HOMEWORK 1 PART 1: DUE WEDNESDAY SEPTEMBER 4

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All of the subsequent problems are either self-contained or from Chapter 1.1 of Hatcher. X will always denote a topological space unless otherwise stated. All arrows denote continuous maps unless otherwise stated.

Exercises 6 - Conjugacy Classes in π_1:

Exercises 9 - Ham Sandwich Theorem:

Two Problems on Constructing Explicit Homotopies

Exercise 10:

Exercise: The fundamental group of a topological group: A topological group is a topological space G that is also a group so that the map $m : G \times G \longrightarrow G$ that sends (g, h) to gh is continuous and so that the map $i : G \longrightarrow G$ that sends g to g^{-1} is continuous. Examples include S^1 and closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ - viewed as an open subset of $\mathbb{R}^{n^2}$. Show that the fundamental group of any topological group is abelian by using the continuity of multiplication to exhibit a homotopy between $\alpha \cdot \beta$ and $\beta \cdot \alpha$ for any two loops α and β based at the identity element e.

Three (Hopefully) Very Short Proofs

Exercises 11, 12, 13:

Two More Perspectives on the Fundamental Group

Exercise: The fundamental group corresponds to path components of the loop space: Given a topological space X with a point $x_0 \in X$, let ΩX be the set of loops in X based at x_0. Give ΩX the compact-open topology. Show that path components of ΩX are in bijective correspondence with elements of $\pi_1(X, x_0)$.

Exercise: Separate continuity vs. Continuity: Say that a map $F : I \times I \longrightarrow X$ is separately continuous if for all $t \in I$ the maps $f_i : I \longrightarrow X$ given by $f_1(s) := F(s, t)$ and $f_2(s) := F(t, s)$ for $i \in \{1, 2\}$ are continuous. Prove or disprove the following: if F is separately continuous then it is continuous.

Fibrations

Exercise: Product spaces are fibrations: Let X_1 and X_2 be two topological spaces. Let $p_1 : X_1 \times X_2 \longrightarrow X_1$ be the projection onto the first factor. Show that this is a fibration.

Exercise: The fundamental group of $\mathbb{R}P^n$: Let $\pi : S^n \longrightarrow \mathbb{R}P^n$ be the quotient of S^n by the antipodal map. Show that π is a covering map and imitate the proof that $\pi_1(S^1) \cong \mathbb{Z}$ to show that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 1$. Notice that a path in $\mathbb{R}\mathrm{P}^n$ corresponds to a continuously varying family of lines in $\mathbb{R}^{n+1}$ - using this description of paths, describe in words what distinguishes the nontrivial element of the fundamental group of $\mathbb{RP}^n$ from the trivial element when $n > 1$ (Hint: "orient" the lines).

Teeing up van Kampen

Exercises 18 and 19: