MODULARITY LIFTING THEOREMS

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1. Introduction

These are live-texed lecture notes from the MIT graduate class on Galois representations given by Sug Woo Shin in Spring of 2014. The class expanded on the notes of Toby Gee's course from the Arizona Winter School in 2014, which it closely followed. The problem sets from the class contained some proofs of results which were stated in class and some of these were later added to the notes. Any mistakes in these notes are due to me and not the lecturer.

2. Lecture 1

Let F be a field and fix an algebraic closure $\overline{F}$ of F. We let $Gal(\overline{F}/F)$ be the absolute Galois group of F.

Definition 2.1. A Galois representation is a continuous group homomoprhism $Gal(\overline{F}/F) \to GL_n(R)$ where R is a topological ring.

Most of the time we will take R to be $\overline{\mathbb{Q}}_l$ where l is a prime, such a representation will be a called an l-adic Galois representation. The first result we need about l-adic representations is that up to conjugation, the image lands in $GL_n(\mathcal{O}_L)$ where L is a finite extension of $\mathbb{Q}_p$. This follows from the next two lemma which actually hold in a more general context:

Lemma 2.2. Let Γ be a compact topological group (in particular a Galois group is such) and let

$$\rho : \Gamma \to GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous homomorphism. Then there exists a finite extension $L/\mathbb{Q}_l$ such that $\rho(\Gamma) \subset GL_n(L)$.

Proof. $\rho(\Gamma)$ is compact Hausdorff, hence by the Baire Category theorem, the intersection of a countable set of open dense subsets of $\rho(\Gamma)$ is dense in $\rho(\Gamma)$.

Now since

$$\rho(\Gamma) = \bigcup_{L/\mathbb{Q}_l \text{ finite}} (\rho(\Gamma) \cap GL_n(L))$$

and each set $\rho(\Gamma) \cap (GL_n(L)$ is closed in $\rho(\Gamma)$, it follows from the above, that some $\rho(\Gamma) \cap GL_n(L)$ contains an open subset, hence $\rho(\Gamma) \cap GL_n(L)$ itself is open. As Γ was profinite, we have that $[\rho(\Gamma) : \rho(\Gamma) \cap GL_n(L)]$ is finite.

If we choose coset representatives $\alpha_1, ..., \alpha_r$ for $\rho(\Gamma)$, there exists $L_i/\mathbb{Q}_l$ a finite extension such that $\rho(\alpha_i) \in GL_n(L_i)$. Taking L to be the compositum of the L_i we obtain the result. □

Lemma 2.3. Let Γ be a compact topological group, $L/\mathbb{Q}_l$ a finite extension with ring of integers $\mathcal{O}_L$ and $\rho : \Gamma \to GL_n(L)$ a continuous representation. Then there exists $g \in GL_n(L)$ such that $g\rho g^{-1}$ has image in $GL_n(\mathcal{O}_L)$

Proof. It is enough to show that there exists a Γ invariant $\mathcal{O}_L$ lattice $\Lambda \subset L^n$. Indeed letting g be the change of basis matrix taking Λ to the standard basis of $\mathcal{O}_L^n$ we get the result.

Define $\Lambda_0 := \mathcal{O}_L^n$ to be the standard lattice. Observe that $GL_{\mathcal{O}_L}(\Lambda_0) \subset GL_n(L)$ is open, so that $\rho^{-1}(GL_{\mathcal{O}_L}(\Lambda_0)) \subset \Gamma$ has finite index.

Let $\alpha_1, ..., \alpha_r$ be a finite set of coset representatives in Γ. We then define

$$\Lambda := \rho(\Gamma)\Lambda_0 = \sum_{i=1}^{r} \alpha_i \Lambda_0$$

which is clearly a $\rho(\Gamma)$ invariant lattice. □

Definition 2.4. Let R be a topological ring and Γ a topological group. Two continuous representations $\rho_1, \rho_2 : \Gamma \to GL_n(R)$ are said to be isomorphic if there exists and isomorphism $\phi : R^n \to R^n$ such that the following diagram commutes:

$$\begin{array}{ccc} R^n & \xrightarrow{\rho_1(\gamma)} & R^n \\ \downarrow \phi & & \downarrow \phi \\ R^n & \xrightarrow{\rho_2(\gamma)} & R^n \end{array}$$

for all $\gamma \in \Gamma$

As a consequence of the Lemma, a representation $\rho : \Gamma \to GL_n(\overline{\mathbb{Q}}_l)$ with Γ compact is isomorphic to one with image in $GL_n(\mathcal{O}_L)$

Our next result will be the Brauer Nestbitt theorem, this gives a condition for when the semi simplification of two representations are isomorphic and which we will need for the study of the mod p reduction of a representation. We first need to define the notion of semisimple representation and semisimplification. As before Γ is a compact topological group.

Definition 2.5. Let k be a field, and $\rho : \Gamma \to GL_n(K) \cong GL_n(V)$. Then ρ is semi simple if

$$\rho \cong \oplus_{i=1}^{r} \rho_i$$

for some irreducible $\rho_i : \Gamma \to GL_{n_i}(R)$.

Definition 2.6. Let $\rho : \Gamma \to GL_n(V)$ and choose

$$0 \subset V_1 \subset ... \subset V_r = V$$

where the V_i are Γ invariant subgroups with V_i/V_{i-1} irreducible. The semi simplification ρ^{ss} is the representation induced by ρ on the vector space $V^{ss} = \oplus_{i=1}^r V_i/V_{i+1}$

It can be shown that this is well defined and does not depend on the choice of the V_i's.

Example 2.7. Consider $\rho : \Gamma \to GL_2(k)$ given by

$$\gamma \mapsto \begin{pmatrix} a(\gamma) & b(\gamma) \\ 0 & d(\gamma) \end{pmatrix}$$

Then $\rho^{ss} = a \oplus d$

Theorem 2.8 (Brauer, Nesbitt)**.** *Let k be a field and Γ a (topological) group, $n_1, n_2, \geq 1$. For $i = 1, 2$ and $\rho_i : \Gamma \to GL_{n_i}(k)$, assume either:*
i) $\forall \gamma \in \Gamma$ we have $\det(1 - \rho_1(\gamma)T) = \det(1 - \rho_1(\gamma)T)$ or
ii) char $k = 0(>> 0)$, and $\forall \gamma \in \Gamma$ we have $tr\rho_1(\gamma) = tr\rho_2(\gamma)$
Then $\rho_1^{ss} = \rho_2^{ss}$

We will need the following lemma:

Lemma 2.9. Let R be an associative k algebras (not necessarily commutative eg. $R = k[\Gamma]$) and let $M_1, ..., M_r$ be non-isomorphic R-simple modules which are finite dimensional over k. Then $\exists e_1, ..., e_r \in R$ such that

$$e_i m_i = m_i \quad \forall i, \forall m_i \in M_i$$

$$e_i m_j = 0 \quad \forall j \neq i, \forall m_j \in M_j$$

Proof. (Sketch) Upon replacing R by its image in $End_k \oplus_{i=1}^r M_i$ wlog. we can assume, k is a finite dimensional semisimple k-algebra.

The Artin Wedderburn theorem then tells us that $R \cong \prod_{i=1}^s M_{n_i}(D_i)$ where D_i is a division algebra over k. One then deduces that $r = s$ and R acts as $M_{n_i}(D_i)$ on M_i for $i = 1, ..., r$ (after reordering). Defining e_i to be 1 on the i^{th} component and 0 elsewhere we obtain the result.

□

Proof of Brauer Nesbitt: Let $\rho_i : \Gamma \to GL_n(V), j = 1, 2$. Since the conditions in the theorem don't change upon taking semisimplifications, we may assume the ρ_i are semisimple.

Let $M_1, ..., M_r$ be the distinct irreducible subrepresentations of $V_1 \oplus V_2$. Let m_i^1, m_i^2 be the multiplicities of M_i in V_1 and V_2 respectively. It suffices to prove that $m_i^1 = m_i^2$ for $i = 1, ..., r$.

Case i) Take e_i as in Lemma 1.9, then the characteristic polynomial of e_i on M_i is $(t-1)^{\dim M_i}$ and $t^{\dim M_j}$ when $j \neq i$. From the equality of characteristic polynomials of $\rho_1(e_i)$ and $\rho_2(e_i)$ one finds the dimensions m_i^1 and m_i^2 match.

Case ii)For $j = 1, 2$, there exists a unique continuous map θ_j which makes the following diagram commute:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho_j"] \arrow[d] & GL_n(V_j) \arrow[r, "tr"] & k \\ k[\Gamma] \arrow[urr, "\theta_j"'] & & \end{tikzcd}$$

It can be shown that $\theta_1(\alpha) = \theta_2(\alpha) \forall \alpha \in k[\Gamma]$. Plugging in e_i, we obtain $\theta_1(e_i) = m_i^1 \dim M_i$ and $\theta_2(e_i) = m_i^2 \dim M_i$.

□

3. Lecture 2

The Brauer Nesbitt Theorem allows us to define the reduction mod l of l-adic representations. As before L is a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}_L$

Definition 3.1. Let Γ be a compact group and $\rho : \Gamma \to GL_n(L)$ be a representation. Choose a conjugate ρ' of ρ such that ρ' has image in $GL_n(\mathcal{O}_L)$ and let $\overline{\rho}'$ be composition with $GL_n(\mathcal{O}_L) \to GL_n(k_L)$. The reduction of ρ mod l is then defined to be

$$\overline{\rho} := (\overline{\rho}')^{ss}$$

Lemma 3.2. ρ is well defined (up to isomorphism), i.e. it does not depends on the choice of ρ'

Proof. Let $\gamma \in \Gamma$. Observe that $char(\rho(\gamma)) = char(\overline{\rho}'(\gamma)) \in k_L[T]$, the Lemma then follows by Theorem 1.8 □

We now recall some basic properties of number fields and local fields. In this course a number field will be a finite extension of $\mathbb{Q}$ and local field will be finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$ so we exclude the function field case.

Definition 3.3. Let F be a number field. A place of F is an equivalence class of valuations on F (where two valuations are considered equivalent if they induce the same topology on F).

A place is finite if the valuation is non-archimedean, otherwise it is an infinite place.

A finite place is p-adic if $|p| < 1$.

We state some basic facts about places:
1)There are 1-1 correspondences between the following sets

$$i) \ \{\text{prime ideals of } \mathcal{O}_F \text{ dividing } p\}$$
$$ii) \qquad \{p\text{-adic places of } F\}$$
$$iii) \quad \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\backslash\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p)$$

The first bijection is given by sending a p-adic place of F to the set $\{a \in \mathcal{O}_F : |a| < 1\}$. The bijection between ii) and iii) is given by sending a $\phi \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\backslash\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p)$ to the place $|.|_p \circ \phi$.

2) The infinite places of F are in one to one correspondence with $\mathrm{Gal}(\mathbb{C}/\mathbb{R})\backslash\mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$, and these are a disjoint union of of real places and complex places, where the real places are the 1 element orbits of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})\backslash\mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$ and the complex places are the two elements orbits.

Now for a place v of F, let F_v to be the completion of F with respect to the topology induced by v.

Definition 3.4. Let F' be an algebraic extension of F, w a place of F' and v a place of F. We say w is above v (or w divides v), write $w|v$ if the restriction of w to F is equivalent to v.

Remark 3.5. From now, when talk about Galois groups of topological fields, we will only consider continuous homomorphisms.

It is straightforward to show that there is a bijection between the places of F' above v and the set $\mathrm{Gal}(\overline{F}_v, F_v)\backslash\mathrm{Hom}_F(F', \overline{F}_v)$. If F'/F is Galois the group $\mathrm{Gal}(F'/F)$ acts on both sides transitively and the bijection is Galois equivariant.

If $w|v$ then $\mathrm{Gal}(F'_w/F_v) \cong \mathrm{Gal}(F'/F)_w := \{\sigma \in \mathrm{Gal}(F'/F) : \sigma(w) = w\}$ where the map is given by $\sigma \mapsto \sigma|_{F'}$. If we pick a different w, the subgroup obtained is conjugate to the above by an element of $\mathrm{Gal}(F'/F)$

Let us now choose an F-algebra embedding $i_v : \overline{F} \hookrightarrow \overline{F}_v$, which indues an map of Galois groups $\mathrm{Gal}(\overline{F}_v/F_v) \hookrightarrow \mathrm{Gal}(\overline{F}/F)$ given by precomposition by i_v. Since changing i_v changes the embedding of Galois groups by conjugation, we obtain a well defined localization of Galois representations. More precisely, letting $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_n(R)$ be a Galois representations, then restricting to $\mathrm{Gal}(\overline{F}_v/F_v)$ we obtain a representation of $\mathrm{Gal}(\overline{F}/F)$ up to isomorphism.

Now let F'/F be a Galois extension of number fields. Recall F'/F is unramified at v if for all (equivalently any) w above v, the reduction map induces a canonical isomorphism $\mathrm{Gal}(F'_w/F_v) \cong \mathrm{Gal}(k_w/k_v)$, where k_w and k_v are the residue fields of $\mathcal{O}_{F'_w}$ and $\mathcal{O}_{F_v}$ respectively.

Thus picking a w we obtain an embedding $\mathrm{Gal}(k_w/k_v) \hookrightarrow \mathrm{Gal}(F'/F)$. The group $\mathrm{Gal}(k_w/k_v)$ has a canonical generator; the *arithmetic frobenius* $frob_w$ which acts on the residue fields by $x \mapsto x^{\#k_v}$. The *geometric frobenius* $Frob_w$ is the inverse of the arithmetic frobenius.

By the above, the conjugacy class $\langle Frob_v \rangle := \{\tau Frob_w \tau^{-1} : \tau \in \mathrm{Gal}(F'/F)\}$ is well defined, i.e. does not depend on $w|v$.

Consequently given $\rho : \mathrm{Gal}(F'/F) \to GL_n(R)$, if F'/F is unramified, the characteristic polynomial, trace, and determinant of the conjugacy class $Frob_v$ is well defined.

Definition 3.6. 1) Let K'/K be an extension of non-archimedean local fields and $I_{K'}$ the inertia subgroup of $\mathrm{Gal}(K'/K)$. The representation $\rho : \mathrm{Gal}(K'/K) \to GL_n(R)$ is unramified if $\rho(I_K)$ is trivial, or equivalently, ρ factors through $\mathrm{Gal}(K^{ur}/K)$ where K^{ur} is the largest unramified extension of K contained in K'.

2) Let F'/F be an extension of number fields. The $\rho : \mathrm{Gal}(F'/F)$ is unramified at v if $\rho|_{\mathrm{Gal}(\overline{F}_v/F)}$ is unramified.

We will now state the Cebotarev density theorem, its proof involves class field theory so we will omit it. It is an important result which allows us to deduce the equivalence of two representations unramified outside a finite set of places S, by the equality of the characteristic polynomials of Frobenius conjugacy classes for places outside S.

Theorem 3.7 (Cebotarev Density)**.** *Let S be a finite subset of the finite places of F and F_S the maximal extension of F in $\overline{F}$ which is unramified outside S.*

1) Given F'/F finite Galois, and $\mathcal{C} \subset Gal(F'/F)$ a conjugacy class, then $\exists$ infinitely many v such that $\langle Frob_v \rangle = \mathcal{C}$

2) Let S be a finite set of places of F, then

$$\bigcup_{v \notin S,\ finite} \langle Frob_v \rangle \subset Gal(F_S/F)$$

is dense.

Remark 3.8. In fact part i) can be strengthened to the statement that the density of such v is equal to $\frac{\#\mathcal{C}}{\#\mathrm{Gal}(F'/F)}$ (for a suitable notion of density).

Theorem 3.9. *Let S be a finite set of places of F. Let $\rho_1, \rho_2 : Gal(\overline{F}/F) \to GL_n(R)$ be two continuous representations unramified outside S, and where R is a topological ring. Assume either*

i) The characteristic polynomials of $\rho_1(Frob_v)$ and $\rho_2(Frob_v)$ are equal for all $v \notin S$ or

ii) $char(k) = 0(> n)$ and the traces of $\rho_1(Frob_v)$ and $\rho_2(Frob_v)$ are equal.

Then $\rho_1^{ss} = \rho_2^{ss}$

Proof. By assumption ρ_1, ρ_2 factor through $\mathrm{Gal}(F_S/F)$. The coefficients of $\rho_2(\gamma)$ are continuous as functions from $\mathrm{Gal}(F_S/F)$ to R, and by part 2) of Theorem 2.7, agree on a dense open subset, hence are equal. Thus by the Brauer Nestbitt Theorem, the two semisimplifications are equal. □

4. Lecture 3

Today we will try to understand l-adic Galois representations over local fields. We begin by discussing the Weil group of a local field. For any field K we let G_K be its absolute Galois group $\mathrm{Gal}(\overline{K}/K)$

Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ be its ring of integers, ϖ a uniformizer and k its residue field. We normalize the valuations $v_K : K^\times \to \mathbb{Z}$ by setting $v_K(\varpi) = 1$

There exists an exact sequence

$$0 \longrightarrow I_K \longrightarrow G_K \longrightarrow G_k \longrightarrow 0$$

where $G_K \to G_k \cong \hat{\mathbb{Z}}$ is also denoted v_K. We normalize the sequence by letting $1 \in \hat{\mathbb{Z}}$ be the geometric Frobenius.

Definition 4.1. The Weil group is defined to be $W_K := v_K^{-1}(\mathbb{Z})$. In other words elements of the Galois group which induce an integral power of Frobenius

We then have an exact sequence

$$0 \longrightarrow I_K \longrightarrow W_K \longrightarrow \mathbb{Z} \longrightarrow 0$$

Although W_K is a subgroup of G_K, we topologize by insisting W_K is homeomorphic to $\coprod_{n\in\mathbb{Z}} v_K^{-1}(n)$ where the sets of the disjoint union are homeomorphic to I_K as a topological space. Indeed this topology is different from the subspace topology coming from G_K. The reason we consider W_K comes from local Langlands; it has a much richer representation theory than G_K.

Let us briefly recall the notion of tamely ramified extension. Recall we have a tower of extensions:

$$K \subset K^{ur} \subset K^{tame} \subset \overline{K}$$

where K^{tame} is the maximal tamely ramified extension of K and K^{ur} the maximal unramified extension of K. The extension $\overline{K}/K^{tame}$ is a totally ramified pro-p extension, its Galois group is called the wild inertia group, and K^{tame}/K^{ur} is a totally ramified prime to p extension whose Galois group is called the tame inertia group. Tamely ramified extensions are generally easier to study than wildy ramified extensions as one can use Kummer theory to study tame extensions. In fact one can prove the following:

$$K^{tame} = \bigcup_{n\geq 1,(n,p)=1} K^{ur}(\varpi^{1/n})$$

There is a canonical isormorphism

$$\mathrm{Gal}(K^{ur}(\varpi^{1/n})/K^{ur}) \cong \mu_n$$

given by $\sigma \mapsto \sigma(\varpi^{1/n})/\varpi^{1/n}$ and there is a non-canonical isomorphism of μ_n with $\mathbb{Z}/n\mathbb{Z}$ which depends on a choice of primitive root.

We have isormorphisms

(4.1)

$$\mathrm{Gal}(K^{tame}/K^{ur}) \cong \lim_{\leftarrow (n,p)=1} \mathrm{Gal}(K^{ur}(\varpi^{1/n})/K^{ur}) \cong \lim_{\leftarrow (n,p)=1} \mu_n \cong \lim_{\leftarrow (n,p)=1} \mathbb{Z}/n\mathbb{Z} \cong \prod_{p' \neq p} \mathbb{Z}_{p'}$$

Composing the isomorphisms with the projections $I_K \to \mathrm{Gal}(K^{tame}/K^{ur})$ and $\prod_{p' \neq p} \mathbb{Z}_{p'} \to \mathbb{Z}_{p'}$ we obtain a map

$$t_{\zeta,p'} : I_K \to \mathbb{Z}_{p'}$$

where $\zeta = (\zeta_n)_{(n,p)=1}$ is a compatible choice of n^{th} roots of unity. This gives the projection of I_K onto its maximal pro-p' quotient; we will show later that this map in fact does not depend on the choice of ζ.

Let $Frob_k \in \mathrm{Gal}(K^{ur}/K)$, this acts by conjugation on I_K^{tame}. More precisely choose a $\phi \in \mathrm{Gal}(K^{tame}/K)$ a lift of $Frob_k$, the action is then given by conjugation by ϕ, this is well defined since $I_K^{tame} = \mathrm{Gal}(K^{tame}/K^{ur})$ is abelian. We have the following lemma:

Lemma 4.2. i) $\forall \tau \in I_K^{tame}$, $\phi^{-1}\tau\phi = \tau^{\#k}$
ii)$\forall \tau \in I_K$, $\sigma \in W_K$, $t_{\zeta,p'}(\sigma^{-1}\tau\sigma) = \#k^{v_K(\sigma)} t_{\zeta,p'}(\tau)$

Proof. i) Let $\eta = \frac{\tau(\varpi^{1/n})}{\varpi^{1/n}}$ a primitive n^{th} root of unity. If ϖ' is another uniformizer, we have $\tau(\frac{\varpi^{1/n}}{\varpi'^{1/n}}) = \frac{\varpi^{1/n}}{\varpi'^{1/n}}$ since $(n,p) = 1$, so $K^{ur}(\frac{\varpi^{1/n}}{\varpi'^{1/n}})$ is an unramified extension, and hence $\frac{\varpi^{1/n}}{\varpi'^{1/n}} \in K^{ur}$. Thus $\frac{\tau(\varpi'^{1/n})}{\varpi'^{1/n}} = \eta$.

We have

$$\frac{\phi^{-1}\tau\phi(\varpi^{1/n})}{\varpi^{1/n}} = \phi^{-1}(\frac{\tau\phi(\varpi^{1/n})}{\phi(\varpi^{1/n})}) = \phi^{-1}(\eta) = \eta^{\#k} = \frac{\tau^{\#k}(\varpi^{1/n})}{\varpi^{1/n}}$$

Since the images of $\varpi^{1/n}$ determines an element of $\mathrm{Gal}(K^{tame}/K^{ur})$ completely, this completes the proof of part i).

ii) This follows from part i) since $t_{\zeta,p'}$ factors through I_K^{tame}. □

An important theorem regarding l-adic representations is Grothendiecks l-adic monodromy theorem, which loosely speaking says any l-adic representation is potentially unipotent. More precisely we have

Theorem 4.3. *$l \neq p$, $L/\mathbb{Q}_l$ a finite extension, and $\rho : G_K \to GL_n(L)$ a continuous representation. Then there exists K'/K a finite extension such that $\rho|I_{K'}$ is unipotent.*

Proof. Step 1: We may assume $\rho(I_K)$ is pro-l. Indeed choose a G_K stable lattice in L^n and consider

$$\overline{\rho} : G_K \to GL_{\mathcal{O}_L}(\Lambda) \to GL(\Lambda/\varpi\Lambda)$$

Then $\ker(\overline{\rho})$ is open and has pro-l image, so we may replace K by $K' = \overline{K}^{\ker(\overline{\rho})}$.

Step 2: Since the wild inertia subgroup of I_K is pro-p, ρ has to factor through the tame inertia quotient, in fact the pro-l part of it, which we denote by $\rho^t : \mathrm{Gal}(K_l^{tame}/K) \to GL(\Lambda)$. Indeed by (3.1) it is enough to show that any continuous homomorphism of a profinite group prime to l to a pro-l subgroup must be trivial.

Step 3: Let $\tau \in I_l^{tame} \cong \mathbb{Z}_l$ be the inverse image of 1 under the isomorphism given by $t_{\zeta,l}$. We show that all eigenvalues of $\rho^t(\tau)$ are l power roots of unity.

Taking ϕ a lift of Frobenius as in Lemma 3.2, equation $\rho^t(\phi^{-1}\tau\phi) = \rho^t(\tau^{\#k}) = \rho^t(\tau)^{\#k}$ shows that the eigenvalues of $\rho^t(\tau)$ are the same as the eigenvalues of $\rho^t(\tau)^{\#k}$ (with multiplicity) so that the roots are all roots of unity. The sequence $\rho^t(\tau), \rho^t(\tau^l), \rho^t(\tau^{l^2})$ converges to 1 so we obtain the claim.

Step 4: Choose $m \geq 1$ such that the eigenvalues of $\rho^t(\tau)$ are contained in μ_{l^m}. We claim that $\forall \sigma \in I_l^{tame}, \rho^t(\sigma)^{l^m}$ are unipotent. We already know this is true for all $\sigma \in \tau^{\mathbb{Z}}$, but the continuity of ρ^t and the fact that $\tau^{\mathbb{Z}} \subset I_l^{tame}$ is dense, we obtain the claim.

Step 5: Take K'/K an extension such that $l^m | e_{K'/K}$ where $e_{K'/K}$ is the ramification index. Then the image of $I_{K'}$ in $\mathbb{Z}_l$ under the map $I_K \to I_l^{tame} \cong \mathbb{Z}_l$ is contained in $l^m\mathbb{Z}_l$. By the above steps, K' satisfies the properties in the Theorem. ☐

Corollary 4.4. *$\exists! N \in End(V)$ nilpotent, $\exists K'/K$ finite, such that*

$$\rho(\sigma) = \exp(t_{\zeta,p}(\sigma)N), \forall \sigma \in I_{K'}$$

N satisfies: $\sigma \in W_K$, $\rho(\sigma)N\rho(\sigma)^{-1} = \#k^{-v_K(\sigma)}N$

Proof. Define $N = \log \rho(\tau)$, where τ is the preimage of 1 in I_l^{tame} under $t_{\zeta,p}$. The second part follows from part ii) of Lemma 3.2. ☐

The point of this Theorem is that $\rho | I_{K'}$ factors through a pro cyclic group like $\mathbb{Z}_l$.

4.1. **Weil Deligne Representations.** Motivated by the previous Theorem and Corollary, we make the following definition:

Definition 4.5. A Weil-Deligne representations over a field Ω of characteristic 0 is a triple (V, ρ, N) where V is a finite dimensional vector space, $\rho : G_K \to GL_\Omega(V)$ a representation and a nilpotent $N \in End_\Omega(V)$, such that the following two conditions hold:

1) $\rho(I_K)$ is finite (equivalently ρ is continuous with respect to the discrete topology on Ω, equivalently $\ker(\rho)$ is open in W_K.

2) $\forall \sigma \in W_K$, $\rho(\sigma)N\rho(\sigma)^{-1} = \#k^{-v_K(\sigma)}N$

The key consequence of this is that it allow us to define a functor $WD_{\zeta,\phi}$ from G_K representations on a vector space over L to Weil-Degligne representations of W_K on a finite dimensional L vector space, by taking (V, ρ) to (V, r, N), where N is the nilpotent element constructed in Corollary 3.4 and $r : W_K \to GL_n(L)$ is the representation given by

$$r(\tau) = \rho(\sigma) \exp(-t_{\zeta,l}(\phi^{-v(\tau)}\tau)N)$$

. In fact this is an equivalence of categories, the N encodes the action of some open subgroup of the pro-l part of the tame inertia of I_K, rest of the G_K representation is then encoded in the Weil representation ρ.

5. Lecture 4

5.1. **Deformations of Galois representations.** Motivation: We would like a bijection between the set of automoprhic forms and l-adic Galois representations. To show the surjectivity of this map we use the following strategy.

Given an l-adic Galois representations ρ_0, define $\overline{\rho}_0$ to be ρ_0 mod l and show that it comes from some ρ_{f_0} associated to some automorphic representations f_0, this is the content of Serre's conjecture. We then study the map

$$\{\text{all } f \text{ s.t. } f \cong f_0 \bmod l\ \} \hookrightarrow \{\text{all } \rho \text{ s.t } \rho \cong \rho_0 \bmod l\ \}$$

$$\mathrm{Spec}T \qquad\qquad\qquad\qquad \mathrm{Spec}R$$

i.e. the deformations paces of f_0 and ρ_0 and we would like to show that these are the same The goal of the next few lectures is to study the right hand side of the above.

When deforming a $\rho : \Gamma \to GL_n(k)$ where k is a finite field, we need to impose a certain finiteness condition on Γ. Let l be prime and Γ a profinite group, and Δ a subgroup of Γ. Define $\Delta^l = \{g^l : g \in \Delta\}$, the condition we impose is the following

Hypothesis: "l finiteness" $\forall \Delta \subset \Gamma$ a finite index subgroup.

i)$\Delta/\overline{\langle[\Delta, \Delta], \Delta^l\rangle}$ is finitely generated.

ii) The maximal pro-l quotient of Δ is topologically finitely generated,

(Recall that a group G is topologically finitely generated if it contains a finitely generated dense subgroup)

Lemma 5.1. 1) i) and ii) are equivalent.

2) Γ topological finitely generated $\Rightarrow$i), ii)

To prove part 1) of this Lemma, we will need the following version of Burnside's basis lemma:

Proposition 5.2. *Let G be a finite group of l power order. If $g_1, ..., g_r$ are elements of G whose images in quotient $G/\langle[G, G], G^l\rangle$ generate it, then the $g_1, ..., g_r$ are a system of generators for G.*

Proof. □

Proof of Lemma. 1) Note that condition ii) of the hypothesis is equivalent to the existence of elements $g_1, .., g_r \in \Gamma$ such that if $\Gamma \to G$ is a surjection onto an l-power group G with open kernel, then the images of $g_1, ..., g_r$ generate G.

