1. Introduction

How does a billiard ball bounce around a polygonal billiard table? One approach to this problem is the unfolding construction of Katok-Zemlyakov [**ZK75**]. Whenever the billiard ball strikes the wall of the polygon, instead of bouncing off it continues in the mirror image of the reflected table. When two reflected copies of the table agree with each other they are identified. If the polygon is a rational polygon, i.e. all angles are rational multiples of π, the process terminates and produces a flat surface. Flow in the flat metric corresponds to billiard flow.

Remarkably, the object created by the unfolding construction - a flat metric on a surface with finitely many singularities and trivial linear holonomy - called a translation surface - is equivalent to specifying a Riemann surface X and a holomorphic one-form ω. The collection of all holomorphic one-forms on genus g Riemann surfaces - $\Omega\mathcal{M}_g$ - is a subbundle of the cotangent bundle of the moduli space of genus g Riemann surfaces - $\mathcal{M}_g$ - an object of central importance to the study of algebraic curves and surface bundles.

The space $\mathcal{M}_g$ is endowed with a natural metric - called the Teichmüller metric - and Teichmüller geodesic flow along with the complex structure of $\mathcal{M}_g$ induces a $\mathrm{GL}(2,\mathbb{R})$ action on $\Omega\mathcal{M}_g$. Studying the dynamics of the $\mathrm{GL}(2,\mathbb{R})$ action is a means of both understanding the geometry and topology of $\mathcal{M}_g$ and of understanding lower dimensional dynamical systems like rational billiards! The deep connections between algebraic geometry, surface bundles, and dynamics in this discipline makes studying the $\mathrm{GL}(2,\mathbb{R})$ action on $\Omega\mathcal{M}_g$ worthwhile.

In the following, I will describe my past research accomplishments and outline projects that build on that work to explore the geometry and dynamics of moduli spaces as well as related questions in billiards and the theory of surface bundles. Specifically, I will describe the following work

- (Section 3) A positive resolution (and in fact strengthening) of a conjecture of Maryam Mirzakhani on orbit closures in hyperelliptic components of strata of $\Omega\mathcal{M}_g$.
- (Section 4) A finiteness result (joint with Alex Wright) on finite blocking problems in rational billiards.
- (Section 5) A study of holomorphic sections of the universal curve over subvarieties of $\mathcal{M}_g$ containing a Teichmüller disk and related work establishing uniqueness of new orbit closures discovered by Eskin-McMullen-Mukamel-Wright and Kumar-Mukamel [**KM16**].
- (Section 6) An upper bound on the Hausdorff dimension of directions that deviate from Birkhoff and Oseledets averages under Teichmüller geodesic flow that applies to every (not just almost-every) point in $\Omega\mathcal{M}_g$ (joint with several coauthors).

At the end of each section I will outline a research proposal based on this work. The final section outlines further research directions that are not directly connected to previously accomplished work.

2. Background

In the sequel, an element of $\Omega\mathcal{M}_g$ will be represented as (X,ω), where X is a Riemann surface and ω is a holomorphic one-form on X. The moduli space $\Omega\mathcal{M}_g$ admits a $\mathrm{GL}(2,\mathbb{R})$-invariant stratification by specifying the number and order of vanishing of the zeros of the holomorphic one-form. Periods of the one-forms (relative to the set of zeros) endow each stratum with a linear structure, i.e. they provide coordinates with transition functions being constant linear maps.

Eskin-Mirzakhani [**EM**] and Eskin-Mirzakhani-Mohammadi [**EMM15**] proved that $\mathrm{GL}(2,\mathbb{R})$ orbit closures are linear submanifolds in period coordinates (this extended work of McMullen [**McM07**] who established this fact in genus two and classified the orbit closures there). This fact is remarkable since orbit closures in a general dynamical system need not have even integer Hausdorff dimension! In the sequel, the phrase "orbit closure" will refer to a $\mathrm{GL}(2,\mathbb{R})$-orbit closure in a stratum of holomorphic one-forms.

