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Several realizations of Fock modules for toroidal $\ddot{U}_{q,d}(\mathfrak{sl}_n)$

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Abstract In this paper, we relate the well-known *Fock representations* of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ to the vertex, shuffle, and 'L-operator' representations of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. These identifications generalize those for the quantum toroidal algebra of $\mathfrak{gl}_1$, which were recently established in Feigin et al. (J. Phys. A **48**(24), 244001, 2015).

Keywords Quantum toroidal algebras · Shuffle algebras · L operators · Fock module · Vertex module

Mathematics Subject Classification (2010) 20G42 · 17B37

1 Introduction

In the recent paper [9], authors proposed a shuffle approach to the Bethe ansatz problem for certain modules over the quantum toroidal algebra of $\mathfrak{gl}_1$, viewing the latter as the Drinfeld double of the *small shuffle* algebra. The general idea behind a shuffle approach is that it frequently allows to interpret complicated concepts in simple terms. As the representation theory of quantum toroidal algebras of $\mathfrak{sl}_n$ is quite similar to that of quantum toroidal algebras of $\mathfrak{gl}_1$ (though technically it is more involved), it is desirable to generalize the aforementioned construction for the former case.

In this article, we identify different families of representations of quantum toroidal algebras of $\mathfrak{sl}_n$. This will be crucial for our arguments in [11], where we diagonalize the commutative subalgebras of the quantum toroidal algebras of $\mathfrak{sl}_n$ studied in [10].

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This paper is organized as follows:

- In Section 2, we recall the definition and key results about the quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n), n \geq 3$. In particular, we recall the relation to the shuffle algebra S (of $A^{(1)}_{n-1}$-type) studied in [10, 20].

  We also discuss three different constructions of their representations:

  - combinatorial representations $\tau^p_{u,\bar{c}}$ introduced in [8],
  - vertex representations $\rho^p_{u,\bar{c}}$ constructed in [21],
  - shuffle representations $\pi^p_{u,\bar{c}}$ introduced in this paper.

  Our construction of $\pi^p_{u,\bar{c}}$ is similar to that of [9] for the quantum toroidal algebra of $\mathfrak{gl}_1$. In particular, the underlying vector space $S_{1,p}(u)$ carries a natural S-bimodule structure, while $\pi^p_{u,\bar{c}}$ is the extension of the left S-action to the action of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, see Proposition 2.18.
- In Section 3, we relate the aforementioned three different families of representations:

  - In Theorem 3.2, we show that $\pi^p_{u,\bar{c}}$ induces an action of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ on the factor of $S_{1,p}(u)$ by the right S'-action (here $S' \subset S$ denotes the augmentation ideal), which is isomorphic to the $\tau^p_{u,\bar{c}}$-action. In Theorems 3.6, 3.7, we generalize this result to some other families of representations.
  - In Theorem 3.8, we show that Miki's isomorphism ϖ of the quantum toroidal algebras intertwines the dual of the combinatorial representation $\tau^p_{u,\bar{c}}$ and the corresponding vertex representation $\rho^p_{u',\bar{c}'}$ for appropriate parameters.
- In Section 4, we study the matrix elements of L operators associated to the vertex representations $\rho^p_{u,\bar{c}}$. In Theorem 4.5, we derive an explicit formula for the matrix element $L^{p,\bar{c}}_{\emptyset,\emptyset}$, whose shuffle realization was obtained in [10]. This allows us to identify the shuffle S-bimodule $S_{1,p}(u)$ with the S-bimodule generated by $L^{p,\bar{c}}_{\emptyset,\emptyset}$, see Proposition 4.7.

2 Basic Definitions and Constructions

2.1 Quantum Toroidal Algebras of $\mathfrak{sl}_n$ for $n \geq 3$

Let $q, d \in \mathbb{C}^\times$ be two parameters. We set $[n] := \{0, 1, \ldots, n-1\}$, $[n]^\times := [n]\backslash\{0\}$, the former viewed as a set of mod n residues. Let $g_m(z) := \frac{q^m z-1}{z-q^m}$. Define $\{a_{i,j}, m_{i,j} | i, j \in [n]\}$ by

$$a_{i,i} = 2,\ a_{i,i\pm 1} = -1,\ m_{i,i\pm 1} = \mp 1,\ \text{and } a_{i,j} = m_{i,j} = 0 \text{ otherwise.}$$

The quantum toroidal algebra of $\mathfrak{sl}_n$, denoted by $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, is the unital associative $\mathbb{C}$-algebra generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi^{-1}_{i,0}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}^{k\in\mathbb{Z}}_{i\in[n]}$ with the following defining relations:

$$[\psi_i^\pm(z), \psi_j^\pm(w)] = 0,\ \gamma^{\pm 1/2} - \text{central}, \tag{T0.1}$$

$$\psi^{\pm 1}_{i,0} \cdot \psi^{\mp 1}_{i,0} = \gamma^{\pm 1/2} \cdot \gamma^{\mp 1/2} = q^{\pm d_1} \cdot q^{\mp d_1} = q^{\pm d_2} \cdot q^{\mp d_2} = 1, \tag{T0.2}$$

$$q^{d_1} e_i(z) q^{-d_1} = e_i(qz),\ q^{d_1} f_i(z) q^{-d_1} = f_i(qz),\ q^{d_1} \psi_i^\pm(z) q^{-d_1} = \psi_i^\pm(qz), \tag{T0.3}$$

$$q^{d_2} e_i(z) q^{-d_2} = q e_i(z),\ q^{d_2} f_i(z) q^{-d_2} = q^{-1} f_i(z),\ q^{d_2} \psi_i^\pm(z) q^{-d_2} = \psi_i^\pm(z), \tag{T0.4}$$

$$g_{a_{i,j}}(\gamma^{-1} d^{m_{i,j}} z/w) \psi_i^+(z) \psi_j^-(w) = g_{a_{i,j}}(\gamma d^{m_{i,j}} z/w) \psi_j^-(w) \psi_i^+(z), \tag{T1}$$

$$e_i(z)e_j(w) = g_{a_{i,j}}(d^{m_{i,j}}z/w)e_j(w)e_i(z), \tag{T2}$$

$$f_i(z)f_j(w) = g_{a_{i,j}}(d^{m_{i,j}}z/w)^{-1}f_j(w)f_i(z), \tag{T3}$$

$$(q-q^{-1})[e_i(z), f_j(w)] = \delta_{i,j}\left(\delta(\gamma w/z)\psi_i^+(\gamma^{1/2}w) - \delta(\gamma z/w)\psi_i^-(\gamma^{1/2}z)\right), \tag{T4}$$

$$\psi_i^\pm(z)e_j(w) = g_{a_{i,j}}(\gamma^{\pm 1/2}d^{m_{i,j}}z/w)e_j(w)\psi_i^\pm(z), \tag{T5}$$

$$\psi_i^\pm(z)f_j(w) = g_{a_{i,j}}(\gamma^{\mp 1/2}d^{m_{i,j}}z/w)^{-1}f_j(w)\psi_i^\pm(z), \tag{T6}$$

$$\mathrm{Sym}_{z_1,z_2}[e_i(z_1),[e_i(z_2),e_{i\pm 1}(w)]_q]_{q^{-1}} = 0,\ [e_i(z),e_j(w)] = 0 \text{ for } j \neq i, i\pm 1, \tag{T7.1}$$

$$\mathrm{Sym}_{z_1,z_2}[f_i(z_1),[f_i(z_2),f_{i\pm 1}(w)]_q]_{q^{-1}} = 0,\ [f_i(z),f_j(w)] = 0 \text{ for } j \neq i, i\pm 1, \tag{T7.2}$$

where we set $[a,b]_x := ab - x\cdot ba$ and define the generating series as follows:

$$e_i(z) := \sum_{k=-\infty}^{\infty} e_{i,k}z^{-k},\ f_i(z) := \sum_{k=-\infty}^{\infty} f_{i,k}z^{-k},\ \psi_i^\pm(z) := \psi_{i,0}^{\pm 1} + \sum_{r>0}\psi_{i,\pm r}z^{\mp r},\ \delta(z) := \sum_{k=-\infty}^{\infty} z^k.$$

It will be convenient to use the generators $\{h_{i,k}\}_{k\neq 0}$ instead of $\{\psi_{i,k}\}_{k\neq 0}$, defined by

$$\exp\left(\pm(q-q^{-1})\sum_{r>0}h_{i,\pm r}z^{\mp r}\right) = \bar{\psi}_i^\pm(z) := \psi_{i,0}^{\mp 1}\psi_i^\pm(z),\quad h_{i,\pm r} \in \mathbb{C}[\psi_{i,0}^{\mp 1}, \psi_{i,\pm 1}, \psi_{i,\pm 2}, \ldots].$$

Then the relations (T5,T6) are equivalent to the following (we use $[m]_q := (q^m - q^{-m})/(q-q^{-1})$):

$$\psi_{i,0}e_{j,l} = q^{a_{i,j}}e_{j,l}\psi_{i,0},\ [h_{i,k}, e_{j,l}] = d^{-km_{i,j}}\gamma^{-|k|/2}\frac{[ka_{i,j}]_q}{k}e_{j,l+k} \text{ for } k\neq 0, \tag{T5$'$}$$

$$\psi_{i,0}f_{j,l} = q^{-a_{i,j}}f_{j,l}\psi_{i,0},\ [h_{i,k}, f_{j,l}] = -d^{-km_{i,j}}\gamma^{|k|/2}\frac{[ka_{i,j}]_q}{k}f_{j,l+k} \text{ for } k\neq 0. \tag{T6$'$}$$

We also introduce $h_{i,0}, c, c'$ via $\psi_{i,0} = q^{h_{i,0}},\ \gamma^{1/2} = q^c,\ c' = \sum_{i\in[n]} h_{i,0}$, so that c, c' are central.

In Sections 2.3–2.4, we will also need to make sense of the elements $q^{\frac{h_{i,0}}{2n}}, \gamma^{\frac{1}{2n}}, q^{\frac{d_2}{n}}$. In such cases, we formally add elements of the form $q^{\frac{h_{i,0}}{N}}, q^{\frac{c}{N}}, q^{\frac{d_1}{N}}, q^{\frac{d_2}{N}}$ for any $N \in \mathbb{Z}_{>0}$.

2.2 Hopf Algebra Structure, Hopf Pairing, and Drinfeld Double

Following [10], we recall some of the basic results on $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ which are relevant to us.

- *Topological Hopf algebra structure on $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.*
  Following [6, Theorem 2.1], we endow $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ with a topological Hopf algebra structure by defining the coproduct Δ, the counit ϵ, and the antipode S as follows:

$$\Delta(\psi_i^\pm(z)) = \psi_i^\pm(\gamma_{(2)}^{\pm 1/2}z)\otimes\psi_i^\pm(\gamma_{(1)}^{\mp 1/2}z),\ \Delta(x) = x\otimes x \text{ for } x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2},$$
$$\Delta(e_i(z)) = e_i(z)\otimes 1 + \psi_i^-(\gamma_{(1)}^{1/2}z)\otimes e_i(\gamma_{(1)}z),\ \Delta(f_i(z)) = 1\otimes f_i(z) + f_i(\gamma_{(2)}z)\otimes\psi_i^+(\gamma_{(2)}^{1/2}z), \tag{H1}$$

$$\epsilon(e_i(z)) = \epsilon(f_i(z)) = 0,\ \epsilon(\psi_i^\pm(z)) = 1,\ \epsilon(x) = 1 \text{ for } x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}, \tag{H2}$$

$$S(e_i(z)) = -\psi_i^-(\gamma^{-1/2}z)^{-1}e_i(\gamma^{-1}z),\ S(f_i(z)) = -f_i(\gamma^{-1}z)\psi_i^+(\gamma^{-1/2}z)^{-1},$$
$$S(x) = x^{-1} \text{ for } x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}, \psi_i^\pm(z), \tag{H3}$$

  where $\gamma_{(1)} := \gamma\otimes 1$ and $\gamma_{(2)} := 1\otimes\gamma$.

- *Sub/quotient-algebras of* $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

  In what follows, we will need the following subalgebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$:

  - $\ddot{U}^{\geq}$ is the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}, \psi_{i,l}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i\in[n]}^{k\in\mathbb{Z}, l\in -\mathbb{N}}$.
  - $\ddot{U}^{\leq}$ is the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{f_{i,k}, \psi_{i,l}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i\in[n]}^{k\in\mathbb{Z}, l\in \mathbb{N}}$.
  - $\ddot{U}^{+}, \ddot{U}^{-}$ are the subalgebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}\}_{i\in[n]}^{k\in\mathbb{Z}}$ and $\{f_{i,k}\}_{i\in[n]}^{k\in\mathbb{Z}}$, respectively.
  - $\ddot{U}^{0}$ is the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{\psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i\in[n]}^{k\in\mathbb{Z}}$.

  We also define two modifications of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$:

  - Let $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ be obtained from $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ by "ignoring" the generator $q^{\pm d_2}$ and taking a quotient by the ideal (c'), i.e., setting $c' = 0$. The subalgebras $\ddot{U}'^{\geq}, \ddot{U}'^{\leq}, \ddot{U}'^{\pm}, \ddot{U}'^{0}$ of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ are defined completely analogously to $\ddot{U}^{\geq}, \ddot{U}^{\leq}, \ddot{U}^{\pm}, \ddot{U}^{0}$ above.
  - Let $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ be obtained from $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ by "ignoring" the generator $q^{\pm d_1}$ and taking a quotient by the ideal (c), i.e., setting $c = 0$. The subalgebras $'\ddot{U}^{\geq}, '\ddot{U}^{\leq}, '\ddot{U}^{\pm}, '\ddot{U}^{0}$ of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ are defined completely analogously to $\ddot{U}^{\geq}, \ddot{U}^{\leq}, \ddot{U}^{\pm}, \ddot{U}^{0}$ above.

- *Hopf pairing and a Drinfeld double realization of* $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

Analogously to the case of quantum affine algebras (see [12]), we have the following result.

Theorem 2.1 *(a) There exists a unique Hopf algebra pairing* $\varphi\colon \ddot{U}^{\geq} \times \ddot{U}^{\leq} \to \mathbb{C}$ *satisfying*

$$\varphi(e_i(z), f_j(w)) = \frac{\delta_{i,j}}{q-q^{-1}} \cdot \delta\left(\frac{z}{w}\right), \varphi(\psi_i^-(z), \psi_j^+(w)) = g_{a_{i,j}}(d^{m_{i,j}} z/w), \varphi(q^{d_2}, q^{d_2}) = q^{\frac{n(n^2-1)}{12}},$$

$$\varphi(e_i(z), x^-) = \varphi(x^+, f_i(z)) = 0 \text{ for } x^{\pm} = \psi_j^{\mp}(w), \gamma^{1/2}, q^{d_1}, q^{d_2},$$

$$\varphi(\gamma^{1/2}, q^{d_1}) = \varphi(q^{d_1}, \gamma^{1/2}) = q^{-1/2},\ \varphi(\psi_i^-(z), q^{d_2}) = q^{-1},\ \varphi(q^{d_2}, \psi_i^+(z)) = q,$$

$$\varphi(\psi_i^-(z), x) = \varphi(x, \psi_i^+(z)) = 1 \text{ for } x = \gamma^{1/2}, q^{d_1},$$

$$\varphi(\gamma^{1/2}, q^{d_2}) = \varphi(q^{d_2}, \gamma^{1/2}) = \varphi(\gamma^{1/2}, \gamma^{1/2}) = \varphi(q^{d_1}, q^{d_1}) = \varphi(q^{d_1}, q^{d_2}) = \varphi(q^{d_2}, q^{d_1}) = 1.$$

(b) The natural Hopf algebra homomorphism from the Drinfeld double $D_{\varphi}(\ddot{U}^{\geq}, \ddot{U}^{\leq})$ *to* $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ *induces the isomorphism*

$$\Xi\colon D_{\varphi}(\ddot{U}^{\geq}, \ddot{U}^{\leq})/I \xrightarrow{\sim} \ddot{U}_{q,d}(\mathfrak{sl}_n) \text{ with } I := (x\otimes 1 - 1\otimes x | x = \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2})_{i\in[n]}.$$

(c) Analogously to (b), the algebras $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ *and* $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ *admit the Drinfeld double realizations via* $D_{\varphi'}(\ddot{U}'^{\geq}, \ddot{U}'^{\leq})$ *and* $D_{'\varphi}('\ddot{U}^{\geq}, '\ddot{U}^{\leq})$*, where* φ' *and* $'\varphi$ *are defined similarly to* φ.

(d) The pairings $\varphi, \varphi', {}'\varphi$ are nondegenerate if and only if q, qd, qd^{-1} are not roots of unity.

(e) If q, qd, qd^{-1} are not roots of unity, then the algebras $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ and ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ admit the universal R-matrices R, R' and ${}'R$, associated to the pairings φ, φ' and ${}'\varphi$, respectively.

2.3 Two Copies of $U_q(\widehat{\mathfrak{sl}}_n)$ Inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$

Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the quantum affine algebra of $\mathfrak{sl}_n$ presented in the *new Drinfeld realization*, see [5]. This is the unital associative $\mathbb{C}$-algebra generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, C^{\pm 1}, \widetilde{D}^{\pm 1}\}_{i\in[n]^\times}^{k\in\mathbb{Z}}$ with the defining relations similar to those of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ (our notation follow [19]):

$$[\psi_i^\pm(z), \psi_j^\pm(w)] = 0,\ C^{\pm 1} - \text{central}, \tag{A0.1}$$

$$\psi_{i,0}^{\pm 1}\cdot\psi_{i,0}^{\mp 1} = C^{\pm 1}\cdot C^{\mp 1} = \widetilde{D}^{\pm 1}\cdot\widetilde{D}^{\mp 1} = 1, \tag{A0.2}$$

$$\widetilde{D}e_i(z)\widetilde{D}^{-1} = qe_i(q^{-n}z),\ \widetilde{D}f_i(z)\widetilde{D}^{-1} = q^{-1}f_i(q^{-n}z),\ \widetilde{D}\psi_i^\pm(z)\widetilde{D}^{-1} = \psi_i^\pm(q^{-n}z), \tag{A0.3}$$

$$g_{a_{i,j}}(C^{-1}z/w)\psi_i^+(z)\psi_j^-(w) = \psi_j^-(w)\psi_i^+(z)g_{a_{i,j}}(Cz/w), \tag{A1}$$

$$e_i(z)e_j(w) = g_{a_{i,j}}(z/w)e_j(w)e_i(z), \tag{A2}$$

$$f_i(z)f_j(w) = g_{a_{i,j}}(z/w)^{-1}f_j(w)f_i(z), \tag{A3}$$

$$(q-q^{-1})[e_i(z), f_j(w)] = \delta_{i,j}\left(\delta(Cw/z)\psi_i^+(Cw) - \delta(Cz/w)\psi_i^-(Cz)\right), \tag{A4}$$

$$\psi_i^+(z)e_j(w) = g_{a_{i,j}}(z/w)e_j(w)\psi_i^+(z),\ \psi_i^-(z)e_j(w) = g_{a_{i,j}}(C^{-1}z/w)e_j(w)\psi_i^-(z), \tag{A5}$$

$$\psi_i^+(z)f_j(w) = g_{a_{i,j}}(C^{-1}z/w)^{-1}f_j(w)\psi_i^+(z),\ \psi_i^-(z)f_j(w) = g_{a_{i,j}}(z/w)^{-1}f_j(w)\psi_i^-(z), \tag{A6}$$

$$\mathrm{Sym}_{z_1,z_2}\,[e_i(z_1), [e_i(z_2), e_j(w)]_q]_{q^{-1}} = 0 \text{ if } a_{i,j} = -1,\ [e_i(z), e_j(w)] = 0 \text{ if } a_{i,j} = 0, \tag{A7.1}$$

$$\mathrm{Sym}_{z_1,z_2}\,[f_i(z_1), [f_i(z_2), f_j(w)]_q]_{q^{-1}} = 0 \text{ if } a_{i,j} = -1,\ [f_i(z), f_j(w)] = 0 \text{ if } a_{i,j} = 0, \tag{A7.2}$$

where the generating series $e_i(z), f_i(z), \psi_i^\pm(z)$ are defined as before.