To show i) $\Rightarrow$ ii), let $g_1, ..., g_r$ be a finite set elements whose images are a set of generators for $\Delta/\overline{\langle [\Delta,\Delta], \Delta^l\rangle}$. Then if $\Gamma \to G$ is a surjection with open kernel onto a l-power subgroup with open kernel, it follows that $G/\langle [G,G], G^l\rangle$ is a quotient of $\Delta/\overline{\langle [\Delta,\Delta], \Delta^l\rangle}$, hence by Burnside's lemma $g_1, ..., g_r$ generate G.

For ii) $\Rightarrow$ i) Let $g_1, ..., g_r$ be elements in Δ whose images topologically generate the maximal pro-l quotient. Note that $\Delta/\overline{\langle [\Delta,\Delta], \Delta^l\rangle}$ is an abelian group which is killed by l, hence is an $\mathbb{F}_l$ vector space, and suppose it is not finite dimensional. As $\overline{\langle [\Delta,\Delta], \Delta^l\rangle}$ is closed, the quotient group is profinte, in fact pro-l since every finite quotient is an l-power group. Thus there exists an open subgroup $H \subset \Delta/\overline{\langle [\Delta,\Delta], \Delta^l\rangle}$ whose quotient is a finite dimensional vector space over $\mathbb{F}_l$ of dimension greater than r. The projection of Γ onto this group has open kernel and hence factors through the maximal pro-l quotient, but the images of $g_1, ..., g_r$ cannot generate this quotient.

2) Γ topologically finitely generated implies ii) and hence i) by part 1). □

Example 5.3. Local: $K/\mathbb{Q}_p$ finite, ($p = l$ or $p \neq l$), $G_K := Gal(\overline{K}/K)$

Global: $F/\mathbb{Q}$ finite $S \subset \{\text{finite places of } F\}$ a finite subset. F_S maximal extension of F unramified outside of S. $G_{F,S} := Gal(F_S/F)$

Lemma 5.4. The hypothesis is satisfied for G_K and $G_{F,S}$. [In DDT, use GCFT]

Proof. For G_K, i) can be rephrased as saying $\forall K'/K$ finite, there exists only finitely many abelian extension of K' of exponent l. It is enough to show this for the case $K = K'$, and for this we can reduce to the case when $\mu_l \subset K$. Kummer theory implies there is a bijection

$$K^\times/(K^\times)^l \to \{\mathbb{Z}/l\mathbb{Z} \text{ extension of } K\} \coprod \{K\}$$

given by $\alpha \mapsto K(\alpha^{1/l})$

To see used why $K^\times/(K^\times)^l$ is finite, note that we have

$$K^\times \cong \varpi_K^{\mathbb{Z}} \times \mathcal{O}_K^\times$$

where $\mathcal{O}_K^\times$ can be decomposed into $\mu(\mathcal{O}_K) \times (1 + \varpi\mathcal{O}_K)$. The statement then follows since $\mu(\mathcal{O}_K)$ is finite and $(1 + \varpi\mathcal{O}_K)$ is isomorphic to $\mathcal{O}_K$ via the exponential (note: slightly different for $p = 2$). □

Suppose we are given a finite extension $L/\mathbb{Q}_l$ with ring of integers $\mathcal{O} = \mathcal{O}_L$, unifomrizer λ and residue field $\mathbb{F} = \mathcal{O}_L/\lambda$. Define $\mathcal{C}_\mathcal{O}$ to be the category

$$\mathrm{ob}(\mathcal{C}_{\mathcal{O}}) = \{\text{complete noetherian local } \mathcal{O}\text{-algebras } A \text{ such that } A/\mathfrak{m}_A \cong \mathbb{F}\}$$

where complete means the canonical morphism

$$A \to \lim_{\leftarrow n} A/\mathfrak{m}_A^n$$

is an isomorphism.

$$\mathrm{Mor}(\mathcal{C}_{\mathcal{O}}) = \{\text{morphisms of local } \mathcal{O}\text{-algebras}\}$$

Here a morphism of local $\mathcal{O}$ algebras $A \to B$ is local if $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$, (in particular this induces an iso. $A/\mathfrak{m}_A \cong B/\mathfrak{m}_B$).

Now fix a continuous representation $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$.

Definition 5.5. $\mathscr{R}^{\square}_{\overline{\rho}} : \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets}$ given by

$$A \mapsto \{\rho : \Gamma \to GL_n(A) \text{ such that } \rho \mod \mathfrak{m}_A = \overline{\rho}\}$$

Remark 5.6. We do not mention isomorphism/ equivalence classes in the definition.

Proposition 5.7. *The functor $\mathscr{R}^{\square}_{\overline{\rho}}$ is representable by some $R^{\square}_{\overline{\rho}} \in \mathcal{C}_{\mathcal{O}}$, called the universal framed lifting ring.*

More explicitly this means $R^{\square}_{\overline{\rho}} \in \mathcal{C}_{\mathcal{O}}$ satisfies one of the 2 equivalent conditions.

1) $\exists$ bijections $\mathscr{R}^{\square}_{\overline{\rho}}(A) \cong \mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(R^{\square}_{\overline{\rho}}, A)$ which are functorial in A. We let $\rho^{\square}_{\overline{\rho}}$ be the element in $\mathscr{R}^{\square}_{\overline{\rho}}(R^{\square}_{\overline{\rho}})$ which is the the inverse image of id on $R^{\square}_{\overline{\rho}}$.

2) $\exists \rho^{\square}_{\overline{\rho}} : \Gamma \to GL_n(R^{\square}_{\overline{\rho}})$ a universal lifting of $\overline{\rho}$ satisfying the universal property:

$\forall A \in \mathcal{C}_{\mathcal{O}}$ and $\rho : \Gamma \to GL_n(A)$ lifting $\overline{\rho}$, $\exists! f_\rho : \mathscr{R}^{\square}_{\overline{\rho}} \to A$ making the following diagram commute

$$\begin{tikzcd}
\Gamma \arrow[r, "\rho"] \arrow[dr, "\rho_{\overline{\rho}}"'] & GL_n(A) \\
& GL_n(R^{\square}_{\overline{\rho}}) \arrow[u, dashed, "f_\rho"']
\end{tikzcd}$$

Warm up exercise: $\Gamma = \hat{\mathbb{Z}}$, the l finiteness hypothesis is satisfied for Γ, then giving $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$ is equivalent to giving $\overline{\rho}(1) \in GL_n(\mathbb{F})$. Fix a lifting $\widetilde{\overline{\rho}(1)} \in GL_n(\mathcal{O})$.

We have

$$\mathscr{R}^{\square}_{\overline{\rho}}(A) = \{\rho : \hat{\mathbb{Z}} \to GL_n(A) : \rho \mod \mathfrak{m}_A = \overline{\rho}\}$$

is in bijection with the set

$$\widetilde{\overline{\rho}(1)} + \mathfrak{m}_A.M_n(A)$$

and this last set is in bijection with $\mathfrak{m}_A^{n^2}$, given by the map $\widetilde{\overline{\rho}(1)}) + (a_{ij}) \mapsto (a_{ij})$.

Therefore

$$\mathscr{R}_{\overline{\rho}}^{\square}(A) \cong \mathfrak{m}_A^{n^2} \cong \mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(\mathcal{O}[[X_{ij}]]_{i,j=1}^n, A)$$

The upshot of this is that $\mathcal{O}[[x_{ij}]]_{i,j=1}^n$ represents the functor $\mathscr{R}_{\overline{\rho}}^{\square}$

Proof of Prop. 5.7. Step 1: We may assume Γ is topologically finitely generated. Let $\Gamma_0 := ker\overline{\rho} \subset \Gamma$, and Δ the kernel of the projection of Γ_0 onto the maximal pro-l quotient of Γ_0

Claim: May replace Γ with Γ/Δ. This is implied by the following

1) Γ/Δ is a profinite group

2) Γ/Δ is topologically finitely generated

3) Every lifting of $\overline{\rho}$ factors through Γ/Δ.

proof of 1: This follows by definition of the maximal pro-l quotient which specifies that the quotient needs to be pro-l, hence profinite, hence Hausdorff. This is equivalent to the closedness of Δ.

proof of 2:

$$1 \longrightarrow \Gamma_0/\Delta \longrightarrow \Gamma/\Delta \longrightarrow \Gamma/\Gamma_0 \longrightarrow 1$$

The group on the left is topologically finitely generated, and the group on the right is finite, hence Γ/Δ is topologically finitely generated.

proof of 3: Let $\rho : \Gamma \to GL_n(A)$ be a lifting of $\overline{\rho}$, so that the following diagram commutes:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho"] \arrow[dr, "\overline{\rho}"'] & GL_n(A) \arrow[d] \\ & GL_n(\mathbb{F}) \end{tikzcd}$$

We have $\rho(\Delta) \subset ker(GL_n(A) \to GL_n(\mathbb{F}))$, which is a pro-$l$ subgroup hence $\rho(\Delta) = \{1\}$

Step 2: When Γ is topologically finitely generated, let $s \geq 1$ and define F_s to be the free group on generators $g_1, ..., g_s$ and $\hat{F}_s$ the pro finite completion of F_s

Choose s large enough so that we have

$$\begin{tikzcd} \hat{F}_s \arrow[r, two heads, "\varphi"] & \Gamma \\ F_s \arrow[u, hook] \arrow[r, "\varphi_0"] & \varphi(F_s) \arrow[u, hook] \end{tikzcd}$$

Define $H = \ker \varphi$ and $H_0 = \ker \varphi_0$. One checks that $H_o \subset H$ is dense. Then giving a lifting

$$\begin{tikzcd} \Gamma \arrow[r] & GL_n(A) \\ \hat{F}_s \arrow[u] \arrow[ur, "\tilde{\rho}"'] & \end{tikzcd}$$

is equivalent to giving $\tilde{\rho}(g_i)$, $i = 1, ..., s$ with the constraint $\tilde{\rho}(\gamma) = 1\ \forall \gamma \in H_0$. Thus $\mathscr{R}_{\overline{\rho}}^{\square}$ is represented by

$$\mathcal{O}[[x_{ij}^{(k)}]]_{i,j=1}^{n}/\text{relations coming from } H_0$$

$\square$

6. Lecture 6

Some primaries on irreducible representations

Let k be a field, Γ is now an abstract group and $\rho : \Gamma \to GL_k(V)$ a representation of Γ on a finite dimensional k-vector space V.

Lemma 6.1 (Schur)**.** ρ irreducible imples $\mathrm{End}(\rho)$ is a finite dimensional division algebra over k.

Proof. Let $0 \neq \alpha \in \mathrm{End}(\rho)$. Since $0 \neq \alpha V \subset V$ is Γ stable, we have $\alpha V = V$, hence $\exists \alpha^{-1}$. Finite dimensionality is clear. $\square$

Lemma 6.2. If $k = \overline{k}$, then for ρ_1, ρ_2 irreducible,

$$\mathrm{Hom}_{k[\Gamma]}(\rho_1, \rho_2) = \begin{cases} k & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. It is easy to see that any finite dimensional division algebra over k is equal to k which shows the first case. When ρ_1 and ρ_2 are not isomorphic, the image of any $\theta \in \mathrm{Hom}_{k[\Gamma]}(\rho_1, \rho_2)$ is Γ invariant, hence is 0 since ρ_2 is irreducible. $\square$

Remark 6.3. • ρ irreducible does not imply $\rho \otimes_k \overline{k} : \Gamma \to GL(V \otimes_k \overline{k})$ irreducible.

• $\mathrm{End}_{k[\Gamma]}(\rho) = k$ does not imply that ρ is irreducible. For example the standard representation of the upper triangular Borel in $GL_2(k)$.

The above lemmas motivate the following definition.

Definition 6.4. A representation ρ is Schur if $\mathrm{End}_{k[\Gamma]}(\rho) = k$. It is absolutely irreducible if $\forall k'/k, \rho \otimes_k k'$ is irreducible.

We have the following fact which can be found in Curtis Reiner:

$$\rho \text{ absolutely irreducible} \Leftrightarrow \rho \otimes_k \overline{k} \text{ is irreducible} \Leftrightarrow \rho \text{ irreducible and Schur}$$

Let us now return to the set up of the last lecture so that $L/\mathbb{Q}_l$ is a finite extension with ring of integers $\mathcal{O}$, uniformizer λ and residue field $\mathbb{F}$. Γ will be a profinite group and Hyp(l) is the l finiteness hypothesis. As be fore $\mathcal{C}_\mathcal{O}$ will be the category of complete Noetherian local $\mathcal{O}$ algebras over A with an isomorphism $A/\mathfrak{m}_A \cong \mathbb{F}$ (which is necessarily unique since $\mathcal{O}$ surjects onto $\mathbb{F}$).

We define a slightly different deformation problem than last time which takes into account isomorphisms.

Definition 6.5. Let $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$ a continuous representation which is Schur, define the deformation problem $\mathscr{R}_{\overline{\rho}}$ to be given by

$$\mathscr{R}_{\overline{\rho}} : A \mapsto \{\rho : \Gamma \to GL_n(A) | \rho \mod \mathfrak{m}_A = \overline{\rho}\}/\cong$$

Note:

$$\rho_1 \cong \rho_2 \Leftrightarrow \exists a \in GL_n(A), a\rho_1 a^{-1} = \rho_2$$
$$\Leftrightarrow \exists a \in \ker(GL_n(A) \to GL_n(\mathbb{F})), a\rho_1 a^{-1} = \rho_2$$

where the second equivalence relies on the fact that ρ_1 and ρ_2 reduce to the $\overline{\rho} \mod \mathfrak{m}_A$.

Proposition 6.6. *If $\overline{\rho}$ is Schur the $\mathscr{R}_{\overline{\rho}}$ is representable say by $R_{\overline{\rho}}^{univ} \in \mathcal{C}_\mathcal{O}$ the universal deformation ring of $\overline{\rho}$.*

Remark 6.7. For Galois representations, deformations were introduced by Mazur, later liftings were introduced by Kisin in [6]

Sketch of Proof. • Mazur: Schlessinger's criterion.
• Can argue as for $\mathscr{R}_{\overline{\rho}}^{\square}$.
• Kisin: $\mathscr{R}_{\overline{\rho}}^{univ} = R_{\overline{\rho}}/PGL_n$ ([Boe] 2.1) □

As before we obtain a universal deformation $\rho_{\overline{\rho}}^{univ}$:

$$\begin{tikzcd} \rho_{\overline{\rho}}^{univ} : \Gamma \arrow[r] \arrow[dr, "\rho"'] & GL_n(R_{\overline{\rho}}^{univ}) \arrow[d, dashed, "\exists!"] \\ & GL_n(A) \end{tikzcd}$$

where the dashed line is induced by a unique map $f_\rho : R_{\overline{\rho}}^{univ} \to A$ in $\mathcal{C}_\mathcal{O}$ and is such that $f_\rho \circ \rho_{\overline{\rho}}^{univ} \cong \rho$ (so the diagram only commutes "up to isomorphism").

Lemma 6.8. $\exists$ a canonical map $R_{\overline{\rho}}^{univ} \to R_{\overline{\rho}}^{\square}$

There are two ways to see this:

1) This map is induced by the map on functors given by $\mathscr{R}^{\square}_{\overline{\rho}}(A) \to \mathscr{R}^{univ}_{\overline{\rho}}(A)$ taking a deformation ρ to its isomorphism class.

2) Viewing $\Gamma \to GL_n(R^{\square}_{\overline{\rho}})$ as a deformation of $\overline{\rho}$ we get an induced map $R^{univ}_{\overline{\rho}} \to R^{\square}_{\overline{\rho}}$

We now collect some linear algebraic lemmas that we'll need.

Lemma 6.9. Let $A \in \mathcal{C}_{\mathcal{O}}$, $\rho : \Gamma \to GL_n(A)$ a continuous representation with $\overline{\rho} := \rho \mod \mathfrak{m}_A$ absolutely irreducible.

1) (Schur's lemma for A coefficients) $a \in GL_n(A)$, $a\rho a^{-1} = \rho \Rightarrow a \in A^{\times}$

2) (Carayol's Lemma) $B \subset A$ closed, $B \in \mathcal{C}_{\mathcal{O}}$, and $tr\rho(\Gamma) \subset B \Rightarrow \exists a \in \ker(GL_n(A) \to GL_n(\mathbb{F}))$ such that $a\rho a^{-1}(\Gamma)$ lands in $GL_n(B) \subset GL_n(A)$.

Proof. The general idea here is to use completeness to reduce to the case of artinian local $\mathcal{O}$ algebras A, and then induct on the length of A. To do this, one proves the base case $A = \mathbb{F}$ and then reduce the induction step to the case $\mathbb{F}[\epsilon]/\epsilon^2$.

1) We may assume A is artinian local. Choose a minimal non-zero ideal $I \subset A$, which must be isomorphic to $\mathbb{F}$ as an $\mathcal{O}$ (or even A) module. Indeed $0 = \mathfrak{m}_A^{e+1} \subsetneq \mathfrak{m}_A^e \subsetneq ... \subsetneq A$ with each quotient is an $\mathbb{F}$ vector space, then choose $I \subset \mathfrak{m}_A^e$ a 1 dimensional subspace.

Let $a \in GL_n(A)$ commuting with ρ. The induction hypothesis implies

$$a \mod I \in \mathrm{End}_{A/I}(\rho \mod I) \cong (A/I)^{\times}$$

therefore we can write $a = \alpha 1_n + a_0$, where $\alpha \in A^{\times}$ and $a_0 \in M_n(I)$.

Then for all $\gamma \in \Gamma$, we have

$$(\alpha 1_n + a_0)\rho(\gamma) = \rho(\gamma)(\alpha 1_n + a_0)$$

which implies

$$a_0\rho(\gamma) = \rho(\gamma)a_0 \text{ in } M_n(I)$$

Since $\mathbb{F} = I$ and A surjects onto $\mathbb{F}$ we have

$$a_0\overline{\rho}(\gamma) = \overline{\rho}(\gamma)a_0 \text{ in } M_n(\mathbb{F})$$

then Schur's lemma implies a_0 is a scalar.

2) Again assume A and B are local artinian, and take $\mathbb{F} \cong I \subset \mathfrak{m}_A$ as before. The induction hypothesis implies that we may assume

$$\rho(\Gamma) \mod I \subset GL_n(B/I \cap B)$$

Then

$$I \cap B \subset I \cong \mathbb{F} \Rightarrow I \cap B = I \text{ or } 0$$

The first case is easy: $\rho(\gamma) \mod I \in GL_n(B/I) \Rightarrow \rho(\gamma) \in GL_n(B)$

For the second case consider the inclusion $B \oplus I\epsilon \hookrightarrow A$ given by $(b, i\epsilon) \mapsto b + i$ where we consider $I\epsilon$ as a square zero ideal. We may assume that it's an isomorphism by the induction hypothesis.

Look for an $a \in \ker(GL_n(A) \to GL_n(A/I))$, $a\rho a^{-1}(\Gamma) \subset GL_n(B)$. Suppose there exists $A \in M_n(I)$ with

$$(1+A)\rho(\gamma)(1+A)^{-1} \mod \mathfrak{m}_B \in GL_n(B/\mathfrak{m}_B)$$

then

$$(1+A)\rho(\gamma)(1+A)^{-1} \in GL_n(B)$$

Thus we reduce to the case $A/\mathfrak{m}_B \cong \mathbb{F} \oplus \mathbb{F}\epsilon = \mathbb{F}[\epsilon]/\epsilon^2$ and $B/\mathfrak{m}_B \cong \mathbb{F}$.

See 7.1 for the proof of this case.

□

Since we are now considering deformations up to isomorphism (read conjugation) it should be possible to study them via their traces. In fact in the absolutely irreducible case, this holds in a strong sense below, providing us with a generalization of Brauer Nesbitt to more general coefficients.

Lemma 6.10 (Brauer Nesbitt for A coefficients)**.** Let $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$ be an absolutely irreducible representation, and $\rho_1, \rho_2 : \Gamma \to GL_n(A)$ are isomorphic to $\overline{\rho} \mod \mathfrak{m}_A$. Then

$$tr\rho_1 = tr\rho_2 \Rightarrow \rho_1 \cong \rho_2$$

(if $\rho_1 = \rho_2 \mod \mathfrak{m}_A$, $\exists a \in \ker(GL_n(A) \to GL_n(\mathbb{F}), \rho_1 = a\rho_2 a^{-1})$

Proof. We can make similar reductions as in the previous lemma. Can use Lemma 6.9(2) when $A = \mathbb{F}[\epsilon/\epsilon^2$. □

Corollary 6.11. *The ring $R^{univ}_{\overline{\rho}}$ is topologically generated over $\mathcal{O}$ by the elements $tr(\rho^{univ}_{\overline{\rho}}(\gamma))$ for γ in a dense subset of Γ.*

Proof. Let S be the closure in $R^{univ}_{\overline{\rho}}$ of the subring generated by the $tr(\rho^{univ}_{\overline{\rho}}(\gamma))$. Then S is an element of $\mathcal{C}_{\mathcal{O}}$ with maximal ideal $\mathfrak{m}_S = \mathfrak{m}_R \cap S$.

Since $tr(\rho^{univ}_{\overline{\rho}}(\gamma)) \in S$ for a dense subset of Γ, it follows that this is so for all of Γ by continuity, hence by part 2) of Lemma 6.9,

$$\exists a \in \ker(GL_n(R^{univ}\overline{\rho}) \to GL_n(\overline{\mathbb{F}}_p))$$

such that

$$a\rho^{univ}_{\overline{\rho}}(\gamma)a^{-1} \in GL_n(S)\ \forall \gamma \in \Gamma$$

This induces a map $R^{univ}_{\overline{\rho}} \to S$ which, since conjugation preserves traces, is a retract of the inclusion $S \hookrightarrow R^{univ}_{\overline{\rho}}$.

Now the inclusion $S \hookrightarrow R^{univ}_{\overline{\rho}}$ induces a morphism of functors $\mathscr{R}^{univ}_{\overline{\rho}} \to \mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(S, -)$ which is surjective by the last paragraph. It is also injective since a deformation is determined by its trace on a dense open subset of Γ. It follows by Yoneda's lemma that the inclusion $S \hookrightarrow R^{univ}_{\overline{\rho}}$ is an isomorphism in $\mathcal{C}_{\mathcal{O}}$ □

7. Lecture 7

Recall last time we reduced Lemma 6.9 down to the following.

Lemma 7.1. Let $\rho : \Gamma \to GL_n(\mathbb{F}[\epsilon]/\epsilon^2)$ a continuous representation of a profinite group such that the reduction $\overline{\rho}$ of $\rho \mod \epsilon$ is absolutely irreducible, and $tr\rho \subset \mathbb{F}$. Then there exists $a \in 1 + M_n(\mathbb{F})\epsilon$ such that $a\rho a^{-1}(\Gamma) \subset GL_n(\mathbb{F})$

Proof.

$$\rho : \mathbb{F}[\Gamma] \to M_n(\mathbb{F}[\epsilon]/\epsilon^2) = M_n(\mathbb{F}) \oplus M_n(\mathbb{F})\epsilon$$

can be written in the following form

$$\rho(\gamma) = \overline{\rho}(\gamma) + \theta(\gamma)\epsilon$$

The map $\theta : \mathbb{F}[\Gamma] \to M_n(\mathbb{F})$ has the following properties:

1) $\theta : \mathbb{F}[\Gamma] \to M_n(\mathbb{F})$ is $\mathbb{F}$ linear.

2) $\theta(\gamma\delta) = \theta(\gamma)\overline{\rho}(\delta) + \overline{\rho}(\gamma)\theta(\delta)$. (This follows from looking at the coefficient of ϵ in $\rho(\gamma\delta) = \rho(\gamma)\rho(\delta)$)

3) $tr(\theta(\gamma)) = 0$.

Claim: $\theta : \mathbb{F}[\Gamma] \to M_n(\mathbb{F})$ factors uniquely as

$$\begin{tikzcd} \mathbb{F}[\Gamma] \arrow[r, "\theta"] \arrow[dr, "\overline{\rho}"'] & M_n(\mathbb{F}) \\ & M_n(\mathbb{F}) \arrow[u, dashed, "\theta'"'] \end{tikzcd}$$

Proof of claim: $\delta \in \ker(\overline{\rho})$. $0 = tr\theta(\gamma\delta) = tr\overline{\rho}(\gamma)\theta(\delta)$

$$\Rightarrow \theta(\delta) = 0$$

since $\overline{\rho}(\gamma)$ can be anything in $M_n(\mathbb{F})$.

Thus we are looking for $a \in 1 + M_n(\mathbb{F})\epsilon$, say $a = 1 + a'\epsilon$, such that :

$$(1 + a'\epsilon)\rho(\gamma)(1 - a'\epsilon) \in M_n(\mathbb{F}), \forall\gamma$$

$$\Leftrightarrow \theta(\gamma) + a'\overline{\rho}(\gamma) - \overline{\rho}(\gamma)a' = 0$$

the last expression is the coefficient of epsilon.

We have thus reduced the problem to the following:

Given $\theta' : M_n(\mathbb{F}) \to M_n(\mathbb{F})$, $\mathbb{F}$-linear and satisfying

•$tr\theta'(\gamma) = 0 \ \forall\gamma \in M_n(\mathbb{F})$

•$\theta'(\gamma\delta) = \theta'(\gamma)\delta + \gamma\theta'(\delta), \gamma\delta \in M_n(\mathbb{F})$

Then we can find an $a' \in M_n(\mathbb{F})$ such that $\theta'(\gamma) = \gamma a' - a'\gamma$. In other words, that every derivation on $\mathfrak{sl}_n$ is by Lie Bracket. Can show: $a' = \sum_{d=1}^{n} \theta'(e_{j1})e_{1j}$ works. $\square$

Before beginning to study the constructed deformation/lifting spaces, we record a result which shows that the formation of the of $R^{\square}_{\overline{\rho}}$ is compatible with base change.

Proposition 7.2. *Let $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$ and absolutely irreducible continuous representation of a profinite group satisfying $Hyp(\Gamma)$. Suppose L'/L is a finite extension and $\mathcal{O}'$ the ring of integers of L'. Let $\mathbb{F}'$ be the residue field of $\mathcal{O}'$ and consider $\overline{\rho}' : \overline{\rho} \otimes_{\mathbb{F}} \mathbb{F}'$, (in other words the composition of $\overline{\rho}$ with the inclusion $GL_n(\mathbb{F}) \subset GL_n(\mathbb{F}')$. There there is a canonical isomorphism*

$$R^{\square}_{\overline{\rho}'} = R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$$

in the category $\mathcal{O}'$.

Proof. Let $\rho' : \Gamma \to A'$ be a deformation of $\overline{\rho}'$. Let A be the preimage in A' of $\mathbb{F}$ under the reduction map $\mathbb{A}' \to \mathbb{F}'$, in particular A is an element of $\mathcal{C}_{\mathcal{O}}$. Since the reduction of ρ mod $\mathfrak{m}'_A$ if actors through $GL_n(\mathbb{F})$, ρ' factors through $GL_n(A)$, hence corresponds to a lifting $\rho : \Gamma \to GL_n(A)$. Since $A \in \mathcal{C}_{\mathcal{O}}$, we obtain a map of $\mathcal{O}$ algebras

$$R^{\square}_{\overline{\rho}} \to A'$$

by composing with the inclusion $A \to A'$, and by extension of scalars, a map of $\mathcal{O}'$ algebras

$$R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}' \to A'$$

Conversely, given a map

$$R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}' \to A'$$

, we obtain a deformation of $\overline{\rho}'$ to A' given by the composition

$$\Gamma \to GL_n(R^{\square}_{\overline{\rho}}) \to GL_n(R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}') \to GL_n(A')$$

It is easily check ed that these constructions are inverse to each other, and hence we obtain a canonical isomorphism

$$R^{\square}_{\overline{\rho}'} \cong R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$$

□

7.1. **Tangent spaces.** Let L be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$, uniformizer λ and residue field $\mathbb{F}$. Recall given a continuous mod l representation $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$, with Γ satisfying Hyp(Γ), we constructed the universal framed lifting ring $R^{\square} = R^{\square}_{\overline{\rho}}$, an element of the category $\mathcal{C}_{\mathcal{O}}$. We let $\mathfrak{m}^{\square}$ denote the maximal ideal of $R^{\square}$.

The adjoint representation is given by the composite

$$\Gamma \longrightarrow GL_n(\mathbb{F}) \overset{\text{ad}}{\longrightarrow} \text{End}(M_n(\mathbb{F}))$$

where the map ad is given by

$$g \mapsto (\phi \mapsto g\phi g^{-1})$$

ad$\overline{\rho}$ also denotes $M_n(\mathbb{F})$ as an $\mathbb{F}[\Gamma]$-module.

Lemma 7.3. There exists a natural bijection between:
1) $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}^{\square}/(\mathfrak{m}^{\square 2},\lambda),\mathbb{F})$
2) $\mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(R^{\square},\mathbb{F}[\epsilon]/\epsilon^2)$
3) $\mathscr{R}(\mathbb{F}[\epsilon]/\epsilon^2)=\{\text{liftings of } \overline{\rho} \text{ to } \mathbb{F}[\epsilon]/\epsilon^2)\}$
4) $Z^1(\Gamma,\mathrm{ad}\overline{\rho})$ continuous 1-cocycles.

Sketch of Proof. (2⇔3) is by definition.

