}\chi(\square_j^l)/z)(1-t_2^{-1}\chi(\square_j^l)/z)(1-t_3^{-1}\chi(\square_j^l)/z)}{(1-t_1\chi(\square_j^l)/z)(1-t_2\chi(\square_j^l)/z)(1-t_3\chi(\square_j^l)/z)}.$$

Therefore, we get

$$\phi^{+}(z)_{|_{\bar{\lambda}}} = \phi^{+}(z)_{|_{\bar{\emptyset}}} \cdot \mathbf{c}_r(z)^{+}_{|_{\bar{\lambda}}}.$$

Applying (♡) once again, we find

$$\phi^{+}(z)_{|_{\bar{\emptyset}}} = (\phi_0^{-} + \sum_{i \geq 0} (1-t_1)(1-t_2)(1-t_3)\gamma_i z^{-i})_{|_{\bar{\emptyset}}}$$

$$= t^r - t^r(1-t_1t_2) \sum_{l=1}^{r} \frac{1}{1-\chi_l^{-1}z^{-1}} \prod_{a \neq l} \frac{\chi_l - t_1t_2\chi_a}{\chi_l - \chi_a} = t^r \prod_{l=1}^{r} \left(\frac{1-t_1t_2\chi_l z}{1-\chi_l z} \right)^{+},$$

where we used the identity

$$1-(1-u)\sum_{l=1}^{r}\frac{1}{1-1/(\chi_l z)}\prod_{a\neq l}\frac{\chi_l-u\chi_a}{\chi_l-\chi_a}=\prod_{l=1}^{r}\frac{u-\frac{1}{\chi_l z}}{1-\frac{1}{\chi_l z}}.$$

This proves $\phi^+(z)=\psi^+(z)$. Analogous arguments also imply $\phi^-(z)=\psi^-(z)$.

The relation (T3) follows from Corollary B.2, while the relations (T4,T5) follow from (12, 13). This completes our proof of Theorem 3.2.

B.2. Sketch of the proof of Theorem 3.4

The proof of Theorem 3.4 is completely analogous to its K-theoretical counterpart. Verification of the relations (Y0,Y1,Y2,Y6) is straightforward. Likewise, it is easy to check that $[e_i,f_j]$ are diagonal in the fixed point basis and depend on $i+j$ only: $[e_i,f_j]([\bar\lambda])=\gamma_{i+j|_{\bar\lambda}}\cdot[\bar\lambda]$. To verify the remaining relations, we will need the following generalization of Lemma 2.4:

Lemma B.4. *We have* $\gamma_{0|_{\bar\lambda}}=\frac{-r}{s_1s_2}$, $\gamma_{1|_{\bar\lambda}}=\frac{1}{s_1s_2}\left(\sum_{a=1}^{r}x_a-\binom{r}{2}(s_1+s_2)\right)$,

$$\gamma_{2|_{\bar\lambda}}=2|\bar\lambda|-\frac{1}{s_1s_2}\left(\sum_{a=1}^{r}x_a^2-(r-1)(s_1+s_2)\sum_{a=1}^{r}x_a+\binom{r}{3}(s_1+s_2)^2\right).$$

Proof. Fix positive integers $L_a>(\lambda^{a*})_1$ for $1\leq a\leq r$. Applying Lemma 3.5(a), we find

$$\begin{aligned}\gamma_{s|_{\bar\lambda}}=&\sum_{l=1}^{r}\sum_{j=1}^{L_l}\frac{1}{s_1^2}(x_j^{(l)})^s\cdot\frac{x_j^{(l)}+s_1+(1-L_l)s_2+x_l}{-x_j^{(l)}+L_ls_2-x_l}\times\\&\prod_{k\neq j}^{k\leq L_l}\frac{(x_j^{(l)}-x_k^{(l)}-s_1-s_2)(x_k^{(l)}-x_j^{(l)}-s_2)}{(x_j^{(l)}-x_k^{(l)}-s_1)(x_k^{(l)}-x_j^{(l)})}\times\\&\prod_{a\neq l}\left(\frac{x_j^{(l)}+s_1+(1-L_a)s_2+x_a}{x_j^{(l)}-L_as_2+x_a}\cdot\prod_{k=1}^{L_a}\frac{(x_j^{(l)}-x_k^{(a)}-s_1-s_2)(x_k^{(a)}-x_j^{(l)}-s_2)}{(x_j^{(l)}-x_k^{(a)}-s_1)(x_k^{(a)}-x_j^{(l)})}\right)-\\&\sum_{l=1}^{r}\sum_{j=1}^{L_l}\frac{1}{s_1^2}(x_j^{(l)}+s_1)^s\cdot\frac{x_j^{(l)}+2s_1+(1-L_l)s_2+x_l}{-x_j^{(l)}+L_ls_2-s_1-x_l}\times\\&\prod_{k\neq j}^{k\leq L_l}\frac{(x_k^{(l)}-x_j^{(l)}-s_1-s_2)(x_j^{(l)}-x_k^{(l)}-s_2)}{(x_k^{(l)}-x_j^{(l)}-s_1)(x_j^{(l)}-x_k^{(l)})}\times\\&\prod_{a\neq l}\left(\frac{x_j^{(l)}+2s_1+(1-L_a)s_2+x_a}{x_j^{(l)}-L_as_2+s_1+x_a}\cdot\prod_{k=1}^{L_a}\frac{(x_k^{(a)}-x_j^{(l)}-s_1-s_2)(x_j^{(l)}-x_k^{(a)}-s_2)}{(x_k^{(a)}-x_j^{(l)}-s_1)(x_j^{(l)}-x_k^{(a)})}\right),\end{aligned}\tag{♠}$$

where $x_k^{(a)} = (\lambda_k^a - 1)s_1 + (k-1)s_2 - x_a$ as before. The right-hand side of (♠) is a degree s rational function in the variables $x_k^{(a)}$. It is easy to see that it has no poles for $s \geq 0$.

(i) For $s = 0$, we therefore see that $\gamma_{0|_{\bar\lambda}}$ must be an element of $\mathbb{C}(s_1, s_2, x_1, \dots, x_l)$ independent of $\bar\lambda$. Evaluating (♠) at $\bar\emptyset$, we find $\gamma_{0|_{\bar\lambda}} = \gamma_{0|_{\bar\emptyset}} = -r/s_1 s_2$.

(ii) For $s = 1$, we therefore see that $\gamma_{1|_{\bar\lambda}}$ is a linear function. But its leading linear part is zero. So $\gamma_{1|_{\bar\lambda}} = \gamma_{1|_{\bar\emptyset}}$. Evaluating (♠) at $\bar\emptyset$, we find $\gamma_{1|_{\bar\lambda}} = \gamma_{1|_{\bar\emptyset}} = \frac{1}{s_1 s_2}\left(\sum_{a=1}^r x_a - \binom{r}{2}(s_1+s_2)\right)$.