This algebra is known to admit a classical *Drinfeld–Jimbo realization* of [4, 15]. To state this explicitly, let $U_q^{\mathrm{DJ}}(\widehat{\mathfrak{sl}}_n)$ be the unital associative $\mathbb{C}$-algebra generated by $\{x_i^\pm, t_i^{\pm 1}, D^{\pm 1}\}_{i\in[n]}$ with the following defining relations:

$$D^{\pm 1}D^{\mp 1} = 1,\ Dt_iD^{-1} = t_i,\ Dx_i^\pm D^{-1} = q^{\pm 1}x_i^\pm,$$

$$t_i^{\pm 1}t_i^{\mp 1} = 1,\ t_it_j = t_jt_i,\ t_ix_j^\pm t_i^{-1} = q^{\pm a_{i,j}}x_j^\pm,$$

$$[x_i^+, x_j^-] = \delta_{i,j}\cdot\frac{t_i - t_i^{-1}}{q-q^{-1}},\ \sum_{s=0}^{1-a_{i,j}}\frac{(-1)^s}{[s]_q![1-a_{i,j}-s]_q!}(x_i^\pm)^s x_j^\pm (x_i^\pm)^{1-a_{i,j}-s} = 0\ (i\ne j),$$

where $[m]_q! := [m]_q[m-1]_q\cdots[1]_q$.

According to [5], there is a $\mathbb{C}(q)$-algebra isomorphism $\Phi\colon U_q^{\mathrm{DJ}}(\widehat{\mathfrak{sl}}_n)\xrightarrow{\sim}U_q(\widehat{\mathfrak{sl}}_n)$ given by

$$x_i^+\mapsto e_{i,0},\ x_i^-\mapsto f_{i,0},\ t_i^{\pm 1}\mapsto\psi_{i,0}^{\pm 1}\ (1\le i\le n-1),\ t_0\mapsto C\cdot(\psi_{1,0}\cdots\psi_{n-1,0})^{-1},\ D\mapsto\widetilde{D},$$

$$x_0^+\mapsto C(\psi_{1,0}\cdots\psi_{n-1,0})^{-1}[\cdots[f_{1,1}, f_{2,0}]_q,\cdots, f_{n-1,0}]_q,$$

$$x_0^-\mapsto [e_{n-1,0},\cdots,[e_{2,0}, e_{1,-1}]_{q^{-1}}\cdots]_{q^{-1}}(\psi_{1,0}\cdots\psi_{n-1,0})C^{-1}.$$

Remark 2.2 (a) The above isomorphism was stated without a proof in [5]. The inverse Φ^{-1} was constructed in [1] by using the braid group action on $U_q^{\mathrm{DJ}}(\widehat{\mathfrak{sl}}_n)$ due to G. Lusztig. The direct verification of the fact that the above assignment gives rise to a $\mathbb{C}(q)$-algebra homomorphism $\Phi\colon U_q^{\mathrm{DJ}}(\widehat{\mathfrak{sl}}_n) \xrightarrow{\sim} U_q(\widehat{\mathfrak{sl}}_n)$ was given in [16] by utilizing the technique of q-commutators, which also plays a key computational role in the current paper. However, proofs of injectivity of Φ^{-1} and Φ in [1, 16] had a gap, which was only recently filled in [3].

(b) In the classical literature, the *grading* elements $D, \widetilde{D}$ satisfy slightly different relations, while our conventions are better adapted to fit into the toroidal story and follow that of [19].

Following [23], we introduce the *vertical* and *horizontal* copies of $U_q(\widehat{\mathfrak{sl}}_n)$ inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. Consider two algebra homomorphisms $h, v\colon U_q(\widehat{\mathfrak{sl}}_n) \to \ddot{U}_{q,d}(\mathfrak{sl}_n)$ defined by

$$h\colon x_i^+ \mapsto e_{i,0},\ x_i^- \mapsto f_{i,0},\ t_i \mapsto \psi_{i,0},\ D \mapsto q^{d_2},$$

$$v\colon e_{i,k} \mapsto d^{ik} e_{i,k},\ f_{i,k} \mapsto d^{ik} f_{i,k},\ \psi_{i,k} \mapsto d^{ik}\gamma^{k/2}\psi_{i,k},\ C \mapsto \gamma,\ \widetilde{D} \mapsto q^{-nd_1} \cdot q^{\sum_{j=1}^{n-1} \frac{j(n-j)}{2} h_{j,0}},$$

where we follow the conventions of Section 2.1 and add elements $q^{h_{j,0}/2}$ to $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

According to [23], both h, v are inclusions. The images of h and v, denoted by $\dot{U}_q^{\mathrm{h}}(\mathfrak{sl}_n)$ and $\dot{U}_q^{\mathrm{v}}(\mathfrak{sl}_n)$, are called the *horizontal* and *vertical* copies of $U_q(\widehat{\mathfrak{sl}}_n)$ inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

Remark 2.3 The injectivity of h, v was stated in [23] without a proof, and was used in numeric literature afterwards. A simple way to see the injectivity is to use the double realization of all algebras involved (here $U_q(\widehat{\mathfrak{sl}}_n)$ is treated as in Theorem 2.1, while $U_q^{\mathrm{DJ}}(\widehat{\mathfrak{sl}}_n)$ is treated with respect to the Drinfeld-Jimbo Borel subalgebras, see e.g. [17]). Both Hopf pairings on $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_q^{\mathrm{DJ}}(\widehat{\mathfrak{sl}}_n)$ are known to be nondegenerate for q not a root of unity. Since h, v respect the pairings, their injectivity follows.

2.4 Miki's Isomorphism

We recall the beautiful result of K. Miki which provides an isomorphism ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n) \xrightarrow{\sim} \ddot{U}'_{q,d}(\mathfrak{sl}_n)$ intertwining the *vertical* and *horizontal* embeddings of quantum affine algebras of $\mathfrak{sl}_n$.

To formulate the main result of this section, we need some more notation.

- Let $U_q(L\mathfrak{sl}_n)$ be obtained from $U_q(\widehat{\mathfrak{sl}}_n)$ by "ignoring" the generator $\widetilde{D}^{\pm 1}$ and taking a quotient by the ideal $(C-1)$, i.e., setting $C = 1$. The algebra $U_q(L\mathfrak{sl}_n)$ is usually called the quantum loop algebra of $\mathfrak{sl}_n$. Analogously to h and v, we have the following inclusions:
$$ {}'\ddot{U}_{q,d}(\mathfrak{sl}_n) \overset{'h}{\hookleftarrow} U_q(\widehat{\mathfrak{sl}}_n) \overset{v'}{\hookrightarrow} \ddot{U}'_{q,d}(\mathfrak{sl}_n)$$
and
$$ {}'\ddot{U}_{q,d}(\mathfrak{sl}_n) \overset{'v}{\hookleftarrow} U_q(L\mathfrak{sl}_n) \overset{h'}{\hookrightarrow} \ddot{U}'_{q,d}(\mathfrak{sl}_n).$$
- Let σ be the antiautomorphism of $U_q(\widehat{\mathfrak{sl}}_n)$ determined by
$$\sigma: x_i^{\pm} \mapsto x_i^{\pm},\ t_i \mapsto t_i^{-1},\ D \mapsto D^{-1}.$$
- Let η be the antiautomorphism of $U_q(\widehat{\mathfrak{sl}}_n)$ determined by
$$\eta\colon e_{i,k} \mapsto e_{i,-k},\ f_{i,k} \mapsto f_{i,-k},\ h_{i,l} \mapsto -C^l h_{i,-l},\ \psi_{i,0} \mapsto \psi_{i,0}^{-1},\ C \mapsto C,\ \widetilde{D} \mapsto \widetilde{D} \cdot \prod_{i=1}^{n-1} \psi_{i,0}^{-i(n-i)}.$$

- Let $'Q$ be the automorphism of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ determined by

$$'Q\colon e_{i,k}\mapsto (-d)^k e_{i+1,k},\ f_{i,k}\mapsto (-d)^k f_{i+1,k},\ h_{i,l}\mapsto (-d)^l h_{i+1,l},\ \psi_{i,0}\mapsto \psi_{i+1,0},\ q^{d_2}\mapsto q^{d_2}.$$

- Let Q' be the automorphism of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ such that it maps the generators other than $\gamma^{\pm 1/2}, q^{\pm d_1}$ as $'Q$, while

$$Q'\colon \gamma^{1/2}\mapsto \gamma^{1/2},\ q^{d_1}\mapsto q^{d_1}\cdot \gamma^{-1}.$$

- Let $'\mathcal{Y}_j$ $(1\leq j\leq n)$ be the automorphism of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ determined by

$$'\mathcal{Y}_j\colon h_{i,l}\mapsto h_{i,l},\ \psi_{i,0}\mapsto \psi_{i,0},\ q^{d_2}\mapsto q^{d_2},$$

$$e_{i,k}\mapsto (-d)^{-n\delta_{i,0}\delta_{j,n}-i\bar{\delta}_{i,j}+i\delta_{i,j-1}} e_{i,k-\bar{\delta}_{i,j}+\delta_{i,j-1}},\ f_{i,k}\mapsto (-d)^{n\delta_{i,0}\delta_{j,n}+i\bar{\delta}_{i,j}-i\delta_{i,j-1}} f_{i,k+\bar{\delta}_{i,j}-\delta_{i,j-1}},$$

where $\bar{\delta}_{i,j}=\begin{cases}1 & \text{if } j\equiv i \pmod n\\ 0 & \text{otherwise}\end{cases}$.

- Let $\mathcal{Y}'_j$ $(1\leq j\leq n)$ be the automorphism of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ such that it maps the generators other than $\psi^{\pm 1}_{i,0}, \gamma^{\pm 1/2}, q^{\pm d_1}$ as $'\mathcal{Y}_j$, while

$$\mathcal{Y}'_j\colon \gamma^{1/2}\mapsto \gamma^{1/2},\ \psi_{i,0}\mapsto \gamma^{-\bar{\delta}_{i,j}+\delta_{i,j-1}}\psi_{i,0},$$

$$\mathcal{Y}'_j\colon q^{d_1}\mapsto q^{d_1}\cdot\gamma^{-\frac{n+1}{2n}}\cdot K_j \text{ with } K_j=\prod_{l=1}^{j-1} q^{\frac{l}{n}h_{l,0}}\prod_{l=j}^{n-1} q^{\frac{l-n}{n}h_{l,0}},$$

where we follow the conventions of Section 2.1 and add elements $\gamma^{\frac{1}{2n}}, q^{\frac{h_{j,0}}{2n}}$ to $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$.

Theorem 2.4 [19, Proposition 1] *There exists an algebra isomorphism*

$$\varpi\colon\ '\ddot{U}_{q,d}(\mathfrak{sl}_n)\xrightarrow{\sim}\ddot{U}'_{q,d}(\mathfrak{sl}_n)$$

satisfying the following properties:

$$\varpi\circ{}'h=v',\ \varpi\circ{}'v\circ\eta\circ\sigma=h',\ Q'\circ\mathcal{Y}'_n\circ\varpi=\varpi\circ{}'\mathcal{Y}_1^{-1}\circ{}'Q.$$

Remark 2.5 (a) Let $U^{\mathrm{tor}}_{q,d}(\mathfrak{sl}_n)$ be obtained from $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ by "ignoring" the generators $q^{\pm d_1}$ and $q^{\pm d_2}$. The construction of ϖ in [19] was based on the previous work [18], where an automorphism $\overline{\varpi}$ of $U^{\mathrm{tor}}_{q,d}(\mathfrak{sl}_n)$ was established.

(b) The generators $x^{\pm}_{i,k}, h_{i,l}, k_i^{\pm 1}, C^{\pm 1}, D^{\pm 1}, \widetilde{D}^{\pm 1}$ from [19] are related to our generators via

$$x^+_{i,k}\leftrightarrow d^{ik}e_{i,k},\ x^-_{i,k}\leftrightarrow d^{ik}f_{i,k},\ h_{i,l}\leftrightarrow d^{il}\gamma^{l/2}h_{i,l},$$

$$k_i^{\pm 1}\leftrightarrow \psi^{\pm 1}_{i,0},\ C^{\pm 1}\leftrightarrow\gamma^{\pm 1},\ D^{\pm 1}\leftrightarrow q^{\pm d_2},\ \widetilde{D}^{\pm 1}\leftrightarrow q^{\mp n d_1}\cdot q^{\pm\sum_{j=1}^{n-1}\frac{j(n-j)}{2}h_{j,0}},$$

while the parameters q,ξ from [19] are related to our parameters q,d via

$$q\leftrightarrow q,\ \xi\leftrightarrow d^{-n}.$$

(c) The aforementioned choice of generators from [19] is convenient as there is no need to add elements $\left\{q^{\frac{h_{j,0}}{N}}, q^{\frac{c}{N}}, q^{\frac{d_1}{N}}, q^{\frac{d_2}{N}}\right\}_{N\in\mathbb{Z}_{>0}}$. However, we prefer the current presentation as it is more symmetric and suitable for the rest of our exposition.

We conclude this subsection by computing images of some generators of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ under ϖ.

Proposition 2.6 *(a) We have*

$$\varpi: e_{i,0}\mapsto e_{i,0},\ f_{i,0}\mapsto f_{i,0},\ \psi^{\pm1}_{i,0}\mapsto\psi^{\pm1}_{i,0}\ \text{for}\ i\in[n]^\times,$$

$$\varpi: \psi^{\pm1}_{0,0}\mapsto\gamma^{\pm1}\cdot\psi^{\pm1}_{0,0},\ q^{\pm d_2}\mapsto q^{\mp nd_1}\cdot q^{\pm\sum_{j=1}^{n-1}\frac{j(n-j)}{2}h_{j,0}},$$

$$\varpi: e_{0,0}\mapsto d\cdot\gamma\psi_{0,0}\cdot[\cdots[f_{1,1},f_{2,0}]_q,\cdots,f_{n-1,0}]_q,$$

$$\varpi: f_{0,0}\mapsto d^{-1}\cdot[e_{n-1,0},\cdots,[e_{2,0},e_{1,-1}]_{q^{-1}}\cdots]_{q^{-1}}\cdot\psi^{-1}_{0,0}\gamma^{-1}.$$

(b) For $i\in[n]^\times$, we have

$$\varpi(h_{i,1})=(-1)^{i+1}d^{-i}\cdot[[\cdots[[\cdots[f_{0,0},f_{n-1,0}]_q,\cdots,f_{i+1,0}]_q,f_{1,0}]_q,\cdots,f_{i-1,0}]_q,f_{i,0}]_{q^2},$$

$$\varpi(h_{i,-1})=(-1)^{i+1}d^{i}\cdot[e_{i,0},[\cdots,[e_{1,0},[e_{i+1,0},\cdots,[e_{n-1,0},e_{0,0}]_{q^{-1}}\cdots]_{q^{-1}}]_{q^{-1}}\cdots]_{q^{-1}}]_{q^{-2}}.$$

(c) For $i=0$, we have

$$\varpi(h_{0,1})=(-1)^n d^{1-n}\cdot[[\cdots[f_{1,1},f_{2,0}]_q,\cdots,f_{n-1,0}]_q,f_{0,-1}]_{q^2},$$

$$\varpi(h_{0,-1})=(-1)^n d^{n-1}\cdot[e_{0,1},[e_{n-1,0},\cdots,[e_{2,0},e_{1,-1}]_{q^{-1}}\cdots]_{q^{-1}}]_{q^{-2}}.$$

(d) We have

$$\varpi(e_{0,-1})=(-d)^n e_{0,1},\ \varpi(f_{0,1})=(-d)^{-n}f_{0,-1}.$$

Proof of Proposition 2.6 (a) Follow straightforwardly by applying the equality $\varpi\circ{}'h=v'$ to the explicit formulas for $\Phi(x_i^\pm)$, $\Phi(t_i)$, $\Phi(D)$ with $\Phi\colon U_q^{\mathrm{DJ}}(\widehat{\mathfrak{sl}}_n)\xrightarrow{\sim}U_q(\widehat{\mathfrak{sl}}_n)$ from Section 2.3.

(b) We will need the following formulas expressing $h_{i,\pm1}$ in the Drinfeld-Jimbo presentation:

$$\Phi^{-1}(h_{i,1})=(-1)^i[[\cdots[[\cdots[x_0^+,x_{n-1}^+]_{q^{-1}},\cdots,x_{i+1}^+]_{q^{-1}},x_1^+]_{q^{-1}},\cdots,x_{i-1}^+]_{q^{-1}},x_i^+]_{q^{-2}},\tag{1}$$

$$\Phi^{-1}(h_{i,-1})=(-1)^i[x_i^-,\cdots,[x_1^-,[x_{i+1}^-,\cdots,[x_{n-1}^-,x_0^-]_q\cdots]_q]_q\cdots]_{q^2}.\tag{2}$$

Formulas (1, 2) are proved by applying iteratively two useful identities involving q-brackets:

$$[a,[b,c]_u]_v=[[a,b]_x,c]_{uv/x}+x\cdot[b,[a,c]_{v/x}]_{u/x},$$

$$[[a,b]_u,c]_v=[a,[b,c]_x]_{uv/x}+x\cdot[[a,c]_{v/x},b]_{u/x},$$

compare to our proof of Theorem 4.5 (we leave verification of details to the interested reader).

Applying the equality $\varpi\circ{}'v\circ\eta=h'\circ\sigma^{-1}$ to formulas (1) and (2), we get the claimed formulas for $\varpi(h_{i,-1})$ and $\varpi(h_{i,1})$, respectively.

(c) Applying the equality $Q'\circ\mathcal{Y}'_n\circ\varpi=\varpi\circ{}'\mathcal{Y}_1^{-1}\circ{}'Q$ to $h_{n-1,\pm1}$ and using the formulas for $\varpi(h_{n-1,\pm1})$ from (b), we obtain the formulas for $\varpi(h_{0,\pm1})$.

(d) It suffices to apply the equality $Q'\circ\mathcal{Y}'_n\circ\varpi=\varpi\circ{}'\mathcal{Y}_1^{-1}\circ{}'Q$ to $e_{n-1,0}$ and $f_{n-1,0}$. $\square$

2.5 Fock and Macmahon Modules

In this section, we recall two interesting classes of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules constructed in [8]. They depend on two parameters: $0 \le p \le n-1$ and $u \in \mathbb{C}^\times$. We also set

$$q_1 := q^{-1}d,\ q_2 := q^2,\ q_3 := q^{-1}d^{-1} \text{ and } \phi(t) := \frac{q^{-1}t - q}{t-1}. \qquad (\diamond)$$

Assumption In the rest of this paper, we assume that q_1, q_2, q_3 are *generic*, that is,

$$q_1^a q_2^b q_3^c = 1 \text{ for some } a, b, c \in \mathbb{Z} \text{ implies } a = b = c. \qquad \text{(G)}$$

Given $\nu \in \mathbb{C}^\times$ and a collection of formal series $\boldsymbol{\phi}(z) = \{\phi_i^\pm(z)\}_{i\in[n]},\ \phi_i^\pm(z) \in \mathbb{C}[[z^{\mp1}]]$, a vector v of an $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-module V is said to have *weight* $(\nu; \boldsymbol{\phi}(z))$ if $q^{d_2}v = \nu \cdot v$ and $\psi_i^\pm(z)v = \phi_i^\pm(z) \cdot v$ for any $i \in [n]$. The module V is called a *lowest weight module* if it is generated by a weight vector v such that $'\ddot{U}^- v = 0$. Such v is called a *lowest weight vector*, and its weight the *lowest weight* of V. Given $\nu \in \mathbb{C}^\times$ and $\boldsymbol{\phi}(z)$ with $\phi_i^+(\infty)\phi_i^-(0) = 1$ for every $i \in [n]$, there is a unique irreducible lowest weight module of that lowest weight. If $\phi_i^\pm(z)$ are expansions of a rational function $\phi_i(z)$ at $z = 0, \infty$, then we write $(\nu; \boldsymbol{\phi}(z)) = (\nu; \phi_i(z))_{i\in[n]}$.

- *Fock modules* $F^{(p)}(u)$.

The most basic lowest weight $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules are the Fock modules $F^{(p)}(u)$ with the basis $\{|\lambda\rangle\}$ labeled by all partitions λ. Given such a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we define $\lambda + 1_l := (\lambda_1, \ldots \lambda_l + 1, \ldots)$, $|\lambda| := \sum_l \lambda_l$, and we denote the transposed partition by $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$. We use $\emptyset$ to denote the partition $(0, 0, \ldots)$. We also write $a \equiv b$ if $a - b$ is divisible by n.

Proposition 2.7 [8, Proposition 3.3] *The $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on $F^{(p)}(u)$ is given by the following formulas:*

$$\langle \lambda + 1_l|e_i(z)|\lambda\rangle = \bar{\delta}_{p+l-\lambda_l,i+1} \prod_{1\le s<l}^{p+s-\lambda_s\equiv i} \phi(q_1^{\lambda_s-\lambda_l-1}q_3^{s-l}) \prod_{1\le s<l}^{p+s-\lambda_s\equiv i+1} \phi(q_1^{\lambda_l-\lambda_s}q_3^{l-s})\delta(q_1^{\lambda_l}q_3^{l-1}u/z),$$

$$\langle \lambda|f_i(z)|\lambda + 1_l\rangle = \bar{\delta}_{p+l-\lambda_l,i+1} \prod_{s>l}^{p+s-\lambda_s\equiv i} \phi(q_1^{\lambda_s-\lambda_l-1}q_3^{s-l}) \prod_{s>l}^{p+s-\lambda_s\equiv i+1} \phi(q_1^{\lambda_l-\lambda_s}q_3^{l-s})\delta(q_1^{\lambda_l}q_3^{l-1}u/z),$$

$$\langle \lambda|\psi_i^\pm(z)|\lambda\rangle = \prod_{s\ge1}^{p+s-\lambda_s\equiv i} \phi(q_1^{\lambda_s-1}q_3^{s-1}u/z) \prod_{s\ge1}^{p+s-\lambda_s\equiv i+1} \phi(q_1^{\lambda_s-1}q_3^{s-2}u/z)^{-1},\ \langle\lambda|q^{d_2}|\lambda\rangle = q^{|\lambda|},$$

while all other matrix coefficients are zero. $F^{(p)}(u)$ is an irreducible lowest weight module of the lowest weight $(1; \phi(z/u)^{\delta_{i,p}})_{i\in[n]}$ and with $|\emptyset\rangle$ being the corresponding lowest weight vector.

Definition 2.8 For $\bar{c} \in (\mathbb{C}^\times)^{[n]}$, let $\tau^p_{u,\bar{c}}$ be the twist of this representation by the automorphism $\chi_{p,\bar{c}}$ of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ defined via $e_{i,k} \mapsto c_i e_{i,k},\ f_{i,k} \mapsto c_i^{-1} f_{i,k},\ \psi_{i,k} \mapsto \psi_{i,k},\ q^{d_2} \mapsto q^{-\frac{p(n-p)}{2}} \cdot q^{d_2}$.

Given a collection $\{(p_k, u_k, \bar{c}_k)\}_{k=1}^r$ with $0 \le p_k \le n-1$, $u_k \in \mathbb{C}^\times$, $\bar{c}_k \in (\mathbb{C}^\times)^{[n]}$, we call it *generic* if for any pair $1 \le s' < s \le r$, there are no $a, b \in \mathbb{Z}$ such that $b-a \equiv p_{s'} - p_s$ and $u_s = u_{s'}q_1^{-a}q_3^{-b}$. We have the following simple result (see [8, Lemma 4.1]).

Lemma 2.9 *For a generic collection $\{(p_k, u_k, \bar{c}_k)\}_{k=1}^r$, the coproduct Δ of* (H1) *endows $\tau^{p_1}_{u_1,\bar{c}_1} \otimes \cdots \otimes \tau^{p_r}_{u_r,\bar{c}_r}$ with a structure of an $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-module. It is an irreducible lowest weight module generated by the lowest weight vector $|\emptyset\rangle \otimes \cdots \otimes |\emptyset\rangle$.*

Remark 2.10 To see the irreducibility, one checks that the action of commuting series $\psi_i^{\pm}(z)$ is diagonalizable and has a simple joint spectrum (here we use q_1, q_2, q_3 being *generic*, see (G)).

- *Macmahon modules $M^{(p)}(u, K)$.*

For $K \in \mathbb{C}^\times$, set $\phi^K(t) := \frac{K^{-1}t-K}{t-1}$. We call K *generic* if $K \notin q^{\mathbb{Z}}d^{\mathbb{Z}}$. For such K, the unique irreducible lowest weight $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-module of the lowest weight $(1; \phi^K(z/u)^{\delta_{i,p}})_{i\in[n]}$ is called the Macmahon module, denoted by $M^{(p)}(u, K)$. They were first studied in [8]. Recall that a collection of partitions $\bar{\lambda} = \{\lambda^{(r)}\}_{r\in\mathbb{Z}_{>0}}$ is called a *plane partition* if

$$\lambda^{(r)}_l \geq \lambda^{(r+1)}_l \text{ for all } r, l \in \mathbb{Z}_{>0} \text{ and } \lambda^{(r)} = \emptyset \text{ for } r \gg 0.$$

Proposition 2.11 [8, Theorem 4.3] *For a generic K, the vector space $M^{(p)}(u, K)$ has a basis $\{|\bar{\lambda}\rangle\}$ (labeled by all plane partitions) with $|\bar{\emptyset}\rangle$ being its lowest weight vector.*

In this paper, we will not need explicit formulas for the $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action in the basis $\{|\bar{\lambda}\rangle\}$.

2.6 Vertex Representations

In this section, we recall a family of vertex $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-representations from [21] generalizing the construction of [7] for quantum affine algebras. Let S_n be the *generalized* Heisenberg algebra generated by $\{H_{i,k} | i \in [n], k \in \mathbb{Z}\backslash\{0\}\}$ and a central element H_0 with the defining relations

$$[H_{i,k}, H_{j,l}] = d^{-km_{i,j}} \frac{[k]_q \cdot [ka_{i,j}]_q}{k} \delta_{k,-l} \cdot H_0.$$

Let S_n^+ be the subalgebra of S_n generated by $\{H_{i,k} | i \in [n], k > 0\} \sqcup \{H_0\}$, and let $\mathbb{C}v_0$ be the S_n^+-representation with $H_{i,k}$ acting trivially and H_0 acting via the identity operator. The induced representation $F_n := \mathrm{Ind}_{S_n^+}^{S_n} \mathbb{C}v_0$ is called the *Fock representation* of S_n.

We denote by $\{\bar{\alpha}_i\}_{i=1}^{n-1}$ the simple roots of $\mathfrak{sl}_n$, by $\{\bar{\Lambda}_i\}_{i=1}^{n-1}$ the fundamental weights of $\mathfrak{sl}_n$, by $\{\bar{h}_i\}_{i=1}^{n-1}$ the simple coroots of $\mathfrak{sl}_n$. Let $\bar{Q} := \bigoplus_{i=1}^{n-1} \mathbb{Z}\bar{\alpha}_i$ be the root lattice of $\mathfrak{sl}_n$, $\bar{P} := \bigoplus_{i=1}^{n-1} \mathbb{Z}\bar{\Lambda}_i = \bigoplus_{i=2}^{n-1} \mathbb{Z}\bar{\alpha}_i \oplus \mathbb{Z}\bar{\Lambda}_{n-1}$ be the weight lattice of $\mathfrak{sl}_n$. We also set

$$\bar{\alpha}_0 := -\sum_{i=1}^{n-1} \bar{\alpha}_i \in \bar{Q},\ \bar{\Lambda}_0 := 0 \in \bar{P},\ \bar{h}_0 := -\sum_{i=1}^{n-1} \bar{h}_i.$$

Let $\mathbb{C}\{\bar{P}\}$ be the $\mathbb{C}$-algebra generated by $e^{\bar{\alpha}_2}, \ldots, e^{\bar{\alpha}_{n-1}}, e^{\bar{\Lambda}_{n-1}}$ with the defining relations:

$$e^{\bar{\alpha}_i} \cdot e^{\bar{\alpha}_j} = (-1)^{\langle \bar{h}_i, \bar{\alpha}_j \rangle} e^{\bar{\alpha}_j} \cdot e^{\bar{\alpha}_i},\ e^{\bar{\alpha}_i} \cdot e^{\bar{\Lambda}_{n-1}} = (-1)^{\delta_{i,n-1}} e^{\bar{\Lambda}_{n-1}} \cdot e^{\bar{\alpha}_i}.$$

For $\alpha = \sum_{i=2}^{n-1} m_i\bar{\alpha}_i + m_n\bar{\Lambda}_{n-1}$, we define $e^{\bar{\alpha}} \in \mathbb{C}\{\bar{P}\}$ via

$$e^{\bar{\alpha}} := (e^{\bar{\alpha}_2})^{m_2} \cdots (e^{\bar{\alpha}_{n-1}})^{m_{n-1}} (e^{\bar{\Lambda}_{n-1}})^{m_n}.$$

Let $\mathbb{C}\{\bar{Q}\}$ be the subalgebra of $\mathbb{C}\{\bar{P}\}$ generated by $\{e^{\bar{\alpha}_i}\}_{i=1}^{n-1}$.

For every $0 \leq p \leq n-1$, define the space

$$W(p)_n := F_n \otimes \mathbb{C}\{\bar{Q}\} e^{\bar{\Lambda}_p}.$$

Consider the operators $H_{i,l}$, $e^{\bar{\alpha}}$, $\partial_{\bar{\alpha}_i}$, $z^{H_{i,0}}$, d acting on $W(p)_n$, which assign to every element

$$v \otimes e^{\bar{\beta}} = (H_{i_1,-k_1} \cdots H_{i_N,-k_N} v_0) \otimes e^{\sum_{j=1}^{n-1} m_j \bar{\alpha}_j + \bar{\Lambda}_p} \in W(p)_n$$

the following values:

$$H_{i,l}(v \otimes e^{\bar{\beta}}) := (H_{i,l} v) \otimes e^{\bar{\beta}},\ e^{\bar{\alpha}}(v \otimes e^{\bar{\beta}}) := v \otimes e^{\bar{\alpha}} e^{\bar{\beta}},\ \partial_{\bar{\alpha}_i}(v \otimes e^{\bar{\beta}}) := \langle \bar{h}_i, \bar{\beta} \rangle v \otimes e^{\bar{\beta}},$$

$$z^{H_{i,0}}(v \otimes e^{\bar{\beta}}) := z^{\langle \bar{h}_i, \bar{\beta} \rangle} d^{\frac{1}{2} \sum_{j=1}^{n-1} \langle \bar{h}_i, m_j \bar{\alpha}_j \rangle m_{i,j}} v \otimes e^{\bar{\beta}},$$

$$\mathrm{d}(v \otimes e^{\bar{\beta}}) := (-\sum k_i + ((\bar{\Lambda}_p, \bar{\Lambda}_p) - (\bar{\beta}, \bar{\beta}))/2) v \otimes e^{\bar{\beta}}.$$

The following result provides a natural structure of an $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-module on $W(p)_n$.

Proposition 2.12 [21, Proposition 3.2.2] *For any $\bar{c} = (c_0, \ldots, c_{n-1}) \in (\mathbb{C}^\times)^{[n]}$, $u \in \mathbb{C}^\times$, and $0 \leq p \leq n-1$, the following formulas define an action of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ on $W(p)_n$:*

$$\rho^p_{u,\bar{c}}(e_i(z)) = c_i \exp\left(\sum_{k>0} \frac{q^{-k/2} u^{-k}}{[k]_q} H_{i,-k} z^k\right) \exp\left(-\sum_{k>0} \frac{q^{-k/2} u^k}{[k]_q} H_{i,k} z^{-k}\right) e^{\bar{\alpha}_i} \left(\frac{z}{u}\right)^{1+H_{i,0}},$$

$$\rho^p_{u,\bar{c}}(f_i(z)) = \frac{(-1)^{n\delta_{i,0}}}{c_i} \exp\left(-\sum_{k>0} \frac{q^{k/2} u^{-k}}{[k]_q} H_{i,-k} z^k\right) \exp\left(\sum_{k>0} \frac{q^{k/2} u^k}{[k]_q} H_{i,k} z^{-k}\right) e^{-\bar{\alpha}_i} \left(\frac{z}{u}\right)^{1-H_{i,0}},$$

$$\rho^p_{u,\bar{c}}(\psi^\pm_i(z)) = \exp\left(\pm(q-q^{-1}) \sum_{k>0} H_{i,\pm k} (z/u)^{\mp k}\right) q^{\pm \partial_{\bar{\alpha}_i}},\ \rho^p_{u,\bar{c}}(\gamma^{1/2}) = q^{1/2},\ \rho^p_{u,\bar{c}}(q^{d_1}) = q^{\mathrm{d}}.$$

$W(p)_n$ is an irreducible $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-module if q, qd, qd^{-1} are not roots of unity.

Remark 2.13 (a) The irreducibility of $\rho^p_{u,\bar{c}}$ follows from the irreducibility of the S_n-module F_n and level one vertex $U_q(\widehat{\mathfrak{sl}}_n)$-modules of [7], established at [2].

(b) The factor $(-1)^{n\delta_{i,0}}$ in $\rho^p_{u,\bar{c}}(f_i(z))$ (missing in [10, 21]) is due to $(e^{\bar{\alpha}_i})^{-1} = (-1)^{n\delta_{i,0}} e^{-\bar{\alpha}_i}$.

2.7 Shuffle Algebra

Consider an $\mathbb{N}^{[n]}$-graded $\mathbb{C}$-vector space

$$\mathbb{S} = \bigoplus_{\bar{k}=(k_0,\ldots,k_{n-1}) \in \mathbb{N}^{[n]}} \mathbb{S}_{\bar{k}},$$

where $\mathbb{S}_{(k_0,\ldots,k_{n-1})}$ consists of $\prod \mathfrak{S}_{k_i}$-symmetric rational functions in the variables $\{x_{i,r}\}_{i \in [n]}^{1 \leq r \leq k_i}$. We also fix an $n \times n$ matrix of rational functions $\Omega = (\omega_{i,j}(z))_{i,j \in [n]} \in \mathrm{Mat}_{n \times n}(\mathbb{C}(z))$ by setting

$$\omega_{i,i}(z) = \frac{z - q^{-2}}{z-1},\ \omega_{i,i+1}(z) = \frac{d^{-1}z - q}{z-1},\ \omega_{i,i-1}(z) = \frac{z - qd^{-1}}{z-1},\ \text{and } \omega_{i,j}(z) = 1 \text{ otherwise.}$$

Let us now introduce the bilinear $\star$ product on $\mathbb{S}$: given $F \in \mathbb{S}_{\bar{k}}$, $G \in \mathbb{S}_{\bar{l}}$, define $F \star G \in \mathbb{S}_{\bar{k}+\bar{l}}$ by

$$(F \star G)(x_{0,1}, \ldots, x_{0,k_0+l_0}; \ldots; x_{n-1,1}, \ldots, x_{n-1,k_{n-1}+l_{n-1}}) :=$$

$$\mathrm{Sym}_{\prod \mathfrak{S}_{k_i+l_i}}\left(F(\{x_{i,r}\}_{i\in[n]}^{1\le r\le k_i})G(\{x_{i',r'}\}_{i'\in[n]}^{k_{i'}<r'\le k_{i'}+l_{i'}})\cdot \prod_{i\in[n]}^{i'\in[n]}\prod_{r\le k_i}^{r'>k_{i'}}\omega_{i,i'}(x_{i,r}/x_{i',r'})\right).$$

Here and afterwards, given a function $f\in\mathbb{C}(\{x_{i,1},\ldots,x_{i,m_i}\}_{i\in[n]})$, we define

$$\mathrm{Sym}_{\prod \mathfrak{S}_{m_i}}(f):=\prod_{i\in[n]}\frac{1}{m_i!}\cdot\sum_{(\sigma_0,\ldots,\sigma_{n-1})\in\mathfrak{S}_{m_0}\times\ldots\times\mathfrak{S}_{m_{n-1}}} f(\{x_{i,\sigma_i(1)},\ldots,x_{i,\sigma_i(m_i)}\}_{i\in[n]}).$$

This endows $\mathbb{S}$ with a structure of an associative unital algebra with the unit $\mathbf{1}\in\mathbb{S}_{(0,\ldots,0)}$. We will be interested only in a certain subspace of $\mathbb{S}$, defined by the *pole* and *wheel conditions*:

- We say that $F\in\mathbb{S}_{\overline{k}}$ satisfies the *pole conditions* if and only if

$$F=\frac{f(x_{0,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i\in[n]}\prod_{r\le k_i}^{r'\le k_{i+1}}(x_{i,r}-x_{i+1,r'})},\ \text{where } f\in(\mathbb{C}[x_{i,r}^{\pm1}]_{i\in[n]}^{1\le r\le k_i})^{\prod\mathfrak{S}_{k_i}}.$$

- We say that $F\in\mathbb{S}_{\overline{k}}$ satisfies the *wheel conditions* if and only if

$$F(\{x_{i,r}\})=0 \text{ once } x_{i,r_1}/x_{i+\epsilon,l}=qd^{\epsilon} \text{ and } x_{i+\epsilon,l}/x_{i,r_2}=qd^{-\epsilon} \text{ for some } \epsilon,i,r_1,r_2,l,$$

  where $\epsilon\in\{\pm1\}$, $i\in[n]$, $1\le r_1,r_2\le k_i$, $1\le l\le k_{i+\epsilon}$ and we use the cyclic notation $x_{n,l}:=x_{0,l}$, $k_n:=k_0$, $x_{-1,l}:=x_{n-1,l}$, $k_{-1}:=k_{n-1}$ as before.

Let $S_{\overline{k}}\subset\mathbb{S}_{\overline{k}}$ be the subspace of all elements F satisfying the above two conditions and set

$$S:=\bigoplus_{\overline{k}\in\mathbb{N}^{[n]}}S_{\overline{k}}.$$

Further $S_{\overline{k}}=\oplus_{r\in\mathbb{Z}}S_{\overline{k},r}$ with $S_{\overline{k},r}:=\{F\in S_{\overline{k}}|\mathrm{tot.deg}(F)=r\}$. The following is straightforward.