(1 ⇔ 2) given $f\in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}^{\square}/(\mathfrak{m}^{\square 2},\lambda),\mathbb{F})$, define a map $a+x\mapsto \overline{a}+f(x)\epsilon$, where $a\in\mathcal{O}$ and $x\in\mathfrak{m}^{\square}$. This is possible since $\mathcal{O}+\mathfrak{m}$ surjects onto $R^{\square}$, and since $\mathcal{O}\cap\mathfrak{m}=(\lambda)$, it is well defined.

Conversely given a map $g\in\mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(R^{\square},\mathbb{F}[\epsilon]/\epsilon^2)$, note that g must map $\mathfrak{m}$ to $\epsilon\mathbb{F}[\epsilon]/\epsilon^2$, which we identify with $\mathbb{F}$. This induces a map $\mathfrak{m}^{\square}/(\mathfrak{m}^{\square 2},\lambda)\to\mathbb{F}$ since $\epsilon^2=0$ and $g(\lambda)=0$ being a map of $\mathcal{O}$ algebras. It is straightforward to check these constructions are mutually inverse.

3) ⇔ 4) The bijection is defined as follows. Given $\rho:\Gamma\to\mathbb{F}[\epsilon]/\epsilon^2$ a lifting of $\overline{\rho}$, write ρ as $\rho(\gamma)=\overline{\rho}(\gamma)+\theta(\gamma)\epsilon$, where $\theta:\Gamma\to M_n(\mathbb{F})$. Using

$$\theta(\gamma\delta)=\theta(\gamma)\overline{\rho}(\delta)+\overline{\rho}(\gamma)\theta(\delta)$$

(cf. Lemma 7.1) it is easy to check that $\gamma\mapsto\theta(\gamma)\overline{\rho}(\gamma)^{-1}$ defines a 1-cocycle in $Z^1(\Gamma,\mathrm{ad}\overline{\rho})$. This inverse construction is then clear. ☐

Corollary 7.4. *There are canonical bijections between the following sets:*
1) $Hom_{\mathbb{F}}(\mathfrak{m}_{\overline{\rho}}^{univ}/(\mathfrak{m}_{\overline{\rho}}^{univ},\lambda),\mathbb{F})$
2) $H^1(\Gamma,ad\overline{\rho})$
3) $Ext^1(\overline{\rho},\overline{\rho})$

Proof. The bijection between 1) and 2) follows from the previous lemma, since if ρ and ρ' are liftings to $\mathbb{F}[\epsilon]/\epsilon^2$, they are isomorphic if and only the corresponding elements in $Z^1(\Gamma,ad\overline{\rho})$ differ by a coboundary. ☐

Define $d:\dim_{\mathbb{F}} Z^1(\Gamma,\mathrm{ad}\overline{\rho})=\dim_{\mathbb{F}}\mathfrak{m}^{\square}/(\mathfrak{m}^{\square 2},\lambda)$

Corollary 7.5. $d=\dim_{\mathbb{F}} H^1(\Gamma,ad\overline{\rho})-\dim_{\mathbb{F}} H^0(\Gamma,ad\overline{\rho})+n^2$.

Proof. There exists an exact sequence of finite dimensional $\mathbb{F}$ vector spaces.

$$0\longrightarrow(\mathrm{ad}\overline{\rho})^{\Gamma}\longrightarrow\mathrm{ad}\overline{\rho}\longrightarrow Z^1(\Gamma,\mathrm{ad}\overline{\rho})\longrightarrow H^1(\Gamma,\mathrm{ad}\overline{\rho})\longrightarrow 0$$

$$\phi\longrightarrow(\gamma\mapsto\gamma\phi-\phi)$$

☐

Consequently, one can choose $\phi:\mathcal{O}[[X]]:=\mathcal{O}[[x_1,...,x_d]]\to R^{\square}$ such that $\phi(x_i)\mathfrak{m}^{\square}$ generate $\mathfrak{m}^{\square}/(\mathfrak{m}^{\square 2},\lambda)$ are $\mathbb{F}$ vector spaces, hence ϕ is onto.

To further control $R^\square$, we analyze $J = \ker \phi$. Observe $\mathfrak{m}J \subset J \subset \mathfrak{m}$, where $\mathfrak{m} = (\lambda, x_1, ..., x_d)$ is the maximal ideal of $\mathcal{O}[[X]]$. We will construct a map

$$\mathrm{Hom}_{\mathbb{F}}(J/\mathfrak{m}J, \mathbb{F}) \to H^2(\Gamma, \mathrm{ad}\overline{\rho})$$

$$f \mapsto [c_f]$$

which will turn out to be an injection, hence allow us to bound the dimension of the deformation ring.

Consider

$$\rho^\square : \Gamma \to GL_n(\mathcal{O}[[X]]/J) \leftarrow GL_n(\mathcal{O}[[X]]/\mathfrak{m}J)$$

For each $\gamma \in \Gamma$ choose a lift $\widetilde{\rho}(\gamma) \in GL_n(\mathcal{O}[[X]]/\mathfrak{m}J)$ which lifts $\rho(\gamma)$. $\widetilde{\rho}$ may not be a homomorphism.

Define

$$c_f(\gamma, \delta) = f(\widetilde{\rho}(\gamma\delta)\widetilde{\rho}(\delta)^{-1}\widetilde{\rho}(\gamma)^{-1} - 1_n) \in M_n(\mathbb{F})$$

We claim that $c_f \in Z^2(\Gamma, \mathrm{ad}\overline{\rho})$, i.e. it is a 2-cocycle.

Proof of claim: Need to show

$$g_1 c_f(g_2, g_3) - c_f(g_1 g_2, g_3) + c_f(g_1, g_2 g_3) - c_f(g_1, g_2) = 0$$

Observe $M_n((J/\mathfrak{m}J), +) \cong (1 + M_n(J/\mathfrak{m}J), .)$ given by $a \mapsto 1 + a$, in particular it is commutative. We need to check:

$$\begin{aligned}&\overline{\rho}(g_1) f(\widetilde{\rho}(g_2 g_3)\widetilde{\rho}(g_3)^{-1}\widetilde{\rho}(g_2)^{-1})\overline{\rho}(g_1)^{-1} + f(\widetilde{\rho}(g_1 g_2 g_3)\widetilde{\rho}(g_2 g_3)^{-1}\widetilde{\rho}(g_1)^{-1}) \\ &= f(\widetilde{\rho}(g_1 g_2 g_3)\widetilde{\rho}(g_3)^{-1}\widetilde{\rho}(g_1 g_2)^{-1} + f(\widetilde{\rho}(g_1 g_2)\widetilde{\rho}(g_2)^{-1}\widetilde{\rho}(g_1)^{-1})\end{aligned}$$

Write $\widetilde{g}$ for $\widetilde{\rho}(g)$, and switching to multiplicative it suffices to show the following equality in $GL_n(\mathcal{O}[[X]]/\mathfrak{m}J)$.

$$(\widetilde{g_1}\widetilde{g_2 g_3}\widetilde{g_3}^{-1}\widetilde{g_2}^{-1}\widetilde{g_1}^{-1})(\widetilde{g_1 g_2 g_3}\widetilde{g_2 g_3}^{-1}\widetilde{g_1}^{-1})$$

$$= \widetilde{g_1 g_2 g_3}\widetilde{g_3}^{-1}\widetilde{g_1 g_2}^{-1}\widetilde{g_1 g_2}\widetilde{g_2}^{-1}\widetilde{g_1}^{-1}$$

We can switch the order of the brackets since these elements lie in $1 + M_n(J/\mathfrak{m}J)$, so the left hand side becomes:

$$\widetilde{g_1 g_2 g_3}\widetilde{g_2 g_3}^{-1}\widetilde{g_1}^{-1}\widetilde{g_1}\widetilde{g_2 g_3}\widetilde{g_3}^{-1}\widetilde{g_2}^{-1}\widetilde{g_1}^{-1}$$

$$= \widetilde{g_1 g_2 g_3}\widetilde{g_3}^{-1}\widetilde{g_2}^{-1}\widetilde{g_1}^{-1}$$

which is visibly equal to the right hand side.

Exercise: c_f is well defined independent of the choice of $\widetilde{\rho}$

$[c_f] = 0 \Leftrightarrow$ there exists a choice of $\widetilde{\rho} : \Gamma \to GL_n(\mathcal{O}[[X]]/\mathfrak{m}J)$ such that $\widetilde{\rho} \mod J_f$ is a homomorphism where $J_f := \ker(J \to J/\mathfrak{m}J \to \mathbb{F})$ (so $\mathfrak{m}J \subset J_f \subset J$)

Lemma 7.6. The map

$$\mathrm{Hom}_{\mathbb{F}}(J/\mathfrak{m}J, F) \to H^2(\Gamma, \mathrm{ad}\rho)$$

given by $f \mapsto c_f$ is injective.

Proof. We show: $\exists\widetilde{\rho}$ as above, which is a homomorphism $\mathrm{mod} J_f$ implies $f = 0$. By the universal property of $R^\square$, we can complete the diagram:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho^\square"] \arrow[dr] & GL_n(\mathcal{O}[[X]]/J) \arrow[d, dashed] \\ & GL_n(\mathcal{O}[[X]]/J_f) \end{tikzcd}$$

We have the composition

$$\mathcal{O}[[X]]/J \hookrightarrow \mathcal{O}[[X]]/J_f \to \mathcal{O}[[X]]/J$$

the first map being given by $x_i \mapsto x_i + a_i$

But $\forall h \in \mathcal{O}[[X]], h = \eta(g)$ for $g(x) = h(x - a)$, hence η is onto which implies that $\mathrm{mod} J$ the map is an isomorphism. Hence $J = J_f$ and so $f = 0$ $\square$

Recap:

$$0 \longrightarrow J \longrightarrow \mathcal{O}[[x_1, ...x_d]] \longrightarrow R^\square \longrightarrow 0$$

$\mathrm{Hom}_F(J/\mathfrak{m}J, \mathbb{F}) \hookrightarrow H^2$
\# generators $= d = dimH^1 - dimH^0 + n^2$
\# relations $= dim_{\mathbb{F}} J/mJ \leq dimH^2$

Corollary 7.7. *i)* $H^2 = 0 \Rightarrow R^\square \cong \mathcal{O}[[x_1..., x_d]]$
ii) In general, $dimR^\square \geq d + 1 - dimH^2$

8. Lecture 8

Correction: situation $\mathfrak{m}J \subset J_f \subset J \subset (\mathfrak{m}, \lambda)$.

$$\mathcal{O}[[x_1, ..., x_d]]/J \xhookrightarrow{x_i \mapsto x_i + a_i} \mathcal{O}[[X]]/J_f \longrightarrow \mathcal{O}[[X]]/J$$

We want $J = J_f$.

$$\forall g(x) \in J \Rightarrow g(x + a) \in J_f$$

$g(x) = g_0 + \sum_{i=1}^d g_i x_i + (deg \geq 2)$, then $g(x + a) - g(x) = \sum_{i=1}^d g_i a_i + (deg \geq 2) \in \mathfrak{m}J \subset J_f$. The first term is in $\mathfrak{m}J$ and so is the second, therefore $g(x) \in J_f \Rightarrow J = J_f$.

8.1. **Generic fibers of universal lifting rings.** Warm-up: Consider $\mathcal{O}[[x]]$ over Spec$\mathcal{O}$. Its geometric points are:

$$\mathrm{Spec}(\mathcal{O}[[X]])(\overline{\mathbb{F}}_l) = \{0\}$$

$$\mathrm{Spec}(\mathcal{O}[[x]](\overline{\mathbb{Q}}_l) = \{x \in \overline{\mathbb{Q}}_l | \; |x| < 1\}$$

Thinking about this geometrically, Spec$(\mathcal{O}[[X]])$ is an open unit ball and the special fiber is the point $x = 0$. The universal framed lifting ring $R^{\square}$ is usually a power series over $\mathcal{O}$ (quotiented by some ideal). Taking its generic fiber should give l-adic representations lifting $\overline{\rho}$.

Fact: 1)The closed points of $R \in \mathcal{C}_{\mathcal{O}}$ are Zariski dense, (eg. when $R = R^{\square}_{\overline{\rho}}$).

2)

$$\{\text{max. ideals of } R[\tfrac{1}{l}]\} \leftrightarrow \{(\phi', l') : L'/L \text{ finite}, \phi' : R \to \mathcal{O}_{L'} \text{ such that } L' = L(\phi'(R))\}$$

$$\ker \phi'[1/l] \mapsto \phi'$$

$$\mathfrak{m} \mapsto R \to R[1/l]/\mathfrak{m} =: L', \phi' \text{ is this map restricted to } R$$

$$\begin{tikzcd} R \arrow[r, "\phi'"] \arrow[d] & \mathcal{O}_{L'} \arrow[d, hook] \\ R[1/l] \arrow[r, "\phi'{[1/l]}"'] & L' \end{tikzcd}$$

So $x \in \mathrm{Spec}R^{\square}[1/l]$ a closed point corresponds to (ϕ_x, L_x), where L_x is the residue field at x.

$$\Gamma \xrightarrow{\rho^{\square}} GL_n(R^{\square}) \xrightarrow{\phi_x} GL_n(\mathcal{O}_{L_x}) \subset GL_n(L_x)$$

Check that ρ_x is continuous with respect to the l-adic topology on L_x.

Lemma 8.1.

$$R^{\square}[1/l]_x^{hat} := \lim_{\leftarrow j} R^{\square}[1/l]/(\ker \phi_x[1/l])^j$$

pro-represents the functor

$$\mathscr{R}^{\square}_{\rho_x} : \mathcal{C}^{Art}_{L_x} = \{\text{Artin local } L_x\text{- algebras with residue field } L_x\} \to (Sets)$$

given by

$$A \mapsto \{\rho : \Gamma \to GL_n(A), (A \text{ with the } l\text{-adic top.}) \text{ such that } \rho \mod \mathfrak{m}_A = \rho_x\}$$

Remark 8.2. $\mathrm{Hom}(R^{\square}[1/l]_x^{hat}, A)$ is defined to be $\lim_{\to j} \mathrm{Hom}(R^{\square}[1/l]^{hat}/(\ker \phi_x[1/l])^j, A)$ and the lemma says that this is equal to $\mathscr{R}^{\square}_{\rho_x}(A)$, functorially in A. i.e. this functor is pro-represented by the object.

Sketch of proof: We explicitly construct the bijection above.

$\rightarrow$ Given $f : R^\square[1/l]^{hat}_x \to A$, we take the composition

$$\rho^\square : \Gamma \to GL_n(R^\square) \to GL_n(R^\square[1/l]^{hat}_x) \to GL_n(A)$$

$\leftarrow$ Now given $(\rho, A) \in \mathscr{R}^\square_{\rho_x}(A)$, $\rho : \Gamma \to GL_n(A)$. Define to be the sub-algebra of A generated by entries of $\rho(\Gamma)$. Then one can show that $A^0 \in \mathcal{C}_{\mathcal{O}_{L_x}}$ and is finitely generated as an $\mathcal{O}_{L_x}$ module.

One checks that the following diagram commute

$$\begin{tikzcd}
\rho : \Gamma \arrow[r] \arrow[dr, "\overline{\rho}_x"'] & GL_n(A^0) \arrow[d, two heads, "\text{mod } A^0"] \\
& GL_n(\mathbb{F}_x)
\end{tikzcd}$$

To construct the map $R^\square[1/l]^{hat}_x \to A$ use universal properties of localization and completion:

$$\begin{tikzcd}
R^\square[1/l]^{hat}_x \arrow[rr, dashed] & & A \\
R^\square[1/l] \arrow[u] \arrow[rr, "Univ.prop.of[1/l]"] & & A^0[1/l] \arrow[u, hook] \\
R^\square \arrow[u] \arrow[rr] \arrow[d] & & A^0 \arrow[u, hook] \\
R^\square_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}_{L_x} \arrow[rr, equal] & & R^\square_{\overline{\rho}_x} \arrow[u]
\end{tikzcd}$$

8.2. **Deformation problems.** We are often interested in liftings ρ which satisfy certain conditions. We would like to have a checklist of these conditions so that we can construct subspaces of the full deformation space with nice properties.

Definition 8.3. Fix $\mathcal{O}$ and a continuous mod l representations $\overline{\rho}$. A deformation problem $\mathcal{D}$ is a collection $\{(A, \rho)\}$ where $A \in \mathcal{C}_{\mathcal{O}}, \rho : \Gamma \to GL_n(A)$, lifting of $\overline{\rho}$ such that:

1) $(\mathbb{F}, \overline{\rho}) \in \mathcal{D}$
2) $f : A \to B$ a morphism in $\mathcal{C}_{\mathcal{O}}$, $(A, \rho) \in \mathcal{D} \Rightarrow (B, f \circ \rho) \in \mathcal{D}$
3) $f : A \hookrightarrow B$ an injection in $\mathcal{C}_{\mathcal{O}}$, then $(A, \rho) \in \mathcal{D}$ if and only if $(B, f \circ \rho) \in \mathcal{D}$.
4) $A_1, A_2, \in \mathcal{C}_{\mathcal{O}}$, with ideals $I_1 \subset A_1, I_1 \subset A_2$ and suppose there is an isomorphism

$$f : A_1/I_1 \cong A_2/I_2$$

such that $f(\rho_1 \mod I_1) = \rho_2 \mod I_2$, then if $(A_1, \rho_1), (A_2, \rho_2) \in \mathcal{D}$, so is $(\{(a_1, a_2) \in A_1 \oplus A_2 : f(a_1 \mod I_1) = a_2 \mod I_2\}, \rho_1 \oplus \rho_2)$.

5) Suppose $I_1 \supset I_2 \supset \ldots$ are a nested sequence of ideals of $A \in \mathcal{C}_{\mathcal{O}}$ such that $\bigcap I_i = 0$. If (A, ρ) lifting of $\overline{\rho}$, such that $(A/I_i, \rho \mod I_i) \in \mathcal{D}$, then $(A, \rho) \in \mathcal{D}$.

6) $(A, \rho) \in \mathcal{D}, a \in \ker(GL_n(A) \to GL_n(\mathbb{F})), \Rightarrow (A, a\rho a^{-1}) \in \mathcal{D}$

Deformation problems in the above sense are related to the universal lifting ring by the following lemma. Note that for $\gamma \in \ker(GL_n(R^\square) \to GL_n(\mathbb{F}))$, the universal property gives us an automorphism $R^\square$ induced by the deformation $\gamma\rho\gamma^{-1}$. In general this is not a group action.

Lemma 8.4. There is a bijection

$$\{\text{deformation problems}\} \leftrightarrow \{\ker(GL_n(R^\square) \to GL_n(\mathbb{F}))\text{-invariant radical ideals of } R^\square\}$$

$$\begin{tikzcd} \mathcal{D} \arrow[r, mapsto] & I(\mathcal{D}) \\ \mathcal{D}(I) & I \arrow[l, mapsto] \end{tikzcd}$$

$I(\mathcal{D})$ is characterized by the property that $(A, \rho) \in \mathcal{D}$ if and only if the map $R^\square \to A$ inducing the lifting (A, ρ) factors through $I(\mathcal{D})$.

The deformation problem $\mathcal{D}(I)$ associated to a $\ker(GL_n(A) \to GL_n(\mathbb{F}))$ is the set of liftings induced by maps factoring through I.

Geometrically speaking, a deformation problem cuts out a closed subscheme of the universal deformation space.

Proof. Once we show well-definedness of the above maps, and the characterization property of $I(\mathcal{D})$, the fact that the maps are inverse bijections will be clear.

First, given a $\ker(GL_n(R^\square) \to GL_n(\overline{\mathbb{F}}_p))$ invariant ideal I, the subspace of liftings factoring through I clearly satisfies $1) - 5)$ in the definition of a deformation problem.

First let's see why $I(\mathcal{D})$ exists uniquely and is $\ker(GL_n(R^\square) \to GL_n(\mathbb{F}))$-invariant.

Define

$$J := \{\text{all ideals } I \subset R^\square \text{ such that } (R^\square/I, \rho^\square \mod I) \in \mathcal{D}\}$$

this is non-empty by 1).

$(A, \rho) \in \mathcal{D} \Leftrightarrow (R^\square/\ker f_\rho, \rho^\square \mod \ker f_\rho) \in \mathcal{D}$

$$\begin{tikzcd} \Gamma \arrow[r] & GL_n(R^\square) \arrow[r] \arrow[d, "\rho"] & GL_n(R^\square/\ker f_\rho) \arrow[dl] \\ & GL_n(A) & \end{tikzcd}$$

J is closed under finite intersection by property 4), and infinite nested intersections by 5) and hence there is a minimal element $I(\mathcal{D})$. By part 2) any ideal J containing $I(\mathcal{D})$

satisfies $(R^\square/J, \rho^\square \mod J) \in \mathcal{D}$, and $I(\mathcal{D})$ is $\ker(GL_n(R^\square) \to GL_n(\mathbb{F}))$ invariant by property 6).

$\square$

9. LECTURE 9

Last time we introduced the notion of deformation problem, and showed that such an object corresponds to a $\ker(GL_n(R^\square_{\overline{\rho}}) \to GL_n(\mathbb{F}))$ invariant ideal of $R^\square_{\overline{\rho}}$.

For example given appropriate $\overline{\rho} : \Gamma \to GL_n(\mathbb{F}_l)$, where $\Gamma = Gal(\overline{K}/K)$, for $p = l$ one could consider all liftings which satisfy of the hypothesis of Fontaine-Laffaile, or liftings which are ordinary. For $p \neq l$, one can consider Taylor-Wiles liftings. See [2] for several more examples.

For applications, it is important to understand the quotient $R^\square_{\overline{\rho}}/I(\mathcal{D})$, i.e. know its Krull dimension, no. of generators/relations etc). Consider

$$Z^1(\Gamma, \mathrm{ad}\overline{\rho}) \cong \mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}^\square/(\mathfrak{m}^{\square 2}, \lambda), \mathbb{F}) \supset \mathrm{Ann}(I(\mathcal{D})) =: \tilde{\mathcal{L}}(\mathcal{D})$$

Let $\mathcal{L}(\mathcal{D})$ be the image of $\tilde{\mathcal{L}}(\mathcal{D})$ in $H^1(\Gamma, \mathrm{ad}\overline{\rho})$

Lemma 9.1.

$\tilde{\mathcal{L}}(\mathcal{D})$ is the full pre image of $\mathcal{L}(\mathcal{D})$

Proof. Suppose $x \in Z^1(\gamma, ad\rho)$ lies in the pre-image of $\mathcal{L}(\mathcal{D})$, then $\exists y \in \tilde{\mathcal{L}}(\mathcal{D})$, mapping to the image of x, in $H^1(\Gamma, \mathrm{ad}\rho)$, i.e. the liftings ρ_x and ρ_y corresponding to x and y under the bijection in Lemma 6.3 are isomorphic. Thus there exists $a \in \ker(GL_n(\mathbb{F}[\epsilon]/\epsilon^2] \to GL_n(\mathbb{F})$ such that $a\rho_x a^{-1} = \rho_y$, so that $\rho_x \in \mathcal{D}$ and $x \in \tilde{\mathcal{L}}(\mathcal{D})$. $\square$

9.1. **Fixing determinants.** Let $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$ be a continuous representation, $\chi : \Gamma \to \mathcal{O}^\times$ the character such that $\chi \otimes_{\mathcal{O}} \mathbb{F} = \det\overline{\rho})$

$\mathscr{R}^\square_{\overline{\rho}}(A) \supset \mathscr{R}^\square_{\overline{\rho},\chi}(A) := \{\rho : \text{lifting of } \overline{\rho} \text{ to A}, \det\rho = \chi\}$ is represented by $R^\square_{\overline{\rho},\chi} \in \mathcal{C}_{\mathcal{O}}$.

$R^\square_{\overline{\rho},\chi}$ is constructed as the quotient of $R^\square_{\overline{\rho}}$ by $\det\rho - \chi$.

When $\overline{\rho}$ is Schur, the existence of a universal deformation ring is also guaranteed: $R^{univ}_{\overline{\rho}} \to R^{univ}_{\overline{\rho},\chi}$. The previous discussion carries over to this setting.

The main change needed here is to replace $\mathrm{ad}\overline{\rho}$ by the subspace $\mathrm{ad}^\circ\overline{\rho}$ consisting of the trace 0 subspace.

9.2. **Global Galois deformations.** $F/\mathbb{Q}$ finite, fixing $\overline{F} \hookrightarrow \overline{F}_v$, we obtain

$$G_F := Gal(\overline{F}/F) \hookleftarrow G_{F_v} := Gal(\overline{F}_v/F_v)$$

Let $L, \mathcal{O}, \mathbb{F}$ as before, S a finite set of primes and $G_{F,S}$ the group $Gal(F_S/F)$ where F_S is the maximal unramified extension outside S.

Definition 9.2. A global Galois deformation problem is $\mathcal{S} = (F, S, \mathcal{O}, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$ where

- $F, S, \mathcal{O}$ are as above.
- $\overline{\rho} : G_{F,S} \to GL_n(\mathbb{F})$ absolutely irreducible (can in fact relax this condition, see eg. Skinner-Wiles, Thorne).
- $\chi : G_{F,S} \to \mathcal{O}^\times$. $\chi \otimes_{\mathcal{O}} \mathbb{F} = \det \overline{\rho}$.
- $\mathcal{D}_v$ is a deformation problem for $\overline{\rho}|G_v$.

Note that we are assuming Hyp(G) is true as is the case for $G_{F,S}$. Using this data one defines the following functor:

Definition 9.3. $T \subset S$ (allow $T = \emptyset$), let $\mathscr{R}_{\mathcal{S}}^{\square,T} : \mathcal{C}_{\mathcal{O}} \to (Sets)$ be the functor given by

$$A \mapsto \{(\rho, \{\alpha_v\}_{v\in T})\}/\sim$$

where $\rho : G \to GL_n(A)$ is a lift of $\overline{\rho}$ and

- $\alpha_v \in \ker(GL_n(A) \to GL_n(\mathbb{F}))$
- $\det \rho = \chi$
- $\rho|G_v \in \mathcal{D}$, $v \in S$.

Here the relation $\sim$ is generated by:

$$(\rho, \{\alpha_v\}) \sim (\beta\rho\beta^{-1}, \{\beta\alpha_v\})$$

$\forall \beta \in \ker(GL_n(A) \to GL_n(\mathbb{F}))$

Lemma 9.4. $\mathscr{R}_{\mathcal{S}}^{\square,T}$ is representable by $R_{\mathcal{S}}^{\square,T} \in \mathcal{C}_{\mathcal{O}}$. (If $T = \emptyset$ write $R_{\mathcal{S}}^{univ}$)

Idea of proof. $\mathcal{D}_v^\circ := \{\text{all liftings of } \overline{\rho}|_{G_v}\} \supset \mathcal{D}_v$

$\mathcal{S}^\circ := (....\{\mathcal{D}_v^\circ\}...)$, we know that $\mathscr{R}_{\mathcal{S}^\circ}^{\square,T}$ is representable by $R_{\mathcal{S}^\circ}^{\square,T} \in \mathcal{C}_{\mathcal{O}} = R_{\overline{\rho},\chi}^{univ}$ if $T = \emptyset$.

Then construct $R_{\mathcal{S}}^{\square,T}$ as $R_{\mathcal{S}^\circ}^{\square,T}$ modulo the minimal ideal I such that ρ factors through $R_{\mathcal{S}^\circ}^{\square,T}/I \Leftrightarrow \rho|_{G_v} \in \mathcal{D}_v, \forall v \in S$. $\square$

9.3. **Presenting $R_{\mathcal{S}}^{\square T}$ over local lifting rings.** $\mathscr{R}_{\mathcal{S}}^{\square,T}$ has a universal object

$$\rho_{\mathcal{S}}^{\square,T} : G \to GL_n(R_{\mathcal{S}}^{\square,T})$$

$$\alpha_v \in \ker(GL_n(R_{\mathcal{S}}^{\square,T}) \to GL_n(\mathbb{F})), v \in T$$

Then $\alpha_v^{-1}\rho_{\mathcal{S}}^{\square,T}\alpha_v \in \mathcal{D}_v, v \in T$, and so we an induced map by the universal property

$$R_{\overline{\rho}|G_v,\chi}^{\square}/I(\mathcal{D}_v) \to R_{\mathcal{S}}^{\square,T}$$

We thus obtain the following diagram:

$$\begin{array}{ccc} G & \longrightarrow & GL_n(R^{\square}_{\overline{\rho}|_{G_v},\chi}) \\ \downarrow & & \downarrow \\ GL_n(R^{\square T}_{\mathcal{S}}) & \overset{\exists}{\dashleftarrow} & GL_n(R^{\square}_{\overline{\rho}_v,\chi}/I(\mathcal{D}_v) \end{array}$$

where $\overline{\rho}_v$ is defined to be the restriction of $\overline{\rho}$ to G_v. This induces a map from the completed tensor product of the local lifting rings to $R^{\square,T}_{\mathcal{S}}$

$$R^{loc}_{\mathcal{S},T} := \hat{\bigotimes}_{v\in T}(R^{\square}_{\overline{\rho}_v,\chi}/I(\mathcal{D}_v)) \to R^{\square,T}_{\mathcal{S}}$$

The completed tensor product is just the pushout in category of complete local $\mathcal{O}$ algebras, in particular it satisfies the usual universal property for tensor products.

The goal of the next few lectures will be to understand this presentation, for example what is the number of generators/ relations for presenting $R^{\square,T}_{\mathcal{S}}$ over $R^{loc}_{\mathcal{S},T}$, to do this we use the techniques already developed of studying the tangent spaces via Galois cohomology.

First thought: Suppose $\mathcal{D}_v$ is the set of all liftings $v \in S$. Write $\mathfrak{m} \subset R^{\square,T}_{\mathcal{S},T}$, $\mathfrak{m}^{loc} \subset R^{loc}_{\mathcal{S},T}$ for the respective maximal ideals.

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/(\mathfrak{m}^2,\lambda),\mathbb{F}) \cong \mathscr{R}^{\square T}_{\mathcal{S}}(\mathbb{F}[\epsilon]/\epsilon^2) \cong \{\rho,(\alpha_v)_{v\in T}\}/\sim \cong (Z^1(G,\mathrm{ad}^{\circ}\overline{\rho}) \oplus \bigoplus_{v\in T}(1+\epsilon M_n(\mathbb{F})))/\sim$$

$$= \mathrm{coker} : \begin{array}{ccc} \mathrm{ad}\overline{\rho} & \xrightarrow{\partial} & Z^1(G,\mathrm{ad}^{\circ}\overline{\rho}) \\ & \searrow^{\oplus_{v\in T} res_v} & \oplus \\ & & \bigoplus_{v\in T}\mathrm{ad}\overline{\rho} \end{array}$$

In general we need two modifications:

1) To consider "the tangent space over $R^{loc}_{\mathcal{S},T}$", replace $\mathfrak{m}/(\mathfrak{m}^2,\lambda)$ by $\mathfrak{m}/(\mathfrak{m}^2,\mathfrak{m}^{loc},\lambda)$, this is tantamount to requiring the trivial lifting at $v \in T$.

More precisely, we require:

$$\rho \in \ker(Z^1(G,\mathrm{ad}^{\circ}\overline{\rho}) \oplus \bigoplus_{v\in T}\mathrm{ad}\overline{\rho} \xrightarrow{\oplus_v res_v \oplus -\partial} \bigoplus_{v\in T} Z^1(G_v,\mathrm{ad}^{\circ}\overline{\rho}))$$

This follows from the following the fact that the maps $R^{\square}_{\overline{\rho}|_{G_v},\chi} \to R^{\square,T}_{\mathcal{S}}$ are given functorially by:

$$(\rho = (1+\phi\epsilon)\overline{\rho}, \{\alpha_v = 1+\psi_v\epsilon\}) \mapsto \{\alpha_v^{-1}\rho\alpha_v\}_{v\in T}$$

where $\psi_v \in M_n(\mathbb{F})$ and ϕ is a 1 cocycle in $Z^1(G, ad^\circ\overline{\rho})$. Hence the requirement is that:

$$\begin{aligned} \alpha_v \rho \alpha_v^{-1} &= (1-\psi_v\epsilon)(1+\phi\epsilon)\overline{\rho}(1+\psi_v\epsilon) \\ &= \overline{\rho} + (\phi\overline{\rho} - \psi_v\overline{\rho} + \overline{\rho}\psi_v)\epsilon = \overline{\rho} \end{aligned}$$

$\phi - \psi_v + \overline{\rho}\psi_v\overline{\rho}^{-1} = 0 \Rightarrow \phi - \partial\psi_v = 0$

2) To allow general $\mathcal{D}_v$ at $v \in S$, we require

$$\rho \in \ker(Z^1(G, ad^\circ\overline{\rho}) \xrightarrow{\oplus_{v\in S\setminus T} res_v} \bigoplus_{v\in S\setminus T} \frac{Z^1(G_v, ad^\circ\overline{\rho})}{\tilde{\mathcal{L}}(\mathcal{D}_v)})$$

In other words, $\rho|_{G_v} \in \mathcal{D}_v$ at $v \in S\setminus T$.

The upshot of this is that

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/(\mathfrak{m}^2, \mathfrak{m}^{loc}, \lambda), \mathbb{F}) \cong$$

$$\begin{tikzcd} H^1(\mathrm{ad}\overline{\rho} \arrow[r] & Z^1(G, ad^\circ\overline{\rho}) \arrow[r] & \bigoplus_{v\in T} Z^1(G_v, ad^\circ\overline{\rho}) \oplus \bigoplus_{v\in S\setminus T} \frac{Z^1(G_v, ad^\circ\overline{\rho})}{\tilde{\mathcal{L}}(\mathcal{D}_v)}) \\ & \oplus & \\ & \bigoplus_{v\in T} \mathrm{ad}\overline{\rho} \arrow[uur] & \end{tikzcd}$$

So we're motivated to consider the complex of $\mathbb{F}[G]$-modules.

$$\begin{tikzcd} \mathcal{C}^\bullet_{\mathcal{S},\mathcal{T}} := \mathcal{C}^0(G, \mathrm{ad}\overline{\rho}) \arrow[r] \arrow[ddr] & \mathcal{C}^1(G, ad^\circ\overline{\rho}) \arrow[rr] \arrow[ddr] & & \mathcal{C}^2(G, ad^\circ\overline{\rho})... \arrow[ddr] & \\ & \oplus & & \oplus & \\ & \bigoplus_{v\in T} \mathcal{C}^0(G_v, \mathrm{ad}\overline{\rho}) \arrow[r] & (\bigoplus_{v\in T} \mathcal{C}^1(G_v, ad^\circ\overline{\rho}) \oplus \bigoplus_{S\setminus T} \mathcal{C}^1(G_v, ad^\circ\overline{\rho})/\tilde{\mathcal{L}}(\mathcal{D}_v)) \arrow[rr] & & \bigoplus_{v\in S} \mathcal{C}^2(G_v, ad^\circ\overline{\rho}) \end{tikzcd}$$

with obvious coboundary maps and where $\mathcal{C}^i(G, M) = \mathrm{Hom}(G \times ... \times G, M)$.

It will be convenient to break this complex into two parts denoted $\mathcal{C}^{\bullet-1}_{\mathcal{S},T,loc}$ and $\mathcal{C}^\bullet_0$, where these complexes are defined as follows:

$$\mathcal{C}^0_{\mathcal{S},T,loc} = \bigoplus_{v\in T} \mathcal{C}^0(G_v, \mathrm{ad}\overline{\rho})$$

$$\mathcal{C}^1_{\mathcal{S},T,loc} = (\bigoplus_{v\in T} \mathcal{C}^1(G_v, ad^\circ\overline{\rho}) \oplus \bigoplus_{S\setminus T} \mathcal{C}^1(G_v, ad^\circ\overline{\rho})/\tilde{\mathcal{L}}(\mathcal{D}_v))$$

$$\mathcal{C}^i_{\mathcal{S},T,loc} = \bigoplus_{v\in S} \mathcal{C}^i(G_v, ad^\circ\overline{\rho}) \text{ for } i > 1$$

$$\mathcal{C}^0_0 = \mathcal{C}^i(G, \mathrm{ad}\overline{\rho})$$

and

$$\mathcal{C}_0^i = \mathcal{C}^i(G, ad^\circ\overline{\rho}) \text{ for } i > 0$$

Next time we will study $H^i_{\mathcal{S},T}(G, \mathrm{ad}^\circ\overline{\rho}) := H^i(\mathcal{C}_{\mathcal{S},T})$

10. Lecture 10

Recall we were studying global deformations problems. We wanted to present the global deformation ring $R_{\mathcal{S}}^{\square T}$ over a completed tensor product of local deformation rings denoted $R^{loc}_{\mathcal{S},T}$, in order to do this we introduced the complex above. We define $H^i_*(G, \mathrm{ad}^\circ\overline{\rho}) := H^i(\mathcal{C}^\bullet_*(G, \mathrm{ad}^\circ\overline{\rho}))$.

Last time we saw

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/(\mathfrak{m}^2, \mathfrak{m}^{loc}, \lambda), \mathbb{F}) \cong H^1_{\mathcal{S},T}(G, \mathrm{ad}^\circ\overline{\rho})$$

.

The same proof as in the local case gives us the following proposition.

Proposition 10.1. *There exists a surjection $R^{loc}_{\mathcal{S},T}[[x_1, ..., x_d]] \to R_{\mathcal{S}}^{\square T}$ where $d = \dim_{\mathbb{F}} H^1_{\mathcal{S},T}$, and the number of relations is $\leq \dim H^2_{\mathcal{S},T}$.*

Next Goal: Compute $H^i_{\mathcal{S},T}(G, \mathrm{ad}^\circ\overline{\rho}), i = 1, 2$, in terms of
- usual local/global cohomology
- dim of $\mathcal{L}(\mathcal{D}_v)$ ($\subset H^1(G_v, \mathrm{ad}^\circ\overline{\rho})$).
- dim of "Dual Selmer group."

10.1. computation of $H^1_{\mathcal{S},T}$.

Assume for technical reasons that:
- $l > 2$
- $l \nmid n$ which implies the exact sequence of G representations

$$0 \to \mathrm{ad}^\circ\overline{\rho} \to \mathrm{ad}\overline{\rho} \to \mathbb{F} \to 0$$

is split. Here G acts trivially on $\mathbb{F}$.
- All places of F above l are in S

Fact: All cohomology are finite dimensional over $\mathbb{F}$ and vanish in sufficiently large degree.

Recall the complexes $\mathcal{C}_0$, $\mathcal{C}_{\mathcal{S},T,loc}$ and $\mathcal{C}_{\mathcal{S},T}$ defined in the previous lecture. The Euler characteristic of each of these complexes is defined to be

$$\chi_*(G, \mathrm{ad}^\circ\overline{\rho}) := \sum_{i\geq 0}(-1)^{i+1} \dim_{\mathbb{F}} H^i_*(G, \mathrm{ad}^\circ\overline{\rho})$$

which is well-defined by the above fact.

Strategy: Step 1 $\chi_{\mathcal{S},T} = \chi_0 - \chi_{\mathcal{S},T,loc}$

This follows from the exact sequence

$$0 \to \mathcal{C}^{\bullet -1}_{\mathcal{S},T,loc} \to \mathcal{C}^\bullet_{\mathcal{S},T} \to \mathcal{C}^\bullet_0 \to 0$$

which gives a long exact sequence in homology and so by our sign convention we obtain the result.

Hence $\chi_{\mathcal{S},T} = \chi(G, \mathrm{ad}^\circ\overline{\rho}) - \chi_{\mathcal{S},T,loc} + \dim(H^0(G, \mathrm{ad}^\circ\overline{\rho})) - \dim H^0(G, \mathrm{ad}\overline{\rho})$. The last term is -1; this follows from $ad = \mathrm{ad}^\circ \oplus \mathbb{F}$.

Step 2: Compute $\chi_{\mathcal{S},T}$ in terms of usual Galois cohomology. By definition,

$$\chi_{\mathcal{S},T,loc} = \sum_{v\in S}\chi(G_v, \mathrm{ad}^\circ\overline{\rho}) + \sum_{v\in T}(\dim H^0(G_v, \mathrm{ad}^\circ\overline{\rho}) - \dim H^0(G_v, \mathrm{ad}\overline{\rho})) + \sum_{v\in S\setminus T}(\dim H^0(G_v, \mathrm{ad}^\circ\overline{\rho}) - \dim \mathcal{L}(\mathcal{D}_v))$$

This implies

$$\chi_{\mathcal{S},T} = \chi - \chi_{\mathcal{S},T,loc} - 1$$
$$= -1 + \#T - \sum_{v\in S}\chi(G_v, \mathrm{ad}^\circ) - \sum_{v\in S\setminus T}(\dim H^0(G_v, \mathrm{ad}^\circ) - \dim \mathcal{L}(\mathcal{D}_v)) + \chi(G, \mathrm{ad}^\circ)$$

Step 3: Apply local/ global Euler Poincare characteristic formulae to $\chi(G, -), \chi(G_v, -)$ to get the formula for $\chi_{\mathcal{S},T}$.

Step 4:Cohomological vanishing implies H^i vanishes for $i > 2$, so we need only consider: $H^0_{\mathcal{S},T}, H^1_{\mathcal{S},T}, H^2_{\mathcal{S},T}, H^3_{\mathcal{S},T}, 0, 0, ...$ In order to get H^2 and H^3 use duality and Poitou-Tate to reduce to H^0 and H^1

Fact: Cohomological vanishing

1) Let K be non archimedan and $G_K := Gal(\overline{K}/K)$ act on a module M, which is finite as an $\mathbb{F}[G_K]$-module. Then $H^i(G_K, M) = 0$ for $i > 2$.

2) $K = \mathbf{R}$, $G_{\mathbf{R}} = \{1, c\}$ acting on M whose order is a power of l, then $H^i(G, M) =$ for $i \geq 0$, when $l > 2$.

3) K a number field, let $G = G_{F,S}$ act on a module M, then $H^i(G, M) = 0$ for $i > 2$, when $l > 2$.

Consequently, it follows directly from the long exact sequence that $H^i_{\mathcal{S},T} = 0$ when $i > 3$.

$$H^2_{\mathcal{S},T} \to H^2(G, \mathrm{ad}^\circ\overline{\rho}) \to H^2_{\mathcal{S},T,loc}$$

$$\to H^3_{\mathcal{S},T} \to H^3(G, \mathrm{ad}^\circ\overline{\rho}) \to H^3_{\mathcal{S},T,loc}$$

Fact 2: Euler Poincare characteristic.

1) K non archimedean (of characteristic 0)

Then $\chi(G_K, M) = \dim_{\mathbb{F}}(\mathcal{O}/\#M) = \dim_{\mathbb{F}} M. \dim_{\mathbb{F}}(\mathcal{O}/\#\mathbb{F}) = \begin{cases} 0 & l \neq p \\ \dim_{\mathbb{F}} M[K:\mathbb{Q}_l] & l = p \end{cases}$

2) F a number field, S a finite set of places and $l > 2$.

$\chi(G_{F,S}, M) = [F:\mathbb{Q}]\dim M - \sum_{v|\infty}\dim H^0(G_v, M)$

Back to Step 3: Thus since S contains all primes above l we have

$$\chi(G,M)-\sum_{v\in S}\chi(G_v,M)=[F:\mathbb{Q}]\dim M-\sum_{v|\infty}\dim H^0(G_v,M)-\sum_{v|l}\dim_{\mathbb{F}}M[F_v,\mathbb{Q}_l]$$
$$=\sum_{v|\infty}\dim H^0(G_v,M)$$

Thus

$$\chi_{\mathcal{S},T}=-1+\#T-\sum_{v\in S\setminus T}(\dim H^0(G_v,\mathrm{ad}^\circ\overline{\rho})-\dim\mathcal{L}(\mathcal{D}_v))+\chi(G,\mathrm{ad}^\circ\overline{\rho})-\sum_{v\in S}\chi(G_v,\mathrm{ad}^\circ\overline{\rho})$$
$$=-1+\#T-\sum_{v\in S\setminus T}(\dim H^0(G_v,\mathrm{ad}^\circ\overline{\rho})-\dim\mathcal{L}(\mathcal{D}_v))-\sum_{v|\infty}\dim H^0(G_v,\mathrm{ad}^\circ\overline{\rho})$$

Step 4: We will need to introduce some notation: Let K be a local/ global field.
M a finite $\mathbb{F}[G_K]$-module.
$M^\vee := \mathrm{Hom}_{\mathbb{F}}(M,\mathbb{F})$
$M^D := M^\vee\otimes_{\mathbb{Z}_l}\mathbb{Z}_l(\epsilon_l)=M^\vee(1)$, where $\epsilon_l : G_K\to\mathbb{Z}_l$ is the l-adic cyclotomic character.

Example 10.2. *$M=\mathrm{ad}^\circ\overline{\rho}$. $M\times M\to\mathbb{F}$ given by $(a,b)\mapsto tr(ab)$ is a perfect pairing.*
Then $M^\vee\cong M, M^D\cong M(1)$

Fact 3: Local duality, K non archimedean.
$H^r(G,M^D)\cong H^{2-r}(G,M)^\vee, r=0,1,2$
Application: $H^1(G_v,M^D)\times H^1(G_v,M)\to\mathbb{F}$. $G_v=G_{F_v}$, $M=\mathrm{ad}^\circ$.
$\mathcal{L}(\mathcal{D}_v)\subset H^1(G_v,M)$ and let $\mathcal{L}(\mathcal{D}_v)^\perp\subset H^1(G_v,M^D)$ be the annihilator of $\mathcal{L}(\mathcal{D}_v)$ under the above pairing.
Fact 4: Poitou Tate theorem: Let $G=G_{F,S}$ and $M=\mathrm{ad}^\circ$, there exists an exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & - \arrow[r] & - \arrow[r] & - \arrow[d] \\
& H^1(G,M^D)^\vee \arrow[d] & \bigoplus_{v\in S} H^1(G_v,M) \arrow[l] & H^1(G,M) \arrow[l] \\
& H^2(G,M) \arrow[r] & \bigoplus_{v\in S} H^2(G_v,M) \arrow[r] & H^0(G,M^D)^\vee \arrow[r] & 0
\end{tikzcd}$$

Back to step 4: $0\to\mathcal{C}_{\mathcal{S},T,loc}\to\mathcal{C}_{\mathcal{S},T}\to\mathcal{C}_0\to 0$ gives us a long exact sequence

$$\to H^1(G,M) \to \bigoplus_{v\in T} H^1(G_v,M) \oplus \bigoplus_{v\in S\setminus T} H^1(G_v,M)/\mathcal{L}(\mathcal{D}_v)$$
$$\to H^2_{\mathcal{S},T} \to H^2(G,M) \to \bigoplus_{v\in S} H^2(G_v,M)$$
$$\to H^3_{\mathcal{S},T} \to 0$$

We modify the appropriate part of the Poitou-Tate sequence as below:

$$\begin{tikzcd}
H^1(G,M) \arrow[r] \arrow[dr] & \bigoplus_{v\in S} H^1(G_v,M) \arrow[r, "\gamma^1"] \arrow[d] & H^1(G,M^D)^\vee \arrow[r] \arrow[d] & H^2(G,M) \\
& \bigoplus_{v\in S} H^1(G_v,M)/N \arrow[r] & H^1(G,M^D)^\vee/\gamma^1(N) \arrow[ur] &
\end{tikzcd}$$

where $N = \bigoplus_{v\in S\setminus T} \mathcal{L}(\mathcal{D}_v)$. The lower part of the diagram remains exact.

$$\operatorname{coker}(N \to H^1(G,M^D)^\vee) = \operatorname{coker}(\bigoplus_{v\in S\setminus T} H^1(G_v,M^D)^\vee$$
$$= \ker(H^1(G,M^D) \to \bigoplus_{v\in S\setminus T} \frac{H^1(G_v,M^D)^{\perp}}{\mathcal{L}(\mathcal{D}_v)}) = H^1_{\mathcal{S},T}(G,M^D)^\vee$$

Comparing the modified Poitou-Tate sequence and the original exact sequence we obtain two exact sequences of the form:

$$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 0$$
$$1 \to 2 \to 3' \to 4 \to 5 \to 6' \to 0$$

This implies $\dim H^2_{\mathcal{S},T} = \dim H^1_{\mathcal{S},T}(G,M^D)$
$\dim H^3_{\mathcal{S},T} = \dim H^0(G,M^D)$
$\dim H^0_{\mathcal{S},T} = \dim H^0(G,\mathrm{ad}\overline{\rho}) = 1$
Using

$$\chi_{\mathcal{S},T} = -1 + \#T - \sum_{v\in S\setminus T}(\dim H^0(G_v,\mathrm{ad}^\circ\overline{\rho}) - \dim \mathcal{L}(\mathcal{D}_v)) - \sum_{v|\infty} \dim H^0(G_v,\mathrm{ad}^\circ\overline{\rho})$$

we obtain the following formula

Proposition 10.3.

$$\dim_{\mathbb{F}} H^1_{\mathcal{S},T}(G, ad^\circ\rho) = \#T - \sum_{v|\infty} \dim_{\mathbb{F}} H^1(G_v, ad^\circ\overline{\rho}) + \sum_{v\in S\setminus T}(\dim_{\mathbb{F}} L(\mathcal{D}_v) - \dim_{\mathbb{F}} H^0(G_v, ad^\circ\overline{\rho}))$$
$$- \dim_{\mathbb{F}} H^1(G, ad^\circ\overline{\rho}(1)) - \dim_{\mathbb{F}} H^0(G, ad^\circ\overline{\rho}(1))$$

In particular, writing d for the above integer, there exists a surjection $R^{loc}_{\mathcal{S},T}[[x_1, ..., x_d]] \to R^{\square}_{\mathcal{S},T}$

As above we can also bound the relations in the above presentation, in the special $T = \emptyset$ we have the following:

Proposition 10.4. *The Krull dimension of* $R^{univ}_{\mathcal{S}}$ *is at least*

$$1 + \sum_{v \in S} (\textit{Krull dim.}(R^{\square}_{\overline{\rho}|_{G_v}, \chi}/I(\mathcal{D}_v)) - n^2) - \sum_{v|\infty} \dim_{\mathbb{F}} H^0(G_v, ad^{\circ}\rho) - \dim H^0(G, ad^{\circ}\rho)$$

Proof. Let d be as in the previous proposition, let J be kernel of the surjection $R := \mathcal{O}[[x_1, ..., x_d]] \to R^{univ}_{\mathcal{S}}$we define a map

$$\mathrm{Hom}(J/\mathfrak{m}^{univ}_{\mathcal{S}} J, \mathbb{F}) \to H^2_{S,\emptyset}(G, \mathrm{ad}^{\circ}\rho)$$

Pick a lift of $\rho^{univ}_{\mathcal{S}}$ to $GL_n(R)$, denoted $\tilde{\rho}$ and define

$$c_f(\gamma, \delta) = f(\tilde{\rho}(\gamma\delta)\tilde{\rho}(\delta)\tilde{\rho}(\gamma) - 1_n)$$

Also for $v \in S$ pick a lift $\hat{\rho}$ of $\rho^{univ}_{\mathcal{S}}|_{G_v}$ and define

$$d_{f,v}(\gamma) = f(\tilde{\rho}\hat{\rho} - 1_n)$$

One shows that this gives a well defined element of $H^2_{\mathcal{S},T}(G, \mathrm{ad}^{\circ}\rho)$ and the associated $f \mapsto [(c_f, d_{f,v})]$ is injective. Hence setting $T = \emptyset$ in the formula in Proposition 9.4 we obtain:

$$\text{Krull dim.}(R^{univ}_{\mathcal{S}}) \geq 1 - \sum_{v|\infty} \dim_{\mathbb{F}} H^1(G_v, \mathrm{ad}^{\circ}\overline{\rho}) + \sum_{v \in S} \dim_{\mathbb{F}} L(\mathcal{D}_v) - \dim_{\mathbb{F}} H^0(G_v, \mathrm{ad}^{\circ}\overline{\rho})$$
$$- \dim_{\mathbb{F}} H^0(G, \mathrm{ad}^{\circ}\overline{\rho}(1))$$

where the 1 comes from the ideal (λ).

Noting that $\dim_{\mathbb{F}} L(\mathcal{D}_v) - \dim_{\mathbb{F}} H^0(G_v, \mathrm{ad}^{\circ}\overline{\rho}) = \text{Krull dim.} R^{\square}_{\overline{\rho}|_{G_v}}/I(\mathcal{D}_v) - n^2$ we obtain the result. ☐

We now prove a proposition concerning maps between global deformation rings. Let F, S and $\overline{\rho}$ be as above and let F' be a finite extension of F and S' a finite subset of primes in F' containing the primes above those in S. Let $G' := G_{F',S'}$, then by restricting to G', we obtain a map of deformation rings:

$$R^{univ}_{\overline{\rho}|_{G'}} \to R^{univ}_{\overline{\rho}}$$

Proposition 10.5. *The above map presents* $R^{univ}_{\overline{\rho}}$ *as a finite* $R^{univ}_{\overline{\rho}|_{G'}}$ *algebra.*

Proof. Write R for $R^{univ}_{\overline{\rho}|_{G'}}$ let $\mathfrak{m}$ be its maximal ideal. By Nakayama's lemma it suffices to show that $R^{univ}_{\overline{\rho}}/\mathfrak{m}R^{univ}_{\overline{\rho}}$ is a finite $\mathbb{F}$ module.

We first show that the image of G in $GL_n(R^{univ}_{\overline{\rho}}/\mathfrak{m}R^{univ}_{\overline{\rho}})$ is finite. Indeed we have

$$\ker(\rho^{univ}_{\overline{\rho}} \mod \mathfrak{m}) \supset \ker(\rho^{univ}_{\overline{\rho}}|_{G'} \mod \mathfrak{m}) \supset \ker(\rho^{univ}_{\overline{\rho}|_{G'}} \mod \mathfrak{m})$$

and the last group has finite index in G.

Let m the order of this image and $\gamma_1, ..., \gamma_m \in G$ generating this image. Define

$$f(T) = \prod_{(\zeta_1,...,\zeta_n)\in\mu_m(\overline{\mathbb{F}}_p)} (T - (\zeta_1 + ... + \zeta_m)) \in \mathbb{F}[T]$$

Let A be the quotient of the ring $\mathbb{F}[X_{i,j}]_{i,j=1,...,n}$ be the equations $(X_{i,j})^m - I_n = 0$ where $(X_{i,j})$ is the matrix with i, j entry $X_{i,j}$. For any prime ideal $\mathfrak{p}$ of A, we have the characteristic polynomial of $X_{i,j}$ are m^{th} roots of unity, hence $f(trX_{i,j}) = 0$ for all $\mathfrak{p}$, hence $f(trX_{i,j})^a = 0$ in A for some $a \in \mathbb{N}$. Suppose now M is any matrix over an $\mathbb{F}$ algebra S such that $M^m = I_n$, then we can find a map $A \to S$ such that $X_{i,j} \mapsto M_{ij} \in S$, and hence $f(trM) = 0$.

It follows that we have a map

$$\mathbb{F}[T_1, ..., T_m]/(f(T_1)^a, ..., f(T_m)^a) \to R^{univ}_{\overline{\rho}}/\mathfrak{m}R^{univ}_{\overline{\rho}}$$

given by $T_i \mapsto tr(\gamma_i)$

This map is dense since the values $tr\gamma$ topologically generated the ring $R^{univ}_{\overline{\rho}}$ over $\mathcal{O}$. But the source is finite, hence the map is surjective and this presents $R^{univ}_{\overline{\rho}}/\mathfrak{m}R^{univ}_{\overline{\rho}}$ as a finite module over $\mathbb{F}$. $\square$

11. Lecture 11

The notation is as above. $K/\mathbb{Q}_p$ is a finite extension and $\overline{\rho} : G_K \to GL_n(\mathbb{F})$ is a continuous representations, and fix a character $\chi : G_K \to \mathcal{O}^\times$. which reduces to $det\overline{\rho}$.

We constructed the ring $R^{\square}_{\overline{\rho},\chi} \in \mathcal{C}_{\mathcal{O}}$ which represents the lifting problem for $\overline{\rho}$ with fixed determinant χ. Its generic fiber $R^{\square}_{\overline{\rho},\chi}[1/l]$ has closed points corresponding to l-adic liftings of $\overline{\rho}$ with determinant χ.

The goal for the next week will be to study the properties (e.g. irreducible components, dimension) of $R^{\square}_{\overline{\rho},\chi}[1/l]$ (or $R^{\square}_{\overline{\rho},\chi}$), we split into the two cases $l \neq p$ and $l = p$ (the second requires some background in p-adic hodge theory).

The motivation for this is as follows:

1) In order to control $H^i_{\mathcal{S},T}$ $(i = 1, 2)$ or Krulll dimension of $R^{\square}_{\mathcal{S},T}$, we saw last that time that we need to know the $\dim_{\mathbb{F}} \mathcal{L}(\mathcal{D}_v)$ or the Krull dimension of $R^{\square}_{\overline{\rho}_v,\chi}/I(\mathcal{D}_v)$.

2) This information enters into the proof of automorphy liftings theorems.

11.1. **Local universal lifting rings $l \neq p$.**

Proposition 11.1. *Spec$R^{\square}_{\overline{\rho},\chi}[1/l]$ has finitely many irreducible components and each irreducible component is generically formally smooth (over L) and of dimension n^2-1.*

Remark 11.2. The same is true of $R^{\square}_{\overline{\rho}}[1/l]$ (here the dimension is n^2) and $R^{univ}_{\overline{\rho}}[1/l]$ if $\overline{\rho}$ is Schur (with dimension =1).

Proof. Define a closed point of Spec$R^{\square}_{\overline{\rho}}[1/l]$ corresponding to the l-adic representation $\rho_x : G_K \to GL_n(L_x)$ to be smooth if $H^0(G_K, \mathrm{ad}^\circ\rho_x(1)) = 0$. It is shown In [BLGGT] (Lemma1.3.2) that the smooth points are Zariski dense. Thus it suffices to prove that $R^{\square}[1/l]^{\wedge}_x \cong L_x[[y_1, ..., y_{n^2-1}]]$. (note this ring is the universal lifting ring for ρ_x with coefficients in $\mathcal{C}^{Art}_{L_x}$).

Idea: Mimic argument for liftings of $\overline{\rho}$ using tangent spaces and Galois cohomology.

Define

$$d := \dim_{L_x}(\text{tangent space at } x)$$

which if we fix determinants is equal to

$$n^2 - 1 - \dim H^0(G_K, \mathrm{ad}\rho_x^\circ) + \dim H^1(G_K, \mathrm{ad}^\circ\rho_x)$$

Hence there exists a surjection

$$\phi : L_x[[x_1, ..., x_d]] \to R^{\square}[1/l]^{\wedge}_x$$

One shows in the same way as before that $\ker\phi = 0$ if $H^2(G_K, \mathrm{ad}^\circ\rho_x) = 0$.