(iii) For $s = 2$, we therefore see that $\gamma_{2|_{\bar\lambda}}$ is a quadratic function. But its leading quadratic part is zero. So $\gamma_{2|_{\bar\lambda}}$ is a linear function. Similarly to the proof of Lemma 2.4, we find that the leading linear part is actually $\frac{2}{s_1}\sum_{a=1}^r \sum_{k=1}^\infty \widetilde{x}_k^{(a)} = 2|\bar\lambda|$, where $\widetilde{x}_k^{(a)} := x_k^{(a)} - (-s_1 + (k-1)s_2 - x_a)$. Hence, $\gamma_{2|_{\bar\lambda}} = 2|\bar\lambda| + \gamma_{2|_{\bar\emptyset}}$. Applying (♠) once again, we recover the last formula. □

Define $\phi_j := [e_j, f_0] \in \mathrm{End}(V^r)$. Explicit computations in the fixed point basis show that $\{\phi_j, e_j, f_j\}_{j\in\mathbb{Z}_+}$ satisfy the relations (Y3,Y4,Y4′,Y5,Y5′) with ψ_j replaced by ϕ_j. This implies

$$\phi(z)_{|_{\bar\lambda+\square_j^l}} = \phi(z)_{|_{\bar\lambda}} \cdot \frac{(z-\chi(\square_j^l)+s_1)(z-\chi(\square_j^l)+s_2)(z-\chi(\square_j^l)+s_3)}{(z-\chi(\square_j^l)-s_1)(z-\chi(\square_j^l)-s_2)(z-\chi(\square_j^l)-s_3)},$$

where $\phi(z) := 1 + \sigma_3 \sum_{j\geq 0} \phi_j z^{-j-1}$. Therefore, $\phi(z)_{|_{\bar\lambda}} = \phi(z)_{|_{\bar\emptyset}} \cdot \mathbf{C}_r(z)_{|_{\bar\lambda}}$. Applying (♠), we find

$$\begin{aligned}\phi(z)_{|_{\bar\emptyset}} &= 1 - \frac{\sigma_3}{s_1 s_2}\sum_{i\geq 0}\sum_{l=1}^r (-x_l)^i z^{-i-1} \prod_{a\neq l} \frac{x_l - x_a - s_1 - s_2}{x_l - x_a} \\ &= 1 - s_3 \sum_{l=1}^r \frac{1}{z+x_l} \prod_{a\neq l} \frac{x_l - x_a + s_3}{x_l - x_a}.\end{aligned}$$

Combining this with the identity $1 + u\sum_{l=1}^r \frac{1}{z+x_l}\prod_{a\neq l}\frac{x_l-x_a-u}{x_l-x_a} = \prod_{l=1}^r \frac{z+x_l+u}{z+x_l}$, we finally get $\phi(z) = \psi(z)$. This completes our proof of Theorem 3.4.

Appendix C. Proofs of Theorems 5.5, 5.9

C.1. Proof of Theorem 5.5

As pointed out in Section 5.4, all the defining relations of the algebra $\ddot{U}_{h_0}(\mathfrak{gl}_1)$ are of Lie-type. Therefore, it is a universal enveloping algebra of the Lie algebra $\ddot{u}_{h_0}$ generated by $\{e_i, f_i, H_i\}_{i\in\mathbb{Z}}$ with the same defining relations. Moreover, $\ddot{u}_{h_0}$ is a $\mathbb{C}\cdot H_0$-central extension of the Lie-algebra $\underline{\ddot{u}}_{h_0}$ generated by $\{e_i, f_i, H_m\}_{i\in\mathbb{Z}}^{m\in\mathbb{Z}^*}$ with the following defining relations:

$$[H_k, H_m] = 0, \tag{u0}$$

$$[e_{i+3}, e_j] - (1+q+q^{-1})[e_{i+2}, e_{j+1}] + (1+q+q^{-1})[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] = 0, \quad \text{(u1)}$$

$$[f_{i+3}, f_j] - (1+q+q^{-1})[f_{i+2}, f_{j+1}] + (1+q+q^{-1})[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] = 0, \quad \text{(u2)}$$

$$[e_i, f_j] = H_{i+j} \text{ for } j \neq -i, \quad [e_i, f_{-i}] = 0, \quad \text{(u3)}$$

$$[H_m, e_i] = -(1-q^m)(1-q^{-m})e_{i+m}, \quad \text{(u4)}$$

$$[H_m, f_i] = (1-q^m)(1-q^{-m})f_{i+m}, \quad \text{(u5)}$$

$$[e_0, [e_1, e_{-1}]] = 0, \quad [f_0, [f_1, f_{-1}]] = 0, \quad \text{(u6)}$$

where $q = e^{h_0} \in \mathbb{C}^*$. Note that $h_0 \notin \mathbb{Q} \cdot \pi\sqrt{-1} \iff q \neq \sqrt{1}$ (q is not a root of 1). Hence, it suffices to check that the corresponding homomorphism $\theta : \ddot{\underline{u}}_{h_0} \to \mathfrak{d}_q^0$ defined by

$$\theta : e_i \mapsto Z^i D, \ f_i \mapsto -D^{-1}Z^i, \ H_m \mapsto (q^{-m}-1)Z^m \ \text{ for } i \in \mathbb{Z}, m \in \mathbb{Z}^*$$

is an isomorphism of the $\mathbb{C}$-Lie algebras for $q \neq \sqrt{1}$.

The Lie algebra $\ddot{\underline{u}}_{h_0}$ is $\mathbb{Z}^2$-graded via $\deg(e_i) := (i,1), \deg(f_i) := (i,-1), \deg(H_m) := (m,0)$. The Lie algebra $\mathfrak{d}_q^0$ is also $\mathbb{Z}^2$-graded via $\deg(Z^i D^j) = (i,j)$. Note that θ is $\mathbb{Z}^2$-graded and surjective. Since $\dim(\mathfrak{d}_q^0)_{i,j} = 1$ for $(i,j) \neq (0,0)$, it suffices to show that $\dim(\ddot{\underline{u}}_{h_0})_{i,j} \leq 1$. This is clear for $j = 0$, while the proof for $j > 0$ will occupy the rest of this section (the case $j < 0$ is completely analogous).

Let $\ddot{\underline{u}}^{\geq}_{h_0}$ be the Lie algebra generated by $\{e_i, H_m\}_{i\in\mathbb{Z}}^{m\in\mathbb{Z}^*}$ with the defining relations (u0,u1,u4,u6). It suffices to show that $\dim(\ddot{\underline{u}}^{\geq}_{h_0})_{i,j} \leq 1$ for $i \in \mathbb{Z}, \ j \in \mathbb{N}$. We prove this by an induction on j.

- *Case* $j = 1$.

It is clear that $(\ddot{\underline{u}}^{\geq}_{h_0})_{N,1}$ is spanned by e_N.

- *Case* $j = 2$.

It is clear that $(\ddot{\underline{u}}^{\geq}_{h_0})_{N,2}$ is spanned by $\{[e_{i_1}, e_{i_2}] | i_1 + i_2 = N\}$. However, (u1) implies

$$[e_{i+2+k}, e_{i+1-k}] = \frac{q^{k+1} - q^{-k}}{q-1}[e_{i+2}, e_{i+1}], \ [e_{i+2+k}, e_{i-k}] = \frac{q^{k+2} - q^{-k}}{q^2-1}[e_{i+2}, e_i].$$

These formulas can be unified in the following way:

$$[e_{i_1}, e_{i_2}] = \frac{q^{i_2} - q^{i_1}}{q^{i_1+i_2} - 1}[e_0, e_{i_1+i_2}] \text{ if } i_1 + i_2 \neq 0, \ [e_i, e_{-i}] = \frac{q^{i+1} - q^{1-i}}{q^2-1}[e_1, e_{-1}]. \quad (14)$$

Therefore, $(\ddot{\underline{u}}^{\geq}_{h_0})_{N,2}$ is either spanned by $[e_0, e_N]$ (if $N \neq 0$) or $[e_1, e_{-1}]$ (if $N = 0$).

- *Case* $j = 3$.

Let us introduce the *length* n *commutator*:

$$[a_1; a_2; \ldots; a_n]_n := [a_1, [a_2, [\ldots [a_{n-1}, a_n] \ldots]]]$$

(we will omit the subscript n when the length is clear). The space $(\ddot{\underline{u}}^{\geq}_{\bar{h}_0})_{N,3}$ is spanned by $\{[e_{i_1}; e_{i_2}; e_{i_3}] | i_1 + i_2 + i_3 = N\}$. Using the automorphism π of the Lie algebra $\ddot{\underline{u}}^{\geq}_{\bar{h}_0}$ determined by $e_i \mapsto e_{i+1}, H_m \mapsto H_m$, we can assume $i_1, i_2, i_3 \in \mathbb{N}$. Due to the above $j = 2$ case, it suffices to show that $[e_k; e_0; e_l]$ is a multiple of $[e_0; e_0; e_{k+l}]$ for any $k, l \in \mathbb{N}$.