Lemma 2.14 *The subspace $S\subset\mathbb{S}$ is $\star$-closed.*

Now we are ready to introduce one of the key actors of this paper:

Definition 2.15 The algebra $(S,\star)$ is called the shuffle algebra (of $A_{n-1}^{(1)}$-type).

Recall the subalgebra $\ddot{U}^+$ of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ from Section 2.2. By standard results,[1] $\ddot{U}^+$ is generated by $\{e_{i,k}\}_{i\in[n]}^{k\in\mathbb{Z}}$ with the defining relations (T2, T7.1). We equip the algebra $\ddot{U}^+$ with the $\mathbb{N}^{[n]}\times\mathbb{Z}$–grading by assigning $\deg(e_{i,k})=(1_i;k)$, where $1_i\in\mathbb{N}^{[n]}$ is the vector with the i-th coordinate 1 and all other coordinates being zero.

The following result is straightforward:

Proposition 2.16 *There exists a unique algebra homomorphism $\Psi\colon\ddot{U}^+\to\mathbb{S}$ such that $\Psi(e_{i,k})=x_{i,1}^k$ for any $i\in[n]$, $k\in\mathbb{Z}$. In particular,* $\mathrm{Im}(\Psi)\subset S$.

The following beautiful result was recently proved by A. Negut:

[1] See [14, Theorem 4.2.1] for the case of finite quantum groups, [13, Theorem 3.2] for the case of quantum affine algebras, and [22, Proposition 1.3] for the case of quantum toroidal of $\mathfrak{gl}_1$.

Theorem 2.17 [20, Theorem 1.1] *The homomorphism $\Psi\colon \ddot{U}^+ \to S$ is an isomorphism of $\mathbb{N}^{[n]} \times \mathbb{Z}$-graded algebras.*

2.8 Shuffle Bimodules

Following the ideas of [9], we introduce three families of S-bimodules.

- *Shuffle modules $S_{1,p}(u)$.*
  For $u \in \mathbb{C}^\times$ and $0 \le p \le n-1$, consider an $\mathbb{N}^{[n]}$-graded $\mathbb{C}$-vector space

$$S_{1,p}(u) = \bigoplus_{\overline{k}=(k_0,\ldots,k_{n-1})\in\mathbb{N}^{[n]}} S_{1,p}(u)_{\overline{k}},$$

where the degree $\overline{k}$ component $S_{1,p}(u)_{\overline{k}}$ consists of $\prod \mathfrak{S}_{k_i}$-symmetric rational functions F in the variables $\{x_{i,r}\}_{i\in[n]}^{1\le r\le k_i}$ satisfying the following three conditions:

(i) *Pole conditions*, that is,

$$F = \frac{f(x_{0,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i\in[n]}\prod_{r\le k_i}^{r'\le k_{i+1}}(x_{i,r}-x_{i+1,r'})\cdot\prod_{r=1}^{k_p}(x_{p,r}-u)}, \text{ where } f \in (\mathbb{C}[x_{i,r}^{\pm1}]_{i\in[n]}^{1\le r\le k_i})^{\prod\mathfrak{S}_{k_i}}.$$

(ii) *First kind wheel conditions*, that is,

$$F(\{x_{i,r}\})=0 \text{ once } x_{i,r_1}/x_{i+\epsilon,l}=qd^{\epsilon} \text{ and } x_{i+\epsilon,l}/x_{i,r_2}=qd^{-\epsilon} \text{ for some } \epsilon, i, r_1, r_2, l,$$

where $\epsilon \in \{\pm1\}$, $i \in [n]$, $1 \le r_1, r_2 \le k_i$, $1 \le l \le k_{i+\epsilon}$ and we use the cyclic notation.

(iii) *Second kind wheel conditions*, that is,

$$f(\{x_{i,r}\}) = 0 \text{ once } x_{p,r_1} = u \text{ and } x_{p,r_2} = q^2u \text{ for some } 1 \le r_1, r_2 \le k_p,$$

where $f(\{x_{i,r}\}) := \prod_{r=1}^{k_p}(x_{p,r}-u)\cdot F(\{x_{i,r}\})$.

Fix $\bar{c} \in (\mathbb{C}^\times)^{[n]}$. Given $F \in S_{\overline{k}}$ and $G \in S_{1,p}(u)_{\bar{l}}$, we define $F \star G, G \star F \in S_{1,p}(u)_{\overline{k}+\bar{l}}$ by

$$(F\star G)(x_{0,1},\ldots,x_{0,k_0+l_0};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+l_{n-1}}) := \prod_{i\in[n]} c_i^{k_i}\times$$
$$\mathrm{Sym}_{\prod\mathfrak{S}_{k_i+l_i}}\left(F\left(\{x_{i,r}\}_{i\in[n]}^{r\le k_i}\right)G\left(\{x_{i',r'}\}_{i'\in[n]}^{r'>k_{i'}}\right)\prod_{i\in[n]}^{i'\in[n]}\prod_{r\le k_i}^{r'>k_{i'}}\omega_{i,i'}(x_{i,r}/x_{i',r'})\prod_{r=1}^{k_p}\phi(x_{p,r}/u)\right) \tag{3}$$

and

$$(G\star F)(x_{0,1},\ldots,x_{0,k_0+l_0};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+l_{n-1}}) :=$$
$$\mathrm{Sym}_{\prod\mathfrak{S}_{k_i+l_i}}\left(G\left(\{x_{i,r}\}_{i\in[n]}^{r\le l_i}\right)F\left(\{x_{i',r'}\}_{i'\in[n]}^{r'>l_{i'}}\right)\prod_{i\in[n]}^{i'\in[n]}\prod_{r\le l_i}^{r'>l_{i'}}\omega_{i,i'}(x_{i,r}/x_{i',r'})\right). \tag{4}$$

These formulas endow $S_{1,p}(u)$ with a structure of an S-bimodule.

Identifying S with $'\ddot{U}^+ \simeq \ddot{U}^+$ via Ψ (see Theorem 2.17), we get two commuting $'\ddot{U}^+$-actions on $S_{1,p}(u)$. Our next result extends one of these to an action of the entire algebra $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

Proposition 2.18 *The following formulas define an action of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ on $S_{1,p}(u)$:*

$$\pi^p_{u,\bar{c}}(q^{d_2})G = q^{-\frac{p(n-p)}{2}+|\bar{k}|}\cdot G,\ \pi^p_{u,\bar{c}}(e_{i,k})G = x_i^k \star G,$$

$$\pi^p_{u,\bar{c}}(h_{i,0})G = (2k_i - k_{i-1} - k_{i+1} - \delta_{i,p})\cdot G,$$

$$\pi^p_{u,\bar{c}}(h_{i,l})G = \left(\frac{1}{l}\sum_{i'\in[n]}\sum_{r'=1}^{k_{i'}}[la_{i,i'}]_q d^{-lm_{i,i'}} x^l_{i',r'} - \delta_{i,p}\frac{[l]_q}{l}q^l u^l\right)\cdot G \text{ for } l\neq 0,$$

$$\pi^p_{u,\bar{c}}(f_{i,k})G = \frac{k_i c_i^{-1}}{q^{-1}-q}\left(\underset{z=0}{\mathrm{Res}}+\underset{z=\infty}{\mathrm{Res}}\right)\frac{z^k G(\{x_{i',r'}\}|_{x_{i,k_i}\mapsto z})}{\prod_{i'}\prod_{r'=1}^{k_{i'}-\delta_{i,i'}}\omega_{i',i}(\frac{x_{i',r'}}{z})}\frac{dz}{z}.$$

Here $k\in\mathbb{Z}$, $\bar{c}=(c_0,\ldots,c_{n-1})\in(\mathbb{C}^\times)^{[n]}$, $G\in S_{1,p}(u)_{\bar{k}}$ and $|\bar{k}|:=\sum_{i\in[n]}k_i$.

Remark 2.19 Formulas of Proposition 2.18 can be equivalently written in the following form

$$\pi^p_{u,\bar{c}}(q^{d_2})G = q^{-\frac{p(n-p)}{2}+|\bar{k}|}\cdot G,\ \pi^p_{u,\bar{c}}(e_i(z))G = \delta\left(\frac{x_i}{z}\right)\star G, \tag{5}$$

$$\pi^p_{u,\bar{c}}(\psi_i^\pm(z))G = \left(\prod_{r=1}^{k_i}\frac{q^2z-x_{i,r}}{z-q^2x_{i,r}}\cdot\prod_{r=1}^{k_{i+1}}\frac{z-qdx_{i+1,r}}{qz-dx_{i+1,r}}\cdot\prod_{r=1}^{k_{i-1}}\frac{dz-qx_{i-1,r}}{qdz-x_{i-1,r}}\cdot\phi(z/u)^{\delta_{i,p}}\right)^\pm\cdot G, \tag{6}$$

$$\pi^p_{u,\bar{c}}(f_i(z))G = \frac{k_ic_i^{-1}}{q^{-1}-q}\cdot\left\{\left(\frac{G(\{x_{i',r'}\}|_{x_{i,k_i}\mapsto z})}{\prod_{i'}\prod_{r'=1}^{k_{i'}-\delta_{i,i'}}\omega_{i',i}(\frac{x_{i',r'}}{z})}\right)^+ - \left(\frac{G(\{x_{i',r'}\}|_{x_{i,k_i}\mapsto z})}{\prod_{i'}\prod_{r'=1}^{k_{i'}-\delta_{i,i'}}\omega_{i',i}(\frac{x_{i',r'}}{z})}\right)^-\right\}, \tag{7}$$

where $g(z)^\pm$ denotes the expansion of a rational function $g(z)$ in $z^{\mp 1}$, respectively.

Proof of Proposition 2.18 We need to check the compatibility of the given assignment $\pi^p_{u,\bar{c}}$ with the defining relations (T0.1–T7.2). The only nontrivial of those are (T3, T4, T6, T7.2). To check (T3, T6), we use formulas (6, 7) together with an obvious identity $\frac{\omega_{i,j}(z/w)}{\omega_{j,i}(w/z)} = g_{a_{i,j}}(d^{m_{i,j}}z/w)$ for any $i,j\in[n]$. The verification of (T7.2) boils down to the identity

$$\underset{z_1,z_2}{\mathrm{Sym}}\left(\frac{\omega_{i,i}(z_1/z_2)^{-1}}{\omega_{i,i\pm1}(z_1/w)\omega_{i,i\pm1}(z_2/w)} - \frac{(q+q^{-1})\omega_{i,i}(z_1/z_2)^{-1}}{\omega_{i,i\pm1}(z_1/w)\omega_{i\pm1,i}(w/z_2)} + \frac{\omega_{i,i}(z_1/z_2)^{-1}}{\omega_{i\pm1,i}(w/z_1)\omega_{i\pm1,i}(w/z_2)}\right) = 0.$$

Finally, the verification of (T4) is based on the observation that the $k_i+1-\delta_{i,j}$ different summands from the symmetrization appearing in $\pi^p_{u,\bar{c}}(e_i(z))\pi^p_{u,\bar{c}}(f_j(w))G$ cancel the $k_i+1-\delta_{i,j}$ terms (out of k_i+1) from the symmetrization appearing in $\pi^p_{u,\bar{c}}(f_j(w))\pi^p_{u,\bar{c}}(e_i(z))G$. □

- *Shuffle modules $S(\underline{u})$.*

The above construction admits a "higher rank" generalization. For any $\bar{r} \in \mathbb{N}^{[n]}$, consider

$$\underline{u} = (u_{0,1}, \ldots, u_{0,l_0}; \ldots; u_{n-1,1}, \ldots, u_{n-1,l_{n-1}}) \text{ with } u_{i,s} \in \mathbb{C}^{\times}.$$

Define $S(\underline{u}) = \oplus_{\overline{k} \in \mathbb{N}^{[n]}} S(\underline{u})_{\overline{k}}$ completely analogously to $S_{1,p}(u)$ with the following modifications:

(i′) *Pole conditions* for a degree $\overline{k}$ function F should read as follows:

$$F = \frac{f(x_{0,1}, \ldots, x_{n-1,k_{n-1}})}{\prod_{i\in[n]} \prod_{r\le k_i}^{r'\le k_{i+1}} (x_{i,r} - x_{i+1,r'}) \cdot \prod_{i\in[n]} \prod_{s=1}^{l_i} \prod_{r=1}^{k_i} (x_{i,r} - u_{i,s})}, \quad f \in (\mathbb{C}[x_{i,r}^{\pm 1}]_{i\in[n]}^{1\le r\le k_i})^{\prod \mathfrak{S}_{k_i}}.$$

(iii′) *Second kind wheel conditions* for such F should read as follows:

$$f(\{x_{i,r}\}) = 0 \text{ once } x_{i,r_1} = u_{i,s} \text{ and } x_{i,r_2} = q^2 u_{i,s} \text{ for some } i \in [n], 1 \le s \le l_i, 1 \le r_1, r_2 \le k_i,$$

where $f(\{x_{i,r}\}) := \prod_{i\in[n]} \prod_{s=1}^{l_i} \prod_{r=1}^{k_i} (x_{i,r} - u_{i,s}) \cdot F(\{x_{i,r}\})$.

Let us endow $S(\underline{u})$ with an S-bimodule structure by applying formulas (3) and (4) with

$$\prod_{r=1}^{k_p} \phi(x_{p,r}/u) \rightsquigarrow \prod_{i\in[n]} \prod_{s=1}^{l_i} \prod_{r=1}^{k_i} \phi(x_{i,r}/u_{i,s}).$$

The resulting left $'\ddot{U}^{+}$-action on $S(\underline{u})$ can be extended to the $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action, denoted $\pi_{\underline{u},\bar{c}}$. The latter is defined by the formulas (5–7) with the following two modifications:

$$\phi(z/u)^{\delta_{i,p}} \rightsquigarrow \prod_{s=1}^{l_i} \phi(z/u_{i,s}), \; q^{-\frac{p(n-p)}{2}} \rightsquigarrow q^{-\sum_{p=0}^{n-1} l_p \cdot \frac{p(n-p)}{2}}.$$

Let $\mathbf{1}_{\underline{u}}$ denote the element $1 \in S(\underline{u})_{(0,\ldots,0)}$. The following is obvious:

Lemma 2.20 *For $X \in {}'\ddot{U}^{+} \cdot {}'\ddot{U}^{0}$, we have $\pi_{\underline{u},\bar{c}}(X)\mathbf{1}_{\underline{u}} = 0$ for all $\underline{u}, \bar{c}$ if and only if $X = 0$.*

- *Shuffle modules $S^{K}_{1,p}(u)$ and $S^{\underline{K}}(\underline{u})$.*

Another generalization of $S_{1,p}(u)$ is provided by the S-bimodules $S^{K}_{1,p}(u)$. As a vector space, $S^{K}_{1,p}(u)$ is defined similarly to $S_{1,p}(u)$ but without imposing the second kind wheel conditions. The S-bimodule structure on $S^{K}_{1,p}(u)$ is defined by the formulas (3) and (4) with the only change

$$\phi(t) \rightsquigarrow \phi^{K}(t) := \frac{K^{-1} \cdot t - K}{t - 1}.$$

The resulting left $'\ddot{U}^{+}$-action can be extended to the $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on $S^{K}_{1,p}(u)$, denoted $\pi^{p,K}_{u,\bar{c}}$, defined by the formulas (5–7) with the only change $\phi \rightsquigarrow \phi^{K}$.

It is clear how to define the "higher rank" generalization $S^{\underline{K}}(\underline{u})$, equip it with an S-bimodule structure, and extend the resulting left $'\ddot{U}^{+}$-action to the $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action $\pi^{\underline{K}}_{\underline{u},\bar{c}}$ on $S^{\underline{K}}(\underline{u})$.

3 Identification of Representations

In this section, we establish relations between representations $\tau^p_{u,\bar{c}}$, $\pi^p_{u,\bar{c}}$, $\rho^p_{u,\bar{c}}$. As before, we assume q_1, q_2, q_3 are generic in the sense of (G).

3.1 Isomorphism $\bar{\pi}^p_{u,\bar{c}} \simeq \tau^p_{u,\bar{c}}$

Fix $0 \le p \le n-1$, $u \in \mathbb{C}^\times$, $\bar{c} \in (\mathbb{C}^\times)^{[n]}$. Recall the action $\pi^p_{u,\bar{c}}$ of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ on $S_{1,p}(u)$ from Proposition 2.18. Define

$$S' := \bigoplus_{\overline{k}\neq(0,\ldots,0)} S_{\overline{k}} \subset S.$$

Consider a $\mathbb{C}$-vector subspace

$$V_0 := S_{1,p}(u) \star S' = \mathrm{span}_{\mathbb{C}}\{G \star F \mid G \in S_{1,p}(u),\ F \in S'\} \subset S_{1,p}(u).$$

The following result is straightforward and its proof is left to the interested reader:

Lemma 3.1 *The subspace V_0 of $S_{1,p}(u)$ is invariant under the action $\pi^p_{u,\bar{c}}$ of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$.*

Let $\bar{\pi}^p_{u,\bar{c}}$ denote the corresponding $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on the factor space $\bar{S}_{1,p}(u) := S_{1,p}(u)/V_0$.

Theorem 3.2 *We have an isomorphism of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules $\bar{\pi}^p_{u,\bar{c}} \simeq \tau^p_{u,\bar{c}}$.*

Corollary 3.3 *If q_1, q_2, q_3 are generic in the sense of (G), then $\bar{\pi}^p_{u,\bar{c}}$ is irreducible.*

Proof of Theorem 3.2 By Proposition 2.7, $\tau^p_{u,\bar{c}}$ is an irreducible $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-representation generated by $|\emptyset\rangle$. Moreover, both $\bar{\mathbf{1}}_u \in \bar{S}_{1,p}(u)$ (the image of $\mathbf{1}_u$) and $|\emptyset\rangle \in F^{(p)}(u)$ are the lowest weight vectors of the same weight. Therefore, it suffices to estimate dimensions of the graded components of $\bar{S}_{1,p}(u)$:

$$\sum_{\overline{k}\in\mathbb{N}^{[n]}}^{|\overline{k}|=m} \dim \bar{S}_{1,p}(u)_{\overline{k}} = p(m) \ \ \forall\, m \in \mathbb{N}, \tag{$\heartsuit$}$$

where $p(m)$ stays for the number of size m partitions.

Descending filtration.

To prove ($\heartsuit$), we equip $S^m_{1,p}(u) := \underset{|\overline{k}|=m}{\oplus} S_{1,p}(u)_{\overline{k}}$ with a filtration $\{S^{m,\lambda}_{1,p}(u)\}_\lambda$ labeled by all size $\le m$ partitions λ. We define $S^{m,\lambda}_{1,p}(u)$ via the specialization maps ρ_λ introduced below as

$$S^{m,\lambda}_{1,p}(u) := \bigcap_{\mu\succ\lambda} \mathrm{Ker}(\rho_\mu) \subset S^m_{1,p}(u),$$

where $\succ$ denotes the lexicographic order on the set of size $\le m$ partitions.