3. The Dynamics and Geometry of the Hyperelliptic Locus

3.1. **Past Accomplishment: Mirzakhani's Conjecture on Orbit Closures.** When does an orbit closure in $\Omega\mathcal{M}_g$ "arise from geometry"? Mirzakhani conjectured (see Wright [**Wri14**]) that for any measure with sufficiently large support, the answer should be always.

To make this conjecture precise, say that a translation surface (X, ω) is imprimitive if there is a holomorphic map $f : X \longrightarrow Y$ to a lower genus Riemann surface Y and a holomorphic one-form η on Y where $\omega = f^*\eta$. An orbit closure is said to be a locus of branched covers if every holomorphic one-form in the orbit closure is imprimitive.

Mirzakhani's conjecture for hyperelliptic components of strata, i.e. components in which all the Riemann surfaces are hyperelliptic, says that every orbit closure of complex-dimension at least four is a locus of branched covers. In Apisa [**Apib**] I proved this statement and in Apisa [**Api17b**] I strengthened the conjecture to the following.

Theorem 1. *In hyperelliptic components of strata of $\Omega\mathcal{M}_g$ for $g > 2$, every orbit is closed, dense, or dense in a locus of branched covers.*

The proof uses a "tree-like" structure of holomorphic one-forms in hyperelliptic components discovered by Lindsey [**Lin15**] to build degenerations of linear submanifolds to the Mirzakhani-Wright boundary (see [**MW17**]). This approach is fundamentally different from the approach used by McMullen [**McM07**] to classify orbit closures in $\Omega\mathcal{M}_2$.

3.2. **Research Proposal 1: Hyperelliptic Translation Surfaces.** To complete the classification of orbit closures in the hyperelliptic components of strata, one may ask the following.

Problem 1. Classify the closed orbits of primitive translation surfaces in hyperelliptic components.

A solution to Problem 1 might combine the work of McMullen [**McM05,McM06b**], which resolved the problem in genus two, with Eskin-Filip-Wright [**EFW17**], which established that there are only finitely many such orbits in genus greater than two. More generally, one may pose the question:

Problem 2. Determine the surface subgroups of the braid group (and more generally the mapping class group) that correspond to holomorphically embedded curves in $\mathcal{M}_g$.

The second natural question to ask in light of Theorem 1 is to what extent hyperellipticity can be used to constrain orbit closures. To pose the question, we must define the rank of a linear submanifold $\mathcal{M}$. Any linear submanifold $\mathcal{M}$ has its tangent space at a point (X, ω) identified with a subspace of relative cohomology $H^1(X, \Sigma; \mathbb{C})$ where Σ is the zero set of ω. The rank of $\mathcal{M}$ is defined to be half the complex-dimension of the projection of this subspace to absolute cohomology. This number is always an integer by Avila-Eskin-Möller [**AEM**]. $\mathcal{M}$ is said to be higher rank if its rank is greater than one. One may then ask,

Problem 3. Do loci of branched covers account for all the higher rank linear submanifolds contained in the hyperelliptic locus?

By Wright [**Wri17**], a positive resolution would imply that the only subgroups of braid groups that correspond to totally geodesic submanifolds of $\mathcal{M}_g$ are braid and surface subgroups. The ingredients needed to approach Problem 3 include constraining the flat geometry of hyperelliptic translation surfaces (analogous to the work of Lindsey [**Lin15**]) and studying the boundary of linear submanifolds of the hyperelliptic locus in both the Mirzakhani-Wright partial compactification [**MW17**] and the compactification in [**BCG$^+$16**].

In ongoing work, Kathryn Lindsey and I hope to extend the strata to which Theorem 1 applies by showing the following.

Problem 4. Show that higher rank linear submanifolds in a component of a stratum with a codimension one hyperelliptic locus must intersect the hyperelliptic locus.

To study low rank, Duc-Manh Nguyen, David Aulicino, and I are adapting the techniques of Apisa [**Api17b**] to classify the rank one orbit closures in genus three that are not closed orbits.