Thus it suffices to prove $\dim H^0 = \dim H^1$ and $H^2 = 0$. This follows from the l-adic version of local duality and the Euler Poincare formula. The first gives us

$$H^2(G_K, \mathrm{ad}^\circ\rho_x) = H^0(\mathrm{ad}^\circ\rho_x(1))^{\wedge} = 0$$

(the second equality follows from smoothness), and the second gives $\chi(\mathrm{ad}^\circ\rho_x) = 0$ and hence the two together implies $\dim H^1 = \dim H^0$. $\square$

Definition 11.3. Let $\emptyset \neq \mathcal{C} \subset \{$irreducible components of Spec$R^{\square}[1/l]\}$ and define $R^{\square}_{\overline{\rho},\chi,\mathcal{C}} = R^{\square}_{\mathcal{C}}$ to be the largest quotient of $R^{\square}_{\overline{\rho},\chi}$ which is

- reduced and l-torsion free
- Spec$R^{\square}_{\overline{\rho},\chi,\mathcal{C}}[1/l] \subset \mathcal{C}$

Here is a rough idea for the construction of this ring. Consider

$$\overline{R}^{\square} := R^{\square}/l\text{-torsion} \hookrightarrow R^{\square}[1/l]$$

Then Spec$R^{\square}[1/l] \subset$ Spec$\overline{R}^{\square}$ is open and dense and $\overline{\mathcal{C}}$ is contained in the latter. Then take the reduced closed subscheme structure on $\overline{\mathcal{C}}$.

Lemma 11.4. 1) $I_{\mathcal{C}} := \ker(R^{\square} \twoheadrightarrow R^{\square}_{\mathcal{C}})$ then $\mathcal{D}(I_{\mathcal{C}})$ is a deformation problem.

2) $R^{\square}_{\mathcal{C}}$ is equidimensional of dimension n^2. (Note $R^{\square}_{I_{\mathcal{C}}} = R^{\square}/I_{\mathcal{C}} = R^{\square}/I(\mathcal{D}(I_{\mathcal{C}}))$

Proof. 1) The non-trivial part is to show that $I_{\mathcal{C}}$ is $\ker(GL_n(R^{\Box}) \to GL_n(\mathbb{F}))$ invariant (this is [BLGGT] Lemma 1.2.2)

2) $\mathrm{Spec}R^{\Box}[1/l]$ is open and dense in $\mathrm{Spec}\overline{R}^{\Box}$. Let Z be an irreducible component of $\mathrm{Spec}\overline{R}^{\Box}$ and define $Z' := Z \cap \mathrm{Spec}R^{\Box}[1/l]$. One checks that Z' is an irreducible component and is non-empty and $\dim Z = \dim Z' + 1$.

Let $Z' = \mathrm{Spec}Y'$ and $Z = \mathrm{Spec}Y$ and take a sequence of ideals:

$$0 \subsetneq \mathfrak{p}_0 \subsetneq ... \subsetneq \mathfrak{p}_{n^2-1} \subsetneq Y'$$

then

$$0 \subsetneq \mathfrak{p}_0 \cap R^{\Box} \subsetneq ... \subsetneq \mathfrak{p}_{n^2-1} \cap R^{\Box} \subsetneq \mathfrak{m}^{\Box} \subsetneq Y$$

since the quotient $Y/\mathfrak{p}_{n^2-1}$ is the ring of integers in a finite extension of L. □

Consdider the map which takes finite dimension Weil-Deligne representations of W_K on L vector space to triples (up to equivalence) (V, r_0, N) where $r_0 : I_K \to GL(V)$, $N \in \mathrm{End}(V)$ nilpotent, and r_0 and N commute, given by restriction to I_K.

Definition 11.5. An inertial type representation is any τ in the image.

A Weil-Deligne representation is of type τ if it lies in the preimage of τ

Remark 11.6. The map isn't onto, cf. the next example.

Example 11.7. Suppose $\psi_1, \psi_2 : I_K \to \mathbb{C}^{\times}$ are two tame characters of G_K, we will prove a criterion for when $\psi_1 \oplus \psi_2$ comes from a WD representation of W_K, i.e. is an inertial type. Suppose $\psi_1 \oplus \psi_2$ is the restriction of a Weil representation ρ to I_K. First suppose that ρ factors through an abelian quotient of W_K, (equivalently $\rho(\sigma)$ commutes with $\rho(\gamma)$ for all $\gamma \in I_K$. This occurs if and only if $\psi_1 = \psi_2$ or $\rho(\sigma)$ factors through the standard diagonal torus T of GL_2.

But $\forall \gamma \in I_K$ the relation

$$\rho(\gamma) = \rho(\sigma)\rho(\gamma)\rho(\sigma^{-1}) = \rho(\gamma)^{\#k}$$

implies that $\psi_1 = \psi_1^{\#k}$ and similarly for ψ_2, so that $\psi_1^{\#k-1} = \psi_2^{\#k-1} = 1$. It can then be checked that if ψ_1 and ψ_2 satisfy these conditions, then $\psi_1 \oplus \psi_2$ will come from a representation of W_K.

Suppose now that ρ does not factor through an abelain quotient of W_K, in particular ψ_1 and ψ_2 are distinct. Then since $\sigma\gamma\sigma^{-1} \in I_K$ for all γ in I_K, it follows that $\rho(\sigma)$ lies in the normalizer of T. As ρ does not factor through an abelian quotient of W_K, it follows that wlog. we may assume

$$\rho(\sigma) = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$$

In this case the commutativity relation gives the following:

$$\begin{pmatrix} \psi_1^{\#k}(\gamma) & 0 \\ 0 & \psi_2^{\#k}(\gamma) \end{pmatrix} = \rho(\gamma^{\#k}) = \rho(\sigma)\rho(\gamma)\rho(\sigma^{-1}) = \begin{pmatrix} \psi_2(\gamma) & 0 \\ 0 & \psi_1(\gamma) \end{pmatrix}$$

This implies $\psi_2 = \psi_1^{\#k}$ and $\psi_1 = \psi_2^{\#k}$, in particular $\psi_2^{\#k^2-1} = 1$. For such characters one checks they extend to representations of W_K, in fact in this case, one can extend to an irreducible representation of W_K. In particular we see that restricting W_K representations to I_K is not s surjective map.

Example 11.8. A representation is of unramified type if it is in the pre image of $\tau = [(1, 0)]$ of any dimension.

Example 11.9. (classification, n=2) Say $\mathbb{C}$ coefficients (or $\overline{\mathbb{Q}}_l$), there are four types:
1) Unramified up to character $\tau = [(\psi \oplus \psi, 0)]$, $\psi : I_K \to \mathbb{C}^\times$ a character.
2) Steinberg: $\tau = [(\psi \oplus \psi, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})]$
3) Split ramified: $\tau = [(\psi_1 \oplus \psi_2, 0)]$ where ψ_1 and ψ_2 are two distinct characters of I_K.
4) Irreducible type: $\tau = [(r_0, 0)]$ comes from an irreducible Weil-Deligne representations of dimension 2.

Theorem 11.10. *([Pil] Section 4) Suppose L is sufficiently large (or we can work with geometrical irreducible components) then*
i) Each irreducible component $\mathcal{C}$ of $SpecR^\square[1/l]$ has associated type $\tau_\mathcal{C}$ such that:
Case (1) (3) (4): Closed points corresponds to ρ_x such that $WD(\rho_x)$ has type $\tau_\mathcal{C}$
Case (2): Closed points correspond to ρ_x which fit into an exact sequence

$$0 \to \psi(1) \to \rho_x \to \psi \to 0$$

for some $\psi : G_K \to L_x^\times$ (type: (2) or (1))
ii) Each case of (1)-(4) occurs in at most one component with the following exception: If $\overline{\rho} = \overline{\psi}_1 \oplus \overline{\psi}_2$ with $\overline{\psi}_1$ and $\overline{\psi}_2$ distinct, and $\overline{\psi}_1\overline{\psi}_2^{-1}$ is unramified, then $\exists$ 2 components in case (3).
iii) Two components intersect only when: (see diagram)
When the two components are case (1) and case (2), the representations in the component corresponding to case (1) are of the form $\rho = \psi' \oplus \chi\psi'$ (χ is the cyclotomic character) and those of case (2) are

$$0 \to \psi(1) \to \rho \to \psi \to 0$$

$WD(\rho) = (\psi(1) \oplus \psi, N = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix})$

As we approach the intersetion point a approaches 0 and ψ' approaches ψ, the representation at the intersection point is $\rho_x = \psi(1) \oplus \psi$.

iv) Each component is formally smooth.

Heuristic for iii): Suppose x the intersection point, then it is not smooth and so the proof of proposition 10.1 implies

$$H^0(G_K, \mathrm{ad}^\circ \rho_x(1)) \neq 0$$

In other words

$$0 \neq \mathrm{ad}\rho_x(1)^{G_K} = \mathrm{Hom}_{L_x}(\rho_x, \rho_x(1))^{G_K} = \mathrm{Hom}_{L_x[G_K]}(\rho_x, \rho_x(1))$$

This implies ρ_x is reducible and fits in an exact sequence

$$0 \to \psi_1 \to \rho_x \to \psi_2 \to 0$$

where $\psi_2 = \psi_1(1)$ and $\rho_x \to \psi_2 \to \psi_1(1) \to \rho_x(1)$, one then shows ρ_x is split, i.e.

$$\rho_x \cong \psi_1 \oplus \psi(1)$$

11.2. **Taylor-Wiles liftings.** In this section we give an explicit description of the deformation ring for a certain type of mod p representation, more importantly these lifting rings are essential for the Taylor-Wiles-Kisin method. Let K be a finite extension of $\mathbb{Q}_p$, write k for its residue field and suppose $\overline{\rho}$ is a continuous representation $G_K \to GL_n(\mathbb{F})$ which is unramified. Suppose also that $\overline{\rho}$ has distinct eigenvalues, $\#k \equiv 1 \mod l$ and that χ is unramified.

Proposition 11.11. *Suppose $\#k$ is exactly divisible by l^m. Then*

$$R^{\square}_{\overline{\rho},\chi} \cong \mathcal{O}[[x, y, B, u]]/((1+u)^{l^m} - 1)$$

and $\rho^{\square}(Frob_p)$ is conjugate to a diagonal matrix.

Proof. Note that $\rho^{\square}$ maps the wild inertia P_K into the pro-l group

$$\ker(GL_n(R^{\square}_{\overline{\rho},\chi}) \to GL_n(\mathbb{F}))$$

hence has trivial image. Thus we may pick $\varphi \in G_K/P_K$ a lift of Frobenius and σ a topological generator of the tame inertia $I_K/P_K \cong \prod_{l \neq p} \mathbb{Z}_l$ satisfying the relation

$$\varphi^{-1}\sigma\varphi = \sigma^{\#k}$$

Since

$$G_K/P_K \cong \langle\sigma\rangle \ltimes \langle\varphi\rangle$$

the representation is completely determined by the images of σ and φ, in fact this is true of any lift of $\overline{\rho}$.

Let $\overline{\alpha}, \overline{\beta}$ be the eigenvalues of $\overline{\rho}(\varphi)$, and let α, β be lifts to $\mathcal{O}$. Then we claim we can write $\rho^{\square}$ as

$$\rho^{\square}(\varphi) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha + B & 0 \\ 0 & \chi(\varphi)/(\alpha + B) \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}$$

$$\rho^{\square}(\sigma) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1+u & 0 \\ 0 & (1+u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}$$

Let $\rho : G_K \to GL_n(A)$ be a lift of ρ, by Hensel's lemma we can pick $a, b \in \mathfrak{m}_A$ such that $\rho(\varphi)$ has characteristic polynomial $(X - (\alpha + a))(X - (\beta + b))$. Then we can find $x, y \in \mathfrak{m}_A$ such that

$$\rho(\varphi) \begin{pmatrix} 1 \\ x \end{pmatrix} = (\alpha + a) \begin{pmatrix} 1 \\ x \end{pmatrix}$$

$$\rho(\varphi) \begin{pmatrix} y \\ 1 \end{pmatrix} = (\beta + b) \begin{pmatrix} y \\ 1 \end{pmatrix}$$

As $\overline{\rho}$ is unramified $\overline{\rho}(\sigma)$ reduces to the identity $\mod \mathfrak{m}_A$ so we may write

$$\begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \rho(\sigma) \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1+u & v \\ w & 1+z \end{pmatrix}$$

where $u, v, w, z \in \mathfrak{m}_A$.

Let n be largest such that $v \in \mathfrak{m}_A^n$, it follows from the commutativity relation $\varphi^{-1}\sigma\varphi = \sigma^{\#k}$ that

$$v \equiv v(\alpha + a)/(\beta + b) \mod \mathfrak{m}_A^{n+1}$$

since α, β are distinct $\mod \mathfrak{m}_A$ it follows that $v \in \mathfrak{m}_A^{n+1}$ so that $v = 0$. Similarly $w = 0$, and since χ is unramified we obtain $1 + z = (1 + u)^{-1}$. Applying the commutativity relation again, we obtain $(1 + u)^{\#k} = 1 + u$ and since $1 + u$ is invertible, we have $(1 + u)^{\#k-1} = 1$.

Let $\gamma = (1+u)^{l^m}$ and $r = (\#k-1)/l^m$, then $(l, r) = 1$, and so by Hensel's Lemma γ is the unique root of the equation $X^r - 1$ which is reducible $\mod \mathfrak{m}_A$ and congruent to $1 \mod \mathfrak{m}_A$, i.e. $\gamma = 1$. Thus $(1 + u)^{l^m} = 1$ and in this way we get a map from $\mathcal{O}[[x, y, B, u]]$ to A such that ρ is the pushforward of $\rho^{\square}$.

$\square$

12. Lecture 12

Last time we studied local lifting rings when $l \neq p$. See[4] for 3.35 for Ihara avoidance.

12.1. **Local lifting rings $l = p$.**

From now on all Galois representations are finite dimensional on L vector spaces. Let $K/\mathbb{Q}_p$ be a finite extension, and $L/\mathbb{Q}_l$ the coefficient field.

The main difference in this case is that there are many more p-adic representations of G_K than l-adic representations ($l \neq p$). This is because $GL_n(\mathcal{O})$ is a mostly pro-l hence any l-adic representations factors through the tame inertia (at least after a finite extension) since wild inertia is pro-p.

The main goal of p-adic Hodge theory is to understand p-adic representations of G_K through categories of (semi)-linear algebraic objects. There is the following hierarchy of representations:

$$\text{(Crystalline)}\subset\text{(Semistable)}\subset\text{(potentially Semistable=de Rham)}$$

$$\subset\text{(Hodge-Tate)}\subset\text{(all representations)}$$

If we think of l-adic representations coming from smooth projective algebraic varieties, the unramified representations "corresponds' to good reduction, representations where I_K acts unipotently corresponds to semistable reduction, potentially unipotent correspond to potentially semistable reduction. By Grothendieck's l-adic monodromy all representations are potentially unipotent and it is conjecture that all smooth projective varieties have potentially semistable reduction.

Definition 12.1. ρ is potentially "blah" it becomes "blah" after a finite extension.

12.2. **Langlands-Fontaine-Mazur philosophy.** If $\rho : G_F \to GL_n(L)$ comes from automorphic forms, then $\rho|_{G_{F_v}}$ for a place v of F dividing l is potentially semistable. Fontaine and Mazur conjecture that the converse also holds if ρ is unramified at almost all places.

To prove instances of this one imposes the potentially semistable condition as a deformation problem so that so called "$R = \mathbb{T}$" theorems have a chance of being true.

In the rest of this lecture we will explain two important invariants attached to potentially semistable representations and explain Kisin's results on various potentially semistable lifting rings.

12.2.1. *WD functor for $l = p$.* Recall we constructed such a functor using Grothendieck's l-adic monodromy theorem in the case $l \neq p$.

For a finite Galois extension K'/K, let K_0' denote the maximal unramified extension of K contained in K' and write ϕ_0 for the absolute frobenius on K_0'. We take L a sufficiently large extension of $\mathbb{Q}_p$ containing the Galois closure of K'. We have the following categories

$$\mathscr{WD}_{K'/K} := (\text{ WD representations of } W_K \text{ such that } (r, N) \text{ is unramified})$$

$$\mathscr{MOD} := ((\phi, N, Gal(K'/K))\text{-module } D)$$

where D is finite free $K_0 \otimes_{\mathbb{Q}_p} L$ module.
$\varphi : D \to D$ bijective and $\phi_0 \otimes 1$ semi-linear (i.e. $\varphi((k \otimes l)d) = (\phi_0(k) \otimes l)\varphi(l))$
$N \in End_{K_0'\otimes L}(D)$ is a nilpotent operator

The $\mathrm{Gal}(K'/K)$ action on D is semi linear and commutes with φ and N, we have the following relation between N and φ:

$$N\varphi = p\varphi N \tag{12.1}$$

There exists the following functors:

$$(G_K\text{representations } \rho \text{ such that } \rho|_{G_{K'}} \text{ is semistable}) \xrightarrow{D_{st,K'/K}} \mathscr{MOD}_{K'/K} \xrightarrow{WD_{K'/K}} \mathscr{WD}_{K'/K}$$

The functor $D_{st,K'/K}$ is Fontaine's functor, which is given by

$$V \mapsto (V \otimes B_{st})^{G_{K'}}$$

where B_{st} is the semistable period ring of Fontaine (cf. [?]).

We will now define $WD_{K'/K,\sigma_0}$ for $\sigma_0 : K'_0 \hookrightarrow L$, then show that it is independent of this embedding.

Define $V := D \otimes_{K'_0 \otimes L} L$ where $K'_0 \otimes L \to L$ is given by $\sigma_0 \otimes 1$. Then for $w \in W_K$ let

$$r(w) := w\varphi^{v_{\mathbb{Q}_p}(w)}$$

where w acts via the $\mathrm{Gal}(K'/K)$ action on D. This gives an action of the Weil group on V and we extend this to a Weil-Deligne representation by defining $N := N \otimes 1$. The condition $r(w)^{-1}Nr(w) = \#kN$ follows from the relation 12.1.

We have that

$$WD_{K'/K,\sigma_0} \cong WD_{K'/K,\tau_0}$$

for τ_0 another embedding.

The upshot of this is that for $L = \overline{\mathbb{Q}}_p$, we obtain a functor from potentially semistable represtatnsion to WD representations of W_K.

Let (r, N):=WD(ρ), we have the following properties.

ρ semistable $\Rightarrow$ r unramified, if ρ is also crystalline, then $N = 0$.

$\rho|_{G'_K}$ semistable $\Leftrightarrow$ $r|_{I_{K'}} = 1$, $r(W_{K'})$ centralizes $r(W_K)$.

12.2.2. *Hodge Tate weights.* Let $\rho : G_K \to GL(V)$ with $N = \dim V$ and $\sigma : K \to L$, we obtain an unordered set of integers $HT_\sigma(\rho) \in \mathbb{Z}^n/S_n$.

The recipe is given as follows: Each $i \in \mathbb{Z}$ in $HT_\sigma(\rho)$ has multiplicity $dim_L(V(i) \otimes_{K,\sigma} \mathbb{C}_p)^{G_K}$, where $V(i) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\epsilon_p^i)$. Here ϵ_p is the p-adic cyclotomic character $G_K \twoheadrightarrow \mathbb{Z}_p^\times$

If ρ is Hodge-Tate then the sum of these dimensions over $i \in \mathbb{Z}$ is equal to n.

Remark 12.2. $\{HT_\sigma(\rho)\}$ can also be read off from some natural filtration data attached to $D_{st,K'/K}(\rho)$

Example 12.3. $\bullet \rho = \epsilon_P : G_K \to \mathbb{Z}_p^\times$, $HT_\sigma(\rho) = \{-1\}$

• A/K an abelian variety $\rho : G_K \to GL_{\mathbb{Q}_l}(T_l A \otimes_{Z_l} \mathbb{Q}_l)$, $HT_\sigma(\rho = \{0, ..., 0, -1, ..., -1\}$

• f an eigenform of weight $k \geq 1$,then the associated Galois representations has Hodge Tate weights $\{0, k-1\}$.

• If ρ has finite image then all Hodge Tate weights are 0.

Remark 12.4. For ρ potentially semistable we obtain $WD(\rho), \{HT_\sigma(\rho)\}$. A natural question to ask then is can we recover ρ from this associated data, and what is the image of this functor.

It turns out ρ can be recovered and the image of this functor corresponds to data which admits an admissible filtration.

Theorem 12.5. *1) $\exists!$ p-torsion free reduced quotient $R^{\square}_{\bar{\rho},\chi,\{H_\sigma\},K'sst(cris)}$ of $R^{\square}_{\bar{\rho},\chi}$ such that for a closed geometric point x of $R^{\square}_{\bar{\rho},\chi}[1/p]$ factors through the quotient if and only if $HT_\sigma(\rho_x) = H_\sigma$ and $\rho_x | G'_K$ is semistable (cristalline).*

2) Spec($R^{\square}_{\bar{\rho},\chi,\{H_\sigma\},K'sst(cris)}$) is equidimensional of dimension $=n^2 + [K : \mathbb{Q}_p](n(n-1)/2$. and the generic fibre is generically smooth

13. Lecture 13

D central simple (division) algebra $/K$, $K/\mathbb{Q}_p$ finite $[D : K] = n^2$

The basic outline of the Jacquet Langlands correspondence is that there should be a functor taking irreducible admissible representations of $D^\times$ to irreducible admissible representations of $GL_n(K)$. The Local Langlands correspondence then states that this last set should be in $1-1$ correspondence with n dimensional frobenius semisimple Weil-Deligne representations of W_K. We will discuss these two topics today as well as touching on the Global Langlands correspondence.

Definition 13.1. A locally profinite group Γ is a topological group where every open neighborhood of $1 \in \Gamma$ contains open compact subgroups.

Proposition 13.2. *Locally profinite $\Leftrightarrow$ locally compact and totally disconnected.*

Proof. cf. profinite $\Leftrightarrow$ compact and totally disconnected. □

Example 13.3. Local: $GL_n(K), D^\times, GL_n(\mathcal{O}_K), \mathcal{O}_D^\times$, ($\mathcal{O}_D \subset D$ maximal order), or the upper triangular Borel subgroup of $GL_2(K)$.

Global: $GL_n(\hat{\mathbb{Z}}), GL_n(\mathbb{A}^\infty)$

Our conventions will be that all representations are on complex vector spaces. In fact we can take any algebraically closed field of characteristic 0

Definition 13.4. (π, V) a representation of Γ means a vector space $V/\mathbb{C}$ (possible infinite dimensional, and

$$\pi : \Gamma \to GL_{\mathbb{C}}(V)$$

a homomorphism.

$V_{sm} := \{v \in V : \text{stab}(v) \subset \Gamma \text{ open}\}$, i.e. the set of $v \in V$ such that $\gamma \mapsto \pi(\gamma)v$ is continuous.

We say V is

1) smooth if $V = V_{sm}$
2) admissible if for all compact open $U \subset \Gamma$ we have $\dim V^U < \infty$

Define also $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and equip it with the action of Γ given by

$$(\pi^*(\gamma)f)(v) = f(\pi(\gamma^{-1})v)$$

Definition 13.5. The smooth dual representation of a smooth representation (π, V) is $(\pi^\vee, V^\vee) := (\pi^*, (V^*)_{sm})$.

Fact: (π, V) is admissible $\Leftrightarrow (\pi^{\vee\vee}, V^{\vee\vee}) \cong (\pi, V)$

$(\pi, V) \mapsto (\pi^\vee, V^\vee)$ is an exact functor.

Definition 13.6. $C_c^\infty(\Gamma) := \{\text{smooth compactly supported functions } \Gamma \to \mathbb{C}\}$

A left right invariant Haar measure is a $\mathbb{C}$-linear functional

$$\mu : C_c^\infty(\Gamma) \to \mathbb{C}$$

denoted

$$f \mapsto \mu(f) = \int_\Gamma f\mu = \int_\Gamma f(\gamma)d\mu$$

such that $\int_\Gamma f(\gamma)d\mu = \int_\Gamma f(\gamma\delta)d\mu_\gamma = \int f(\delta\gamma)d\mu_\gamma$

Conceretely if we write

$$f = \sum_{i=1}^r c_i 1_{U_i}$$

where $c_i \in \mathbb{C}$ and $U_i \subset \Gamma$ are compact open subsets. Then $\int_\Gamma f(\gamma)d\mu = \sum_{i=1}^r c_i\mu(1_{U_i}) = \sum_{i=1}^r c_i \text{vol}(U_i)$

Then we define the Hecke algebra to be

$$\mathcal{H}(\Gamma) := C_c^\infty(\Gamma)$$

with the product given by convolution:

$$(f * g)(\gamma) := \int_\Gamma f(\delta)g(\delta^{-1}\gamma)d\mu$$

This turn $\mathcal{H}(\Gamma)$ into an associative algebra, but in general it is not unital nor commutative (unless Γ is abelian).

We say Γ is unimodular if $\exists$ a bi-invariant Haar measure, which if it exists is necessarily unique up to scalar. Assume for now such a measure exists.

Variant: $\mathcal{H}(\Gamma/U) = C_c^\infty(U\backslash\Gamma/U)$ consisting of the smooth compactly supported functions which are bi-invariant under the compact open subgroup U.

For (π, V) a smooth representation and $f \in \mathcal{H}(\Gamma)$ define $\pi(f) \in \mathrm{End}_{\mathbb{C}}(V)$ by

$$\pi(f)v = \int_\Gamma f(\gamma)\pi(\gamma)v d\mu$$

if $U \subset stab(v)$ and f is right U-invariant, this equals

$$= \sum_{\gamma\in\Gamma/U} f(\gamma)(\pi(\gamma)v)\mu(U)$$

Lemma 13.7. $\pi(f * g) = \pi(f)\pi(g)$

Proof.

$$\pi(f * g)(v) = \int_\Gamma f * g(\gamma)\pi(\gamma)v d\mu_\gamma$$

$$= \int_\Gamma \left(\int_\Gamma f(\delta)g(\delta^{-1}\gamma)d\mu_\delta \right) \pi(\gamma)v d\mu_\gamma$$

Writing $\alpha = \delta^{-1}\gamma$ and changing the order of integration, we obtain:

$$= \int_\Gamma \left(\int_\Gamma f(\delta)g(\alpha)\pi(\delta\alpha)v d\mu_\alpha \right) d\mu_\delta$$

$$= \int_\Gamma f(\delta)\pi(\delta)\pi(g)v d\mu_\delta$$

$$= \pi(f)\pi(g)v$$

□

It follows that π gives a homomorphism $\mathcal{H}(\Gamma) \to End_{\mathbb{C}}(V)$. In fact this extends to a functor from smooth representations of Γ to $\mathcal{H}(\Gamma)$ modules.

Lemma 13.8. The functor constructed above is fully faithful

Proof. □

If f is left U-invariant, then $\pi(f)v \in V^U$ because

$$(\pi(u)(\pi(f)v) = \int_\Gamma f(\gamma)\pi(u\gamma)v d\mu = \int_\Gamma f(u^{-1}\gamma)\pi(\gamma)v d\mu = \pi(f)v$$

Consequently: V^U is $\mathcal{H}(\Gamma/U)$-module

(π, V) admissible implies $\dim(im(\pi(f)) < \infty$ since any f is bi-invariant under a small enough open compact subgroup U, in particular the trace of $\pi(f)$ is well defined.

We have there is an equivalence of categories between smooth representations (π, V) and "smooth" $\mathcal{H}(\Gamma)$-modules. Therefore in the theory of smooth admissible representations the Hecke algebra $\mathcal{G}(\Gamma)$ plays the role of the group algebra $\mathbb{C}[\Gamma]$.

13.1. **Representation theory of $GL_n(K)$.** Now we specialize to the case $\Gamma = GL_n(K)$ where $K/\mathbb{Q}_p$ finite. Then $GL_n(K)$ is unimodular, and Schur's Lemma holds, i.e. for (π, V) is irreducible admissible $\mathrm{End}_{\mathbb{C}} \cong \mathbb{C}$. To prove this, one uses the fact that $GL_n(K)/U$ is countable for any open compact subgroup U. This allows us to define the central character of (π, V), $\omega_\pi : K^\times \to \mathbb{C}^\times$ given by

$$\pi(z) \in \mathrm{End}(\pi, V) \cong \mathbb{C}$$

Harish-Chandra proved that there exists a character $\Theta_\pi : GL_n(K)_{reg} \to \mathbb{C}$ which is locally constant and characterized by

$$tr(\pi(f)) = \int_{GL_n(K)_{reg}} \Theta_\pi(\gamma) f(\gamma) d\mu$$

Here $GL_n(K)_{reg}$ is the subset of regular semi simple elements. Some fundamental problems in the subject are given below:

1) Construct and classify irreducible representations. We have the following coarse classification:

$$\{\text{ irred. adm. }\} \supset \{\text{ square int. }\} \supset \{\text{ supercuspidal}\}$$

2) Relate the irreducible representations of $G(K)$ and $G'(K)$ where $G \subset G'$ or $G \times G' \subset H$

Langlands functorialty, ("L-morphism" ${}^LG \to {}^LG'$)

Jacquet Langlands/ base change

3) Classify by Galois data (Local Langlands)

14. Lecture 14

Recall that we were studying the representation theory of $GL_n(K)$ for $K/\mathbb{Q}_p$ finite.

