PRACTICE MIDTERM - MATH 544A

INSTRUCTOR: PAUL APISA

The test will be graded out of 60 points and there are 63 possible points that can be earned.

Problem 1 (10 points): Compute the fundamental group of the following spaces:

(1) (5 points) Let T be the solid torus - i.e. T is homeomoprhic $S^1 \times \mathbb{D}^2$ where $\mathbb{D}^n = \{x \in \mathbb{R}^{n+1} : |x| \leq 1\}$ - with its usual embedding in $\mathbb{R}^3 \subseteq S^3$ (here S^3 may be taken to be the one-point compactification of $\mathbb{R}^3$). Compute the fundamental group of $S^3 - T$.

(2) (5 points) Take two copies of $X = S^3 - \text{int}(T)$ where int denotes the interior of the solid torus T. The boundary of X is homeomorphic to the boundary of T, which (the boundary) is homeomorphic to $\mathbb{R}^2/\mathbb{Z}^2$. Since $\text{SL}(2, \mathbb{Z})$ - the collection of 2×2 invertible matrices with entries in $\mathbb{Z}$ - preserves $\mathbb{Z}^2$, it acts by homeomorphisms on $\mathbb{R}^2/\mathbb{Z}^2$. Fix $A \in \text{SL}(2, \mathbb{Z})$ and take this to be a homeomorphism of ∂T as described above. Let Y be the space formed by taking two copies of X and identifying their boundaries by gluing a point x in the boundary of the first copy to $A(x)$ in the boundary of the second copy. Compute the fundamental group of Y.

Problem 2 (10 points): How many conjugacy classes of index three subgroups of the free group on two generators (F_2) are there? Choose a representative from each conjugacy class and write down a finite collection of elements of F_2 that generate the subgroup.

Problem 3 (10 points): Let $G = \langle a_1, \ldots, a_n | w_1, \ldots, w_m \rangle$ be a finite presentation of a group. Recall that this notation means that G is the quotient of the free group F generated by $a_1, \ldots, a_n$ by the smallest normal subgroup of F containing the elements $w_1, \ldots, w_m$.

(1) (3 points) Build a CW complex X_G with one 0-cell, n 1-cells, and m 2-cells so that X_G is path connected and has fundamental group isomorphic to G.

(2) (7 points) Take two copies of X_G. In each copy delete a disk from a 2-cell (this forms a circular boundary on each copy of X_G). Glue the two copies of X_G together along their circular boundaries. Compute the fundamental group of the resulting space (be careful! the answer may depend on the 2-cells that were chosen).

Problem 4 (5 points): Show that the only finite groups that act freely on S^1 are cyclic.

Problem 5 (10 points):

(1) (5 points) Let X be a space for which all reduced homology groups vanish. Let x be a point in X with a neighborhood homeomorphic to $\mathbb{R}^n$ for $n > 1$. Show that there is an isomorphism $H_k(X - x) \cong H_k(S^{n-1})$ for every integer k.

(2) (5 points) Let $f : X \longrightarrow X$ be a homeomorphism of a space X and let $M_f := X \times I/((x, 0) \sim (f(x), 1))$ be the mapping torus. Let X_0 denote the image of $X \times \{0\}$ in the mapping torus. Compute the relative homology groups $H_n(M_f, X_0)$ in terms of $H_n(X)$.

Problem 6 (10 points): Let T be the unit square with opposite sides identified by translation (this is a concrete way of describing the torus $S^1 \times S^1$). Let $f : T \to T$ be the map from the torus to itself given by rotating the unit square by 180 degrees. Let X be the mapping torus of $f : T \to T$ - i.e. the quotient of the space $T \times [0, 1]$ under the equivalence relation $(x, 0) \sim (f(x), 1)$. Put a Δ-complex structure on X and compute its homology groups.

Problem 7 (5 points): Let $f : S^n \longrightarrow S^n$ and $g : S^n \longrightarrow S^n$ be continuous maps. Suppose that there is some point $y_0 \in S^n$ so that $g^{-1}(y_0) = \{x_1, x_2\}$ and so that for $i \in \{1, 2\}$ there are open neighborhoods U_i of x_i so that the restriction of g to U_i is a homeomorphism onto its image. Show that $g \circ f$ has a fixed point.

Problem 8 (3 points): Compute the fundamental group of $\mathrm{GL}_2(\mathbb{C})$.

Write answers and work in the test booklet