4. Rational Billiards

4.1. **Past Accomplishment: The Finite Blocking Problem.** Given a polygonal billiard table, one may ask the following.

Problem 5 (Finite Blocking Problem)**.** Given two points on the billiard table is there a finite collection of points S so that all billiard shots between the two points pass through S?

In the case of billiards on rational polygons, Wright and I used the unfolding construction to establish the following finiteness result in [**AW17**] for the finite blocking problem.

Theorem 2. *If B is a rational billiard table with connected boundary, then*

(1) If B is tiled by rectangles or by $30-60-90$ or $45-45-90$ triangles, then any two points are finitely blocked from each other.
(2) If all angles in B are multiples of $\frac{\pi}{2}$ and B is not tiled by rectangles, then any point is only finitely blocked from finitely many other points.
(3) Otherwise, only finitely many pairs of points are finitely blocked from each other.

Theorem 2 is often strong enough to completely solve the finite blocking problem on specific polygons.

In [**AW17**], Wright and I used Filip's result [**Fil16**] on algebraicity of linear submanifolds in $\Omega\mathcal{M}_g$ to connect the finite blocking problem on a rational triangle to torsion in a factor of the Jacobian of its unfolding. The unfolding is always a cyclic cover of a thrice-punctured sphere, and torsion in the Jacobians of such Riemann surfaces was studied in Coleman [**Col89**] and Coleman-Tamagawa-Tzermias [**CTT98**]. This allowed us to resolve the finite blocking problem for infinitely many rational triangles.

4.2. **Research Proposal 2: Unfolding Billiard Tables and Torsion in Factors of the Jacobian.** To solve the finite blocking problem for all rational triangles, I propose the following,

Problem 6. Extend Coleman's p-adic integration technique to study torsion in factors of the Jacobians of cyclic covers of the thrice punctured sphere. As a corollary, solve the finite blocking problem in rational triangles.

A solution would contribute new tools to solving Diophantine equations while developing a connection between number theory and dynamics. Another approach to the finite blocking problem on rational triangles is the following:

Problem 7. Classify all orbit closures of the unfoldings of rational triangles.

A solution could help determine the asymptotic number of closed billiard paths as in Veech [**Vee89**] and Athreya-Eskin-Zorich [**AEZ16**]. A solution would likely use work of Mirzakhani-Wright [**MW16**], McMullen [**McM13**], and the Ahlfors variational formula [**Ahl60**]. Using these tools and ideas from Apisa [**Apib**], I have sketched a partial solution to Problem 7 for rational isosceles triangles.

Finally, in the proof of Theorem 2, a new criterion for recognizing loci of branched covers was developed. In ongoing work, Alex Wright and are using this method to study the following.

Problem 8. If a linear submanifold $\mathcal{M}$ has a locus of branched covers as a codimension one component of its Mirzakhani-Wright boundary, then when is $\mathcal{M}$ itself a locus of branched covers?

5. Holomorphic Sections of the Universal Curve Defined over Subvarieties and Marked Points on Translation Surfaces

Given a holomorphically varying family of Riemann surfaces, is there a holomorphically varying collection of points defined over the family? To precisely pose this problem, make the following definition. For any cover of $\mathcal{M}_g$ corresponding to a torsionfree subgroup of the mapping class group, the universal curve is the surface bundle whose fiber over a point is the Riemann surface to which that point corresponds. We ask the following:

Problem 9. Given a finite index torsionfree subgroup Γ of the mapping class group and a subvariety C of the cover of $\mathcal{M}_g$ corresponding to Γ, classify all holomorphic sections of the universal curve restricted to C.

To provide a partial answer to this problem we define a periodic point p on a translation surface (X, ω) to be any point (excluding zeros of ω) where the orbit closure of the marked translation surface $(X, \omega; p)$ has the same dimension as the orbit closure of (X, ω). For example, if X is hyperelliptic, then all Weierstrass points are either zeros of ω or periodic points.

5.1. Past Accomplishment: Classifying Holomorphic Sections of the Universal Curve.

In Apisa [**Apia**], I showed the following:

Theorem 3. *Let C be as in Problem 9 and suppose that it contains the Teichmüller disk generated by the translation surface (X, ω). Any holomorphic section of the universal curve defined on C must mark periodic points or zeros of (X, ω) over X.*

The proof uses Kobayashi hyperbolicity to promote a holomorphic section to a $\mathrm{GL}(2, \mathbb{R})$-equivariant one defined on the orbit closure of (X, ω). Theorem 3 is often enough to classify holomorphic sections of the universal curve. For instance, in Apisa [**Apia**] I showed that,

Theorem 4. *If (X, ω) has dense orbit in a component of a stratum of holomorphic one-forms, then there is a periodic point p on (X, ω) if and only if (X, ω) is hyperelliptic and p is a Weierstrass point that is not a zero of ω.*

As a corollary, holomorphic sections of the universal curve defined over the projections of any stratum of holomorphic one-forms to $\mathcal{M}_g$ may be classified. Projections of strata have been considered by various authors, for instance Gendron [**Gen15**], Mullane [**Mul17**], and Farkas-Verra [**FV13**]. The results of Apisa [**Apia**] were used to prove Mirzakhani-Wright [**MW16**, Theorem 1.1].

5.2. Past Accomplishment: Periodic points in genus two and two new orbit closures.

Given Theorem 4 one might be tempted to conjecture that the only periodic points on a primitive translation surface in genus two are Weierstrass points. Indeed, Möller [**Möl06**] established this for primitive genus two translation surfaces with closed orbits. However, Kumar-Mukamel [**KM16**] discovered two periodic points, dubbed the "golden points", on a family of primitive genus two translation surfaces that do not coincide with Weierstrass points. Shortly, thereafter, Eskin-McMullen-Mukamel-Wright announced the discovery of a new higher rank nonarithmetic linear submanifold in the minimal stratum in genus four. In Apisa [**Api17a**] I showed that these new orbit closures were unique. Specifically, I extended Möller's result to the following,

Theorem 5. *The only periodic points on primitive genus two translation surfaces are Weierstrass points or golden points.*

In the same work, I showed that Theorem 5 resolves the finite blocking problem (Problem 5) in genus two. Using degeneration arguments, I went on to use Theorem 5 to show,

Theorem 6. *There is at most one higher rank non-arithmetic linear submanifold in the minimal stratum in genus four.*

By the work of Eskin-McMullen-Mukamel-Wright there is exactly one such submanifold.

5.3. **Research Proposal 3: Holomorphic Sections and Periodic Points.** In light of Theorem 3, it is natural to ask the following:

Problem 10. Under what conditions do the topological sections of the universal curve, defined over a subvariety as in Problem 9, coincide with the holomorphic ones?

Similarly, what can be said about holomorphic sections of the universal curve defined over subvarieties that do not contain a Teichmüller disk? Building on the classification of non-generic points in genus two, I propose to solve the following -

Problem 11. Can non-generic points be classified for the Prym eigenforms discovered by McMullen [**McM06a**]? What new orbit closures in higher genus do these points lead to?

A solution would reveal constraints on holomorphic sections (making progress on Problem 9) and potentially provide new examples of higher dimensional orbit closures, of which very few are known.

6. Deviations of Ergodic Averages

While results in ergodic theory are often stated for "almost every point", a body of work stretching from Ratner's theorems to Chaika-Eskin [**CE15**] has developed techniques to analyze the behavior of every point in a dynamical system. The work described in this section is joint with Hamid al-Saqban, Alena Erchenko, Osama Khalil, Shahriar Mirzadeh, and Caglar Uyanik [**aSAE$^+$17**].