14.1. **Parabolic induction.** $r \geq 1$. $\underline{n} = (n_1, ..., n_r)$ such that $\sum n_i = n$

Let $P_{\underline{n}} \subset GL_n$ be the block upper triangular parabolic subgroup whose blocks are of size $n_1, .., n_r$. This contains the Levi subgroup $M_{\underline{n}}$ consisting of block diagonal matrices of size $n_1, ..., n_r$, and define $N_{\underline{n}}$ to be the unipotent radical of $P_{\underline{n}}$, this gives a decomposition $P_{\underline{n}} = N_{\underline{n}} \ltimes M_{\underline{n}}$.

$P_{\underline{n}}$ has the modulus character $\delta(n) : P_n \to \mathbf{R}_{>0}$ modulus character given by

$$g \mapsto |det(Ad(g)|_{Lie(N_{\underline{n}})})|_K$$

Example 14.1. $n = 2, \underline{n} = (1, 1)$

Let $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a basis of $Lie(N_{\underline{n}})$, and so since

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & -b/da \\ 0 & d^{-1} \end{pmatrix} = \begin{pmatrix} 0 & a/d \\ 0 & 0 \end{pmatrix}$$

Thus $\delta(d) = |a/d|$

We now construct the parabolic induction functor which takes smooth representations of $M_{\underline{n}}$ to smooth representations of $GL_n(K)$, denoted by

$$(\pi_{\underline{n},V_{\underline{n}}}) \mapsto (n - \mathrm{ind}_{P_{\underline{n}}(K)}^{GL_n(K)}, V)$$

To construct $n - \mathrm{ind}_{P_{\underline{n}}(K)}^{GL_n(K)}, V)$, we inflate $\pi_{\underline{n}}$ from $M_{\underline{n}}$ to $P_{\underline{n}}$, twist by $\delta_{\underline{n}}^{1/2}$ and finally induce the representation up to $GL_n(K)$.

Explicitly we have

$$V = \{\phi : GL_n(K) \to V_{\underline{n}} | \phi(mng) = \delta_{\underline{n}}^{1/2}(m)\pi_{\underline{n}}(m)\phi(g), m \in M_{\underline{n}}, n \in N_{\underline{n}}, g \in GL_n(K)\}$$

And the action by $GL_n(K)$ is given by right translation:

$$g\phi(h) = \phi(hg)$$

Fact: 1) $n - \mathrm{ind}$ is an exact functor (it has an exact left adjoint, the Jacquet module functor)

2) $\pi_{\underline{n}}$ finite length $\Rightarrow n - \mathrm{ind}\pi_{\underline{n}}$ is also finite length.

Definition 14.2. $\pi_i \in Irr^{adm}(GL_{n_i}(K)), i = 1, ..., r$. Set $\pi_{\underline{n}} := \pi_1 \otimes ... \otimes \pi_r \in Irr^{adm}(M_{\underline{n}}(K))$.

Then define

$$\pi_1 \times ... \times \pi_r := n - \mathrm{ind}_{P_{\underline{n}}}^{GL_n} \pi_{\underline{n}}$$

which is a finite length admissible representation of $GL_n(K)$.

Fact: $JH(\pi_1, ..., \pi_r) = JH(\pi_{\sigma(1)} \times, ..., \times \pi_{\sigma(r)})$ as a multi-set, where $\sigma \in S_r$.

Example 14.3. $n = 2, \underline{n} = (1, 1)$. $\pi_1 = |.|^{-1/2}, \pi_2 = |.|^{1/2}$, where $|.| : K^\times \to \mathbf{R}_{>0}^\times$ is the absolute value.

Then

$$\pi_1 \times \pi_2 = \{\phi : GL_2(K) \to \mathbb{C} | \phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \delta \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{1/2} |a|^{-1/2}|d|^{1/2}\phi(g)\}$$

$$= \{\phi : GL_2(K) \to \mathbb{C} | \phi \text{ is left invariant under} P_{(1,1)}\}$$

We have the exact sequence

$$0 \longrightarrow \{\text{constant fns.}\} \longrightarrow \pi_1 \times \pi_2 \longrightarrow Sp_2(|.|^{-1/2}) \longrightarrow 0$$

Can twist by any smooth character $\chi : K^\times \to \mathbb{C}^\times$ to get

$$0 \longrightarrow \chi \circ \det \longrightarrow |.|^{1/2}\chi \times |.|^{1/2}\chi \longrightarrow Sp_2(\chi|.|^{-1/2}) \longrightarrow 0$$

Dualizing, $|.|^{1/2}\chi \times |.|^{-1/2}\chi$ has length 2 and the 1-dimensional quotient $\chi \circ \det$

14.2. **Discrete series and supercuspidals.**

Definition 14.4. $(\pi, V) \in Irr^{adm}(GL_n(K))$, $V^\vee$ its smooth dual. A matrix coefficient $\phi_{v,f} : GL_n(K) \to \mathbb{C}$ for $v \in V, f \in V^\vee$ is defined to be the function $\phi_{v,f}(g) = f(\pi(g)v)$. One can check that $\phi_{v,f}(zg) = \omega_\pi(z)\phi(g), z \in K^\times$.

Definition 14.5. (π, V) is supercuspidal if all matrix coefficients $\phi_{v,f}$ have compact support mod $K^\times$ and (π, V) is square integrable (discrete series) if every $\phi_{v,f}$ is "l^2 mod center:"

$$\int_{GL_n(K)/K^\times} |\phi_{v,f}(g)\omega_\pi(det(g))^{-1/n}|^2 d\mu < \infty$$

Observe: supercuspidal implies discrete series, however the converse does not hold unless $n = 1$.

Define $Irr^{sc}(GL_n) \subset Irr^2(GL_n) \subset Irr^{adm}(GL_n)$

Proposition 14.6. *1) $\pi \in Irr^{sc}(GL_m), m, s \geq 1$, $|det| : GL_m(K) \to K^\times \to \mathbb{C}^\times$*
Then

$$\pi \times \pi|det| \times ... \times \pi|det|^{s-1}$$

has a unique irreducible quotient, which is a discrete series of $GL_ms(K)$, we write $Sp_s(m)$ for this representation.

14.3. **Langlands classification.** For $\pi \in Irr^2(GL_n)$, the modulus of its central character $|\omega_\pi(.)|$ is of the form $|.|_K^{-a_\pi n}$ for a unique real number $a_\pi \in \mathbf{R}$. $\mathbf{R}_{>0}^\times$ valued character on $K^\times/\mathcal{O}_K^\times \cong \mathbb{Z}$

Proposition 14.7. *$\pi_i \in Irr^2(GL_{n_i}) i = 1, ..., r, a_{\pi_1} \leq ... \leq a_{\pi_r}$. Then $\exists!$ irreducible quotient of $\pi_1 \times ... \times \pi_r$, which is called the Langlands quotient.*

Remark 14.8. Without the condition $a_{\pi_1} \leq ... \leq a_{\pi_r}$, we must order π_i suitably.

Definition 14.9. Write $\pi_1 \boxplus ... \boxplus \pi_r$ for this unique quotient.

Fact: It's well defined (if $a_{\pi,i} = a_{\pi,i+1}$ we can swap π_i, π_{i+1})

Proposition 14.10. *$Irr^{adm}(GL_n)$ is in 1-1 correspondence with*

$$\{(r, \{\pi_i\}_{i=1}^r | r \geq 1, \pi_i \in Irr^2(GL_{n_i}), \sum n_i = n\}$$

The correspondence is given by sending $(r, \{\pi_i\})$ to $\boxplus_{i=1}^r \pi_i$

Fact: $(\boxplus \pi_i)^\vee = \boxplus(\pi_i^\vee)$ and $\boxplus$ is associative

Example 14.11.

$$0 \longrightarrow \chi \circ det \to \chi|.|^{-1/2} \times \chi|.|^{1/2} \to Sp_2(\chi|.|^{-1/2}) \longrightarrow 0$$

$a_{\chi|.|^{-1/2}} = 1/2, a_{\chi|.|^{1/2}} = -1/2.$

$$\chi|.|^{1/2} \times \chi|.|^{-1/2} \to \chi \circ det = \chi|.|^{1/2} \boxplus \chi|.|^{-1/2}$$

14.4. Local Langlands Correspondences.

Definition 14.12. A Weil-Deligne representation is Frobenius semi-simple if $\forall$ lift of (geometric) Frobenius ϕ, $r(\phi)$ is semi simple. $(W_K \to W_K/I_K \cong \mathbb{Z} \to GL(V))$

Write

$$Rep_n^{irr}(WD_K) \subset Rep_n^{indec}(WD_K) \subset Rep_n^{Fss}(WD_K)$$

Recall that $Rep_n^{indec}(WD_K)$ is in 1-1 correspondence with

$$\{(s,\rho) : s \geq 1, s|n, \rho \in Rep_{n/s}(WD_K)\}$$

The correspondence being given by

$$(s,\rho) \mapsto Sp_s(\rho) = (\rho \oplus ... \oplus \rho|.|^{s-1}, N)$$

where N is the matrix with 1's on the upper diagonal and 0's elsewhere.

Then $Rep^{Fss}(WD_K)$ is in 1-1 correspondence with

$$\{(r, \{\rho_i\})|r \geq 1, \rho_i \in Rep_{n_i}^{indec}(WD_K), \sum n_i = n\}$$

15. Lecture 15

Recall we were discussing the local Langlands correspondences, which relates smooth admissible representations of $GL_n(K)$ and Weil Deligne representations. More precisely we have the following theorem.

Theorem 15.1. *(Harris-Taylor, Henniart, Scholze) Let $K/\mathbb{Q}_p$ be a finite extension. Then $\exists!$ bijection*

$$rec_K^n : Irr(GL_n(K)) \to Rep_(WD_K)^{Fss}$$

such that rec_K^1 comes from local class field theory. i.e. for $\chi : K^\times \to \mathbb{C}^\times$, $rec_K^1(\chi)$ is given by composition with the reciprocity map $Art_K : W_K \to W_K^{ab} \to K^\times$.

2) rec_K^n preserves L and ϵ factors in pairs:

$$L(s, \pi_1 \times \pi_2) = L(s, \rho_1 \otimes \rho_2)$$
$$\epsilon(s, \pi_1 \times \pi_2) = \epsilon(s, \rho_1 \otimes \rho_2)$$

Fact: Further properties of rec_K^n:

3) It preserves conductors

4) If π corresponds to ρ then

$\pi^\vee$ corresponds to $\rho^\vee$

The central character ω_π corresponds to $det\rho$ under rec_K^1

(Compatibility with twisting)$\pi \otimes (\chi \circ \det)$ corresponds to $\rho \otimes rec_K^1(\chi)$

5) If π_i and ρ_i correspond under rec_K, then $\pi_1 \boxplus \pi_2$ corresponds to $\rho_1 \oplus \rho_2$

If $\pi \in Irr^{sc}(GL_n(K))$ corresponds to ρ, then $Sp_s(\pi)$ corresponds to $Sp_s(\rho)$ for all $s \geq 1$.

In fact,

$$Irr(GL_n(K)) \longrightarrow Rep_n^{Fss}(WD_K)$$

$$Irr^2(GL_n(K)) \longrightarrow Rep_n^{indec}(WD_K)$$

$$Irr^{sc}(GL_n(K)) \longrightarrow Rep_n^{irr}(WD_K)$$

The properties imply that that once we establish the correspondence $Irr^{sc} \leftrightarrow Rep^{irr}$ satisfying 1) and 2), then we get the entire local Langlands correspondence once we have checked 1) and 2) for all pairs. (Note that for any Weil Deligne representation (r, N), $\ker N$ is a W_K invariant subspace, if (r, N) is irreducible then N must be 0, so that supercuspidal representations should correspond to irreducible Weil group representations).

15.1. **The Satake Isomorphism.** Let $B = P(1, ..., 1)$ be the standard Borel subgroup of upper triangular matrices of GL_n, T the maximal torus of diagonal matrices and N the unipotent radical of B.

We have the following isomorphism:

$$\mathcal{H}(T(K)/T(\mathcal{O}_K)) \cong \mathbb{C}[x_1^{\pm}, ..., x_n^{\pm}]$$

which takes the characteristic function of $diag(\varpi^{a_1}, ..., \varpi^{a_n})T(\mathcal{O}_K)$ to $x_1^{a_1}....x_n^{a_n}$. (Note we are normalizing the Haar measure so that $T(\mathcal{O}_K) = 1$).

Fix also a Haar measure on $GL_n(K)$ such that $GL_n(\mathcal{O}_K) = 1$

Definition 15.2. The Satake Transform is the map

$$\mathcal{S} : \mathcal{H}^{ur}(GL_n(K)) := \mathcal{H}(GL_n(\mathcal{O}_K)\backslash GL_n(K)/GL_n(\mathcal{O}_K)) \to \mathcal{H}(T(K)/T(\mathcal{O}_K)) \cong \mathbb{C}[x_i^{\pm}]$$

The map is given by $f \mapsto (t \mapsto \delta_B^{1/2}(t) \int_N f(tn)\mu_n)$

Theorem 15.3. *$\mathcal{S} : \mathcal{H}^{ur}(GL_n(K)) \cong \mathbb{C}[x_1^{\pm}, ..., x_n^{\pm}]^{S_n}$*

$T_i = q^{i(n-i)} tr \Lambda^i diag(x_1, ..., x_n)$, then the algebra on the right is isomorphic to $\mathbb{C}[T_1, ..., T_{n-1}, T_n^{\pm}]$ and the isomorphism is given by:

$\mathcal{S}(\mathbb{1}_{GL_n(\mathcal{O}_K) diag(\varpi I_i, I_{n-i}) GL_n(\mathcal{O}_K)}) = T_i$

See [4] Exercise 4.7 for a special case.

15.2. **Unramified Local Langlands.**

Definition 15.4. $\pi \in Irr(GL_n(K))$ is unramified if $\pi^{GL_n(\mathcal{O}_K)} \neq 0$

Observe π unramified implies $\dim \pi^{GL_n(\mathcal{O}_K)} = 1$. Indeed π irreducible implies $\pi^{GL_n(\mathcal{O}_K)} \neq 0$ is irreducible as an $\mathcal{H}^{ur}$ module, but since $\mathcal{H}^{ur}$ is commutative, we have $\dim \pi^{GL_n(\mathcal{O}_K)} = 1$.

$\mathcal{H}^{ur}$ acts on $\pi^{GL_n(\mathcal{O}_K)}$, since the latter space is isomorphic to $\mathbb{C}$, we obtain a map $S_\pi : \mathcal{H}^{ur} \to End(\pi^{GL_n(\mathcal{O}_K)}) = \mathbb{C}$, such a map is called a Satake parameter.

Alternatively $S_\pi \in MaxSpec\mathcal{H}^{ur} = (\mathbb{C}^\times)^n/S_n$

Remark 15.5. S_π is determined $S_\pi(T_i), i = 1, ..., n$.

Proposition 15.6. *(Unramified Local Langlands) rec_K^n induces a bijection between $Irr^{ur}(GL_n(K))$ and $Rep_n^{ur}(WD_K)^{Fss}$ making the following diagram commute:*

$$\begin{tikzcd}
Irr^{ur}(GL_n(K)) \arrow[rr] & & Rep_n^{ur}(WD_K)^{Fss} \\
& (\mathbb{C}^\times)^n/S_n \arrow[ul, "S \mapsto \boxplus_i \chi(s_i)"] \arrow[ur, "S \mapsto \oplus \sigma(s_i)"'] &
\end{tikzcd}$$

where $S = \{s_1, ..., s_n\}$ and $\chi(s_i)$ and $\sigma(s_i)$ correspond via rec_K^1

Note that in this correspondence the characteristic polynomial of Frobenius on ρ is equal to the Hecke polynomial of $\pi = \prod(x - s_i)$

Proof. $\boxplus\chi(s_i)$ is unramified, and irreducible if and only if $s_i \neq q^{\pm}s_j, \forall i \neq j$, $\pi \mapsto \pi^{GL_n(\mathcal{O}_K)}$

Using the Iwasawa decomposition, $GL_n(K) = B(K)GL_n(\mathcal{O}_K)$ implies $\chi(s_1) \times ... \times \chi(s_n)^{GL_n(\mathcal{O}_K)}$ is 1-dimensional $\implies \exists!$ irreducible subquotient which is unramified and identify as $\boxplus\chi(s_i)$ by induction. ∎

$Irr^{ur}(GL_n(K)) \to \{irred.\mathcal{H}^{ur} - mod\}$, $\pi \mapsto \pi^{GL_n(\mathcal{O}_K)}$ is a bijection.

15.3. **Local Base Change.** Let $K'/K/\mathbb{Q}_p$ be finite extensions. Fact: If $K'/\mathbb{Q}_p$ is Galois, then K'/K is solvable.

Corollary 15.7 (of Local Langlands)**.** *The following diagram commutes*

$$\begin{tikzcd}
Irr(GL_n(K')) \arrow[r, "rec_{K'}"] & Rep(WD'_K)^{Fss} \\
Irr(GL_n(K)) \arrow[r, "rec_K"] \arrow[u] & Rep(WD_K)^{Fss} \arrow[u]
\end{tikzcd}$$

where the vertical maps are given by base change on the left and restriction on the right.

$n = 1$ $\chi : K^\times \to \mathbb{C}^\times$ base changes to $\chi \circ N_{K'/K} : K'^\times \to \mathbb{C}^\times$
$n = 2$ Langlands

$n > 2$ Arthur-Clozel [1] ,the main tool for these proofs is the trace formula.

We actually need an alternative characterization of Base Change in terms of trace characters. (Θ_π and its twisted version)

When the extension K'/K is cyclic, we have the following properties of the local base change functor which follows from the properties of the Local Langlands correspondence.

Proposition 15.8. *Let $Gal(K'/K) \cong \langle\sigma\rangle$ and let BC denote the base change map*

$$Irr^{adm}(GL_n(K)) \to GL_n(K')$$

i) A representation π lies in the image of BC if and only if $\pi \cong \pi \circ \sigma$, where $\pi \circ \sigma$ is the representation of $GL_n(K')$ with same underlying space as π, but with g acting via $\pi(\sigma(g))$

ii) $BC(\pi_1) \cong BC(\pi_2)$ if and only if $\pi_1 \cong \pi_2 \otimes \chi \circ \det$, for some $\chi : K^\times \to \mathbb{C}^\times$ smooth character which factors through $K^\times/N_{K'/K}(K'^\times)$

iii) $BC(\pi)$ is supercuspidal if and only if π is supercuspidal and $\pi \ncong \pi \circ \chi \det$ for some non-trivial χ as in part ii)

iv) $\omega_{BC(\pi)} = \omega_\pi \circ N_{K'/K}$

Proof. i)[TO DO]

ii) Let ρ_1, ρ_2 correspond to π_1, π_2 under rec^n, then ρ_1, ρ_2 have then $BC(\pi_1) \cong BC(\pi_2)$ if and only if $\rho_1|W_{K'} \cong \rho_2|_{W_{K'}}$. If $\pi_1 \cong \pi_2 \otimes \chi \circ \det$ as in the statement, then by the compatibility with twisting, property, we have $\rho_1 \cong \rho_2 \otimes rec^1(\chi)$ and hence their restrictions to $W_{K'}$ are isomorphic.

Conversely suppose $BC(\pi_1) = BC(\pi_2)$, fix an isomorphism $\rho_1|W_{K'} \cong \rho_2|_{W_{K'}}$. It suffices to show that $\rho_1(\sigma)\rho_2(\sigma)^{-1}$ is a scalar, since ρ

iii) Let $rec_K(\pi) = (r, N)$, then $rec_{K'}(BC(\pi)) = (r|_{W_{K'}}, N) := (r', N)$. Then π (resp. $BC(\pi)$) is irreducible if and only if $N = 0$ and r is irreducible (resp. $N = 0$ and r' is irreducible). Thus we need to show r' is irreducible if and only if r is irreducible and $\pi \neq \pi \otimes \chi \circ \det$ for some χ in the statement. This last condition is equivalent to $r \neq r \otimes \chi$ for $\chi : W_K \to \mathbb{C}^\times$ a character which is trivial on $W_{K'}$.

Suppose r' is irreducible, then r is irreducible. Let V be the underlying space of r, then suppose $\varphi : r \cong \chi$ is an isomorphism. Then since χ is trivial on $W_{K'}$ φ gives an isomorphism $r' \cong r'$ which by Schur's Lemma is a scalar. Then clearly $\varphi : r \cong \chi$ can only be an isomorphism if χ is trivial.

Conversely suppose r' is irreducible. Let $W \subset V$ be an irreducible subrepresentation and let $\sigma \in W_K$ be a lift of the generator of $\mathrm{Gal}(K'/K)$. Then $\sigma(W)$ is also an irreducible representation of W_K' and we have

$$V = W \oplus \sigma(W) \oplus ... \oplus \sigma^{n-1}(W)$$

where n divvies the order of $\mathrm{Gal}(K'/K)$. Choose χ a non-trivial character of $\mathrm{Gal}(K'/K)$ which is trivial on σ^n. Now the map $V \to V$ given by

$$v_0 + ... + v_{n-1} \mapsto v_0 + \chi(\sigma)v_0 + ... + \chi(\sigma^{n-1})v_{n-1}$$

where $v_i \in \sigma^i(W)$ gives an isomorphism from r to $r \otimes \chi$.

iv) This follows immediately from property 4) of the Local Langlands correspondences, i.e. that the central character ω_π corresponds to $\det(rec^1(\pi))$ and the explicit description of the the base change map in the case $n = 1$. □

Remark 15.9. 1) There exists an archimedean analogue which was proved much earlier.

2) There exists a global analogue.

3) Similarly there exists automorphic induction if we replace $res^{W_K}_{W_{K'}}$ by $ind^{W_K}_{W_{K'}}$.

16. Lecture 16

Last time we talked about Local Base change, see PS8 for its properties.

16.1. **Local Jacquet Langlands.**

Let D/K be a central division algebra with

$$[D : K] = n^2$$

Consider the composition

$$D \longrightarrow D \otimes_K \overline{K} \cong M_n(\overline{K}) \xrightarrow{det} \overline{K}$$

which we denote by N_D the reduced norm, its image actually lies in $K \subset \overline{K}$.

There exists a map which takes

$$D^\times_{reg}/\sim \hookrightarrow GL_n(K)_{reg}/\sim$$

$$\delta \mapsto \gamma$$

such that $\mathrm{char}(\delta) = \mathrm{char}(\gamma)$ where regular means the eigenvalues are distinct and $\sim$ is the equivalence given by conjugation. Note that for $\delta \in D$, we have $char(\delta) \in K[X]$ is of degree n.

The image of this map is

$$\{\gamma \text{ s.t. } \mathrm{char}(\gamma) \text{ is irred } /K\}$$

we call these elliptic elements.

Theorem 16.1 (Jacquet-Langlands, Rogawski, Deligne-Kazhdan-Vigneras)**.** *There exists a unique 1-1 map*

$$JL : Irr(D^\times) \cong Irr^2(GL_n(K))$$

such that $\forall \delta, \gamma$ *as above,* $\Theta_{\pi_D}(\delta) = (-1)^{n-1}\Theta_{JL(\pi_D)}(\gamma)$

(Recall: Θ_π was defined so that $tr\pi(f) = \int_{GL_n(K)_{reg}} \Theta_\pi(g)f(g)d\mu_g \;\forall f \in \mathcal{H}(GL_n(K))$)

Remark 16.2. When $n = 2$, $JL(\chi \circ N_D) = Sp_2(|\chi|.|^{-1/2})$, for $\chi : K^\times \to \mathbb{C}^\times$ a smooth character. $\dim(\pi_D) > 1 \Leftrightarrow JL(\pi_D)$ is supercuspidal.

Fact: Every $\pi_D \in Irr(D^\times)$ is finite dimensional, this is because $D^\times/K^\times$ is compact.

Example 16.3. $(n = 2, \chi = 1)$, suppose δ maps to $\gamma \in GL_2(K)$, char(γ) irreducible $/K$.

$$0 \longrightarrow 1 \longrightarrow |.|^{-1/2} \times |.|^{1/2} \to Sp_2(|.|^{-1/2}) \longrightarrow 0$$

Remark 16.4. There exists a non-archimendean analogue, $Irr(D^\times) \cong Irr^2(GL_2(\mathbf{R}))$, here D are the Hamiltonians over $\mathbf{R}$

16.2. **Global preliminaries.** Let F be a finite extension of $\mathbb{Q}$ and S a finite set of places of F, S_∞ the infinite places.

Define

$$F_S := \prod_{v \in S} F_v, \mathcal{O}_{F,S} = \prod_{v \in S} \mathcal{O}_{F_v}$$

$$\hat{\mathcal{O}}_F^S := \prod_{v \notin S \cup S_\infty} \mathcal{O}_{F_v}$$

For example, when $S = \emptyset$, $\hat{\mathcal{O}}_F$ is the profinite completion of $\mathcal{O}_F$

$$\mathbb{A}_F^S := \prod_{v \notin S}{}' F_v = \lim_{\to T \ fin. T \cap S = \emptyset} F_T \times \hat{\mathcal{O}}_F^{S \cup T}$$

When $(S = \emptyset)$, this is just the usual adeles $\mathbb{A}_F$.

Write $\mathbb{A}_F^\infty = \mathbb{A}_F^{S_\infty}, F_\infty = F_{S_\infty}$

For D/F a central simple algebra $/F$, $[D : F] = n^2$, let

$$S(D) := \{v \text{ places of } F | D \otimes_F F_v \neq M_n(F)\}$$

it is a finite set.

Fact: $(n = 2)$ Giving D is equivalent to giving $S(D)$ s.t. $S(D)$ is even., we view $G := D^\times = GL_1(D)$ as an algebraic space over F, i.e. for all F algebras R, $G(R) = (D \otimes_F R)^\times$

Note: G_D in [4] is $Res_{F/\mathbb{Q}} G$.

$G(\mathbb{A}_F^S) = \prod'_{v \notin S} G(F_v) = \lim_{\to T \ fin. T \cap S = \emptyset} G(F_T) \times G(\hat{\mathcal{O}}_F^{S \cup T})$.

Since $G(\mathbb{A}_F^S)$ (with $S \supset S_\infty$) is a locally profinite group, admissible/ smooth representations make sense.

Given $\{\pi_v \in Irr(G(F_v)\}_{v \notin S}$ such that π_v is unramified almost everywhere, define

$$S_{ram} := \{v \notin S \cup S(D) | \pi_v \text{ ramified}\}$$

again this is a finite set. Put $\sigma_v := \pi_v^{G(\mathcal{O}_{F_v})}$, a 1 dimensional space for $v \notin S_{ram}$.

Define

$$\bigotimes_{v \notin S}' \pi_v := \lim_{\to T fin. T\cap S = \emptyset, T \supset S_{ram} \cup S(D)} (\pi_T \otimes_{\mathbb{C}} \sigma^{T\cup S})$$

where $\pi_D := \otimes_{v\in T}\pi_v$, $\sigma^{T\cup S} = \otimes_{v\notin T\cup S}\sigma_v$ (here we usually choose a basis $0 \neq w_v \in \sigma_v$ to construct this tensor product). This space is equipped with an action $G(\mathbb{A}_F^S)$.

The transition maps are given by $\pi_T \otimes \sigma^{T\cup S} \to \pi_{T'} \otimes \sigma^{T\cup S}$ for $T \subset T'$ are induced by $\sigma_{T'\setminus T} \hookrightarrow \pi_{T'\setminus T}$.

Fact: ([Flath] 1979) $irr^{adm}(G(\mathbb{A}_F^S))$ are in 1-1 correspondence with the set of $\pi_v \in Irr^{adm}(G(F_v))$ which are unramfieid almost everywhere, the correspondence being given by $\{\pi_v\} \mapsto \otimes'\pi_v$. (In fact this stays true without assuming $S \supset S_\infty$).

A key question that one could ask now is which representations on the left hand side are automoprhic?

16.3. **Automorphic forms on Quaternion algebras.** Loosely speaking, the space of automorphic forms on $G(\mathbb{A}_F)$ is equipped with an action of $G(\mathbb{A}_F)$ by right translations, and automorphic representations are the irreducible representations appearing in this space.

Assume now that $n = 2, F$ is totally real, and $G = GL_1(D)/F$ as before, and that $S(D) \supset S_\infty$

Definition 16.5. 1)

$$\mathcal{S}_D := \{G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}, \text{smooth}\}$$

$$= \lim_{\to U \subset G(\mathbb{A}_F^\infty) \text{ open compact}} \{G(F)\backslash G(\mathbb{A}_F)/U \to \mathbb{C} | \text{ smooth in real variables}\}$$

$\mathcal{S}_D$ is acted on by $G(\mathbb{A}_F)$ by right translation.

2)Vector valued forms. Let $(\tau, W) \in Irr(G(F_\infty))$, $W = \otimes_{v\in S_\infty} W_v$ is a finite dimensional space acted on by $G(\mathbf{R}) \cong \mathbb{H}^\times$, the Hamiltonians.

$$\mathcal{S}_{D,W} := \mathrm{Hom}_{G(F_\infty)}(W^\vee, \mathcal{S}_D)$$

$$= \{G(F)\backslash G(\mathbb{A}_F) \to W | \phi(gu_\infty) = \tau(u_\infty)^{-1}\phi(g), u_\infty \in G(F_\infty)\}$$

The second equality is given by $\varphi \mapsto (g \mapsto (w^\vee \mapsto \varphi(w^\vee)(g)))$

Fact: $G(F)\backslash G(\mathbb{A}_F^\infty)$ is compact, this follows from the fact that $S(D) \supset S_\infty$. This implies that $\forall U \subset G(\mathbb{A}_F^\infty)$, $G(F)\backslash G(\mathbb{A}_F^\infty)/U$ is finite, and hence that $\mathcal{S}_{D,W} = \lim_{\to U} \mathcal{S}_{D,W}^U$ is admissible as a $G(\mathbb{A}_F^\infty)$ representation. It turns out that $\mathcal{S}_{D,W}$ is a semi simple $G(\mathbb{A}_F^\infty)$ representations.