Consider the $[n]$-coloring of the Young diagram λ by assigning $c(\Box) := p - a + b \pmod n \in [n]$ to a box $\Box = (a, b) \in \lambda$ located at the b-th row and a-th column ($1 \le b \le \lambda'_1$, $1 \le a \le \lambda_b$). Define

$$\overline{k}^\lambda := (k^\lambda_0, \ldots, k^\lambda_{n-1}) \in \mathbb{N}^{[n]}, \ \text{where } k^\lambda_i = \#\{\Box \in \lambda \mid c(\Box) = i\}.$$

Remark 3.4 We denote $\tau^p_{u,(1,\dots,1)}$ simply by τ^p_u. Note that the map $|\lambda\rangle \mapsto \prod_{\square\in\lambda} c_{c(\square)} \cdot |\lambda\rangle$ induces an isomorphism of ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-representations $\tau^p_u \xrightarrow{\sim} \tau^p_{u,\bar{c}}$ for any $\bar{c} \in (\mathbb{C}^\times)^{[n]}$.

Let us fill the boxes of λ by entering $q_1^{a-1} q_3^{b-1} u$ into the box $(a, b) \in \lambda$. For $F \in S_{1,p}(u)_{\bar{k}}$, we would like to specialize $\bar{k}^\lambda$ variables to the corresponding entries of λ. Such a naive substitution produces zeroes in numerators and denominators, so we need to modify it properly to get ρ_λ.

Specialization maps ρ_λ.

For $F \in S_{1,p}(u)_{\bar{k}}$, we set $\rho_\lambda(F) = 0$ if $\bar{k} - \bar{k}^\lambda \notin \mathbb{N}^{[n]}$. If $\bar{l} := \bar{k} - \bar{k}^\lambda \in \mathbb{N}^{[n]}$, we do the following:

- First, we consider the corner box $\square = (1, 1) \in \lambda$ of color p and specialize $x_{p,k_p} \mapsto u$. Since F has the first order pole at $x_{p,k_p} = u$, the following is well-defined:

$$\rho_\lambda^{(1)}(F) := [(x_{p,k_p} - u) \cdot F]_{|x_{p,k_p} \mapsto u}.$$

- Next, we specialize more variables to the entries of the remaining boxes from the first row and the first column. For every box $(a+1, 1) \in \lambda$ $(0 < a < \lambda_1)$ of color $p-a$, we choose an unspecified yet variable of the $(p-a)$-th family $\{x_{p-a,\cdot}\}$ and set it to $q_1^a u$. Likewise, for every box $(1, b+1) \in \lambda$ $(0 < b < \lambda'_1)$, we choose an unspecified yet variable of the $(p+b)$-th family $\{x_{p+b,\cdot}\}$ and set it to $q_3^b u$. We perform this procedure step-by-step moving from $(1, 1)$ to the right and then from $(1, 1)$ up. We denote the resulting specialization of F by $\rho_\lambda^{(\lambda_1+\lambda'_1-1)}(F)$.
- If $(2, 2) \notin \lambda$, we set $\rho_\lambda(F) := \rho_\lambda^{(\lambda_1+\lambda'_1-1)}(F)$. If λ contains $(2, 2)$, we would like to specify another variable of the p-th family, say x_{p,k_p-1}, to $q_1 q_3 u$. Due to the first kind wheel conditions, the function $\rho_\lambda^{(\lambda_1+\lambda'_1-1)}(F)$ has zero at $x_{p,k_p-1} = q_1 q_3 u$. Hence, the following is well-defined:

$$\rho_\lambda^{(\lambda_1+\lambda'_1)}(F) := \left[\frac{1}{x_{p,k_p-1} - q_1 q_3 u} \cdot \rho_\lambda^{(\lambda_1+\lambda'_1-1)}(F)\right]_{|x_{p,k_p-1} \mapsto q_1 q_3 u}.$$

- Next, we start moving from $(2, 2)$ to the right and then from $(2, 2)$ up. On each step, we specialize the corresponding $x_{\cdot,\cdot}$-variable to the prescribed entry of the diagram. However, due to the first kind wheel conditions, we have to eliminate order 1 zeros as above.
- Performing this procedure $|\lambda|$ times, we finally obtain $\rho_\lambda^{(|\lambda|)}(F) \in \mathbb{C}\left(\{x_{i,r}\}_{i\in[n]}^{1\le r\le l_i}\right)$. Set

$$\rho_\lambda(F) := \rho_\lambda^{(|\lambda|)}(F).$$

Key properties of ρ_λ.

Tracing back the contribution of the first and second kind wheel conditions, we find that

$$\rho_\lambda \colon S_{1,p}(u)_{\bar{k}^\lambda + \bar{l}} \longrightarrow S_{\bar{l}} \cdot G_{\bar{l},\lambda} := \{F' \cdot G_{\bar{l},\lambda} | F' \in S_{\bar{l}}\},$$

where

$$G_{\bar{l},\lambda}=\prod_{r=1}^{l_p}\frac{x_{p,r}-q^2u}{x_{p,r}-u}$$

$$\times\frac{\prod_{\square=(a,b)\in X_\lambda^+}\prod_{r=1}^{l_{c(\square)}}(x_{c(\square),r}-q_1^{a-1}q_3^{b-1}u)\cdot\prod_{\square=(a,b)\in X_\lambda^-}\prod_{r=1}^{l_{c(\square)}}(x_{c(\square),r}-q_1^{a-1}q_3^{b-1}u)}{\prod_{\square=(a,b)\in\lambda}\left\{\prod_{r=1}^{l_{c(\square)-1}}(x_{c(\square)-1,r}-q_1^{a-1}q_3^{b-1}u)\prod_{r=1}^{l_{c(\square)+1}}(x_{c(\square)+1,r}-q_1^{a-1}q_3^{b-1}u)\right\}}.$$

Here the set $X_\lambda^+\subset\mathbb{Z}^2$ consists of those $(a,b)\in\mathbb{Z}^2$ such that $(a+1,b)\&(a+1,b+1)\in\lambda$ or $(a,b+1)\&(a+1,b+1)\in\lambda$, while $X_\lambda^-\subset\mathbb{Z}^2$ consists of those $(a,b)\in\mathbb{Z}^2$ such that $(a-1,b)\&(a-1,b-1)\in\lambda$ or $(a,b-1)\&(a-1,b-1)\in\lambda$.

For $F\in S_{1,p}^{|\lambda|+|\bar{l}|,\lambda}(u)_{\bar{k}^\lambda+\bar{l}}$, we further have $\rho_\lambda(F)\in S_{\bar{l}}\cdot G_{\bar{l},\lambda}Q_{\bar{l},\lambda}$, where

$$Q_{\bar{l},\lambda}=\prod_{r=1}^{l_{p-\lambda_1}}(x_{p-\lambda_1,r}-q_1^{\lambda_1}u)\cdot\prod_{b\geq 1}^{\lambda_{b+1}<\lambda_b}\prod_{r=1}^{l_{p-\lambda_{b+1}+b}}(x_{p-\lambda_{b+1}+b,r}-q_1^{\lambda_{b+1}}q_3^bu).$$

Our next result establishes two crucial properties of ρ_λ.

Lemma 3.5 *(a) If $\bar{k}-\bar{k}^\lambda\notin\mathbb{N}^{[n]}$, then $\rho_\lambda(S_{1,p}(u)_{\bar{k}}\star S_{\bar{l}})=0$ for any $\bar{l}\in\mathbb{N}^{[n]}$.*

(b) We have $\rho_\lambda(S_{1,p}^{|\lambda|+|\bar{l}|,\lambda}(u)_{\bar{k}^\lambda+\bar{l}})=\rho_\lambda(S_{1,p}(u)_{\bar{k}^\lambda}\star S_{\bar{l}})$ for any $\bar{l}\in\mathbb{N}^{[n]}$.

Proof of Lemma 3.5 (a) For $F_1\in S_{1,p}(u)_{\bar{k}}$ and $F_2\in S_{\bar{l}}$, let us evaluate the ρ_λ-specialization of any summand from $F_1\star F_2$. In what follows, we say that $q_1^aq_3^bu$ gets into F_2 in the chosen summand if the $x_{\cdot,\cdot}$-variable which is specialized to $q_1^aq_3^bu$ enters F_2 rather than F_1. If u gets into F_2, we automatically get zero once we apply $\rho_\lambda^{(1)}$. A simple inductive argument shows that if at least one of the variables $\{q_1^au\}_{a=1}^{\lambda_1-1}\cup\{q_3^bu\}_{b=1}^{\lambda'_1-1}$ gets into F_2, we also obtain zero after applying $\rho_\lambda^{(a+1)}$ or $\rho_\lambda^{(\lambda_1+b)}$ since the corresponding $\omega_{\cdot,\cdot}$-factor is zero. If q_1q_3u gets into F_2, but all the entries from the first hook of λ get into F_1, then there are two zero $\omega_{\cdot,\cdot}$-factors, and so we get zero after applying $\rho_\lambda^{(\lambda_1+\lambda'_1)}$, etc. However, not all the specialized variables get into F_1 as $\bar{k}-\bar{k}^\lambda\notin\mathbb{N}^{[n]}$. Hence, the ρ_λ-specialization of this summand is zero, and so $\rho_\lambda(F_1\star F_2)=0$.

(b) For $F_1\in S_{1,p}(u)_{\bar{k}^\lambda}$, $F_2\in S_{\bar{l}}$, the specialization $\rho_\lambda(F_1\star F_2)$ is a sum of ρ_λ-specializations applied to each summand from $F_1\star F_2$. According to (a), only one such specialization is nonzero and we have $\rho_\lambda(F_1\star F_2)=\prod_{i\in[n]}\frac{k_i^{|\lambda|}!l_i!}{(k_i^{|\lambda|}+l_i)!}\rho_\lambda(F_1)\cdot F_2(\{x_{i,r}\}_{i\in[n]}^{1\leq r\leq l_i})\cdot P$, where P denotes the product of the corresponding $\omega_{\cdot,\cdot}$-factors: $P=\prod_{\square=(a,b)\in\lambda}\prod_{i\in[n]}\prod_{r=1}^{l_i}\omega_{c(\square),i}(q_1^{a-1}q_3^{b-1}u/x_{i,r})$. It is straightforward to check that $P=\nu\cdot G_{\bar{l},\lambda}Q_{\bar{l},\lambda}$ with $\nu\in\mathbb{C}^\times$. To complete the proof of (b), it remains to provide $F_1\in S_{1,p}(u)_{\bar{k}^\lambda}$ such that $\rho_\lambda(F_1)\neq 0$. To achieve this, we set $F_1=K_{\bar{\lambda}'_{\lambda_1}}\star\cdots\star K_{\bar{\lambda}'_1}\cdot\prod_{r=1}^{\bar{k}_p^\lambda}(x_{p,r}-u)^{-1}$, where $\bar{\lambda}'_r\in\mathbb{N}^{[n]}$ is prescribed by the coloring of the r-th column of λ and $K_{\overline{m}}:=\prod_{i\in[n]}\prod_{1\leq r\neq r'\leq m_i}(x_{i,r}-q^{-2}x_{i,r'})\cdot\prod_{i\in[n]}\prod_{1\leq r\leq m_i}^{1\leq r'\leq m_{i+1}}\frac{x_{i,r}-q_1x_{i+1,r'}}{x_{i,r}-x_{i+1,r'}}$. □

Proof of ($\heartsuit$).

Now we are ready to deduce ($\heartsuit$), completing our proof of Theorem 3.2. Note that

$$\dim S^m_{1,p}(u) = \sum_{\lambda:|\lambda|\le m} \dim \mathrm{gr}_\lambda(S^m_{1,p}(u)),\quad \dim \bar{S}^m_{1,p}(u) = \sum_{\lambda:|\lambda|\le m} \dim \mathrm{gr}_\lambda(\bar{S}^m_{1,p}(u)),$$

where the filtration $\{\bar{S}^{m,\lambda}_{1,p}(u)\}_\lambda$ on $\bar{S}^m_{1,p}(u)$ is induced by the filtration $\{S^{m,\lambda}_{1,p}(u)\}_\lambda$ on $S^m_{1,p}(u)$. The ρ_λ-specialization identifies $\mathrm{gr}_\lambda(S^m_{1,p}(u))$ with $\rho_\lambda(S^{m,\lambda}_{1,p}(u))$. This observation and Lemma 3.5 imply that $\mathrm{gr}_\lambda(\bar{S}^m_{1,p}(u))$ is zero if $|\lambda| < m$ and is 1-dimensional if $|\lambda| = m$. This proves ($\heartsuit$). □

3.2 Generalizations to $S(\underline{u})$ and $S^{\underline{K}}(\underline{u})$

The result of Theorem 3.2 can be generalized in both directions mentioned in Section 2.8. Recall the ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action $\pi_{\underline{u},\bar{c}}$ on the space $S(\underline{u})$, which preserves the subspace $S(\underline{u}) \star S'$ (see Lemma 3.1). Let $\bar{\pi}_{\underline{u},\bar{c}}$ denote the induced ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on $\bar{S}(\underline{u}) := S(\underline{u})/(S(\underline{u}) \star S')$. We call $\underline{u} = \{u_{i,s}\}_{i\in[n]}^{1\le s\le l_i}$ *generic* if $\{(i, u_{i,s}, (1,\ldots,1))\}$ is *generic* in the sense of Section 2.5.

Theorem 3.6 *For a generic* $\underline{u} = \{u_{i,s}\}_{i\in[n]}^{1\le s\le l_i}$*, we have an isomorphism of* ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$*-modules*

$$\bar{\pi}_{\underline{u},\bar{c}} \simeq \otimes_{i=0}^{n-1} \otimes_{s=1}^{l_i} \tau^i_{u_{i,s}}.$$

Proof of Theorem 3.6 The proof of this theorem goes along the same lines as for the case $\sum l_i = 1$ from above. According to Lemma 2.9, $\otimes_{i=0}^{n-1} \otimes_{s=1}^{l_i} \tau^i_{u_{i,s}}$ is a well-defined, irreducible, lowest weight representation generated by the lowest weight vector $|\emptyset\rangle_{\underline{u}} := \otimes_{i=0}^{n-1} \otimes_{s=1}^{l_i} |\emptyset\rangle$. On the other hand, the vector $\bar{\mathbf{1}}_{\underline{u}} \in \bar{S}(\underline{u})$ is the lowest weight vector of the same weight as $|\emptyset\rangle_{\underline{u}}$. Therefore, it suffices to compare the dimensions. This can be accomplished as above by using the specialization maps $\rho_{\underline{\lambda}}$ with $\underline{\lambda} = \{\lambda^{(0,1)},\ldots,\lambda^{(n-1,l_{n-1})}\}$ (they are defined similarly to ρ_λ, but the entry of $\square = (a,b) \in \lambda^{(i,s)}$ is set to be $q_1^{a-1}q_3^{b-1}u_{i,s}$, while its color is $c(\square) := i - a + b \pmod{n} \in [n]$). □

Another generalization of Theorem 3.2 establishes an isomorphism of $\bar{\pi}^{\underline{K}}_{\underline{u},\bar{c}}$ and tensor products of Macmahon modules for generic parameters. For simplicity of our exposition, we restrict attention to the case of $\pi^{p,K}_{u,c}$ for *generic* K (that is, $K \notin q^{\mathbb{Z}}d^{\mathbb{Z}}$). Let $\bar{\pi}^{p,K}_{u,c}$ denote the induced ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on the factor space $\bar{S}^K_{1,p}(u) := S^K_{1,p}(u)/(S^K_{1,p}(u) \star S')$.

Theorem 3.7 *We have an isomorphism of* ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$*-modules* $\bar{S}^K_{1,p}(u) \simeq M^{(p)}(u, K)$.

Proof of Theorem 3.7 We apply same proof as for Theorem 3.2, while now the filtration is parametrized by plane partitions $\bar{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots)$. Here, we fill the boxes of $\bar{\lambda}$ by entering $q_1^{a-1}q_2^{c-1}q_3^{b-1}u$ into the box $\square = (a,b) \in \lambda^{(c)}$ and define the specialization maps $\rho_{\bar{\lambda}}$ as before. Whence the arguments from our proof of Theorem 3.2 apply word by word. Note that the only place where we used the second kind wheel conditions was the appearance of the factor $\prod_{r=1}^{l_p}(x_{p,r} - q^2u)$ in $G_{\bar{l},\lambda}$. This is now compensated by a change of $Q_{\bar{l},\lambda}$–the factor which keeps track of the filtration depth. □

3.3 Isomorphism $\rho^{p,\varpi}_{v,\bar{c}} \simeq \tau^{*,p}_u$

Given a representation ρ of an algebra B on a vector space V and an algebra homomorphism $\sigma\colon A \to B$, we use ρ^σ to denote the corresponding representation of A on V: $\rho^\sigma(x) = \rho(\sigma(x))$. To simplify our notation, we define $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules $\rho^{p,\varpi}_{v,\bar{c}} := (\rho^p_{v,\bar{c}})^\varpi$ and $\tau^{*,p}_u := {}^*(\tau^p_u)$. Actually, the left dual and the right dual modules of τ^p_u are isomorphic: ${}^*(\tau^p_u) \simeq (\tau^p_u)^*$.

Theorem 3.8 *For any* $0 \le p \le n-1, v \in \mathbb{C}^\times, \bar{c} \in (\mathbb{C}^\times)^{[n]}$, *we have an isomorphism of* $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$*-modules* $\rho^{p,\varpi}_{v,\bar{c}} \simeq \tau^{*,p}_u$, *where* $u := (-1)^{\frac{(n-2)(n-3)}{2}} q^{-1} d^{-p-(n-1)\delta_{p,0}} \cdot (c_0 \cdots c_{n-1})^{-1}$.

Corollary 3.9 *For any* $0 \le p \le n-1,\ v, v' \in \mathbb{C}^\times,\ \bar{c}, \bar{c}' \in (\mathbb{C}^\times)^{[n]}$ *with* $\prod_{i\in[n]} c_i = \prod_{i\in[n]} c'_i$, *we have an isomorphism of* $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$*-representations* $\rho^p_{v,\bar{c}} \simeq \rho^p_{v',\bar{c}'}$.

Proof of Theorem 3.8 Our proof consists of three steps. First, we verify that both $v_0 \otimes e^{\bar{\Lambda}_p}$ and $|\emptyset\rangle^*$ have the same eigenvalues with respect to the *finite Cartan subalgebra* $\mathbb{C}[\psi^{\pm 1}_{0,0}, \ldots, \psi^{\pm 1}_{n-1,0}, q^{\pm d_2}]$. Second, we show that both vectors are annihilated by $e_{i,k}$-action for any $i \in [n], k \in \mathbb{Z}$. Finally, we prove that they have the same eigenvalues with respect to $\psi_{i,l}$-action for any $i \in [n], l \in \mathbb{Z}\backslash\{0\}$.

Step 1: Comparing weights with respect to $\mathbb{C}[\psi^{\pm 1}_{0,0}, \ldots, \psi^{\pm 1}_{n-1,0}, q^{\pm d_2}]$.