For $m \in \mathbb{Z}^*$, define $\mathbf{h}_m := -\frac{H_m}{(1-q^m)(1-q^{-m})}$, so that $\mathrm{ad}(\mathbf{h}_m)(e_i) = e_{i+m}$ for any i. Set $A_1 := \mathrm{ad}(\mathbf{h}_1)$. Then

$$A_1([e_k; e_0; e_l]) = [e_{k+1}; e_0; e_l] + [e_k; e_1; e_l] + [e_k; e_0; e_{l+1}].$$

Assuming $[e_k; e_0; e_l]$ is a multiple of $[e_0; e_0; e_{k+l}]$, we get $[e_{k+1}; e_0; e_l]$ is a linear combination of $[e_0; e_0; e_{k+l+1}]$, $[e_1; e_0; e_{k+l}]$, due to (14). It remains to consider the $k = 1$ case.

We prove by an induction on $N > 1$ that $[e_1; e_0; e_{N-1}] = \frac{(q^{N-1}-q^2)(q^{N-1}-1)}{(q^N-1)^2}[e_0; e_0; e_N]$. This is equivalent to $[e_1; e_0; e_{N-1}]$ being a multiple of $[e_0; e_0; e_N]$, since we can recover the constant $\lambda_{N,3} = \frac{(q^{N-1}-q^2)(q^{N-1}-1)}{(q^N-1)^2}$ by comparing $\theta([e_1; e_0; e_{N-1}])$ and $\theta([e_0; e_0; e_N])$.

∘ *Case* $N = 2$.

Analogously to Proposition 1.2, the relation (u6) combined with (u4) imply

$$\mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}; e_{i_2+1}; e_{i_3-1}] = 0 \quad \text{for any } i_1, i_2, i_3 \in \mathbb{Z}. \tag{u6′}$$

Plugging in $i_1 = 1, i_2 = 1, i_3 = 0$, we get $[e_1; e_2; e_{-1}] + [e_1; e_1; e_0] + [e_0; e_2; e_0] = 0$. Combining this with (14), we find $[e_0; e_0; e_2] = -\frac{(q+1)^2}{q}[e_1; e_0; e_1] \Rightarrow [e_1; e_0; e_1] = \lambda_{2,3}[e_0; e_0; e_2]$.

∘ *Case* $N = 3$.

Plugging in $i_1 = 1$, $i_2 = 2$, $i_3 = 0$ into (u6′), we get

$$[e_1; e_3; e_{-1}] + [e_2; e_2; e_{-1}] + [e_2; e_1; e_0] + [e_0; e_3; e_0] + [e_0; e_2; e_1] = 0.$$

Applying (14), we get $-(q+2+q^{-1})[e_2; e_0; e_1] - (q+q^{-1})[e_1; e_0; e_2] - (1+\frac{q^2-q}{q^3-1})[e_0; e_0; e_3] = 0$. Meanwhile, applying A_1 to the above equality $(q+1)^2[e_1; e_0; e_1] + q[e_0; e_0; e_2] = 0$, we find $(q+1)^2[e_2; e_0; e_1] + (q^2+3q+1)[e_1; e_0; e_2] + (q - \frac{q^2-q^3}{q^3-1})[e_0; e_0; e_3] = 0$. Combining these two linear combinations of $[e_2; e_0; e_1]$, $[e_1; e_0; e_2]$, $[e_0; e_0; e_3]$, we get $[e_1; e_0; e_2] = 0 = \lambda_{3,3}[e_0; e_0; e_3]$.

∘ *Case* $N = k + 2$, $k > 1$.

By the induction assumption $[e_1; e_0; e_k] - \lambda_{k+1,3}[e_0; e_0; e_{k+1}] = 0$. Applying A_1, we get

$$\begin{aligned}&([e_2; e_0; e_k] + [e_1; e_1; e_k] + [e_1; e_0; e_{k+1}])\\ &\quad - \lambda_{k+1,3}([e_1; e_0; e_{k+1}] + [e_0; e_1; e_{k+1}] + [e_0; e_0; e_{k+2}]) = 0.\end{aligned}$$

Consider the linear operator $A_2 := \frac{1}{2}(\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2))$ acting on $\ddot{\underline{u}}^{\geq}_{\bar{h}_0}$. Then

$$A_2([e_{i_1}; e_{i_2}; e_{i_3}]) = [e_{i_1+1}; e_{i_2+1}; e_{i_3}] + [e_{i_1+1}; e_{i_2}; e_{i_3+1}] + [e_{i_1}; e_{i_2+1}; e_{i_3+1}].$$

By the induction assumption $[e_1; e_0; e_{k-1}] - \lambda_{k,3}[e_0; e_0; e_k] = 0$. Applying A_2, we get

$$([e_2;e_1;e_{k-1}]+[e_2;e_0;e_k]+[e_1;e_1;e_k])$$
$$-\lambda_{k,3}([e_1;e_1;e_k]+[e_1;e_0;e_{k+1}]+[e_0;e_1;e_{k+1}])=0.$$

Applying (14), we find two linear combinations of $[e_2;e_0;e_k],[e_1;e_0;e_{k+1}],[e_0;e_0;e_{k+2}]$ which are zero. It is a routine verification to check that they are not proportional for $q\neq\sqrt{1}$. Therefore, we can eliminate $[e_2;e_0;e_k]$, which proves that $[e_1;e_0;e_{k+1}]$ is a multiple of $[e_0;e_0;e_{k+2}]$.

• *Case* $j=n>3$.

Analogously to the previous case, it suffices to show that $[e_1;e_0;\dots;e_0;e_{N-1}]_n$ is a multiple of $[e_0;\dots;e_0;e_N]_n$. This is equivalent to

$$[e_1;e_0;\dots;e_0;e_{N-1}]_n=\lambda_{N,n}\cdot[e_0;\dots;e_0;e_N]_n \quad \text{with}$$
$$\lambda_{N,n}=\frac{(q^{N-1}-1)^{n-2}(q^{N-1}-q^{n-1})}{(q^N-1)^{n-1}},$$

where $\lambda_{N,n}$ is computed by comparing the images of these length n commutators under θ.

We will need the following generalization of (u6), which follows from Proposition 7.5:

$$[e_0;e_1;e_0;\dots;e_0;e_{-1}]_n=0. \tag{u7n}$$

Analogously to Proposition 1.2, one can see that (u7n) combined with (u4) imply

$$\mathrm{Sym}_{\mathfrak{S}_n}[e_{i_1};e_{i_2+1};e_{i_3};\dots;e_{i_{n-1}};e_{i_n-1}]_n=0 \quad \text{for any } i_1,\dots,i_n\in\mathbb{Z}. \tag{u7′n}$$

Now we proceed to the proof of the aforementioned result by an induction on $N>1$.

◦ *Case* $N=2$.

Applying A_2 to the equality (u7n), we get

$$[e_1,A_1([e_1;e_0;\dots;e_0;e_{-1}]_{n-1})]+[e_0;A_2([e_1;e_0;\dots;e_0;e_{-1}]_{n-1})]=0.$$

By the induction assumption on length $n-1$ commutators, this can be simplified to $a_n\cdot[e_1;e_0;\dots;e_0;e_1]_n+b_n\cdot[e_0;\dots;e_0;e_2]_n=0$. Computing the images under θ, we find $a_n=\frac{(-1)^n(1-q^{n-1})^2}{q^{n-2}(1-q)^2}\neq 0$.

◦ *Case* $N=3$.