Definition 16.6. A regular algebraic automorphic representation of $G(\mathbb{A}_F^\infty)$ is an irreducible direct summant of $\mathcal{S}_{D,W}$ for W of the form.

$$(\tau, W) = \bigotimes_{v \in S_\infty} (Sym^{k_v - 2}(\mathbb{C}^2) \otimes \det(\mathbb{C}^2)^{\eta_v})$$

such that $k_v, \eta_v \in \mathbb{Z}$, $k_v \geq 2$, and $k_v + 2\eta_v - 1$ is same for all $v \in S_\infty$.

We say $(k, \eta) := (k_v, \eta_v)_{v \in S_\infty}$ is the weight. $\mathcal{S}_{D,k,\eta} = \mathcal{S}_{D,W}$

Remark 16.7. There exists a similar more complicated definition of the cusp forms $\mathcal{S}_{D,k,\eta} \subset \mathcal{M}_{D,k,\eta}$ without the assumption that $S(D) \supset S_\infty$. (For $S(D) = \emptyset$, we obtain GL_2/F). See [4] for more details.

17. Lecture 17

Last time we introduced automorphic forms on Quaternion algebras. In the following we provide classification of such.

For D/F a quaternion central simple algebra over a totally real field we let $G = GL_1(D)/F$. Recall $S(D)$ was the set of places of F where D ramifies and we defined

$$(\tau, w_{k,\eta}) : \bigotimes_{v \in S_\infty} (Sym^{k_v - 2}(\mathbb{C}^2) \otimes \det(\mathbb{C}^2)^{\eta_v}), k_v \geq 2, \eta_v \in \mathbb{Z}$$

$$\mathcal{S}_{D,k,\eta} := Hom_{G(F_\infty)}(W^\vee, \mathcal{S}_D)$$

where $\mathcal{S}_D = \{\phi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C} | \phi \text{ smooth}\}$. Then S_D is equipped with an actions of $G(\mathbb{A}_F)$ and this turns $\mathcal{S}_{D,k,\eta}$ into a semi simple admissible $G(\mathbb{A}_F^\infty)$ representation. We also made the following definition

Definition 17.1. For $\pi \subset \mathcal{S}_{D,k,\eta}$ is regular algebraic if $k_v + 2\eta_v - 1$ is independent of $v \in S_\infty$

In fact a better definition would be to specify that $k_v \mod 2$ is independent of v, these are in fact equivalent.

(Sketch of proof) Let us assume $S(D) \supset S_\infty$ for simplicity. Look at central character χ of $\pi \otimes W^\vee$ (this is an automorphic representation of $Z(\mathbb{A}_F)$)

$$\chi = \prod_v \chi_v : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$$

Weil proved that $\chi = |.|_{\mathbb{A}_F^\times}^{-w}$.finite character. Looking at χ_v for $v|\infty$, we have $w = k_v + 2\eta_v - 1, \forall v|\infty$.

Let us see why this seemingly mysterious definition is actually a generalization of the classical space of modular forms

Example 17.2. Let $F = \mathbb{Q}$, $S(D) = \emptyset$, $k_v = k \geq 2$, $\eta_v = 0$. Define the compact open subgroup of $GL_2(\mathbb{A}^\infty)$

$$U_1(N) = \{g \in GL_2(\hat{\mathbb{Z}}) | g = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod N\}$$

Let $GL_2(\mathbb{Q})^+$ denote the set of matrices in $GL_2(\mathbb{Q})$ with positive determinant, we first show that $GL_2(\mathbb{Q})^+ \cap U_1(N) = \Gamma_1(N)$. The inclusion "$\supset$" is clear.

For the reverse inclusion, suppose $\gamma \in GL_2(\mathbb{Q})^+ \cap U_1(N)$. Then since $\hat{\mathbb{Z}} \cap \mathbb{Q} = \{\pm 1\}$, we have $\det(\gamma) = 1$, and hence $\gamma \in SL_2(\mathbb{Z})$ and so a fortiori $\det(\gamma) \equiv 1 \mod N$, thus

$$\gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N$$

so that $\gamma \in \Gamma_1(N)$.

Lemma 17.3. $GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) U_1(N) GL_2(\mathbf{R})^+$

Proof. Since $\det(GL_2(\mathbb{Q}) U_1(N) GL_2(\mathbf{R})^+) = \mathbb{Q}^\times . \hat{\mathbb{Z}}^\times . \mathbf{R}_{>0} = \mathbb{A}^\times$, it follows that given $g \in GL_2(\mathbb{A})$ we have find $h \in GL_2(\mathbb{Q}) U_1(N) GL_2(\mathbf{R})^+$ such that $\det(h) = g$, thus wlog. we may assume $\det(g) = 1, \in SL_2(\mathbb{A})$. By the strong approximation theorem for SL_2, we have $SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{A}^\infty)$, thus g can be modified to lie in $SL_2(\mathbf{R})$, and hence g lies in $GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) U_1(N) GL_2(\mathbf{R})^+$. ☐

It now follows that $S_{D,k,\eta}^{U_1(N)}$ can be identified with a space of functions

$$\Gamma_1(N) \backslash GL_2(\mathbf{R})^+ \to \mathbb{C}$$

satisfying $\phi(g u_\infty) = j(u_\infty, i)^{-k} \phi(g)$ for all $u \in \mathbf{R}^\times SO_2(\mathbf{R})$.

Given a $\varphi : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) \to \mathbb{C}$, we define $\phi : \Gamma_1(N) \backslash GL_2(\mathbf{R})^+$ setting

$$\phi(h) = \varphi(1, h) \det(h)^{-1} \text{ where } (1, h) \in GL_2(\mathbb{A}) = GL_2(\mathbb{A}^\infty) \times GL_2(\mathbf{R})^+$$

For $u \in \mathbf{R}^\times SO_2(\mathbf{R})$ we have

$$\begin{aligned} \phi(gu) &= \varphi(1, hu) \det(hu)^{-1} \\ &= \varphi(1, h) \det h^{-1} j(u, i)^{-k} \\ &= \phi(g) j(u, i)^{-k} \end{aligned}$$

To go in the other direction, suppose we have a ϕ as above. Let $(g, h) \in GL_2(\mathbb{A})$, then from the above there exists $g_1 \in GL_2(\mathbb{Q}), g_2 \in U_1(N)$ such $g = g_1 g_2$. We then set

$$\varphi(g, h) = \phi(g_1^{-1} h) \det(g h^{-1})$$

One checks that is well defined and satisfies the correct transformation property.

Since the stabilizer of i in $GL_2(\mathbf{R})^+$ is $\mathbf{R}^\times SO_2(\mathbf{R})$, given ϕ as above we may define a function

$$f : \mathcal{H} \to \mathbb{C}$$

by setting

$$hi \mapsto \phi(h)j(h,i)^k$$

That this is well defined follows from the cocycle condition $j(gu,i) = j(u,i)j(h,ui)$. Then for $g \in \Gamma_1(N)$, we have

$$f(ghi) = \phi(gh)j(gh,i)^k = \phi(h)j(g,hi)^k j(h,i) = f(hi)j(g,hi)$$

but this is just the usual transformation formula for modular forms of level $\Gamma_1(N)$, the conditions of holomorphicity and vanishing at infinity follow from the other two conditions. The inverse construction is then clear.

Remark 17.4. The general definition of regular algebraic representations is due to Clozel and Buzzard-Gee [5]

If $k_v \mod 2$ is not the same, we still have $\mathcal{S}_{D,k,\eta} \neq 0$ if we allows $\eta_v \in \frac{1}{2}\mathbb{Z}$. Under the Jacquet Langlands correspondence, these correspond to mixed parity Hilbert modular forms.

$$\mathcal{A}^{ra}(G) \supsetneq \mathcal{A}^{ra}_{cusp}(G) := \{\pi \text{irred.} \subset \mathcal{S}_{D,k,\eta}, k_v \text{same parity}\}$$

For $G = GL_2$, $\mathcal{A}^{ra}_{cusp}(G)$ are just the Hilbert cusp new forms with weights of the same parity.

We now briefly discuss the Multiplicity one Theorems. These comprise of the following statements

1) (Weak) For given k, η, D, $\pi \subset \mathcal{S}_{D,k,\eta}$ has multiplicity one i.e. $\mathcal{S}_{D,k,\eta}$ decomposes as

$$\pi \oplus (\bigoplus_{\neq \pi} \text{irred.})$$

2) (Strong) Weak + if $(\pi, k, \eta), (\pi, k', \eta') \in \mathcal{A}^{ra}_{cusp}(G)$ such that $\pi^S \cong \pi^S$ for a finite set of finite places, then $\pi \cong \pi', k = k', \eta = \eta'$

Remark 17.5. This is true for $G = GL_n$ over any fixed number field (or any inner form of GL_2).

17.1. **Global Jacquet Langlands.** $n = 2$, $G = D^\times = GL_1(D)$ for D a quaternion algebra over a totally real field F.

Theorem 17.6 (Jacquet-Langlands, Badulescu)**.** *$\exists!$ map*

$$JL : \mathcal{A}^{ra}(D^\times) \hookrightarrow \mathcal{A}^{ra}(GL_2)$$

$$\pi_D \mapsto \pi, k, \eta \mapsto k, \eta$$

Such that

1) $\dim \pi_D = 1$*,* $\pi_{D,v} = \chi_v \circ N_D \mapsto \pi_v = \chi_v \circ \det$*, note that* $\pi_v \neq JL(\pi_{D,v}) = Sp_2(\chi|.|^{-1/2})$*, this last representation is a discrete series and hence infinite dimensional.*

2) $\dim \pi_D > 1$ *(Actually ∞-dimensional.)*
$\pi_v = JL(\pi_{D,v})$*, JL is the local Jacquet Langlands functor.*

Fact: When $S(D) \subset S_\infty$ then $\mathcal{A}^{ra}(D^\times) \cong \mathcal{A}^{ra}(GL_2)$. (In general, the image of JL is characterised by local obstructions).

17.2. **Global base Change.**

Remark 17.7. This is known for GL_n over any number field F'/F such that F'/F is solvable ([1] 89).

Theorem 17.8. *(Cyclic base change) Let E/F be cyclic extension of prime degree with $Gal(E/F) = \langle\sigma\rangle$ and let δ be a generator of the dual abelian group $Hom(Gal(E/F), \mathbb{C}^\times)$. Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F^\infty)$ of weight (k, η), then there is a cuspidal automorphic representation $BC_{E/F}(\pi)$ of $GL_2(\mathbb{A}_E^\infty)$ of weight $(BC_{E/F}(k), BC_{E/F}(\eta))$ such that*

i) For all finite place v of E, we have $rec_{E_v}(BC(\pi)_v) = rec_{F_v}(\pi_{v|_F})|_{W_{E_v}}$ so that by the Cebotarev density theorem we have $r_\lambda(BC(\pi)) = r_\lambda(\pi)|_{G_E}$ (see next section for definition of r_λ).

ii) $BC(k)_v = k_{v|_F}, BC(\eta)_v = \eta_{v|_F}$

iii) $BC(\pi) = BC(\pi')$ *if and only if* $\pi \cong \pi' \otimes (\delta^i \circ Art_F \circ \det)$

iv) A cuspidal automorphic automorphic representation of $GL_2(\mathbb{A}_E^\infty)$ is in the image of BC if and only if $\pi = \pi \circ \sigma$.

17.3. **Global Langlands Correspondence.** $F/\mathbb{Q}$ finite and l a prime, fix an isomorphism $i : \overline{\mathbb{Q}}_l \cong \mathbb{C}$

Conjecture: There exists a unique bijection

$$\{\text{cusp. aut. reps of } GL_n(\mathbb{A}_F) \text{ algebraic at } \infty\} \leftrightarrow \{G_F \to GL_n(\overline{\mathbb{Q}}_l)\text{algebraic at } l \text{ (de Rham at all } v|l)\}$$

such that if $\pi \leftrightarrow r$:
(Local global compatibility) $rec_{F_v}(\pi_v \otimes |det|^{\frac{1-n}{2}}) \cong WD(r|_{G_{F_v}})^{Fss}$, $\forall v \nmid \infty$
For all $v|\infty$ parameter (like k_v, η_v) for $\pi_v \leftrightarrow HT_{i^{-1}v}(r|_{G_{F_v}})$, where $v : F \to \mathbb{C} \cong \overline{\mathbb{Q}}_l$

Remark 17.9. $\pi \leftrightarrow r$ is pinned down by LGC at almost all v by the strong multiplicity one theorems and the Cebotarev density theorem.

Apply to varying l, i we get a compatible system.

Definition 17.10. Let F and L be number fields, S a finite set of places of F and n a positive integer. By a weakly compatible system of n-dimensional l-adic representations of G_F defined over L and unramified outside S we mean a family of continuous Galois representations

$$r_\lambda : G_F \to GL_n(\overline{L}_\lambda)$$

where λ runs over the finite places of L such that:

1) For all v a finite place of F with $v \notin S$ and all λ no dividing the residue characteristic of l. The representation $r_\lambda|_{G_{K_v}}$ is unramified, and the characteristic polynomial of Frob_v lies in $L[X]$.

2) Each representation r_λ is de Rham at any place v above the residue characteristic of λ and in fact crystalline if $v \notin S$.

3) For each embedding $i : F \to \overline{L}$ the i-HT weights of r_λ are independent of L.

Best know results: In the forward direction:

Harris-Lan-Taylor-Thorne, Scholze (They proved it for all n, π regular algebraic and F a totally real field, however missing LGC at bad places)

In the reverse direction there are many forefronts, BLGGT, Calegari-Geraghty. We focus on the case $n = 2$ F totally real.

$\to$ today, $\leftarrow$ next 3-4 classes

17.4. **GLC for GL_2/F.**

Theorem 17.11. *(Carayol, Wiles, Taylor, Blasius-Rogawski, T. Saito, Skinner)* $(\pi, k, \eta) \in \mathcal{A}^{ra}_{cusp}(GL_2)$ *There exists a CM field L_π with $\overline{L}_\pi \subset \mathbb{C}$ and*

$$\{r_\lambda(\pi) : G_F \to GL_2(\overline{L}_{\pi,\lambda}\})$$

λ a finite place of L_π such that $\forall i : \overline{L}_{\pi,\lambda} \cong \mathbb{C}$ extending $\overline{L}_\pi \subset \mathbb{C}$

i)LGC: $\forall v \nmid \infty, rec_K(\pi_v \otimes |\det|^{-1/2}) \cong WD(r_\pi(\lambda)|_{G_{F_v}})^{Fss}$. Consequently $\forall v \nmid l$, π_v unramified $\Leftrightarrow r_\pi(\lambda)$ is unramfied at v.

$$char(r_\lambda(\pi)(Frob_v)) = Heckepoly(\mathcal{S}_{\pi_v|\det|^{-1/2}})$$
$$= X^2 - q_v T_v(S_{\pi_v|\det|^{-1/2}})X + S_v(S_{\pi_v}|\det|^{-1/2})$$
$$= X^2 - T_v(S_{\pi_v})X + q_v S_v(S_{\pi_v})$$

ii) $char(r_\lambda(\pi)(Frob_v)) \in L_\pi[X]$

iii) For all $v|l$, $r_\lambda(\pi)|_{G_{F_v}}$ is de Rham with τ-HT weights $\eta_\tau, \eta_\tau + k_i - 1$, where

$$\tau : F \to \overline{L}_\pi \subset \mathbb{C}$$

is an embedding lying over v. Moreover $r_\lambda(\pi)|_{G_{F_v}}$ is crystalline if π_v is unramified.

iv) For all $v|\infty$ $\det(r_\lambda(c_v)) = -1$ where c_v is the class of complex conjugation, i.e. r_λ is "Totally odd"

Definition 17.12. A representation $\rho : G_F \to GL_2(\overline{\mathbb{Q}}_l)$ is modular (of weight (k, η) if it is over the form $i \circ r_\lambda(\pi))$ for some cuspidal automorphic representation π (of weight (k, η) and where $i : L_\pi \to \overline{\mathbb{Q}}_l$

Proposition 17.13. *Let E/F be a finite solvable extension, then $r : G_F \to GL_2(\overline{\mathbb{Q}}_l)$ is modular if and only if $r|_{G_E}$ is modular.*

Proof. An easy induction allows us to reduce to the case E/F is cyclic of prime degree. One direction then follows immediately from the cyclic base change theorem, (Thm 16.9).

Suppose that $r|_{G_E}$ is modular with $r|_{G_E} \cong i(r_\lambda(\pi))$ with $i : L_\pi \to \overline{\mathbb{Q}}_l$. [TO DO] $\square$

18. Lecture 18

18.1. **Integral Theory of automorphic forms.** The goal of this section is to develop the theory from the previous lecture but with $\mathcal{O}$ coefficients (rather than $\mathbb{C}$). We must first fix a set of data with which to define these objects. Let D/F be a quaternion algebra over a totally real field, $G = GL_1(D)$ and $Z \subset G$ its center. Assume that $S(D) = S_\infty$, so that $[F : \mathbb{Q}]$ is even, and $\mathcal{A}^{ra}(D^\times) \cong \mathcal{A}^{ra}_{cusp}(GL_2(F))$ via the Jacquet Langlands correspondence. Let L be a finite extension of $\mathbb{Q}_l$ sufficiently large so that it contains the images of all embeddings $F \to \overline{L}$, $\mathcal{O}$ its ring of integers with uniformizer λ and residue field $\mathbb{F}$.

For $v \nmid \infty$, $G(F_v) \cong GL_2(F_v)$, and fix a subgroup $G(\mathcal{O}_{F_v})$ corresponding to $GL_2(\mathcal{O}_{F_v})$, then $G(\mathbb{A}_F^\infty) \cong GL_2(\mathbb{A}_F^\infty)$

Fix also the following data, which should be thought of as the data of a level.

- An isomorphism $i : \overline{\mathbb{Q}}_l \cong \mathbb{C}$
- A weight $(k, \eta) = ((k_v), (\eta_v))$ where $w = k_v + 2\eta_v - 1$ is independent of v.
- $S \subset \{\text{finite places of } F, v \nmid l\}$ a finite set of places.
- $U = \prod_{v \nmid \infty} U_v = U_S.U^S \subset G(\mathbb{A}_F^\infty)$ an open compact subgroup, assume that $U^S = \prod_{v \notin S \cup S_\infty} GL_2(\mathcal{O}_F)$.
- A central character $\chi_0 : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ such that χ_0 unramified outside S, and $\chi_0|_{(F_\infty^\times)^0}(z) = z^{1-w}$

Note as in [4] 2.41, this defines character $\chi_{0,i} : (\mathbb{A}_F^\infty)^\times/F^\times(F_\infty^\times)^0 \to \overline{L}^\times$:

$$\chi_{0,i}(x) = i^{-1}(\prod_{\tau:F\to\mathbb{C}} \tau(x_\infty)^{1-w})\chi_0(x) \prod_{\tau\to L} \tau(x_p)^{1-w}$$

Let Λ be a representation of $GL_2(\mathcal{O}_{F,l}) := \prod_{v|l} GL_2(\mathcal{O}_{F,v})$ on a finite free $\mathcal{O}$ module. In practice we will take Λ to be the representation

$$\otimes_{\tau:F\to\mathbb{C}} \mathrm{Sym}^{k_\tau-2}(\mathcal{O}^2) \otimes (\wedge^2\mathcal{O}^2)^{\eta_\tau}$$

where the action of $GL_2(\mathcal{O}_{F,l})$ on the τ factor is given by $i^{-1}\tau$.

Definition 18.1. Let A be a finitely generated $\mathcal{O}$ module (eg. $A = \mathcal{O}, \mathbb{F}$), define

$$\mathcal{S}(U, A) = \mathcal{S}_{k,\eta,\chi_0,i}(U, A)$$

$$= \{\phi : G(F)\backslash G(\mathbb{A}_F^\infty) \to \Lambda \otimes_\mathcal{O} A | \phi(guz) = \chi_{0,i}(z)u_l^{-1}.\phi(g)\}$$

where $g \in G(\mathbb{A}_F^\infty), u \in U, z \in Z(\mathbb{A}_F^\infty)$, and $u_l^{-1}.\phi$ is the image of ϕ under the action of u_l^{-1}, for $u_l \in GL_2(\mathcal{O}_{F,l}) = \prod_{v|l} GL_2(\mathcal{O}_{F_v})$, the l component of U. Note also that $G(F)\backslash G(\mathbb{A}_F^\infty) \cong D^\times\backslash GL_2(\mathbb{A}_F^\infty)$, so we can consider functions on this set.

Lemma 18.2. 0) $\mathcal{S}(U,\mathcal{O})$ is a finite free $\mathcal{O}$-module.
1) $\mathcal{S}(U,\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C} \cong \mathcal{S}^U_{k,\eta,\chi_0}$, the U invariants of the space of complex automorphic forms with central character χ_0).
2) $\mathcal{S}(U,A) \cong \mathcal{S}(U,\mathcal{O}) \otimes_{\mathcal{O}} A$
3) $V \subset U$ open, $\mathcal{S}(U,A) \hookrightarrow \mathcal{S}(V,A)$

Proof. For simplicity assume that $\Lambda = \mathcal{O}$ (trivial representation).
Fact:

$$|D^\times\backslash GL_2(\mathbb{A}_F^\infty)/UZ(\mathbb{A}_F^\infty)| < \infty$$

There fore we may pick a finite set of representatives $g_i, i \in I$ of these double cosets and we have

$$GL_2(\mathbb{A}_F^\infty) = \coprod_{i\in I} D^\times g_i U Z(\mathbb{A}_F^\infty)$$

0) $\mathcal{S}(U,A) \cong \bigoplus_{i\in I} A$, given by $\phi \mapsto \{\phi(g_i)\}$.
1) Is routine, 2), 3) are obvious.
For a general Λ, we must consider when $g_i = \delta g_i u z$, since in this case we have

$$\phi(g_i) = \phi(\delta g_i u z) = \chi_{0,i}(z) u_l^{-1} \phi(g_i)$$

so that the image of the map $\phi \mapsto \{\phi(g_i)$ lies in $\bigoplus_{i\in I}(\Lambda \otimes_{\mathcal{O}} A)^{U.Z(\mathbb{A}_F^\infty)\cap g_i^{-1}D^\times g_i)/F^\times}$. In fact it easy to see that this map is actually an isomorphism so we obtain

$$\mathcal{S}(U,A) \cong \bigoplus_{i\in I} (\Lambda \otimes_{\mathcal{O}} A)^{(U.Z(\mathbb{A}_F^\infty)\cap g_i^{-1}D^\times g_i)/F^\times}$$

Denote the term in the exponent by G_i (it is a finite group since $D^\times$ is discrete in $G_D(\mathbb{A}^\infty)$). Since $l \geq 5$, we have $(|G_i|, l) = 1$, indeed let $g_i^{-1}\delta g_i \in G_i$ for some $\delta \in D^\times$, then $\delta = g_i u g_i^{-1} z$ for some $u \in U$ and $z \in Z(\mathbb{A}^\infty)$. Since $\det z = z^2$ (here we identify $Z(\mathbb{A}^\infty)$ with $\mathbb{A}^\times$ via the diagonal embedding), we have $\delta^2/\det\delta \in D^\times \cap g_i U g_i^{-1} \det U$ the intersection of a discrete set and a compact set, hence is finite. Thus $\delta^2/\det\delta$ is a root of unity in $D^\times$, however any element of D generates and extension of degree at most 2 and since we have assumed $[F(\zeta_l) : F] > 2$ it must be a root unity of order prime to l, hence $\exists N$ prime to l such that $\delta^N \in F^\times$, so that $g_i^{-1}\delta g_i$ has order prime to l. It follows that $(\Lambda \otimes_{\mathcal{O}} A)^{G_i} = \Lambda^{G_i} \otimes_{\mathcal{O}} A$. □

18.2. **Global Hecke algebra.** (note this is not equal to the double coset action of $\mathcal{H}(G(\mathbb{A}_F^\infty)//U)$).

Keep the notation from before, consider

$$\tilde{\mathbb{T}} := \mathcal{O}[T_v, S_v | v \notin S, v \nmid l] \to End(\mathcal{S}(U,\mathcal{O}))$$

the natural maps, (for $v \notin S, v \nmid l$, the representation is unramfied and this map sends S_v and T_v to the usual coset operators).

Let $\mathbb{T}_U$ denote the image of $\tilde{\mathbb{T}}$.

Observe: $\mathbb{T}_U$ acts faithfully on $\mathcal{S}(U, \mathcal{O})$, it is a finite free $\mathcal{O}$-module, and a commutative $\mathcal{O}$ algebra.

Lemma 18.3.

$$\mathbb{T}_U \otimes_{\mathcal{O}} \mathbb{C} \cong \coprod_{\pi \subset \mathcal{S}_{k,\eta,\chi_0}} \mathbb{C}$$

where (for $v \notin S, v \nmid l$) the map is given by

$$T_v \mapsto T_v(s_{\pi_v}), S_v \mapsto S_v(s_{\pi_v}), (s_{\pi_v} \text{ a Satake parameter})$$

Another way to define this is to notice that $\{\mathbb{T}_U \to \mathbb{C}\}$ is in 1-1 correspondence with $\{\pi \text{ on RHS}\}$

Proof.

$$\mathcal{S}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C} \cong \mathcal{S}^U_{k,\eta,\chi_0} \cong \bigoplus_{\pi \subset \mathcal{S}_{k,\eta,\chi_0}, \pi^U \neq 0} \pi_S^{U_S} \otimes \big(\bigotimes_{v \notin S, v \nmid \infty} \Pi_v^{GL_2(\mathcal{O}_{F_v})} \big)$$

$\mathbb{T}_U$ acts on both sides (on the right hand side via $(T_v(s_{\pi_v}), S_v(s_{\pi_v}))$ and the map is equivariant for this action.

Thus the lemma follows by strong multiplicity one. $\square$

$\mathbb{T}_U$ is a finite free over the complete discrete valuation ring $\mathcal{O}$, hence it is semi-local and there is a decomposition

$$\mathbb{T}_U \cong \prod_{\mathfrak{m} \subset \mathbb{T}_U} \mathbb{T}_{U,\mathfrak{m}}$$

Fix a maximal ideal $\mathfrak{m}$, let's construct Galois representations

$$\overline{\rho}_{\mathfrak{m}} : G_F \to GL_2(\mathbb{T}_U/\mathfrak{m})$$

and if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, a lift

$$\rho_{\mathfrak{m}} : G_F \to GL_2(\mathbb{T}_{U,\mathfrak{m}})$$

Step 1) ρ^{mod}, construct

$$\rho^{mod} : G_F \to GL_2(\mathbb{T}_U \otimes_{\mathcal{O}} \overline{L}) \cong \prod_{\pi} GL_2(\overline{L})$$

by first starting with π on $D^\times$, then applying JL to obtain a representation on GL_2, and we can associate to this a Galois representation $G_F \to GL_2(\overline{L})$ with

$$tr\rho_\pi(Frob_v) = T_v(s_{\pi_v})$$
$$det(\rho_\pi Frob_v) = q_v S_v(s_{\pi_v})$$

.

Observe that for $v \notin S, v \nmid l$,

$$tr\rho^{mod}(Frob_v) = T_v$$

$$det\rho^{mod}(Frob_v) = S_v q_v$$

because they are equal in $\prod_\pi \overline{L}$ (use previous lemma).

Step 2) $\overline{\rho}_{\mathfrak{m}}$, choose a minimal prime $\mathfrak{p} \subsetneq \mathfrak{m} \subsetneq \mathbb{T}_U$ and an injection $\mathbb{T}_U/\mathfrak{p} \hookrightarrow \overline{L} \cong \mathbb{C}$, then from the Lemma we get a π.

Associate $\rho_\pi : G_F \to GL_2(\overline{L})$ as above and taking its mod l reduction we obtain $\overline{\rho}_\pi : G_F \to GL_2(\overline{\mathbb{F}})$ (in fact this is realized over a finite extension of $\mathbb{F}$).

We have

$$tr(\rho_\pi) \in \mathbb{T}_U/\mathfrak{p} \subset \mathcal{O}_{\overline{L}} \Rightarrow tr\overline{\rho}_\pi \in \mathbb{T}_U/\mathfrak{m} \subset \overline{\mathbb{F}}$$

As the Brauer group of a finite field is 0, $\overline{\rho}_\pi$ can be conjugated to

$$\overline{\rho}_{\mathfrak{m}} : G_F \to GL_2(\mathbb{T}_U/\mathfrak{m})$$

Step 3) $\rho_{\mathfrak{m}}^{mod}$ when $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Localising at the ideal $\mathfrak{m}$ we obtain

$$\rho_{\mathfrak{m}}^{mod} : G_F \to GL_2(\mathbb{T}_{U,\mathfrak{m}} \otimes \overline{L}) \cong \prod_\pi GL_2(\overline{L})$$

where the product is over π whose associated Galois representation reduces to $\overline{\rho}_m$. Then we may conjugate $\rho_{\mathfrak{m}}^{mod}$ to have image in $\prod_\pi GL_2(\mathcal{O}_{\overline{L}})$. Upon further conjugation we may conjugate $\rho_{\mathfrak{m}}^{mod}$ to lie in the subring of elements whose reductions modulo $\mathfrak{m}$ lies in $\mathbb{T}_U/\mathfrak{m}$.