According to Proposition 2.12, elements $\psi_{i,0}, \gamma, q^{d_1}$ act on $v_0 \otimes e^{\bar{\Lambda}_p}$ via multiplication by $q^{\langle \bar{h}_i, \bar{\Lambda}_p\rangle}, q, 1$, respectively. Combining this with Proposition 2.6(a), we get

$$\rho^{p,\varpi}_{v,\bar{c}}(q^{d_2}) v_0 \otimes e^{\bar{\Lambda}_p} = q^{\frac{p(n-p)}{2}} \cdot v_0 \otimes e^{\bar{\Lambda}_p},\ \rho^{p,\varpi}_{v,\bar{c}}(\psi_{i,0}) v_0 \otimes e^{\bar{\Lambda}_p} = q^{\delta_{i,p}} \cdot v_0 \otimes e^{\bar{\Lambda}_p}\ \ \forall\, i \in [n].$$

We also have $\tau^p_u(q^{d_2})|\emptyset\rangle = q^{-\frac{p(n-p)}{2}} \cdot |\emptyset\rangle$ and $\tau^p_u(\psi_{i,0})|\emptyset\rangle = q^{-\delta_{i,p}} \cdot |\emptyset\rangle$. Therefore, the vectors $v_0 \otimes e^{\bar{\Lambda}_p} \in \rho^{p,\varpi}_{v,\bar{c}}$ and $|\emptyset\rangle^* \in \tau^{*,p}_u$ have the same weights with respect to $\mathbb{C}[\psi^{\pm 1}_{0,0}, \ldots, \psi^{\pm 1}_{n-1,0}, q^{\pm d_2}]$.

Remark 3.10 This explains the appearance of $q^{-\frac{p(n-p)}{2}}$ in the formulas for $\tau^p_{u,\bar{c}}(q^{d_2}),\ \pi^p_{u,\bar{c}}(q^{d_2})$.

Step 2: Verifying an annihilation property with respect to $'\ddot{U}^+$.

First, we prove $\rho^{p,\varpi}_{v,\bar{c}}(e_{i,0}) v_0 \otimes e^{\bar{\Lambda}_p} = 0$ for $i \in [n]$. For $i \ne 0$, this is clear as $\langle \bar{h}_i, \bar{\Lambda}_p\rangle + 1 > 0$, while $H_{i',k} v_0 = 0$ for all $i' \in [n], k > 0$. For $i = 0$, $\varpi(e_{0,0}) = d\gamma\psi_{0,0}[\cdots[f_{1,1}, f_{2,0}]_q, \cdots, f_{n-1,0}]_q$ by Proposition 2.6(a), and the equality $\rho^{p,\varpi}_{v,\bar{c}}(e_{0,0}) v_0 \otimes e^{\bar{\Lambda}_p} = 0$ follows from our next result.

Lemma 3.11 $\rho^p_{v,\bar{c}}([\cdots[f_{1,1}, f_{2,0}]_q, \cdots, f_{n-1,0}]_q) v_0 \otimes e^{\bar{\Lambda}_p} = 0$.

Proof of Lemma 3.11 It suffices to show that any summand $f_{i_{n-1},r_{n-1}} \cdots f_{i_1,r_1}$ of the above multicommutator (here $\{i_1, \ldots, i_{n-1}\} = [n]^\times$ and $r_k = \delta_{i_k,1}$) acts trivially on $v_0 \otimes e^{\bar{\Lambda}_p}$. Since $-\langle \bar{h}_i, \bar{\Lambda}_p\rangle + 1 > 0$ for $i \ne p$, we see that $\rho^p_{v,\bar{c}}(f_{i_1,r_1}) v_0 \otimes e^{\bar{\Lambda}_p} = 0$ unless $i_1 = p \ne 1$. For $i_1 = p \ne 1$, we get $\rho^p_{v,\bar{c}}(f_{i_1,r_1}) v_0 \otimes e^{\bar{\Lambda}_p} = \pm c^{-1}_{i_1} v_0 \otimes e^{\bar{\Lambda}^{(1)}_p}$ with $\bar{\Lambda}^{(1)}_p := \bar{\Lambda}_p - \bar{\alpha}_{i_1}$. The key property of this weight is $-\langle \bar{h}_i, \bar{\Lambda}_p^{(1)}\rangle + 1 \geq 0$. In particular, $\rho^p_{v,\bar{c}}(f_{i_2,r_2})v_0 \otimes e^{\bar{\Lambda}_p^{(1)}} = 0$ unless $i_2 = p-1 \neq 1$ or $i_2 = p+1 \neq 1$. In the latter two cases, the result is $\pm c_{i_2}^{-1} v_0 \otimes e^{\bar{\Lambda}_p^{(2)}}$ with $\bar{\Lambda}_p^{(2)} := \bar{\Lambda}_p^{(1)} - \bar{\alpha}_{i_2}$ satisfying a similar property. Continuing in the same way, we finally get to the k-th place with $i_k = 1$ and $r_k = 1$. As $-\langle \bar{h}_1, \bar{\Lambda}_p^{(k-1)}\rangle + 1 \geq 0$, we have $\rho^p_{v,\bar{c}}(f_{i_k,r_k} \cdots f_{i_1,r_1})v_0 \otimes e^{\bar{\Lambda}_p} = 0$. □

This completes our proof of the equality $\rho^{p,\varpi}_{v,\bar{c}}(e_{i,0})v_0 \otimes e^{\bar{\Lambda}_p} = 0$ for any $i \in [n]$.

According to (‡) from the next step, we have $\rho^{p,\varpi}_{v,\bar{c}}(h_{j,\pm 1})v_0 \otimes e^{\bar{\Lambda}_p} = 0$ for $j \neq p$. Combining this formula with the relation (T5′) $[h_{j,\pm 1}, e_{i,k}] = d^{\mp m_{j,i}} \gamma^{-1/2}[a_{j,i}]_q \cdot e_{i,k\pm 1}$, one gets

$$\rho^{p,\varpi}_{v,\bar{c}}(e_{i,k})v_0 \otimes e^{\bar{\Lambda}_p} = 0 \ \text{ for any } \ i \in [n], k \in \mathbb{Z}.$$

On the other hand, the identity $S(e_i(z)) = -\psi_i^-(\gamma^{-1/2}z)^{-1} e_i(\gamma^{-1}z)$ combined with the formulas of Proposition 2.7 imply a similar equality $\tau^{*,p}_u(e_{i,k})|\emptyset\rangle^* = 0$ for any $i \in [n], k \in \mathbb{Z}$.

Step 3: Comparing weights with respect to $'\ddot{U}^0$.

Let us now prove that both $v_0 \otimes e^{\bar{\Lambda}_p} \in \rho^{p,\varpi}_{v,\bar{c}}$ and $|\emptyset\rangle^* \in \tau^{*,p}_u$ are eigenvectors with respect to the generators $\{\psi_{i,l}\}_{i\in[n]}^{l\neq 0}$ and have the same eigenvalues. By the definition of τ^p_u, we have

$$\tau^{*,p}_u(\psi_i^{\pm}(z))|\emptyset\rangle^* = \phi(z/u)^{-\delta_{i,p}}|\emptyset\rangle^* \Rightarrow \tau^{*,p}_u(\psi_{i,\pm r})|\emptyset\rangle^* = \pm\delta_{i,p}(q-q^{-1})(q^2u)^{\pm r}|\emptyset\rangle^*$$

for any $i \in [n], r \in \mathbb{Z}_{>0}$. Therefore, it remains to show

$$\rho^{p,\varpi}_{v,\bar{c}}(\psi_{i,\pm r})v_0 \otimes e^{\bar{\Lambda}_p} = \pm\delta_{i,p}(q-q^{-1})(q^2u)^{\pm r} v_0 \otimes e^{\bar{\Lambda}_p} \text{ for any } i \in [n], r \in \mathbb{Z}_{>0}. \quad (‡)$$

Our proof of (‡) is based on the following technical result.

Lemma 3.12 *We have the following equalities:*

$$\rho^{p,\varpi}_{v,\bar{c}}(f_{i,0})v_0 \otimes e^{\bar{\Lambda}_p} = \delta_{i,p}c_p^{-1} \cdot \lambda^{\delta_{p,0}} \cdot v_0 \otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}, \quad (8)$$

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0 \otimes e^{\bar{\Lambda}_p} = \delta_{i,p}q^{-1}u^{-1} \cdot v_0 \otimes e^{\bar{\Lambda}_p}, \quad (9)$$

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{i,1})v_0 \otimes e^{\bar{\Lambda}_p} = \delta_{i,p}qu \cdot v_0 \otimes e^{\bar{\Lambda}_p}, \quad (10)$$

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,-1})v_0 \otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p} = -q^{-3}u^{-1} \cdot v_0 \otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}, \quad (11)$$

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,1})v_0 \otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p} = -q^{3}u \cdot v_0 \otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}, \quad (12)$$

where $\mathfrak{c} := \prod_{j\in[n]} c_j$, $\lambda := (-1)^{\frac{(n-2)(n-3)}{2}} v^{-1}q^{-1}d^{-1}\mathfrak{c}$, $u := (-1)^{\frac{(n-2)(n-3)}{2}} q^{-1} d^{-p-(n-1)\delta_{p,0}}\mathfrak{c}^{-1}$.

Proof of Lemma 3.12

- For $i \neq 0$, we have $\varpi(f_{i,0}) = f_{i,0}$ and $-\langle \bar{h}_i, \bar{\Lambda}_p\rangle + 1 = 1 - \delta_{i,p} \geq 0$, so that

$$\rho^{p,\varpi}_{v,\bar{c}}(f_{i,0})v_0 \otimes e^{\bar{\Lambda}_p} = \delta_{i,p}c_p^{-1} \cdot v_0 \otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}. \checkmark$$

  For $i = 0$, we apply the formula for $\varpi(f_{0,0})$ from Proposition 2.6(a) to get

$$\rho^{p,\varpi}_{v,\bar{c}}(f_{0,0})v_0 \otimes e^{\bar{\Lambda}_p} = q^{-\delta_{p,0}}d^{-1} \cdot \rho^p_{v,\bar{c}}([e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{q^{-1}} \cdots]_{q^{-1}})v_0 \otimes e^{\bar{\Lambda}_p}.$$

As $\rho^p_{v,\bar{c}}(e_{j,0})v_0 \otimes e^{\bar{\Lambda}_p} = 0$ for $j \neq 0$, we see (by rewriting the above multicommutator) that

$$\rho^p_{v,\bar{c}}([e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{q^{-1}} \cdots]_{q^{-1}})v_0 \otimes e^{\bar{\Lambda}_p} = \rho^p_{v,\bar{c}}(e_{n-1,0}) \cdots \rho^p_{v,\bar{c}}(e_{2,0})\rho^p_{v,\bar{c}}(e_{1,-1})v_0 \otimes e^{\bar{\Lambda}_p}.$$

For $p \neq 0$, the same argument as before implies $\rho^p_{v,\bar{c}}(e_{p,0}) \cdots \rho^p_{v,\bar{c}}(e_{2,0})\rho^p_{v,\bar{c}}(e_{1,-1})$ $v_0 \otimes e^{\bar{\Lambda}_p} = 0$, while for $p = 0$ we have

$$\rho^p_{v,\bar{c}}(e_{n-1,0}) \cdots \rho^p_{v,\bar{c}}(e_{2,0})\rho^p_{v,\bar{c}}(e_{1,-1})v_0 \otimes e^{\bar{\Lambda}_p} = v^{-1}(c_1 \cdots c_{n-1}) \cdot v_0 \otimes (e^{\bar{\alpha}_{n-1}} \cdots e^{\bar{\alpha}_1}).$$

Since $e^{\bar{\alpha}_{n-1}} \cdots e^{\bar{\alpha}_1} = (-1)^{\frac{(n-2)(n-3)}{2}} e^{-\bar{\alpha}_0}$, we finally get

$$\rho^{0,\varpi}_{v,\bar{c}}(f_{0,0})v_0 \otimes e^0 = (-1)^{\frac{(n-2)(n-3)}{2}} v^{-1}q^{-1}d^{-1}\mathfrak{c} \cdot c_0^{-1} v_0 \otimes e^{-\bar{\alpha}_0}. \checkmark$$

In what follows below, we assume $p \neq 0$.

- Combining the formula for $\varpi(h_{0,-1})$ from Proposition 2.6(c) with

$$\rho^p_{v,\bar{c}}(e_{0,1})v_0 \otimes e^{\bar{\Lambda}_p} = \rho^p_{v,\bar{c}}(e_{2,0})v_0 \otimes e^{\bar{\Lambda}_p} = \cdots = \rho^p_{v,\bar{c}}(e_{n-1,0})v_0 \otimes e^{\bar{\Lambda}_p} = 0,$$

we get

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{0,-1})v_0 \otimes e^{\bar{\Lambda}_p} = (-1)^n d^{n-1} \cdot \rho^p_{v,\bar{c}}(e_{0,1})\rho^p_{v,\bar{c}}(e_{n-1,0}) \cdots \rho^p_{v,\bar{c}}(e_{2,0})\rho^p_{v,\bar{c}}(e_{1,-1})v_0 \otimes e^{\bar{\Lambda}_p}.$$

The latter is zero, since $\rho^p_{v,\bar{c}}(e_{p-1,0}) \cdots \rho^p_{v,\bar{c}}(e_{1,-1})v_0 \otimes e^{\bar{\Lambda}_p} = \pm v^{-1}c_1 \cdots c_{p-1} \cdot v_0 \otimes e^{\bar{\Lambda}_p + \bar{\alpha}_1 + \cdots + \bar{\alpha}_{p-1}}$ and $\langle \bar{h}_p, \bar{\Lambda}_p + \bar{\alpha}_1 + \cdots + \bar{\alpha}_{p-1} \rangle + 1 > 0$. Thus $\rho^{p,\varpi}_{v,\bar{c}}(h_{0,-1})v_0 \otimes e^{\bar{\Lambda}_p} = 0$ for $p \neq 0$. $\checkmark$

For $i \neq 0$, the formula for $\varpi(h_{i,-1})$ combined with $\rho^p_{v,\bar{c}}(e_{j,0})v_0 \otimes e^{\bar{\Lambda}_p} = 0$ $(j \neq 0)$ implies

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0 \otimes e^{\bar{\Lambda}_p} = (-1)^{i+1} d^i \rho^p_{v,\bar{c}}(e_{i,0}) \cdots \rho^p_{v,\bar{c}}(e_{1,0})\rho^p_{v,\bar{c}}(e_{i+1,0}) \cdots \rho^p_{v,\bar{c}}(e_{n-1,0})\rho^p_{v,\bar{c}}(e_{0,0})v_0 \otimes e^{\bar{\Lambda}_p}.$$

If $i < p$, then $\langle \bar{h}_p, \bar{\Lambda}_p + \bar{\alpha}_0 + \sum_{j=p+1}^{n-1} \bar{\alpha}_j \rangle + 1 > 0$ and so

$$\rho^p_{v,\bar{c}}(e_{p,0}) \cdots \rho^p_{v,\bar{c}}(e_{n-1,0})\rho^p_{v,\bar{c}}(e_{0,0})v_0 \otimes e^{\bar{\Lambda}_p} = 0 \Rightarrow \rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0 \otimes e^{\bar{\Lambda}_p} = 0.$$

If $i > p$, then $\langle \bar{h}_p, \bar{\Lambda}_p + \bar{\alpha}_0 + \sum_{j=i+1}^{n-1} \bar{\alpha}_j + \sum_{j=1}^{p-1} \bar{\alpha}_j \rangle + 1 > 0$ and hence $\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0 \otimes e^{\bar{\Lambda}_p} = 0$. If $i = p$, then we get

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,-1})v_0 \otimes e^{\bar{\Lambda}_p} = (-1)^{p+1} d^p (c_0 \cdots c_{n-1}) v_0 \otimes (e^{\bar{\alpha}_p} \cdots e^{\bar{\alpha}_1} e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}) =$$
$$(-1)^{\frac{(n-2)(n-3)}{2}} d^p \mathfrak{c} \cdot v_0 \otimes e^{\bar{\Lambda}_p} \ \text{ as } \ e^{\bar{\alpha}_p} \cdots e^{\bar{\alpha}_1} e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} = (-1)^{\frac{n(n-1)}{2}+p} \cdot e^0. \checkmark$$

- Due to similar arguments (though this case is a bit more tedious), we have $\rho^{p,\varpi}_{v,\bar{c}}(h_{i,1})v_0 \otimes e^{\bar{\Lambda}_p} = 0$ if $i \neq p$. For $i = p$, we get

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,1})v_0 \otimes e^{\bar{\Lambda}_p} = (-1)^{n+p+1} d^{-p} (c_0^{-1} \cdots c_{n-1}^{-1}) v_0 \otimes (e^{-\bar{\alpha}_0} e^{-\bar{\alpha}_{n-1}} \cdots e^{-\bar{\alpha}_{p+1}} e^{-\bar{\alpha}_1} \cdots e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}) =$$
$$(-1)^{\frac{(n-2)(n-3)}{2}} d^{-p} \mathfrak{c}^{-1} \cdot v_0 \otimes e^{\bar{\Lambda}_p} \ \text{ as } \ e^{-\bar{\alpha}_0} e^{-\bar{\alpha}_{n-1}} \cdots e^{-\bar{\alpha}_{p+1}} e^{-\bar{\alpha}_1} \cdots e^{-\bar{\alpha}_p} = (-1)^{\frac{n(n+1)}{2}+p} \cdot e^0. \checkmark$$

- Arguing as above, only one summand of the corresponding multicommutator acts nontrivially:

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,-1})v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p} =$$

$$(-1)^{p+1}d^p(-q^{-2})\rho^p_{v,\bar{c}}(e_{p-1,0})\cdots\rho^p_{v,\bar{c}}(e_{1,0})\rho^p_{v,\bar{c}}(e_{p+1,0})\cdots\rho^p_{v,\bar{c}}(e_{0,0})\rho^p_{v,\bar{c}}(e_{p,0})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p} = (-1)^{1+\frac{(n-2)(n-3)}{2}}d^pq^{-2}\mathfrak{c}\cdot v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}. \checkmark$$

- Arguing as above, only one summand of the corresponding multicommutator acts nontrivially:

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,1})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p} =$$
$$(-1)^{p+1}d^{-p}(-q^2)\rho^p_{v,\bar{c}}(f_{p,0})\rho^p_{v,\bar{c}}(f_{0,0})\cdots\rho^p_{v,\bar{c}}(f_{p+1,0})\rho^p_{v,\bar{c}}(f_{1,0})\cdots\rho^p_{v,\bar{c}}(f_{p-1,0})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p} =$$
$$(-1)^{1+\frac{(n-2)(n-3)}{2}}d^{-p}q^2\mathfrak{c}^{-1}\cdot v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}. \checkmark$$

The proofs of (9–12) for $p=0$ are analogous and are left to the interested reader. □

Proof of (‡).
Note that $\rho^{p,\varpi}_{v,\bar{c}}(e_{p,0})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p} = c_p\lambda^{-\delta_{p,0}}\cdot v_0\otimes e^{\bar{\Lambda}_p}$. Combining this with the identity $[h_{p,\pm 1}, e_{p,\pm r}] = \gamma^{-1/2}(q+q^{-1})e_{p,\pm(r+1)}$ and the equalities (9–12) of Lemma 3.12, we get

$$\rho^{p,\varpi}_{v,\bar{c}}(e_{p,\pm r})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p} = c_p\lambda^{-\delta_{p,0}}(q^2u)^{\pm r}\cdot v_0\otimes e^{\bar{\Lambda}_p} \text{ for } r\in\mathbb{Z}_{>0}.$$

On the other hand, we have

$$\rho^{p,\varpi}_{v,\bar{c}}(\psi_{i,\pm r}) = \pm(q-q^{-1})[\rho^{p,\varpi}_{v,\bar{c}}(e_{i,\pm r}), \rho^{p,\varpi}_{v,\bar{c}}(f_{i,0})] \text{ for } r\in\mathbb{Z}_{>0}.$$

Since $\rho^{p,\varpi}_{v,\bar{c}}(e_{i,\pm r})v_0\otimes e^{\bar{\Lambda}_p} = \rho^{p,\varpi}_{v,\bar{c}}(f_{i,0})v_0\otimes e^{\bar{\Lambda}_p} = 0$ for $i\neq p$, we get $\rho^{p,\varpi}_{v,\bar{c}}(\psi_{i,\pm r})v_0\otimes e^{\bar{\Lambda}_p} = 0$ if $i\neq p$. The equality (‡) follows now from

$$\rho^{p,\varpi}_{v,\bar{c}}(\psi_{p,\pm r})v_0\otimes e^{\bar{\Lambda}_p} = \pm(q-q^{-1})\rho^{p,\varpi}_{v,\bar{c}}(e_{p,\pm r})\rho^{p,\varpi}_{v,\bar{c}}(f_{p,0})v_0\otimes e^{\bar{\Lambda}_p} = \pm(q-q^{-1})(q^2u)^{\pm r}v_0\otimes e^{\bar{\Lambda}_p}.$$

The irreducibility of $\rho^p_{v,\bar{c}}$ and τ^p_u (which is guaranteed by the assumption (G), see Propositions 2.7, 2.12) implies that both $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-representations $\rho^{p,\varpi}_{v,\bar{c}}$ and $\tau^{*,p}_u$ are irreducible. Moreover, they are generated by the vectors $v_0\otimes e^{\bar{\Lambda}_p}$ and $|\emptyset\rangle^*$, which are the highest weight vectors of the same weight, due to Steps 1–3. Theorem 3.8 follows. □

4 Matrix Elements of L Operators

In this section, we study matrix elements of L operators associated to $\rho^p_{u,\bar{c}}$. Let us denote $\rho^p_{1,\bar{c}}$ simply by $\rho^p_{\bar{c}}$. It suffices to work only with $\rho^p_{\bar{c}}$ as $\rho^p_{u,\bar{c}}\simeq\rho^p_{\bar{c}}$ for any $u\in\mathbb{C}^\times$, due to Corollary 3.9. We provide a new realization of the S-bimodule $S_{1,p}(u)$ as a bimodule generated by $L^{p,\bar{c}}_{\emptyset,\emptyset}$.