Applying A_1 to the equality $[e_1;e_0;\dots;e_0;e_1]_n=\lambda_{2,n}[e_0;\dots;e_0;e_2]_n$, we get

$$[e_2;e_0;\dots;e_0;e_1]_n+[e_1,A_1([e_0;\dots;e_0;e_1]_{n-1})]=$$
$$\lambda_{2,n}([e_1;e_0;\dots;e_0;e_2]_n+[e_0,A_1([e_0;\dots;e_0;e_2]_{n-1})]).$$

By the induction assumption on length $n-1$ commutators, this can be simplified to

$$[e_2;e_0;\dots;e_0;e_1]_n+c_n\cdot[e_1;e_0;\dots;e_0;e_2]_n+d_n\cdot[e_0;e_0;\dots;e_0;e_3]_n=0.$$

Computing the images under θ, we find $c_n=\frac{(1-q)^{n-2}(1+2q-2q^{n-1}-q^n)}{(1-q^2)^{n-1}}$.

Define the linear operator $A_3 := \mathrm{ad}(\mathbf{h}_1)\,\mathrm{ad}(\mathbf{h}_2) - \mathrm{ad}(\mathbf{h}_3)$. Applying A_3 to (u7n), we get

$$[e_2, \mathrm{ad}(\mathbf{h}_1)[e_1; e_0; \dots; e_0; e_{-1}]_{n-1}] + [e_1, \mathrm{ad}(\mathbf{h}_2)[e_1; e_0; \dots; e_0; e_{-1}]_{n-1}] + [e_0, A_3([e_1; e_0; \dots; e_0; e_{-1}]_{n-1})] = 0.$$

By the induction assumption on length $n-1$ commutators, this can be simplified to

$$a'_n \cdot [e_2; e_0; \dots; e_0; e_1]_n + c'_n \cdot [e_1; e_0; \dots; e_0; e_2]_n + d'_n \cdot [e_0; e_0; \dots; e_0; e_3]_n = 0.$$

Computing the images under θ, we find the following formulas

$$a'_n = \frac{(-1)^n(1-q^{n-1})^2}{q^{n-2}(1-q)^2},\ c'_n = \frac{(-1)^n(1-q)^{n-3}(1-q^{n-1})(1-q^{2n-2})}{q^{n-2}(1-q^2)^{n-1}}.$$

It remains to notice that $c'_n \neq a'_n c_n$ for $q \neq \sqrt{1}$. Therefore, eliminating $[e_2; e_0; \dots; e_0; e_1]_n$, we see that $[e_1; e_0; \dots; e_0; e_2]_n$ is a multiple of $[e_0; e_0; \dots; e_0; e_3]_n$.

∘ *Case* $N = k+2$, $k > 1$.

By the induction: $[e_1; e_0; \dots; e_0; e_k]_n = \lambda_{k+1,n}[e_0; \dots; e_0; e_{k+1}]_n$. Applying A_1, we get

$$[e_2; e_0; \dots; e_0; e_k]_n + [e_1, A_1([e_0; \dots; e_0; e_k]_{n-1})] = \lambda_{k+1,n}([e_1; e_0; \dots; e_0; e_{k+1}]_n + [e_0, A_1([e_0; \dots; e_0; e_{k+1}]_{n-1})]).$$

By the induction assumption on length $n-1$ commutators, this can be simplified to

$$[e_2; e_0; \dots; e_0; e_k]_n + v_n \cdot [e_1; e_0; \dots; e_0; e_{k+1}]_n + w_n \cdot [e_0; \dots; e_0; e_{k+2}]_n = 0.$$

Computing the images under θ, we find $v_n = \frac{(1-q^k)^{n-2}(q^{n+k}-2q^{k+1}-2q^{n-1}+q^n+q^k+1)}{(1-q^{k+1})^{n-1}(1-q)}$.

Meanwhile, by the induction assumption $[e_1; \dots; e_0; e_{k-1}]_n = \lambda_{k,n}[e_0; \dots; e_0; e_k]_n$. Applying A_2 to this equality, we find

$$[e_2, A_1([e_0; \dots; e_0; e_{k-1}]_{n-1})] + [e_1, A_2([e_0; \dots; e_0; e_{k-1}]_{n-1})] = \lambda_{k,n}([e_1, A_1([e_0; \dots; e_0; e_k]_{n-1})] + [e_0, A_2([e_0; \dots; e_0; e_k]_{n-1})]).$$

By the induction assumption on length $n-1$ commutators, this can be simplified to

$$u'_n \cdot [e_2; e_0; \dots; e_0; e_k]_n + v'_n \cdot [e_1; e_0; \dots; e_0; e_{k+1}]_n + w'_n \cdot [e_0; \dots; e_0; e_{k+2}]_n = 0.$$

Computing the images under θ, we find

$$u'_n = \frac{(1-q^{n-1})(1-q^{k-1})^{n-2}}{(1-q)(1-q^k)^{n-2}},$$

$$v'_n = \frac{(1-q^{k-1})^{n-2}(1-q^{n-1})}{(1-q^{k+1})^{n-2}(1-q)}\left(\frac{q-q^{n-1}}{1-q^2} + \frac{q^{k-1}-q^{n-1}}{1-q^k}\right).$$

Since $v'_n \neq u'_n v_n$ for $q \neq \sqrt{1}$, eliminating $[e_2; e_0; \dots; e_0; e_k]_n$, we see that $[e_1; e_0; \dots; e_0; e_{k+1}]_n$ is a multiple of $[e_0; \dots; e_0; e_{k+2}]_n$. This completes our proof of $\dim(\underline{\ddot{u}}^{\geqslant}_{h_0})_{i,j} \leq 1$ for $j > 0$.

C.2. Proof of Theorem 5.9

As pointed out in Section 5.5, all the defining relations of the algebra $\ddot{Y}_{h_0}(\mathfrak{gl}_1)$ are of Lie-type. Therefore, it is a universal enveloping algebra of the Lie algebra $\ddot{y}_{h_0}$ generated by $\{e_j, f_j, \psi_j\}_{j\in\mathbb{Z}_+}$ with the same defining relations. Moreover, $\ddot{y}_{h_0}$ is a $\mathbb{C}\cdot\psi_0$-central extension of the Lie-algebra $\underline{\ddot{y}}_{h_0}$ generated by $\{e_j, f_j, \psi_{j+1}\}_{j\in\mathbb{Z}_+}$ with the following defining relations:

$$[\psi_k, \psi_l] = 0, \tag{y0}$$

$$[e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] - h_0^2([e_{i+1}, e_j] - [e_i, e_{j+1}]) = 0, \tag{y1}$$

$$[f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] - h_0^2([f_{i+1}, f_j] - [f_i, f_{j+1}]) = 0, \tag{y2}$$

$$[e_0, f_0] = 0,\ [e_i, f_j] = \psi_{i+j} \text{ for } i + j > 0, \tag{y3}$$

$$[\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}] - h_0^2([\psi_{i+1}, e_j] - [\psi_i, e_{j+1}]) = 0, \tag{y4}$$

$$[\psi_1, e_j] = 0,\ [\psi_2, e_j] = 2h_0^2 e_j, \tag{y4$'$}$$

$$[\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}] - h_0^2([\psi_{i+1}, f_j] - [\psi_i, f_{j+1}]) = 0, \tag{y5}$$

$$[\psi_1, f_j] = 0,\ [\psi_2, f_j] = -2h_0^2 f_j, \tag{y5$'$}$$

$$\mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = 0,\quad \mathrm{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2}, f_{i_3+1}]] = 0. \tag{y6}$$

Hence, it suffices to check that the corresponding homomorphism $\vartheta : \underline{\ddot{y}}_{h_0} \to \mathfrak{D}_{h_0}$ defined by

$$\vartheta : e_j \mapsto x^j\partial,\ f_j \mapsto -\partial^{-1}x^j,\ \psi_{j+1} \mapsto ((x-h_0)^{j+1} - x^{j+1})\partial^0 \ \text{ for } j \in \mathbb{Z}_+$$

is an isomorphism of the $\mathbb{C}$-Lie algebras for $h_0 \neq 0$. The surjectivity of ϑ is clear.