6.1. **Past Accomplishment: Deviations of Birkhoff and Oseledets Averages.** Fix a unit-area translation surface (X, ω) whose orbit closure is $\mathcal{M}$ and let $\mathcal{M}_1$ be the locus of unit-area translation surfaces in $\mathcal{M}$. Let (g_t) be time t Teichmüller geodesic flow. The arguments in Masur [**Mas82**] and Veech [**Vee82**] show that Teichmüller geodesic flow on $\mathcal{M}_1$ is ergodic with respect to normalized Lebesgue measure μ given by period coordinates. Fix a bounded Lipschitz function $f : \mathcal{M}_1 \longrightarrow \mathbb{R}$. The Birkhoff ergodic theorem says that for any almost any point $x \in \mathcal{M}_1$, the average of f along $(g_t x)_{t=0}^T$ converges to $\int_{\mathcal{M}_1} f d\mu$ as $T \longrightarrow \infty$. But, for a given $\epsilon > 0$, must the set

$$S := \left\{ \theta \in [0, 2\pi) : \limsup_{T \longrightarrow \infty} \frac{1}{T} \int_0^T f\left(g_t \cdot \left(X, e^{i\theta}\omega\right)\right) dt > \epsilon + \int_{\mathcal{M}_1} f d\mu \right\}$$

have measure zero? In other words, if the $\mathrm{GL}(2,\mathbb{R})$ orbit of (X, ω) is dense in $\mathcal{M}$ is it necessarily the case that the Birkhoff averages of Teichmüller geodesic flow in almost every direction θ converge to $\int_{\mathcal{M}_1} f d\mu$? Chaika-Eskin [**CE15**] showed that this is indeed the case. In [**aSAE$^+$17**] we strengthened that result to show the following -

Theorem 7. *The Hausdorff dimension of S is strictly less than one.*

The work applies the height functions constructed by Athreya [**Ath06**] and Eskin-Mirzakhani-Mohammadi [**EMM15**] and the integral inequality techniques of Eskin-Mozes-Margulis [**EMM98**]. Similar results also apply to Oseledets averages.

6.2. **Past Accomplishment and Research Problem: Divergence on Average.** In the same work we studied the set of directions θ so that Teichmüller geodesic flow applied to $(X, e^{i\theta}\omega)$ spends asymptotically zero percent of its time in any compact set. These directions are said to be directions that diverge on average. Masur [**Mas92**] and Masur-Smillie [**MS91**] connected these directions to directions of nonergodic flow on the translation surface. We showed the following in [**aSAE$^+$17**].

Theorem 8. *For any translation surface, the set of directions that diverge on average has Hausdorff dimension bounded above by* $1/2$.

The result is sharp by Cheung [**Che11**] and strengthens a result of Masur [**Mas92**]. The proof uses the techniques of Eskin-Mozes-Margulis [**EMM98**] to extend work of Kadyrov-Kleinbock-Lindenstrauss-Margulis [**KKLM14**] to the setting of dynamics on moduli space. Conversely, I propose studying the following question:

Problem 12. Show that for all translation surfaces, the Hausdorff dimension of the set of directions that diverge on average is exactly $1/2$.

This work would likely use methods of Cheung [**Che11**] to compute lower bounds on Hausdorff dimension, the work of Minsky-Weiss [**MW14**] (as applied in Athreya-Chaika [**AC15**]), and Masur's results on complexes [**Mas90**].

7. Further Research Directions

7.1. **Modular Forms and Translation Surfaces.** Filip [**Fil16**] proved that every linear submanifold of $\Omega\mathcal{M}_g$ is an algebraic variety consisting of holomorphic one-forms (X, ω) where ω is an eigenform for the action of an order in a number field on the Jacobian of X. Since this structure is reminiscent of the decomposition of the Jacobian of modular curves into Hecke-eigenspaces, it is natural to ask the following.

Problem 13. What are the $\mathrm{GL}(2, \mathbb{R})$ orbit closures of translation surfaces corresponding to weight two Hecke eigenforms?