4.1 Matrix Elements

For any $w\in W(p)^*_n$ and $v\in W(p)_n$, we consider

$$L^{p,\bar{c}}_{w,v} := \langle 1\otimes w|(1\otimes\rho^p_{\bar{c}})(R')|1\otimes v\rangle,$$

the matrix element of the universal R-matrix R' with respect to the second component. We will mainly work with the cases $v=|\emptyset\rangle := v_0\otimes e^{\bar{\Lambda}_p}\in W(p)_n$ or $w=\langle\emptyset|$–the dual of

$|\emptyset\rangle$. In what follows, we abbreviate $|\emptyset\rangle$ and $\langle\emptyset|$ simply by $\emptyset$ when they appear as indexes of matrix elements.

Lemma 4.1 *For $i \in [n]$, $r \in \mathbb{Z}_{>0}$, $v \in W(p)_n$, we have*

$$[h_{i,-r}, L^{p,\bar{c}}_{\emptyset,v}]_{q^{-r}} = (\gamma/q)^{r/2} \cdot L^{p,\bar{c}}_{\emptyset,\rho^p_{\bar{c}}(h_{i,-r})v}.$$

Proof of Lemma 4.1 We combine $\Delta(h_{i,-r}) = h_{i,-r} \otimes \gamma^{-r/2} + \gamma^{r/2} \otimes h_{i,-r}$ with $R'\Delta(h_{i,-r}) = \Delta^{\mathrm{op}}(h_{i,-r})R'$ and apply $1 \otimes \rho^p_{\bar{c}}$ to the resulting equality. Comparing the matrix elements between $\langle\emptyset|$ and v (with respect to the second component) recovers the claimed identity. $\square$

Our first goal is to compute explicitly $L^{p,\bar{c}}_{\emptyset,\emptyset}$. The shuffle-type formula for $L^{p,\bar{c}}_{\emptyset,\emptyset}$ was obtained in [10, Theorem 4.8(a)]. To state the result, let $\Psi^{\geq} \colon \ddot{U}'^{\geq} \xrightarrow{\sim} S^{\geq}$ be the natural extension of the isomorphism $\Psi \colon \ddot{U}'^{+} \xrightarrow{\sim} S$ from Theorem 2.17.

Theorem 4.2 *The image of $L^{p,\bar{c}}_{\emptyset,\emptyset}$ under $\Psi^{\geq}$ has the following form:*

$$\Psi^{\geq}(L^{p,\bar{c}}_{\emptyset,\emptyset}) = \sum_{N=0}^{\infty} a_{p,N}\mathfrak{c}^{-N} q^{-d_1} q^{\bar{\Lambda}_p} \Gamma^0_{p,N},$$

where $a_{p,0} = 1$, $a_{p,N} \in \mathbb{C}[q^{\pm 1}, d^{\pm 1}]$ and the shuffle elements $\Gamma^0_{p,N} \in S_{(N,\ldots,N)}$ are defined via

$$\Gamma^0_{p,N} = \frac{\prod_{i\in[n]}\prod_{1\leq r\neq r'\leq N}(x_{i,r} - q^{-2}x_{i,r'}) \cdot \prod_{i\in[n]}\prod_{r=1}^{N} x_{i,r}}{\prod_{i\in[n]}\prod_{1\leq r,r'\leq N}(x_{i,r} - x_{i+1,r'})} \cdot \prod_{r=1}^{N}\frac{x_{0,r}}{x_{p,r}}.$$

Recall the Hopf pairing $'\varphi \colon {}'\ddot{U}^{\geq} \times {}'\ddot{U}^{\leq} \to \mathbb{C}$ from Theorem 2.1(d). Clearly, the generators $h_{j,r}$ $(r \in \mathbb{Z}_{>0})$ are orthogonal to all generators of $'\ddot{U}^{\geq}$ except for $h_{i,-r}$. Moreover, we have

$$'\varphi(h_{i,-r}, h_{j,r}) = \frac{[ra_{i,j}]_q d^{rm_{i,j}}}{r(q-q^{-1})}.$$

Note that the matrix $([ra_{i,j}]_q d^{rm_{i,j}})_{i,j\in[n]}$ is nondegenerate if q, qd, qd^{-1} are not roots of unity.

Definition 4.3 Let $\{h^{\perp}_{i,r}\}_{i\in[n]}$ be the basis of $\mathrm{span}_{\mathbb{C}}\langle h_{0,-r}, \cdots, h_{n-1,-r}\rangle$, which is dual to $\{h_{i,r}\}_{i\in[n]}$ with respect to $'\varphi$. In other words, $'\varphi(h^{\perp}_{i,r}, h_{j,s}) = \delta_{i,j}\delta_{r,s}$ for any $i, j \in [n]$, $r, s \in \mathbb{Z}_{>0}$.

Our next result provides the first insight toward the elements $\Psi^{-1}(\Gamma^0_{p,N})$.

Lemma 4.4 *We have $\Psi^{-1}(\Gamma^0_{p,1}) = -(q^{-1}-q)^{-n}\varpi(h^{\perp}_{p,1})$.*

Proof of Lemma 4.4 Applying Ψ to the formulas for $\varpi(h_{k,-1})$ of Proposition 2.6(b,c), we find:

$$\Psi(\varpi(h_{k,-1})) = (q^{-1}-q)^{n-1} \cdot \frac{\prod_{i\in[n]} x_i}{\prod_{i\in[n]}(x_i - x_{i+1})} \cdot \left\{(q+q^{-1})\frac{x_0}{x_k} - d^{-1}\frac{x_0}{x_{k+1}} - d\frac{x_0}{x_{k-1}}\right\}.$$

Rewriting this as $\Psi(\varpi(h_{k,-1})) = -(q^{-1}-q)^n \sum_{p\in[n]}{}' \varphi(h_{k,-1}, h_{p,1})\Gamma^0_{p,1}$, we get the claim. $\square$

Now we are ready to state the main result of this section.

Theorem 4.5 *Given* $0 \le p \le n-1,\ \bar{c} \in (\mathbb{C}^\times)^{[n]}$, *define* $u \in \mathbb{C}^\times$ *as in Theorem 3.8 via*

$$u := (-1)^{\frac{(n-2)(n-3)}{2}} q^{-1} d^{-p-(n-1)\delta_{p,0}} \mathfrak{c}^{-1} \text{ with } \mathfrak{c} = c_0 \cdots c_{n-1}.$$

(a) *For any* $i \neq p$, *we have*

$$\varpi(e_i(z)) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(e_i(z)),$$

$$\varpi(f_i(z)) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(f_i(z)),$$

$$\varpi(\psi_i^{\pm}(z)) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(\psi_i^{\pm}(z)).$$

(b) *We have*

$$(z-u)\cdot\varpi(e_p(z)) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(e_p(z))\cdot(q^{-1}z-qu),$$

$$(q^{-1}z-qu)\cdot\varpi(f_p(z)) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(f_p(z))\cdot(z-u),$$

$$\varpi(\psi_p^{\pm}(z)) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(\psi_p^{\pm}(z)).$$

(c) *We have the following explicit formula*

$$L^{p,\bar{c}}_{\emptyset,\emptyset} = q^{-d_1} q^{\bar{\Lambda}_p} \exp\left(\sum_{r=1}^{\infty} \frac{[r]_q}{r} (qu)^r \varpi(h^{\perp}_{p,r})\right). \tag{$\sharp$}$$

Remark 4.6 An analogous computation for the representation $\tau_u^{*,p}$ is much simpler. The corresponding matrix element $L^{\tau_u^{*,p}}_{\emptyset,\emptyset} := \langle 1\otimes\emptyset|(1\otimes\tau_u^{*,p})('R)|1\otimes\emptyset\rangle$ equals

$$\langle 1\otimes\emptyset|(1\otimes\tau_u^{*,p})(q^{\frac{1}{n}(d_2-\sum_{j=1}^{n-1}\bar{\Lambda}_j)\otimes c' + c'\otimes\frac{1}{n}(d_2-\sum_{j=1}^{n-1}\bar{\Lambda}_j)+\sum_{j=1}^{n-1}\bar{\Lambda}_j\otimes h_{j,0}} \cdot \exp(\sum_{i\in[n]}\sum_{r=1}^{\infty} h^{\perp}_{i,r}\otimes h_{i,r}))|1\otimes\emptyset\rangle,$$

since $\tau_u^p('\ddot{U}^-)|\emptyset\rangle = 0$. As $\tau_u^{*,p}(h_{i,r})|\emptyset\rangle^* = \delta_{i,p}\frac{[r]_q q^r u^r}{r}|\emptyset\rangle^* (r>0)$, $\tau_u^{*,p}(h_{i,0})|\emptyset\rangle^* = \delta_{i,p}|\emptyset\rangle^*$, we get

$$L^{\tau_u^{*,p}}_{\emptyset,\emptyset} = q^{\frac{1}{n}(d_2-\sum_{j=1}^{n-1}\bar{\Lambda}_j)+\bar{\Lambda}_p} \exp\left(\sum_{r=1}^{\infty}\frac{[r]_q}{r} q^r u^r h^{\perp}_{p,r}\right).$$

In particular, $\varpi(L^{\tau_u^{*,p}}_{\emptyset,\emptyset})$ coincides with the right-hand side of ($\sharp$). However, we are not aware of the conceptual reason for $L^{p,\bar{c}}_{\emptyset,\emptyset} = \varpi(L^{\tau_u^{*,p}}_{\emptyset,\emptyset})$ (though it would immediately imply Theorem 4.5).

4.2 Proof of Theorem 4.5

Our proof is based on the equality

$$\langle 1\otimes w|(1\otimes\rho^p_{\bar{c}})(R'\Delta(x))|1\otimes v\rangle = \langle 1\otimes w|(1\otimes\rho^p_{\bar{c}})(\Delta^{\mathrm{op}}(x)R')|1\otimes v\rangle \tag{$\star$}$$

for any $x \in \ddot{U}'_{q,d}(\mathfrak{sl}_n),\ v \in W(p)_n,\ w \in W(p)^*_n$.

Notation Given a collection of elements $\beta_1, \cdots, \beta_N \in \{\pm\bar{\alpha}_0, \cdots, \pm\bar{\alpha}_{n-1}\}$ and $0 \le p \le n-1$, consider $v_0 \otimes e^{\beta_1} \cdots e^{\beta_N} e^{\bar{\Lambda}_p}$–an element of $W(p)_n$. We will also use the same notation for a dual element of $W(p)_n^*$, when writing it in the matrix coefficients of L operators.

- *Case* $p \neq 0$.

  (a) We need to show that $L^{p,\bar{c}}_{\emptyset,\emptyset}$ commutes with $\{\varpi(e_{i,k}), \varpi(f_{i,k}), \varpi(h_{i,k})\}_{i\neq p}^{k\in\mathbb{Z}}$.

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{i,0})] = 0$ *and* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{i,0})] = 0$ *for* $i \neq 0, p$.
Due to (H1), we have

$$\Delta(e_{i,k}) = e_{i,k} \otimes 1 + \psi_{i,0}^{-1}\gamma^{-k} \otimes e_{i,k} + \sum_{r>0} \psi_{i,-r}\gamma^{-k-r/2} \otimes e_{i,k+r},$$

$$\Delta(f_{i,k}) = 1 \otimes f_{i,k} + f_{i,k} \otimes \psi_{i,0}\gamma^{-k} + \sum_{r>0} f_{i,k-r} \otimes \psi_{i,r}\gamma^{-k+r/2}.$$

Evaluating both sides of ($\star$) at $v = |\emptyset\rangle$, $w = \langle\emptyset|$ and $x = e_{i,0}$ or $x = f_{i,0}$, we immediately get $[L^{p,\bar{c}}_{\emptyset,\emptyset}, e_{i,0}] = 0$ and $[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{i,0}] = 0$. It remains to use $\varpi(e_{i,0}) = e_{i,0}$, $\varpi(f_{i,0}) = f_{i,0}$ for $i \neq 0$. ✓

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,-1})] = 0$ *and* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,1})] = 0$.
Evaluating both sides of ($\star$) at $v = |\emptyset\rangle$, $w = \langle\emptyset|$ and $x = e_{0,1}$ or $x = f_{0,-1}$, we immediately get $[L^{p,\bar{c}}_{\emptyset,\emptyset}, e_{0,1}] = 0$ and $[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{0,-1}] = 0$, respectively. It remains to apply the equalities $\varpi(e_{0,-1}) = (-d)^n e_{0,1}$ and $\varpi(f_{0,1}) = (-d)^{-n} f_{0,-1}$ from Proposition 2.6(d). ✓

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(h_{i,-1})] = 0$ *for any* $i \in [n]$.

It suffices to prove $[\Psi^{\ge}(L^{p,\bar{c}}_{\emptyset,\emptyset}), \Psi^{\ge}(\varpi(h_{i,-1}))] = 0$. According to Lemma 4.4, $\Psi(\varpi(h_{i,-1}))$ is a linear combination of $\Gamma^0_{p',1}$. On the other hand, $\Psi^{\ge}(L^{p,\bar{c}}_{\emptyset,\emptyset})$ is a linear combination of $q^{-d_1}q^{\bar{\Lambda}_p}\Gamma^0_{p,N}$, due to Theorem 4.2. The commutativity of the elements $\{\Gamma^0_{p',m}\}^{m\in\mathbb{N}}_{p'\in[n]}$ has been established in [10], while $q^{-d_1}q^{\bar{\Lambda}_p}$ obviously commutes with $\Gamma^0_{p',1}$. The result follows. ✓

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(h_{i,1})] = 0$ *for* $i \neq 0, p$.
According to Proposition 2.6(b), it suffices to prove that $E = 0$, where E is defined via

$$E := [L^{p,\bar{c}}_{\emptyset,\emptyset}; [f_{i,0}, [f_{i-1,0}, \cdots, [f_{1,0}, [f_{i+1,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-2}}]_1. \tag{13}$$

In what follows, we assume $i < p \le n-1$ leaving the case $0 < p < i$ to the interested reader. Applying iteratively the q-commutator identity (mentioned in our proof of Proposition 2.6(b))

$$[a, [b, c]_u]_v = [[a, b]_x, c]_{uv/x} + x \cdot [b, [a, c]_{v/x}]_{u/x} \tag{$\diamond$}$$

together with $[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{j,0}] = 0$ for $j \neq 0, p$, we reduce to a stronger equality $E_1^{(1)} + E_2^{(1)} = 0$ with

$$E_1^{(1)} := [[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{p,0}]_{q^{-1}}, [f_{p+1,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_1,$$

$$E_2^{(1)} := q^{-1}[f_{p,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, [L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{0,0}]_q]_{q^{-1}} \cdots]_{q^{-1}}]_1.$$

Evaluating both sides of ($\star$) for appropriate v, w, x step-by-step, we obtain an explicit formula

$$E_2^{(1)} = -(-q)^{p-n} \cdot \frac{\psi_{p+1,0} \cdots \psi_{n-1,0}\psi_{0,0}}{c_p \cdots c_{n-1}c_0} \cdot L^{p,\bar{c}}_{v_0 \otimes e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}.$$

Let us now compute $E_1^{(1)}$. Evaluating both sides of ($\star$) at $v = |\emptyset\rangle,\ w = \langle\emptyset|,\ x = f_{p,0}$, we find

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{p,0}]_{q^{-1}} = -q^{-1}c_p^{-1} \cdot L^{p,\bar{c}}_{\emptyset, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}.$$

Evaluating both sides of ($\star$) at $v = v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}, w = \langle\emptyset|, x = f_{j,0}$, we find $[L^{p,\bar{c}}_{\emptyset, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}, f_{j,0}] = 0$ for $p+1 < j \leq n-1$. Applying iteratively ($\diamond$), we get $E_1^{(1)} = E_1^{(2)} + E_2^{(2)}$ with

$$E_1^{(2)} := -q^{-1}c_p^{-1}[[L^{p,\bar{c}}_{\emptyset, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}, f_{p+1,0}]_{q^{-1}}, [f_{p+2,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_1,$$

$$E_2^{(2)} := -q^{-2}c_p^{-1}[f_{p+1,0}, [f_{p+2,0}, \cdots, [f_{n-1,0}, [L^{p,\bar{c}}_{\emptyset, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}, f_{0,0}]_q]_{q^{-1}} \cdots]_{q^{-1}}]_1.$$

Evaluating both sides of ($\star$) for appropriate v, w, x step-by-step, we obtain an explicit formula

$$\begin{aligned} E_2^{(2)} &= (-q)^{p-n} \cdot \frac{\psi_{p+1,0} \cdots \psi_{n-1,0}\psi_{0,0}}{c_p \cdots c_{n-1}c_0} \cdot L^{p,\bar{c}}_{v_0 \otimes e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}} \\ &\quad - (-q)^{p-n} \cdot \frac{\psi_{p+2,0} \cdots \psi_{n-1,0}\psi_{0,0}}{c_p \cdots c_{n-1}c_0} \cdot L^{p,\bar{c}}_{v_0 \otimes e^{\bar{\alpha}_{p+2}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}, v_0 \otimes e^{-\bar{\alpha}_{p+1}} e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}. \end{aligned}$$

The first summand cancels $E_2^{(1)}$, while the second summand is very similar to $E_2^{(1)}$. Evaluating $E_1^{(2)}$, we get a similar formula $E_1^{(2)} = E_1^{(3)} + E_2^{(3)}$ with

$$\begin{aligned} E_2^{(3)} &= (-q)^{p-n} \cdot \frac{\psi_{p+2,0} \cdots \psi_{n-1,0}\psi_{0,0}}{c_p \cdots c_{n-1}c_0} \cdot L^{p,\bar{c}}_{v_0 \otimes e^{\bar{\alpha}_{p+2}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}, v_0 \otimes e^{-\bar{\alpha}_{p+1}} e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}} \\ &\quad - (-q)^{p-n} \cdot \frac{\psi_{p+3,0} \cdots \psi_{n-1,0}\psi_{0,0}}{c_p \cdots c_{n-1}c_0} \cdot L^{p,\bar{c}}_{v_0 \otimes e^{\bar{\alpha}_{p+3}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}, v_0 \otimes e^{-\bar{\alpha}_{p+2}} e^{-\bar{\alpha}_{p+1}} e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}. \end{aligned}$$

Proceeding further in the same way, we see that all nontrivial summands in the formula for E split into pairs of opposite terms. Hence, $E = 0$ and so $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(h_{i,1})] = 0$ for $i \neq 0, p$. ✓

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{i,k})] = 0$ *and* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{i,k})] = 0$ *for* $i \neq p$ *and any* $k \in \mathbb{Z}$.