The Lie algebra $\underline{\ddot{y}}_{h_0}$ is $\mathbb{Z}$-graded via $\deg_2(e_j) := 1, \deg_2(f_j) := -1, \deg_2(\psi_{j+1}) := 0$ and $\mathbb{Z}_+$-filtered as a quotient of the free algebra $\mathbb{C}\langle e_j, f_j, \psi_{j+1}\rangle$ graded via $\deg_1(e_j) := j$, $\deg_1(f_j) := j$, $\deg_1(\psi_{j+1}) := j$. The Lie algebra $\mathfrak{D}_{h_0}$ is also $\mathbb{Z}$-graded via $\deg_2(x^i\partial^j) = j$ and $\mathbb{Z}_+$-filtered as a quotient of $\mathbb{C}\langle x, \partial^{\pm 1}\rangle$ with $\deg_1(x) = 1$, $\deg_1(\partial^{\pm 1}) = 0$. Note that ϑ is $\mathbb{Z}$-graded and preserves the $\mathbb{Z}_+$-filtration, while $\dim(\mathfrak{D}_{h_0})_{\leq i,j} = \dim(\mathfrak{D}_{h_0})_{\leq i-1,j} + 1$. Hence, it suffices to prove $\dim(\underline{\ddot{y}}_{h_0})_{\leq i,j} - \dim(\underline{\ddot{y}}_{h_0})_{\leq i-1,j} \leq 1$. This is clear for $j = 0$, while we consider $j > 0$ below (the case $j < 0$ is completely analogous).

Let $\ddot{\underline{y}}^{\geq}_{h_0}$ be the Lie algebra generated by $\{e_j, \psi_{j+1}\}_{j\geq 0}$ subject to the relations (y0,y1,y4,y4′,y6). It suffices to prove $\dim(\ddot{\underline{y}}^{\geq}_{h_0})_{\leq i,j} - \dim(\ddot{\underline{y}}^{\geq}_{h_0})_{\leq i-1,j} \leq 1$. Let $W(n;N)$ be the subspace of $\ddot{\underline{y}}^{\geq}_{h_0}$ spanned by $\{[e_0;\ldots;e_0;e_M]_n | 0 \leq M \leq N\}$. Let $V(n;N)$ be the subspace of $\ddot{\underline{y}}^{\geq}_{h_0}$ spanned by $\{[e_{i_1};\ldots;e_{i_n}]_n | i_1+\ldots+i_n \leq N\}$. Given $x, y \in V(n;N)$, we write $x \sim y$ if $x - y \in V(n;N-1)$. Given $x \in W(n;N)$, we write $x \equiv \nu \cdot [e_0;\ldots;e_0;e_N]_n$ if $x - \nu \cdot [e_0;\ldots;e_0;e_N]_n \in W(n;N-1)$.

The required estimate on dimensions of $(\ddot{\underline{y}}^{\geq}_{h_0})_{\leq i,j}$ follows from the following result:

Proposition C.1. *We have $V(n;N) = W(n;N)$ for any $n, N \in \mathbb{N}$.*

Proof. It is clear that $W(n;N) \subset V(n;N)$. We prove $V(n;N) \subset W(n;N)$ by an induction on n.

• *Case $n = 1, 2$.*

The case $n = 1$ is obvious. Let us now consider the case $n = 2$. The relation (y1) implies

$$[e_{i+2+k}, e_{i+1-k}] \sim (2k+1)[e_{i+2}, e_{i+1}] \text{ and } [e_{i+2+k}, e_{i-k}] \sim (k+1)[e_{i+2}, e_i].$$

These formulas can be unified in the following way:

$$[e_i, e_j] \sim \frac{j-i}{i+j}[e_0, e_{i+j}] \quad \text{for } i, j \in \mathbb{N}. \tag{15}$$

Assuming by induction $V(2;i+j-1) \subset W(2;i+j-1)$, we find $V(2;i+j) \subset W(2;i+j)$.

• *Case $n = 3$.*

Define $\mathbf{h}_1 := \frac{\psi_3}{6h_0^2}$, $\mathbf{h}_2 := \frac{\psi_4 - h_0^2\psi_2}{12h_0^2}$, so that $[\mathbf{h}_1, e_j] = e_{j+1}$, $[\mathbf{h}_2, e_j] = e_{j+2}$, due to (y4, y4′). Consider the linear operators $A_1 := \mathrm{ad}(\mathbf{h}_1), A_2 := \frac{1}{2}(\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2)) \in \mathrm{End}(\ddot{\underline{y}}^{\geq})$.

By the induction assumption for $n = 2$, it suffices to prove $[e_k;e_0;e_l] \in W(3;k+l)$. Assuming by induction on k that $[e_k;e_0;e_l] = \sum_{M=0}^{k+l} \nu_M [e_0;e_0;e_M]$ and applying A_1 to this equality, we find (as in Appendix C.1) that $[e_{k+1};e_0;e_l]$ can be expressed as a linear combination of $\{[e_0;e_0;e_M]\}_{M\leq k+l+1} \cup \{[e_1;e_0;e_M]\}_{M\leq k+l}$. Therefore, it remains to prove $[e_1;e_0;e_{N-1}] \in W(3;N)$. This is equivalent to $[e_1;e_0;e_{N-1}] \equiv \frac{N-4}{N}[e_0;e_0;e_N]$, where the constant $\beta_{N,3} = \frac{N-4}{N}$ can be recovered by comparing $\vartheta([e_1;e_0;e_{N-1}])$ and $\vartheta([e_0;e_0;e_N])$.

◦ *Case $N = 1, 2$.*

We have $[e_1;e_0;e_0] = 0 = [e_0;e_0;e_1]$. Applying A_1 to the latter equality, we also find $[e_1;e_0;e_1] = -[e_0;e_0;e_2]$.

◦ *Case $N = k+1$, $k > 1$.*

By the induction assumption: $[e_1;e_0;e_{k-1}] \equiv \beta_{k,3}[e_0;e_0;e_k]$. Applying A_1, we find

$$[e_2;e_0;e_{k-1}] + [e_1;e_1;e_{k-1}] + [e_1;e_0;e_k] \equiv \beta_{k,3}([e_1;e_0;e_k] + [e_0;e_1;e_k] + [e_0;e_0;e_{k+1}]).$$

Applying (15), we get $[e_2;e_0;e_{k-1}]+\frac{k+2}{k}[e_1;e_0;e_k]\in W(3;k+1)$. On the other hand, applying A_2 to $[e_1;e_0;e_{k-2}]\equiv\beta_{k-1,3}[e_0;e_0;e_{k-1}]$, we find

$$\begin{aligned}&[e_2;e_1;e_{k-2}]+[e_2;e_0;e_{k-1}]+[e_1;e_1;e_{k-1}]\\&\quad\equiv\beta_{k-1,3}([e_1;e_1;e_{k-1}]+[e_1;e_0;e_k]+[e_0;e_1;e_k]).\end{aligned}$$

Applying (15), we get $\frac{2(k-2)}{k-1}[e_2;e_0;e_{k-1}]+\frac{8-k}{k}[e_1;e_0;e_k]\in W(3;k+1)$. Comparing those two linear combinations of $[e_2;e_0;e_{k-1}]$, $[e_1;e_0;e_k]$, we find $[e_1;e_0;e_k]\in W(3;k+1)$ unless $k=3$. The latter case will be considered in the greater generality below.

• *Case* $n>3$.