To perform computer experiments, I implemented an algorithm in SAGE that takes a q-expansion for a weight two cusp form and outputs a translation surface to arbitrary precision. This problem could suggest a means of studying Teichmüller geodesics using positive characteristic techniques - an approach also suggested by Mukamel [**Muk17**].

7.2. **Density of Thurston-Veech surfaces in the collection of pseudo-Anosovs.** The pseudo-Anosov homeomorphims of the mapping class group correspond to closed geodesics in $\mathcal{M}_g$. A pseudo-Anosov homeomorphism is said to be Thurston-Veech if it is the product of two non-commuting Dehn multi-twists. In ongoing work, Wright, Lanneau, and I have established the existence of a dense collection of translation surfaces fixed by a Thurston-Veech homeomorphism in every linear submanifold of $\Omega\mathcal{M}_g$. It is natural to ask,

Problem 14. Is the typical pseudo-Anosov homeomorphism Thurston-Veech? Can Thurston-Veech surfaces be used to compute volumes of orbit closures as in Eskin-Okounkov [**EO01**]?

7.3. **Dynamics on Infinite Measure Systems.** Quotients of Teichmüller space by infinite index subgroups naturally arise in Teichmüller theory - for instance the quotient by the Torelli group arose in the study of the isoperiodic foliation in Calsamiglia-Deroin-Francaviglia [**CDF15**]. However, the techniques of Eskin-Mirzakhani [**EM**] do not apply to infinite measure dynamical systems. Nevertheless, McMullen-Mohammadi-Oh [**MMO17**] established a version of Ratner's theorem for rigid acylindrical hyperbolic 3-manifolds of infinite measure. Let $\mathrm{Teich}_{g,n}$ denote the Teichmüller space of genus g surfaces with n punctures where $3g - 3 + n > 0$.

Problem 15. For which infinite index subgroups Γ of the mapping class group do Ratner-type results continue to hold for the $\mathrm{GL}(2, \mathbb{R})$ action on the strata of holomorphic one-forms over $\mathrm{Teich}_{g,n}/\Gamma$?

Apart from the Torelli group, it would be interesting to determine whether Problem 15 could be applied to subgroups of the mapping class group in the Johnson filtration.

7.4. **Double Kodaira Fibrations.** How many ways can a compact four-manifold be written as a surface bundle over a surface where the base and fiber surfaces both have genus greater than one? Johnson [**Joh99**] showed that the number of such fiberings is always finite and Salter [**Sal15**] showed that for any n there is a four-manifold with at least n such fiberings. If the four-manifold M is a complex surface and the projection to the base surface is a holomorphic map, then the fibering is called a holomorphic-fibering. The following question is open:

Problem 16. Do examples of complex surfaces that holomorphically fiber in three ways exist?

If E is a complex surface with holomorphic fiberings over curves $p_i : E \longrightarrow B_i$ for $i = 1, 2$, then Chen **[Che17]** showed that E cannot fiber in three ways if $(p_1, p_2)^* : H^{1,0}(B_1 \times B_2) \longrightarrow H^{1,0}(E)$ is an isomorphism. Since each element of $H^{1,0}(E)$ restricted to fibers is an isoperiodic family of translation surfaces, this suggests an approach using work of **[CDF15]** on the isoperiodic foliation.

7.5. **Dynamics on Character Varieties.** In Benoist-Quint **[BQ11]**, the stationary measures for random walks on finite measure locally symmetric spaces are classified. Eskin and Mirzakhani **[EM]** illustrated a method of recovering these results in a nonlinear setting - in particular for the $\mathrm{SL}(2,\mathbb{R})$-action on $\Omega\mathcal{M}_g$. Goldman-Forni **[FG17]** showed that the $\mathrm{SL}(2,\mathbb{R})$ action on strata can be combined with character varieties to build an $\mathrm{SL}(2,\mathbb{R})$ action on a combined dynamical system. One may ask,

Problem 17. Can one establish a nonlinear Benoist-Quint theorem for the $\mathrm{SL}(2,\mathbb{R})$ action on the dynamical systems constructed by Goldman-Forni.

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