Choose $j \neq 0, p$ such that $a_{j,i} \neq 0$. Combining the commutator relations

$$[\varpi(h_{j,\pm1}), \varpi(e_{i,k})] = d^{\mp m_{j,i}}[a_{j,i}]_q \cdot \varpi(e_{i,k\pm1}),\ [\varpi(h_{j,\pm1}), \varpi(f_{i,k})] = -d^{\mp m_{j,i}}[a_{j,i}]_q \cdot \varpi(f_{i,k\pm1})$$

with

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{i,-\delta_{i,0}})] = 0,\ [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{i,\delta_{i,0}})] = 0,\ [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(h_{j,\pm1})] = 0$$

established above, we get $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{i,k})] = 0$ and $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{i,k})] = 0$ by induction on k. ✓

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(\psi_{i,k})] = 0$ *for* $i \neq p$ *and any* $k \in \mathbb{Z}$.

For $k \neq 0$, this follows immediately from the defining relation (T4) and the previous step. For $k = 0$, it suffices to prove $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \psi_{i,0}] = 0$ for any $i \in [n]$. This equality follows by evaluating both sides of ($\star$) at $w = \langle\emptyset|,\ v = |\emptyset\rangle,\ x = \psi_{i,0}$. ✓

(b) The first two equalities of (b) are equivalent to the following identities:

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,k+1})]_q = q^2 u \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,k})]_{q^{-1}} \ \ \forall\, k \in \mathbb{Z}, \tag{14}$$

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,k-1})]_q = u^{-1} \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,k})]_{q^{-1}} \ \ \forall\, k \in \mathbb{Z}. \tag{15}$$

It suffices to check (14) and (15) for single values of k as we can derive the general equalities by commuting further iteratively with $\varpi(h_{p+1,\pm1})$.

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,1})]_q = q^2 u \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,0})]_{q^{-1}}$.

Evaluating both sides of ($\star$) at $v = |\emptyset\rangle$, $w = \langle\emptyset|$, $x = e_{p,0}$, we find $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,0})]_{q^{-1}} = c_p \cdot L^{p,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p},\emptyset}$. As $\varpi(e_{p,0}) = e_{p,0}$, we finally get

$$q^2 u \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,0})]_{q^{-1}} = q^2 c_p u \cdot L^{p,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p},\emptyset}.$$

To compute $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,1})]_q$, let us first evaluate $\varpi(e_{p,1})$. Due to (T5′), we have

$$[h_{p,1}, e_{p,0}] = (q+q^{-1})e_{p,1} \Rightarrow \varpi(e_{p,1}) = -(q+q^{-1})^{-1} \cdot [\varpi(e_{p,0}), \varpi(h_{p,1})],$$

where $\varpi(e_{p,0}) = e_{p,0}$ and

$$\varpi(h_{p,1}) = (-1)^{n+p} d^{-p} q^n \cdot \left[f_{p,0}, \cdots, [f_{1,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots\right]_{q^{-2}}.$$

Applying iteratively the equality ($\diamond$) together with the relation (T4), we finally get

$$\begin{aligned}\varpi(e_{p,1}) \ &= \ (-1)^{n+p+1} d^{-p} q^{n-2} \psi_{p,0} \\ &\quad [f_{p-1,0}, \cdots, \left[f_{1,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}\right]_{q^{-1}} \cdots \Big]_{q^{-1}}\end{aligned}$$

Therefore, it remains to evaluate

$$E := [L^{p,\bar{c}}_{\emptyset,\emptyset}, [f_{p-1,0}, \cdots, [f_{1,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}]_q.$$

Applying iteratively the equality ($\diamond$) together with $[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{j,0}] = 0$ for $j \neq 0, p$, we get

$$E = [f_{p-1,0}, \cdots, [f_{1,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, [L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{0,0}]_q]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}.$$

To compute this multicommutator, we apply the equality ($\star$) with an appropriate choice of v, w, x step-by-step. Leaving details to the interested reader, let us present the final formula

$$E = (-1)^n q^{3-n} \prod_{i \neq p} \frac{\psi_{i,0}}{c_i} \cdot L^{p,\bar{c}}_{v_0\otimes e^{\bar{\alpha}_{p-1}} \cdots e^{\bar{\alpha}_1} e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p},\emptyset}.$$

Since $\prod_{i\in[n]} \psi_{i,0} = 1$ in $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$, we finally get

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,1})]_q = (-1)^{\frac{(n-2)(n-3)}{2}} d^{-p} q c_p \mathfrak{c}^{-1} \cdot L^{p,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p},\emptyset},$$

where we used the following identity in $\mathbb{C}\{\bar{P}\}$

$$e^{\bar{\alpha}_{p-1}} \cdots e^{\bar{\alpha}_1} e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} = (-1)^{\frac{n(n-1)}{2}+p} e^{-\bar{\alpha}_p}.$$

The equality $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,1})]_q = q^2 u \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,0})]_{q^{-1}}$ follows. ✓

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,-1})]_q = u^{-1} \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,0})]_{q^{-1}}$.

Evaluating both sides of ($\star$) at $v = |\emptyset\rangle$, $w = \langle\emptyset|$, $x = f_{p,0}$, we find $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,0})]_{q^{-1}} = \frac{-1}{qc_p} \cdot L_{\emptyset, v_0\otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}$. As $\varpi(f_{p,0}) = f_{p,0}$, we finally get

$$u^{-1} \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,0})]_{q^{-1}} = -q^{-1} c_p^{-1} u^{-1} \cdot L^{p,\bar{c}}_{\emptyset, v_0\otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}.$$

To evaluate $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,-1})]_q$, let us first compute $\varpi(f_{p,-1})$. Due to (T6′), we have

$$[h_{p,-1}, f_{p,0}] = -(q+q^{-1}) f_{p,-1} \Rightarrow \varpi(f_{p,-1}) = (q+q^{-1})^{-1} \cdot [\varpi(f_{p,0}), \varpi(h_{p,-1})],$$

where $\varpi(f_{p,0}) = f_{p,0}$ and

$$\varpi(h_{p,-1}) = (-1)^{p+1} d^p \cdot [e_{p,0}, \cdots, [e_{1,0}, [e_{p+1,0}, \cdots, [e_{n-1,0}, e_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-2}}.$$

Applying iteratively the equality ($\diamond$) together with the relation (T4), we finally get

$$\varpi(f_{p,-1}) = (-1)^{p+1} d^p \cdot [e_{p-1,0}, \cdots, [e_{1,0}, [e_{p+1,0}, \cdots, [e_{n-1,0}, e_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}} \cdot \psi^{-1}_{p,0}.$$

Therefore, it remains to evaluate

$$E := [L^{p,\bar{c}}_{\emptyset,\emptyset}, [e_{p-1,0}, \cdots, [e_{1,0}, [e_{p+1,0}, \cdots, [e_{n-1,0}, e_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}]_q.$$

Applying iteratively the equality ($\diamond$) together with $[L^{p,\bar{c}}_{\emptyset,\emptyset}, e_{j,0}] = 0$ for $j \neq 0, p$, we get

$$E = [e_{p-1,0}, \cdots, [e_{1,0}, [e_{p+1,0}, \cdots, [e_{n-1,0}, [L^{p,\bar{c}}_{\emptyset,\emptyset}, e_{0,0}]_q]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}.$$

To compute this multicommutator, we apply the equality ($\star$) with an appropriate choice of v, w, x step-by-step. Leaving details to the interested reader, let us present the final formula

$$E = -c_p^{-1} \mathfrak{c} \cdot L^{p,\bar{c}}_{\emptyset, v_0\otimes e^{\bar{\alpha}_{p-1}} \cdots e^{\bar{\alpha}_1} e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}} \cdot \psi_{p,0}.$$

Therefore,

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,-1})]_q = (-1)^{\frac{n(n-1)}{2}} d^p c_p^{-1} \mathfrak{c} \cdot L^{p,\bar{c}}_{\emptyset, v_0\otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}.$$

The equality $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,-1})]_q = u^{-1} \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,0})]_{q^{-1}}$ follows. ✓

– *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(\psi^{\pm}_p(z))] = 0$.

Define $\widetilde{\psi}_{p,N} \in {}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ as the coefficient of z^{-N} in $\psi^+_p(z) - \psi^-_p(z)$, so that $[e_{p,a}, f_{p,b}] = \frac{\widetilde{\psi}_{p,a+b}}{q-q^{-1}}$ for any $a, b \in \mathbb{Z}$. Set $X_N := [\varpi(\widetilde{\psi}_{p,N}), L^{p,\bar{c}}_{\emptyset,\emptyset}]$. Combining the equalities

$$(\varpi(e_{p,k+1}) - u\varpi(e_{p,k}))L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset}(q^{-1}\varpi(e_{p,k+1}) - qu\varpi(e_{p,k})),$$

$$(q^{-1}\varpi(f_{p,l+1}) - qu\varpi(f_{p,l}))L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset}(\varpi(f_{p,l+1}) - u\varpi(f_{p,l})),$$

we get the following recursive relation: $q^{-1}X_{k+l+2} - u(q+q^{-1})X_{k+l+1} + u^2qX_{k+l} = 0$. As $X_{-1} = X_0 = 0$, we get $X_k = 0$ for any $k \in \mathbb{Z}$. This proves $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(\psi^{\pm}_p(z))] = 0$. ✓

(c) The unique element satisfying conditions (a,b) of Theorem 4.5 and whose shuffle interpretation has a form as in Theorem 4.2 (we only need to know that it lives in an appropriate completion and its 'purely Cartan part' equals $q^{-d_1}q^{\bar{\Lambda}_p}$) is given by the right-hand side of (♯).

- *Case* $p = 0$.

Parts (a) and (c) are proved completely analogously to the case $p \neq 0$. Since the last equality in (b) follows from the former two, it suffices to check (14) and (15) for some $k \in \mathbb{Z}$.

– *Proof of* $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,0})]_q = q^2u \cdot [L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,-1})]_{q^{-1}}$.

According to Proposition 2.6(d), we have $\varpi(e_{0,-1}) = (-d)^n e_{0,1}$. Evaluating both sides of (⋆) at $v = |\emptyset\rangle$, $w = \langle\emptyset|$, $x = e_{0,1}$, we get $[L^{0,\bar{c}}_{\emptyset,\emptyset}, e_{0,1}]_{q^{-1}} = (-1)^n c_0 \cdot L^{0,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_0},\emptyset}$. Therefore

$$q^2u \cdot [L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,-1})]_{q^{-1}} = uq^2d^nc_0 \cdot L^{0,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_0},\emptyset}.$$

Next, we evaluate the left-hand side of the claimed equality. According to Proposition 2.6(a)

$$\varpi(e_{0,0}) = d(-q)^{n-2}\gamma\psi_{0,0} \cdot [f_{n-1,0}, \cdots, [f_{2,0}, f_{1,1}]_{q^{-1}} \cdots]_{q^{-1}}.$$

Applying iteratively (◇) together with $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \psi_{0,0}] = 0$ and $[L^{0,\bar{c}}_{\emptyset,\emptyset}, f_{j,0}] = 0$ for $j \neq 0$, we get

$$[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,0})]_q = d(-q)^{n-2}\gamma\psi_{0,0} \cdot [f_{n-1,0}, \cdots, [f_{2,0}, [L^{0,\bar{c}}_{\emptyset,\emptyset}, f_{1,1}]_q]_{q^{-1}} \cdots]_{q^{-1}}.$$

Evaluating this multicommutator step-by-step as before, we finally get

$$[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,0})]_q = qdc_0\mathfrak{c}^{-1} \cdot L^{0,\bar{c}}_{v_0\otimes e^{\bar{\alpha}_{n-1}}\cdots e^{\bar{\alpha}_1},\emptyset} = (-1)^{\frac{(n-2)(n-3)}{2}} qdc_0\mathfrak{c}^{-1} \cdot L^{0,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_0},\emptyset}.$$

The equality $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,0})]_q = q^2u \cdot [L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,-1})]_{q^{-1}}$ follows. ✓

– *Proof of* $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,0})]_q = u^{-1}[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,1})]_{q^{-1}}$.

According to Proposition 2.6(d), we have $\varpi(f_{0,1}) = (-d)^{-n} f_{0,-1}$. Evaluating both sides of ($\star$) at $v = |\emptyset\rangle,\ w = \langle\emptyset|,\ x = f_{0,-1}$, we get $[L^{0,\bar{c}}_{\emptyset,\emptyset}, f_{0,-1}]_{q^{-1}} = \frac{(-1)^{n+1}}{qc_0} \cdot L^{0,\bar{c}}_{\emptyset, v_0\otimes e^{-\bar{\alpha}_0}}$. Hence

$$u^{-1} \cdot [L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,1})]_{q^{-1}} = -q^{-1} d^{-n} c_0^{-1} u^{-1} \cdot L^{0,\bar{c}}_{\emptyset, v_0\otimes e^{-\bar{\alpha}_0}}.$$

Let us now evaluate the left-hand side of the claimed equality. According to Proposition 2.6(a)

$$\varpi(f_{0,0}) = d^{-1} \cdot [e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{q^{-1}} \cdots]_{q^{-1}} \cdot \psi_{0,0}^{-1} \gamma^{-1}.$$

Applying iteratively ($\diamond$) together with $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \psi_{0,0}] = 0$ and $[L^{0,\bar{c}}_{\emptyset,\emptyset}, e_{j,0}] = 0$ for $j \neq 0$, we get

$$[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,0})]_q = d^{-1} \cdot [e_{n-1,0}, \cdots [e_{2,0}, [L^{0,\bar{c}}_{\emptyset,\emptyset}, e_{1,-1}]_q]_{q^{-1}} \cdots]_{q^{-1}} \cdot \psi_{0,0}^{-1} \gamma^{-1}.$$

Evaluating this multicommutator step-by-step as before, we finally get

$$[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,0})]_q = -d^{-1} c_0^{-1} \mathfrak{c} \cdot L^{0,\bar{c}}_{\emptyset, v_0\otimes e^{\bar{\alpha}_{n-1}} \cdots e^{\bar{\alpha}_1}} = -(-1)^{\frac{(n-2)(n-3)}{2}} d^{-1} c_0^{-1} \mathfrak{c} \cdot L^{0,\bar{c}}_{\emptyset, v_0\otimes e^{-\bar{\alpha}_0}}.$$

The equality $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,0})]_q = u^{-1}[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,1})]_{q^{-1}}$ follows. ✓

This completes our proof of Theorem 4.5 for any $p \in [n]$.

4.3 Bimodule $\mathcal{S}(p, \bar{c})$

Let $'\ddot{U}^{\geq,\wedge}$ (resp. $'\ddot{U}^{+,\wedge}$) be the completion of $'\ddot{U}^{\geq}$ (resp. $'\ddot{U}^{+}$) with respect to the $\mathbb{Z}$–grading on $'\ddot{U}^{\geq}$ (resp. $'\ddot{U}^{+}$) defined by assigning $\deg(e_{i,k}) = -k$, $\deg(h_{i,k}) = -k$, $\deg(q^{d_2}) = 0$. Note that $L^{p,\bar{c}}_{\emptyset,\emptyset} \in \varpi('\ddot{U}^{\geq,\wedge})$, due to Theorem 4.5. Consider the $'\ddot{U}^{+}$-bimodule $\mathcal{S}(p, \bar{c})$ defined as

$$\mathcal{S}(p, \bar{c}) := \varpi('\ddot{U}^{+}) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi('\ddot{U}^{+}) \subset \varpi('\ddot{U}^{\geq,\wedge}),$$

where both $'\ddot{U}^{+}$-actions are in conjunction with ϖ. We conclude this section with the following result analogous to [9, Lemma 4.4].

Proposition 4.7 *There exists an isomorphism of $'\ddot{U}^{+}$-bimodules*

$$\iota\colon S_{1,p}(u) \xrightarrow{\sim} \mathcal{S}(p, \bar{c}) \text{ with } \mathbf{1} \mapsto L^{p,\bar{c}}_{\emptyset,\emptyset},$$

where $u = (-1)^{\frac{(n-2)(n-3)}{2}} q^{-1} d^{-p-(n-1)\delta_{p,0}} (c_0 \cdots c_{n-1})^{-1}$ *as before.*

Proof of Proposition 4.7 Any element $H \in S_{1,p}(u)$ can be written as $H = \sum_l F_l \star \mathbf{1} \star G_l$ with $F_l, G_l \in S$, due to Theorem 3.2. Set $\iota(H) := \sum_l \varpi(a_l) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(b_l)$, where $a_l := \Psi^{-1}(F_l), b_l := \Psi^{-1}(G_l) \in {'\ddot{U}^{+}}$. We must show that ι is well-defined. Applying Theorem 4.5(a, b), we find

$$\varpi(e_{i,k}) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(\widetilde{e}_{i,k}), \text{ where } \widetilde{e}_{i,k} = \begin{cases} e_{i,k} & \text{if } i \neq p \\ q^{-1} e_{i,k} + (q^{-1}-q) \sum_{r=1}^{\infty} u^r \cdot e_{i,k-r} & \text{if } i = p \end{cases}.$$

Let ϱ be the automorphism of $'\ddot{U}^{+,\wedge}$ such that $\varrho(e_{i,k}) = \widetilde{e}_{i,k}$. Extending Ψ to an isomorphism of completions $\Psi\colon {}'\ddot{U}^{+,\wedge} \xrightarrow{\sim} S^\wedge$, we use $\widetilde{\varrho}$ to denote the induced automorphism of $S^\wedge$. Clearly $\widetilde{\varrho}(\Psi(X)) = \Psi(\varrho(X))$ and $Y \star \mathbf{1} = \mathbf{1} \star \widetilde{\varrho}(Y)$ for any $X \in' \ddot{U}^{+,\wedge}$, $Y \in S^\wedge$. Therefore

$$\sum_l F_l \star \mathbf{1} \star G_l = 0 \Leftrightarrow \sum_l \widetilde{F}_l G_l = 0 \Leftrightarrow \sum_l \widetilde{a}_l b_l = 0 \Leftrightarrow \sum_l \varpi(a_l) \cdot L^{p,\bar{c}}_{\emptyset,\emptyset} \cdot \varpi(b_l) = 0.$$

Thus, the linear map $\iota\colon S_{1,p}(u) \to \mathcal{S}(p,\bar{c})$ is well-defined and injective. It is clear that ι is surjective and is an S-bimodule homomorphism. This completes the proof. $\square$

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