Analogously to the previous case, it suffices to show $[e_1;e_0;\dots;e_0;e_{N-1}]_n\in W(n;N)$, which is equivalent to $[e_1;\dots;e_0;e_{N-1}]_n\equiv\beta_{N,n}\cdot[e_0;\dots;e_0;e_N]_n$ with $\beta_{N,n}=\frac{N-2n+2}{N}$.

We will need the following generalization of (y6), which follows from Proposition 7.8:

$$[e_0;\dots;e_0;e_{n-2}]_n=0. \tag{y7n}$$

Now we proceed to the proof of the aforementioned result by an induction on N.

∘ *Case* $N\leq n-1$.

If $N<n-1$, then $[e_0;\dots;e_0;e_{N-1}]_{n-1}=0=[e_0;\dots;e_0;e_N]_n$.

Applying A_1 to $[e_0;\dots;e_0;e_{n-2}]_n=0$, we find $[e_1;\dots;e_0;e_{n-2}]_n\in W(n;n-1)$.

∘ *Case* $N=k+1, k>n-2$.

Applying A_1 to $[e_1;e_0;\dots;e_0;e_{k-1}]_n\equiv\beta_{k,n}[e_0;\dots;e_0;e_k]_n$, we find

$$\begin{aligned}&[e_2;e_0;\dots;e_0;e_{k-1}]_n+[e_1,A_1([e_0;\dots;e_0;e_{k-1}]_{n-1})]\\&\quad\equiv\beta_{k,n}([e_1;e_0;\dots;e_0;e_k]_n+[e_0,A_1([e_0;\dots;e_0;e_k]_{n-1})]).\end{aligned}$$

Combining this with the induction assumption for length $n-1$ commutators, we get

$$k[e_2;e_0;\dots;e_0;e_{k-1}]_n+((n-2)k-(n-1)(n-4))[e_1;e_0;\dots;e_0;e_k]_n\in W(n;k+1). \tag{16}$$

Applying A_2 to $[e_1;e_0;\dots;e_0;e_{k-2}]_n\equiv\beta_{k-1,n}[e_0;\dots;e_0;e_{k-1}]_n$ and using the induction assumption, we find $P[e_2;e_0;\dots;e_0;e_{k-1}]_n+Q[e_1;e_0;\dots;e_0;e_k]_n\in W(n;k+1)$, where $P=\frac{(n-1)(k-n+1)}{k-1}$, $Q=\frac{n-1}{2k(k-1)}(k^2(n-4)-k(2n^2-13n+12)+(n^3-9n^2+18n-8))$. Comparing those two linear combinations, we get $[e_1;e_0;\dots;e_0;e_k]_n\in W(n;k+1)$ for $k\neq n$.

It remains to consider the case $k=n$. Define $\mathbf{h}_3:=\frac{\psi_5}{20h_0^2}-\frac{\psi_3}{12}$, so that $[\mathbf{h}_3,e_j]=e_{j+3}$. Applying $\mathrm{ad}(\mathbf{h}_1)\,\mathrm{ad}(\mathbf{h}_2)-\mathrm{ad}(\mathbf{h}_3)$ to $[e_0;\dots;e_0;e_{n-2}]_n=0$, we find

$$n[e_2;e_0;\dots;e_0;e_{n-1}]_n+2[e_1;e_0;\dots;e_0;e_n]_n\in W(n;n+1).$$

Combining this with (16), we get $[e_1;e_0;\dots;e_0;e_n]_n\in W(n;n+1)$. This completes our proof. □

Appendix D. Proof of Proposition 7.5

Proposition 7.5 follows from Theorem 7.4(c) and the following two lemmas:

Lemma D.1. *The elements* $\{L_n^m\}_{n\in\mathbb{N}}$ *belong to* $\mathcal{A}^m$.

Lemma D.2. *The elements* $\{L_n^m\}_{n\in\mathbb{N}}$ *are algebraically independent.*

Proof of Lemma D.1. According to Theorem 7.4(a), it suffices to prove that $\partial^{(\infty,k)}L_n^m$ exist for all k. We have

$$L_n^m = \mathrm{Sym}_{\mathfrak{S}_n}\left\{\left(\sum_{l=0}^{n-2}(-1)^l\binom{n-2}{l}\frac{x_1}{x_{n-l}} - \sum_{l=0}^{n-2}(-1)^l\binom{n-2}{l}\frac{x_n}{x_{n-l-1}}\right)\prod_{i<j}\omega^m(x_j,x_i)\right\}. \tag{17}$$

Our goal is to show that the right-hand side of (17) has a finite limit as $x_{n-k+1}\mapsto \xi\cdot x_{n-k+1},\dots,x_n\mapsto\xi\cdot x_n$ with $\xi\to\infty$. Note that $\frac{x_{\sigma(i)}}{x_{\sigma(j)}}$ has a finite limit as $\xi\to\infty$ unless $\sigma(j)\le n-k<\sigma(i)$, while it has a linear growth in the latter case. On the other hand, $\omega^m(x_j,x_i)$ has a finite limit as $\xi\to\infty$ for any i,j. Moreover: $\omega^m(\xi\cdot x,y)=1+O(\xi^{-1})$, $\omega^m(y,\xi\cdot x)=1+O(\xi^{-1})$ as $\xi\to\infty$. Therefore, it remains to prove the equality $A_1=A_2$, where A_1,A_2 are given by

$$A_1 := \sum_{s=n-k+1}^{n}\sum_{\sigma\in\mathfrak{S}_n}^{\sigma(1)=s\ \sigma(n-l)\le n-k}\sum_{l}(-1)^l\binom{n-2}{l}\frac{x_s}{x_{\sigma(n-l)}} \times\prod_{i<j}^{j\le n-k}\omega_\sigma^m(x_j,x_i)\prod_{i<j}^{n-k<i}\omega_\sigma^m(x_j,x_i),$$

$$A_2 := \sum_{s=n-k+1}^{n}\sum_{\sigma\in\mathfrak{S}_n}^{\sigma(n)=s\ \sigma(n-l-1)\le n-k}\sum_{l}(-1)^l\binom{n-2}{l}\frac{x_s}{x_{\sigma(n-l-1)}} \times\prod_{i<j}^{j\le n-k}\omega_\sigma^m(x_j,x_i)\prod_{i<j}^{n-k<i}\omega_\sigma^m(x_j,x_i).$$

Here we set $\omega_\sigma^m(x_j,x_i)=\omega^m(x_j,x_i)$ if $\sigma^{-1}(i)<\sigma^{-1}(j)$ and $\omega_\sigma^m(x_j,x_i)=\omega^m(x_i,x_j)$ otherwise.

If $k=1$, then $s=n$ in both sums and the map $(\sigma,l)\mapsto(\sigma',l)$ with $\sigma'(i):=\sigma(i+1)$ (for $1\le i\le n-1$) establishes a bijection between equal summands in A_1 and A_2, so that $A_1=A_2$.

For $k>1$, there is no such bijection. Instead, we prove $A_1=0$ (the proof of $A_2=0$ is analogous). Let us group the summands in A_1 according to s, $\sigma(n-l)$ and also the ordering of $\{\sigma^{-1}(1),\dots,\sigma^{-1}(n-k)\}$ and $\{\sigma^{-1}(n-k+1),\dots,\sigma^{-1}(n)\}$, which are given by elements $\sigma_1^{-1}\in\mathfrak{S}_{n-k}$ and $\sigma_2^{-1}\in\mathfrak{S}_k$. Define

$$\omega^m_{\sigma_1,\sigma_2}(x_1;\dots;x_n) := \prod_{i<j}^{j\le n-k} \omega^m_{\sigma_1}(x_j,x_i)\cdot \prod_{i<j}^{n-k<i} \omega^m_{\sigma_2}(x_j,x_i).$$

Then A_1 can be written in the form

$$A_1 = \sum_{t\le n-k}\sum_{\sigma_1\in\mathfrak{S}_{n-k}}\sum_{\sigma_2\in\mathfrak{S}_k} A_{t,\sigma_1,\sigma_2}\frac{x_{\sigma_2(1)}}{x_t}\omega^m_{\sigma_1,\sigma_2}(x_1;\dots;x_n) \text{ with } A_{t,\sigma_1,\sigma_2}\in\mathbb{Z}.$$

We claim that all these constants A_{t,σ_1,σ_2} are zero. As an example, we compute $A_{t,1_{n-k},1_k}$:

$$\begin{aligned} A_{t,1_{n-k},1_k} &= \sum_{l=n-k-t}^{n-t-1}(-1)^l\binom{n-2}{l}\binom{l}{n-k-t}\binom{n-l-2}{t-1} \\ &= \frac{(-1)^{n-k-t}(n-2)!(1-1)^{k-1}}{(t-1)!(k-1)!(n-k-t)!}. \end{aligned}$$

Thus $A_{t,1_{n-k},1_k} = 0$ since $k > 1$. Analogously $A_{t,\sigma_1,\sigma_2} = 0$ for any t,σ_1,σ_2. Hence, $A_1 = 0$. □

Proof of Lemma D.2. The elements L^m_n correspond to nonzero multiples of $\theta_{n,0}$ via the identification $S^m \simeq \widetilde{\mathcal{E}}^+$. An algebraic independence of $\{\theta_{n,0}\}_{n\in\mathbb{N}}$ follows from an analogue of Proposition 1.3(b) for $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$.

Alternatively, note that $\{e_0\}\cup\{[e_1;e_0;\dots;e_0;e_{-1}]_n\}_{n=2}^{\infty}$ correspond to nonzero multiples of $\{D^n\}_{n=1}^{\infty}$ via the identification $\ddot{U}_h(\mathfrak{gl}_1)\simeq U(\bar{\mathfrak{d}}^0_q)$. The result follows from the PBW theorem applied to $U(\bar{\mathfrak{d}}^0_q)$. □

Appendix E. Proofs of Theorems 8.5, 8.6

E.1. Proof of Theorem 8.5

In the fixed point basis, we have $v^K_r = \sum_{\bar\lambda} a_{\bar\lambda}\cdot[\bar\lambda]$ with $a_{\bar\lambda} = \prod_{w\in T_{\bar\lambda}} M(r,|\bar\lambda|)(1-w)^{-1}$. Hence, it suffices to prove the following equality for any r-tuple of diagrams $\bar\lambda$:

$$C_{j,-n} = \sum_{\bar\lambda'}\frac{a_{\bar\lambda'}}{a_{\bar\lambda}}\cdot K^{(m;j)}_{-n\,|[\bar\lambda',\bar\lambda]}, \tag{18}$$

where the sum is over all r-tuples of diagrams $\bar\lambda'$ such that $\bar\lambda\subset\bar\lambda'$ and $|\bar\lambda'| = |\bar\lambda| + n$.

For such a pair $(\bar\lambda,\bar\lambda')$, define a collection of positive integers

$$j_{1,1}\le j_{1,2}\le\dots\le j_{1,l_1},\ j_{2,1}\le j_{2,2}\le\dots\le j_{2,l_2},\ \dots,\ j_{r,1}\le j_{r,2}\le\dots\le j_{r,l_r} \tag{19}$$

(with $\sum_{a=1}^r l_a = n$) via the following equality:

$$\bar\lambda' = \bar\lambda + \square^1_{j_{1,1}} + \dots + \square^1_{j_{1,l_1}} + \square^2_{j_{2,1}} + \dots + \square^2_{j_{2,l_2}} + \dots + \square^r_{j_{r,1}} + \dots + \square^r_{j_{r,l_r}}.$$

We also consider the sequence of r-tuples of diagrams $\bar\lambda = \bar\lambda^{[0]} \subset \bar\lambda^{[1]} \subset \cdots \subset \bar\lambda^{[n]} = \bar\lambda'$, where $\bar\lambda^{[\mathrm{q}]}$ is obtained from $\bar\lambda$ by adding the first q boxes from above. For $1 \leq \mathrm{q} \leq n$, the qth box from above has a form $\square^{s_\mathrm{q}}_{j_{s_\mathrm{q}}, i_\mathrm{q}}$ and we denote its character by $\chi(\mathrm{q})$.

For any $F \in (S_n^m)^{\mathrm{opp}}$, we have the following formula for the matrix coefficient $F_{|[\bar\lambda', \bar\lambda]}$:

$$F_{|[\bar\lambda', \bar\lambda]} = \frac{F(\chi(1), \dots, \chi(n))}{\prod_{a<b} \omega^m(\chi(a), \chi(b))} \cdot \prod_{\mathrm{q}=1}^{n} f_{0|[\bar\lambda^{[\mathrm{q}]}, \bar\lambda^{[\mathrm{q}-1]}]}.$$

In particular, we get

$$K^{(m;j)}_{-n}{}_{|[\bar\lambda', \bar\lambda]} = \prod_{1\leq a<b\leq n} \frac{(\chi(a)-\chi(b))(\chi(b)-t_1\chi(a))}{(\chi(a)-t_2\chi(b))(\chi(a)-t_3\chi(b))} \cdot \prod_{\mathrm{q}=1}^{n} \chi(\mathrm{q})^j \cdot \prod_{\mathrm{q}=1}^{n} f_{0|[\bar\lambda^{[\mathrm{q}]}, \bar\lambda^{[\mathrm{q}-1]}]}.$$

As an immediate consequence, we find $K^{(m;j)}_{-n}{}_{|[\bar\lambda', \bar\lambda]} = 0$ if $\bar\lambda' \backslash \bar\lambda$ contains two boxes in the same row of its ith component, $1 \leq i \leq r$. Therefore, the sum in (18) should be taken only over those $\bar\lambda'$ which correspond to collections $\{j_{1,1}, \dots, j_{r,l_r}\}$ from (19) with strict inequalities.

We also split $\frac{a_{\bar\lambda'}}{a_{\bar\lambda}}$ into the product over consequent pairs: $\frac{a_{\bar\lambda'}}{a_{\bar\lambda}} = \prod_{\mathrm{q}=1}^{n} \frac{a_{\bar\lambda^{[\mathrm{q}]}}}{a_{\bar\lambda^{[\mathrm{q}-1]}}}$. According to the Bott–Lefschetz fixed point formula, we have

$$\frac{a_{\bar\lambda^{[\mathrm{q}]}}}{a_{\bar\lambda^{[\mathrm{q}-1]}}} \cdot f_{0|[\bar\lambda^{[\mathrm{q}]}, \bar\lambda^{[\mathrm{q}-1]}]} = T \cdot e_{-r|[\bar\lambda^{[\mathrm{q}-1]}, \bar\lambda^{[\mathrm{q}]}]}, \text{where } T = (-t)^{r-2}\chi_1^{-1} \cdots \chi_r^{-1}.$$

For two r-tuples of diagrams $(\bar\mu, \bar\mu')$ such that $\bar\mu' = \bar\mu + \square^l_j$, the matrix coefficient $e_{-r|[\bar\mu, \bar\mu']}$ is computed by Lemma 3.3(a):

$$e_{-r|[\bar\mu, \bar\mu']} = \frac{1}{1-t_1} \prod_{a=1}^{r} \frac{1}{t_1\chi_j^{(l)} - t_2^{L_a}\chi_a^{-1}} \prod_{(a,k)\neq(l,j)}^{k\leq L_a} \frac{\chi_j^{(l)} - t_2\chi_k^{(a)}}{\chi_j^{(l)} - \chi_k^{(a)}},$$

where $\{L_a\}_{a=1}^r$ are chosen to satisfy $L_a \geq (\mu^{a*})_1 + 1$.

Combining these formulas together, we finally get

$$\frac{a_{\bar\lambda'}}{a_{\bar\lambda}} \cdot K^{(m;j)}_{-n}{}_{|[\bar\lambda', \bar\lambda]} = T^n \cdot \prod_{1\leq \mathrm{q}\leq n} \left\{ \frac{(-t_1t_2)^{\mathrm{q}-1}}{1-t_1} \cdot \prod_{a=1}^{r} \frac{1}{\chi(\mathrm{q}) - t_2^{L_a}\chi_a^{-1}} \cdot \prod \frac{\chi(\mathrm{q}) - t_1t_2\chi_k^{(a)}}{\chi(\mathrm{q}) - t_1\chi_k^{(a)}} \cdot \chi(\mathrm{q})^j \right\},$$

with the last product taken over pairs $(a,k) \notin \{(s_\mathrm{q}, j_{s_\mathrm{q}, i_\mathrm{q}})\}_{\mathrm{q}=1}^n$, $k \leq L_a$ with $L_a \geq (\lambda^{a*})_1 + n$.

Let us denote the RHS of this equality by $C_{\mathbf{j}}$, where $\mathbf{j} = \{j_{1,1}, \dots, j_{r,l_r}\}$ is defined in (19). Note that $C_{\mathbf{j}} = 0$ if the corresponding $\bar\lambda'$ fails to be a collection of r Young

diagrams. Hence, (18) is reduced to $C_{j,-n} = \sum C_{\mathbf{j}}$, where the sum is over all $\mathbf{j}$ from (19) with strict inequalities.

It is easy to check that $\sum C_{\mathbf{j}}$ is a rational function in $\chi_k^{(a)}$ with no poles for $0 \leq j \leq r$. The degree estimate implies that $\sum C_{\mathbf{j}}$ is an element of $\mathbb{F}_r$ independent of $\bar{\lambda}$. Thus v_r^K is indeed an eigenvector with respect to $K_{-n}^{(m;j)}$. To compute its eigenvalue, we evaluate $\sum C_{\mathbf{j}}$ at $\bar{\lambda} = \bar{\emptyset}$. This sum is actually taken over all partitions $(l_1, \dots, l_r)$ of n with $j_{a,b} = b$, and it equals

$$\frac{T^n(-t_1t_2)^{n(n-1)/2}}{(1-t_1)^n} \sum_{l_1+\ldots+l_r=n} \prod_{a,b=1}^{r} \frac{1}{(\chi_b^{-1} - t_2^{l_a}\chi_a^{-1}) \cdots (t_2^{l_b-1}\chi_b^{-1} - t_2^{l_a}\chi_a^{-1})}$$
$$\times \prod_{b=1}^{r} (t_2^{\frac{l_b(l_b-1)}{2}} \chi_b^{-l_b})^j = \frac{(-t)^{(r-2)n}(-t_1t_2)^{\frac{n(n-1)}{2}}}{(1-t_1)^n}$$
$$\times \sum_{l_1+\ldots+l_r=n} \prod_{a,b=1}^{r} \frac{1}{(\chi_a - t_2^{l_a}\chi_b) \cdots (\chi_a - t_2^{l_a-l_b+1}\chi_b)} \prod_{b=1}^{r} (t_2^{-\frac{l_b(l_b-1)}{2}} \chi_b^{l_b})^{r-j}.$$

It is straightforward to check that latter expression is a rational function in χ_a with no poles for $0 \leq j \leq r-1$. Together with the degree estimate, we see that it is independent of χ_a. To compute this constant, we let $\chi_1 \to \infty$. Then the only nonzero contribution comes from the collection $(l_1, l_2, \dots, l_r) = (n, 0, \dots, 0)$ and the result equals $C_{j,-n}$.

For $j = r$, the product of the above expression and $(\chi_1 \cdots \chi_r)^n$ is a rational function in χ_a with no poles and of total degree 0, hence, it is independent of χ_a. To compute this constant, we let $\chi_1 \to \infty$. The only nonzero contributions come from $(l_1, \dots, l_r)$ with $l_1 = 0$. For these terms, we let $\chi_2 \to \infty$, etc. The result follows from straightforward computations.

E.2. Sketch of the proof of Theorem 8.6

In the fixed point basis, we have $v_r^H = \sum_{\bar{\lambda}} b_{\bar{\lambda}} \cdot [\bar{\lambda}]$ with $b_{\bar{\lambda}} = \prod_{w \in T_{\bar{\lambda}} M(r,|\bar{\lambda}|)} w^{-1}$. Hence, it suffices to prove the following equality for any r-tuple of diagrams $\bar{\lambda}$:

$$D_{j,-n} = \sum_{\bar{\lambda}'} \frac{b_{\bar{\lambda}'}}{b_{\bar{\lambda}}} \cdot K_{-n}^{(a;j)}{}_{|[\bar{\lambda}',\bar{\lambda}]}, \tag{20}$$

where the sum is over all r-tuples of diagrams $\bar{\lambda}'$ such that $\bar{\lambda} \subset \bar{\lambda}'$ and $|\bar{\lambda}'| = |\bar{\lambda}| + n$.

Analogously to the K-theoretical case, we have

$$\frac{b_{\bar{\lambda}'}}{b_{\bar{\lambda}}} \cdot K_{-n}^{(a;j)}{}_{|[\bar{\lambda}',\bar{\lambda}]}$$
$$= \prod_{1 \leq \mathrm{q} \leq n} \left\{ \frac{(-1)^{\mathrm{q}}}{s_1} \cdot \prod_{a=1}^{r} \frac{1}{\chi(\mathrm{q}) - L_a s_2 + x_a} \cdot \prod \frac{\chi(\mathrm{q}) - x_k^{(a)} - s_1 - s_2}{\chi(\mathrm{q}) - x_k^{(a)} - s_1} \cdot \chi(\mathrm{q})^j \right\},$$

with the last product taken over pairs $(a,k) \notin \{(s_q, j_{s_q,i_q})\}_{q=1}^n$, $k \leq L_a$ with $L_a \geq (\lambda^{a*})_1 + n$.

Let us denote the RHS of this equality by $D_{\mathbf{j}}$. Then $\sum_{\mathbf{j}} D_{\mathbf{j}}$ is a rational function in $x_k^{(a)}$ and it has no poles for $j \geq 0$. The degree estimate implies that it is independent of $x_k^{(a)}$ for $0 \leq j \leq r$. Thus v_r^H is indeed an eigenvector with respect to $\{K_{-n}^{(a;j)}\}_{0\leq j\leq r}^{n>0}$. To compute the corresponding eigenvalues, we evaluate $\sum_{\mathbf{j}} D_{\mathbf{j}}$ at $\bar{\lambda} = \bar{\emptyset}$. This sum equals

$$\frac{(-1)^{n(n+1)/2}}{s_1^n} \times \sum_{l_1+\ldots+l_r=n} \left\{ \prod_{a,b=1}^{r} \prod_{k=1}^{l_b} (x_a - x_b - (l_a - k + 1)s_2)^{-1} \prod_{b=1}^{r} \prod_{k=1}^{l_b} ((k-1)s_2 - x_b)^j \right\}.$$

It is straightforward to check that this sum is a rational function in x_a with no poles. Together with the degree estimate for $j \leq r-1$, we see that it is independent of x_a. To compute this constant, we let $x_1 \to \infty$. For $j < r-1$, all the summands tend to 0. For $j = r-1$, the only nonzero contribution comes from $(l_1, l_2, \ldots, l_r) = (n, 0, \ldots, 0)$ and equals $D_{r-1,-n}$.

Remark E.1. An explicit formula for the eigenvalues $D_{r,-n}$ $(n > 1)$ was first obtained in [20].

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