We have the diagram:

$$\begin{tikzcd} G_F \arrow[r, "\rho_{\mathfrak{m}}^{mod}"] \arrow[dr, "\overline{\rho}_{\mathfrak{m}}"'] & GL_2(\mathbb{T}_{U,\mathfrak{m}} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{L}}) = \prod_\pi GL_2(\mathcal{O}_{\overline{L}}) \arrow[d] \\ & GL_2(\mathbb{T}_U/\mathfrak{m}) \end{tikzcd}$$

We have that:

- $tr\rho_{\mathfrak{m}}^{mod} \subset \mathbb{T}_{U,\mathfrak{m}} \subset R := \mathbb{T}_{U,\mathfrak{m}} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{L}}$
- $\rho_{\mathfrak{m}}^{mod}$ has coefficients in R by (2).
- $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

It then follows from Carayol's Lemma (Lemma 6.9) that we may conjugate $\rho_{\mathfrak{m}}^{mod}$ to lie in $GL_2(\mathbb{T}_{U,\mathfrak{m}})$

19. Lecture 19

19.1. **Automorphy lifting theorem.** Let $F/\mathbb{Q}$ be a totally real field, recall we have a map:

$$\mathcal{A}^{ra}_{cusp}(GL_{2/F}) \to (L/\mathbb{Q}, \{r(\lambda) : G_F \to GL_2(\overline{L}_\lambda) | \lambda \text{ a fin. place of L}\})$$
$$\pi \mapsto (L_\pi, \{r_\pi(\lambda)\})$$

Although the L_π is not unique, the $r_\pi(\lambda)$ are unique up to $\cong$.

Definition 19.1. A semisimple $\rho : G_F \to GL_2(\overline{\mathbb{Q}}_l)$ is automorphic if $\rho \cong r_\pi(\lambda)$ (or $\rho \cong \chi_1 \oplus \chi_2$, it's automorphic by CFT)

A semisimple representation $\overline{\rho} : G_F \to GL_2(\overline{\mathbb{F}}_l)$ is automorphic if $\overline{\rho} \cong (r_\pi(\lambda) \mod l)^{ss}$

Ideally, the sort of statement we'd like to prove is the following.

Conjecture 19.2. *Suppose $\rho : G_F \to GL_2(\overline{\mathbb{Q}}_l)$ is unramified almost everywhere and de Rham (potentially semistable at all $v|l$) with distinct Hodge Tate weights for all $i : F \to \overline{\mathbb{Q}}_l$ (this is known as the regular algebraic condition).*

If $(\rho \mod l)^{ss}$ is automorphic, then ρ is automorphic. (i.e. for ρ, ρ_0 as above, $\rho \equiv \rho_0 \mod l$ (up to ss.) and ρ_0 automorphic, implies ρ automorphic)

Remark 19.3. In reality, to prove such a result, we will need several additional hypotheses.

Definition 19.4. Let $\rho, \rho_0 : G_F \to GL_2(L)$ be two Galois representations.

$\det(\rho) = \det \rho_0 =: \chi$

$\rho \mod \lambda \equiv \rho_0 \mod \lambda =: \overline{\rho}$

$(\rho, \rho_0 : G_F \to GL_2(\mathcal{O})$, well-defined up to $GL_2(\mathcal{O})$-conjugacy).

For $v \nmid \infty$, say $\rho_v \sim \rho_{0,v}$ ("ρ_v connects to $\rho_{0,v}$) if $\rho_v, \rho_{0,v}$ belong to the same irreducible component of $Spec R^{\square}_{\overline{\rho}_v,\chi}[1/l]$ (note we can remove if L sufficiently large).

The main theorem that we will prove in the next few lectures will be the following.

Theorem 19.5. *(Minimal ALT)*

$[F(\zeta_l) : F] > 2$ (so that $l \geq 5$). Let $\rho, \rho_0 : G_F \to GL_2(L)$ be unramified outside $S \coprod T_l$, such that ρ_0 is automorphic, $\rho \mod \lambda \cong \rho_0 \mod \lambda =: \overline{\rho}$ is absolutely irreducible and we have:

•$HT(\rho) = HT(\rho_0) =: HT$ are distinct integers for all embeddings $F \to \overline{\mathbb{Q}}_l$

•ρ, ρ_0 are cystalline, for all $v|l$

•$\rho_v \sim \rho_{0,v}$ for all finite places $v \in S \cup T_l$ (this is why its called minimal)

Then ρ is automorphic.

Remark 19.6. ρ_0 is automorphic implies $\det \rho_0(c_v) = -1$, where c_v is the conjugacy class of complex conjugation.

19.2. **Proof of minimal ALT.** Some reductions: By Proposition 17.13, we know that automorphy can be proved after a finite solvable base change. Together with the following fact from class field theory we may impose some additional assumptions.

Proposition 19.7. *Let F be number field, and S a finite set of primes of K. For each $v \in S$ let L_v be a finite Galois extension of F_v, then there is a finite solvable extension M such that for each place w of M above v, there is an isomorphism $L_v \cong M_w$ of K_v algebras.*

We may thus make the following assumption:
- $[F : \mathbb{Q}]$ even
- $\overline{\rho}$ is unramified outside l.
- For all primes $v \nmid p$ both $\rho(I_{F_v})$ and $\rho_0(I_{F_v})$ are unipotent.
- If ρ or ρ_0 are ramified at some place $v \nmid l$, then $\overline{\rho}|_{G_{F_v}}$ is trivial and $\#k(v) \equiv 1 \mod l$.
- $\det(\rho) = \det(\rho_0) =: \chi$

Let us briefly mention how some of the above assumptions can be realized. To assume $[F : \mathbb{Q}]$ even, we replace F with a totally real quadratic extension.

Now let S be the finite set of primes not dividing l at which ρ or ρ_0 is ramified. For $v \in S$, let $L_v = \overline{F}_v^{\ker \overline{\rho}}$ and for $v|\infty$, let $L_v := F$. Replacing F by M as in Proposition 19.7, we see that $\overline{\rho}$ is trivial at any place $w|v$. In particular the second condition and the first part of the fourth condition is satisfied. To get the last part of condition 4, we may take an unramified extension of degree $\mathrm{ord}\#k(v)$ at each $v \in S$ where the order is $\#k(v)$ considered as an element of $(\mathcal{O}/\lambda)^\times$ and apply 19.7 again.

The third condition follows easily from the l-adic monodromy theorem (Proposition 4.3).

Our setup now is as follows. On the Galois side we let

$$T_l := \{v|l\}$$

$$T_r := \{v \nmid \infty, l : \rho \text{ or } \rho_0 \text{ is ramified at } v\}$$

Set $T = T_l \coprod T_r$.

Consider the global deformation problem $(F, T, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v \in T})$, i.e. deformations which are unramfied outside T with fixed determinant χ and satisfying the local deformation conditions $\mathcal{D}_v$, where for $v \in T_r$, $\mathcal{D}_v$ is the set of all lifts of $\overline{\rho}_v$ with det$=\chi_v$, and for $v \in T_l$, $\mathcal{D}_v \leftrightarrow ker(R^{\square}_{\overline{\rho}_v,\chi_v} \to R^{\square}_{\overline{\rho}_v,\chi_v,cr,\{HT\}})$. Here $R^{\square}_{\overline{\rho}_v,\chi_v,cr,\{HT\}}$ is the maximal quotient which is l-torsion free and reduced and whose characteristic 0 points are representations with HT weights $\{HT\}$.

$R^{univ} := R^{univ}_{\mathcal{S}} = R^{\square,\emptyset}_{\mathcal{S}}$, the universal deformation ring for $\mathcal{S}$, (cf. $R^{\square,T}_{\mathcal{S}}$ T framed version).

On the automorphic side we use the integral theory of automorphic forms on $D^\times$, where D is a quaternion algebra, such that $S(D) = \{v|\infty\}$ (this if fine since $[F:\mathbb{Q}]$ is even).The idea is due to Diamond and Fujiwara.

We take our level to be

$$U := \prod_v U_v \subset GL_2(\mathbb{A}_F^\infty)(= D^\times(\mathbb{A}_F^\infty))$$

an open compact subgroup such that

$$U_v = \begin{cases} GL_2(\mathcal{O}_{F_v}) & \text{for a finite } v \notin T_r \\ \text{Any compact open s.t. } \pi_{0,v}^{U_v} \neq 0 & \text{for } v \in T_r \end{cases}$$

Here π_0 is the automorphic representation corresponding to ρ_0 (which exists since we are assuming ρ_0 is modular). By assumption π_0 is unramified outside T.

We define our space of automorphic forms

$$\mathcal{S} := \mathcal{S}(U, \mathcal{O}) = \mathcal{S}^U_{k,\eta,\chi|\cdot|_{\mathbb{A}_F^\times}}$$

which is a finite free $\mathcal{O}$ module. This has an action of the Hecke algebra $\mathbb{T}_U$ a commutative $\mathcal{O}$ algebra which is reduced and finite free as an $\mathcal{O}$ module.

We have a 1-1 correspondence

$$\{\mathbb{T} \to \mathbb{C}\} \leftrightarrow \{\pi \subset \mathcal{S}_{k,\eta,\chi}, \text{ s.t. } \pi^U \neq 0\}$$

Let $\mathcal{O}_0 : \mathbb{T} \to \mathbb{C}$ correspond to π_0 under this correspondence.

$$\mathbb{T} \supset \mathfrak{m} := \langle \lambda, tr\overline{\rho}(Frob_v) - T_v, \det \overline{\rho}(Frob_v) - q_v S_v \rangle$$

This is a proper maximal ideal since it's the kernel of

$$\mathbb{T}_U \to \mathcal{O} \to \mathbb{F}$$
$$T_v \mapsto tr\rho_0(Frob_v)$$
$$S_v \mapsto q^{-1} \det \rho_0(Frob_v)$$

where the first map $\mathbb{T}_U \to \mathcal{O}$ comes from π_0. It follows from the definitions that $\overline{\rho}_\mathfrak{m} \cong \overline{\rho}$.

Recall we constructed the representation

$$\rho_\mathfrak{m}^{mod} : G_F \to GL_2(\mathbb{T}_{U,\mathfrak{m}})$$

which is a lift of $\overline{\rho}$, and for which the trace of $Frob_v$ is T_v and the determinant of $Frob_v$ is $q_v^{-1} S_v$.

One can check that that $\rho_\mathfrak{m}^{mod}$ is of type $\mathcal{S}$ so that the universal property of R^{univ} gives us a map

$$R^{univ} \to \mathbb{T}_{U,\mathfrak{m}}$$
$$tr\rho^{univ}(Frob_v) \mapsto T_v$$
$$\det \rho^{univ}(FRob_v) \mapsto q_v^{-1} S_v$$

These identies show that the map is onto.

If ρ is of type $\mathcal{S}$ deforming $\overline{\rho}$ is equivalent to giving a map $R^{univ} \to \mathcal{O}$, then to prove the theorem, it suffices to show we may complete the diagram

$$\begin{tikzcd} R \;:= R^{univ} \arrow[r] \arrow[dr, "f_\rho"'] & \mathbb{T}_{U,\mathfrak{m}} \;:=\; T \arrow[d, dashed, "\exists \mathcal{O}_\rho"] \\ & \mathcal{O} \end{tikzcd}$$

Why is this sufficient? The map

$$\mathcal{O}_\rho : \mathbb{T}_{U,\mathfrak{m}} \to \mathcal{O} \subset \mathbb{C}$$

gives us an element $\pi_\rho \in \mathcal{A}^{ra}_{cusp}$

It is then easy to see that the Galois representation attached to π_ρ is isomorphic to ρ, since by definition, for $v \notin T$, the eigenvalues of $Frob_v$ are equal to those of ρ and hence we can conclude they are isomorphic by the Cebotarev density theorem.

Thus if we manage to prove that $R \twoheadrightarrow T$ induces an iso from $R^{red} \twoheadrightarrow T$ then the proof will be complete. This is because $\mathcal{O}$ is reduced and hence any map $R \to \mathcal{O}$ must factor through R^{red} and hence through T.

20. Lecture 20

Recall we were trying to prove the minimal ALT theorem:

Theorem 20.1. *$F(\zeta_l : F] > 2$ (so that $l \geq 5$). Let $\rho, \rho_0 : G_F \to GL_2(L)$ be unramified outside $S \coprod T_l$, such that ρ_0 is automorphic, $\rho \mod \lambda \cong \rho_0 \mod \lambda =: \overline{\rho}$ (absolutely irreducible) and such that:*
- *$HT(\rho) = HT(\rho) =: HT$ are distinct integers for all embeddings $F \to \overline{\mathbb{Q}}_l$*
- *ρ, ρ_0 are cystalline, for all $v|l$*
- *$\rho_v \sim \rho_0, v$ for all finite places $v \in S \cup T_l$ Then ρ is automorphic.*

We made the following reductions:

Reduction 1: We may assume F satisfies $[F : \mathbb{Q}]$ even, $\det \rho = \det \rho_0 = \chi$ etc. (see Lecture 19 for a complete list)

Reduction 2: We defined the deformation problem $\mathcal{S} = (F, T, \overline{\rho}, \chi\{\mathcal{D}_v\}_{v \in T})$ and showed it suffices to prove that for every Galois deformation ρ to $\mathcal{O}$ corresponding to $f_\rho : R^{univ}_{\mathcal{S}} \to \mathcal{O}$ we can complete the diagram:

$$\begin{tikzcd} R = R^{univ}_{\mathcal{S}} \arrow[r, two heads] \arrow[dr, "f_\rho"'] & \mathbb{T} := \mathbb{T}_{U,\mathfrak{m}} \arrow[d, dashed, "\exists"] \\ & \mathcal{O} \end{tikzcd}$$

20.1. **Reduction via Patching.** The traditional method take surjections

$$\begin{tikzcd} J_\infty \arrow[r] & R_\infty \arrow[r, two heads] \arrow[d, two heads] & \mathbb{T}_\infty \arrow[d, two heads] \\ & R \arrow[r, two heads] & \mathbb{T} \end{tikzcd}$$

where J_∞ is a power series ring over $\mathcal{O}$ and we show

$$R_\infty \cong \mathbb{T}_\infty \Rightarrow R^{red} \cong \mathbb{T}$$

The later approach is better described as a patching "patching scenario:"

Let J_∞ be a power series ring over $\mathcal{O}$ together with an ideal $\mathfrak{a}_\infty \lhd J_\infty$ and a map $J_\infty \to R_\infty$ such that the $R_\infty/\mathfrak{a}_\infty \cong R$. Suppose S_∞ is an R_∞ module such that $S \cong S_\infty/\mathfrak{a}_\infty$ such that we have the following diagram.

$$\begin{tikzcd} J_\infty \arrow[r] & R_\infty \arrow[r, two heads, "\text{mod } \mathfrak{a}_\infty"] & R \\ & \text{acts on} & \text{acts on} \\ & S_\infty \arrow[r, two heads] & S := S(U, \mathcal{O}) \end{tikzcd}$$

Suppose the following conditions are satisfied:

i) $\dim J_\infty = \dim R_\infty =: d$

ii) S_∞ is a finite free J_∞ module.

Once we are in this situation the proof of ALT will follow from pure commutative algebra.

Definition 20.2. Let M be an A module. The support of M is defined to be

$$supp_A M = \{\mathfrak{p} \in Spec A : M_\mathfrak{p} \neq 0\}$$

When A is noetherian and M is finitely generated, $Supp_A M$ consists of the ideals in the closed subscheme defined the by ideal $Ann_A(M)$.

The output of the patching argument is the following:

Proposition 20.3. *i) $Supp_{R_\infty} S_\infty$ is a union of irreducible components of $Spec R_\infty$*
ii) If $Supp_{R_\infty} S_\infty \cong Spec R_\infty$, then $Supp_R S = Spec R$ and hence obtain $R^{red} \cong \mathbb{T}$

Proof. i) Let $\mathfrak{p} \in Supp_{R_\infty} S_\infty$ be a minimal prime. We want to show that $\mathfrak{p}$ is a minimal prime of $Spec R_\infty$

$$d = \dim R_\infty \geq \dim R_\infty/\mathfrak{p} \geq^{(1)} \operatorname{depth}_{R_\infty} S_\infty \geq^{(2)} \operatorname{depth}_{J_\infty} S_\infty = d$$

Inequality 1) follows is a general property of modules over a ring. If M is a module over a Noetherian local ring A, we have

$$\operatorname{depth}_A M \leq \dim_A M$$

Recall, if $\mathfrak{m}$ denotes the maximal ideal of A, $\mathrm{depth}_A M$ is the maximal length of a regular sequence $x_1, ...x_r \in \mathfrak{m}$ for M, and $\dim_A M$ is the Krull dimension of $A/\mathrm{Ann}_A M$ Thus equality holds throughout and $\mathfrak{p}$ is a minimal prime of $SpecR_\infty$.

ii) The first part follows from the following commutative algebra result:

Lemma 20.4. Let M be a finitely generated module over a Noetherian ring R and I an ideal of R, then

$$Supp_R(M/IM) = V(Ann(M) + I)$$

Proof. The inclusion $Supp_R(M/IM) \subset V(Ann(M) + I)$ is clear. For the other direction, let $\mathfrak{p}$ be a prime containing $Ann(M)+I$, then since $\mathfrak{p} \supset Ann(M)$, we have $M_\mathfrak{p} \neq 0$. We have $(M/IM)_\mathfrak{p} = M_\mathfrak{p}/I_\mathfrak{p}M_\mathfrak{p}$, hence if $(M/IM)_\mathfrak{p} = 0$, we have $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = 0$, and hence by Nakayama's lemma, we have $M_\mathfrak{p} = 0$ which is a contradiction. $\square$

Applying this to R_∞ and its ideal $\mathfrak{a}_\infty$ we obtain $Supp_R S = SpecR$. We have $\ker(R \to T) \subset Ann(M)$ and hence is contained in the nilradical of R. Conversely any x in the nilradical of R must map to 0 in T since T is reduced, thus $R^{red} \cong T$. $\square$

Bad news: It's rarely true that $Supp_{R_\infty} S_\infty = SpecR_\infty$. Major reason is that if take the deformation problem to be $\mathcal{D}_v = \{all\ lifts\}, v \in T_r$, the R_∞ we construct is too big. However we can still deduce the required result.

The idea is to introduce another global deformation problem:

$$\mathcal{S}' = (F, T, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v \in T})$$

.

$v \in T_l$ (resp. T_r), $\mathcal{D}'_v$ corresponds to the unique irreducible components of $R^{\square}_{\overline{\rho}_v, \chi_v, cr, \{H_\tau\}}$ containing $\rho_{0,v}, \rho_v$ (resp. $R^{\square}_{\overline{\rho}_v, \chi_v}$), in same connected component.

Since ρ is of type $\mathcal{S}'$, it corresponds to a map $f'_\rho : R' := R^{univ}_{\mathcal{S}'} \to \mathcal{O}$.

$$\begin{tikzcd}
R' \arrow[dr, "f'_\rho"'] & R \arrow[l] \arrow[r] \arrow[d, "f_\rho"] & \mathbb{T} \arrow[dl, dashed] \\
& \mathcal{O} &
\end{tikzcd}$$

We can deduce from the previous proposition that f_ρ factors through $R \to \mathbb{T}$. Indeed it suffices to prove $SpecR' \subseteq Supp_{R_\infty} S_\infty$, since then Lemma 20.4 implies we have

$$Supp_{R'} S_\infty \otimes_{R_\infty} R' = SpecR'$$

Hence for $x \in \ker(R \to T)$, the image of x in R' lies in the nilradical of R', but $\mathcal{O}$ is reduced so $f_\rho(x) = 0$ and hence f_ρ factors through T.

Let us now explain the outline of the patching argument, the idea is encoded in the following diagram which we will elaborate on below

$$\begin{tikzcd}
& J_\infty \arrow[dl] \arrow[dr] & & & \\
R_\infty \arrow[dr] & & J[\Delta_Q] \arrow[r] \arrow[d] & \mathcal{O}[\Delta_Q] \arrow[d] & \\
& R\hat{\otimes}_{\mathcal{O}} J \cong & R_Q^{\square} \arrow[r] & R_Q \arrow[r] & R \\
& & \text{acts on} & \text{acts on} & \text{acts on} \\
& S_Q\hat{\otimes}_{\mathcal{O}} J \cong & S_Q^{\square} \arrow[r] & S_Q \arrow[r] & S
\end{tikzcd}$$

Here Q is an auxiliary set of places of F, $J := \mathcal{O}[[X_{v,i,j}]]_{1\leq i,j,\leq 2}/(X_{v_0,1,1})$ for some fixed $v_0 \in T$ and let $\mathfrak{a}$ denote the ideal $\langle X_{v,i,j}\rangle$. Then R_∞ and J_∞ are fixed power series rings over R^{loc} and J respectively. $S_Q, S_Q^{\square}$ are finite free modules over $\mathcal{O}[\Delta_Q], J[\Delta_Q]$ which are constructed using automorphic forms and the $R_Q^{\square}, R_Q$ are certain framed (resp. non framed) deformation rings. For certain sets of places Q (known as Taylor-Wiles primes) we will construct such a diagram and let Q vary. The S_∞ will then be constructed using the $S_Q^{\square}$ as Q varies.

In order to do this we will need to consider each of the following:

1) Galois side
2) Automorphic side
3) Patching (i.e how to choose $Q's$)

We start by discussing the Galois side and the deformation problems that we impose. We let Q be a finite set of finite places of F such that $T \cap Q = \emptyset$ and such that $\forall v \in Q$ $q_v \cong 1 \mod l$ and $\overline{\rho}(Frob_v)$ has distinct eigenvalues $\overline{\alpha}_v \neq \overline{\beta}_v$. Consider the deformation problem

$$\mathcal{S}_Q := (F, T \cup Q, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v \in T\cup Q})$$

where the local deformation problems are as follows. For $v \in T_l$, $\mathcal{D}_v =$ parametrizes cystalline lifts, with HT weights $\{H_\tau\}$, and for $v \in T_r \cup Q$ we allow all lifts. Of course all this is under the blanket assumption $\det \rho = \chi$.

We obtain the deformation rings

$$R_Q^{\square} := R_{\mathcal{S}_Q}^{\square T} \to R_Q := R_{\mathcal{S}_Q}^{univ} \to R_{\mathcal{S}_\emptyset}^{univ} = R_{\mathcal{S}}^{univ} = R$$

We have $R_Q^{\square} \cong R_Q \hat{\otimes}_Q J$ induced by the T framed deformation of type $\mathcal{S}_Q$ given by $(\rho_Q^{univ}, \{1 + (X_{v,i,j})\}_{v\in T})$

Lemma 20.5. 1)$\forall v \in Q$, $\rho_Q^{univ}|_{G_{F_v}} \cong \chi_\alpha \oplus \chi_\beta$ where $\chi_\alpha, \chi_\beta : G_{F_v} \to R_Q^\times$ satisfies

$$\chi_\alpha \mod \mathfrak{m}_{R_Q}(Frob_v) = \overline{\alpha}_v$$

$$\chi_\beta \mod \mathfrak{m}_{R_Q}(Frob_v) = \overline{\beta}_v$$

2) $\chi_\alpha|_{I_{F_v}}$ (also $\chi_\beta|_{I_{F_v}}$) factors through

$$I_{F_v} \to I_{F_v}^{tame} \to k(v)^\times \to \Delta_v$$

where Δ_v maximal l-group quotient of $k(v)^\times$

Proof. 1) This follows from the proof of Proposition 11.11. Note that we have assumed $\overline{\rho}$ is unramified at v so that modulo $\mathfrak{m}$, the Frobenius is well defined.

2) i) $\chi_\alpha(I_{F_v}^{wild}) \subset \ker(R_Q^\times \to (R_Q/\mathfrak{m}_Q)^\times)$ which is l-adic. ☐

The lemma implies that $\forall v \in Q$, we have a character $\chi_{\alpha,v} : \Delta_v \to R_Q^\times$, and hence we obtain a map

$$\prod_{v\in Q} \chi_{\alpha,v} : \Delta_Q := \prod_{v\in Q} \Delta_v \to R_Q^\times$$

We get a map

$$\mathcal{O}[\Delta_Q] \to R_Q$$

defined by sending $\gamma \in \Delta_v$ to $\chi_{\alpha,v}(\gamma) - 1$. Similarly we obtain a map

$$J[\Delta_Q] \to R_Q^\square \cong J\hat{\otimes}_{\mathcal{O}} R_Q$$

Lemma 20.6. 1) $(R_Q)_{\Delta_Q} \cong R$

2) $(R_Q^\square)_{\mathfrak{a}} \cong R_Q$

Proof. 1) It follows from the proof of 11.11 $\chi_\alpha(\sigma)\chi_\beta(\sigma) = 1$ for any $\sigma \in I_{F_v}$. Thus

$$G_{F,Q\cup T} \to GL_2(R_Q) \to GL_w(R/\Delta_v)$$

is unramified at $v \in Q$. By the universal property of R we obtain a map $R \to R/\Delta_v$ which is an isomorphism.

2) This follow from definitions of the universal deformation/lifting rings. ☐

21. Lecture 21

Recall the outline from last time.

Let Q be a set of finite places of F such that

- $Q \cap T = \emptyset$, ($T = T_l \coprod T_r$, where $T_l = \{v|l\|\}$).
- $\forall v \in Q$, $\overline{\rho}(Frob_v)$ has eigenvalue $\overline{\alpha_v} \neq \overline{\beta}_v$
- $q_v \equiv 1 \mod l$

21.1. **Automorphic side.** We need to choose level subgroups for $v \nmid \infty$

Consider the compact subgroups $U(v) \subset U_0(v) \subset GL_2(\mathcal{O}_{F_v})$ where we define

$$U_0(v) = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod \varpi_v\}$$

$$U(v)\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(v) | \frac{a}{d} \in (\mathcal{O}_{F_v}/\varpi)^\times \text{maps to 1 in } \Delta_v\}$$

Furthermore we set

$$U_Q := \prod_{v\nmid\infty} U_{Q,v} \subset U_{Q,0} := \prod_{v\nmid\infty} U_{Q,0,v} \subset GL_2(\mathbb{A}_F^\infty)$$

For $v \notin Q, U_{Q,v} = U_{Q,0,v} = U_v = GL_2(\mathcal{O}_F)$. For $v \notin T_r$, let U_{Q_v} be any fixed open compact with $\pi_{0,v}^{U_{Q_v}} \neq 0$

For $v \in Q, U_{Q,v} := U(v) \subset U_{Q,0,v} := U_0(v) \subset U_v := GL_2(\mathcal{O}_F)$

$U_{Q,0}/U_Q \cong \Delta_Q$ the maximal quotient of $k(v)^\times$, the map being given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a}{d}$$

21.2. **Automorphic forms and Hecke Algebras.** Fix a weight (k, η) the weight of π_0 and a central character χ.

$U_{Q,0}, U_Q$ acts by right translation on $S(U_Q, \mathcal{O}) \supset S(U_{Q,0}, \mathcal{O}) \supset S(U, \mathcal{O})$. The first group also has an action of U_{ϖ_v} and $\mathbb{T}_{U_Q} \supset \mathfrak{m}_Q$, these guys together give $\tilde{\mathbb{T}}_Q \supset \tilde{\mathfrak{m}}$. The last space has an action of $\mathbb{T}_U \supset \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal corresponding to $\overline{\rho}$.

Here we define

$$\mathbb{T}_{U_Q} := \langle T_v, S_v | v \notin T \cup Q \rangle \subset End(S(U_Q, \mathcal{O}))$$

$$U_{\varpi_v} := [U(v) \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} U(v)]$$

$$\tilde{\mathbb{T}} := \langle \mathbb{T}_{U_Q}, U_{\varpi_v}, v \in Q \rangle$$

$S(U_Q, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C} = \oplus_{\pi, \text{wt } (k,\eta), \text{cent. char. } \chi} ({\bigotimes}'_{v \notin Q} \pi_v^{U_{Q,v}}) \otimes (\bigotimes_{v \in Q} \pi_v^{U_v})$

We want to show that there is a 1-1 correspondence between

$$\{\mathbb{T}_{U_Q} \to \mathbb{C}\}$$

and

$$\{\pi \text{as in RHS + choice of } U_{\varpi_v} \text{ e-value}\}$$

Lemma 21.1.

Proof. Use local Global compatibility and explicit calculation cf [4] p 39. $\square$

Define $\mathfrak{m}_{U_Q} := \langle \lambda, T_v - tr\overline{\rho}(Frob_v), S_v - q_v^{-1}det\overline{\rho}(Frob_v), v \notin T \cup Q\rangle$, $\tilde{\mathfrak{m}}_{U_Q} := \langle \mathfrak{m}_{U_Q}, U_v - \overline{\alpha}_v\rangle$.

These are maximal ideals. (eg. $\tilde{m}_{U_Q} = \ker(\tilde{\mathbb{T}}_{U_Q} \to \mathcal{O} \to \mathbb{F})$) $\mathbb{T}_0$ and U_{ϖ_v} eigenvalues reducing to $\overline{\alpha}_v$.

$S_Q = S(U_Q, \mathcal{O})_{\tilde{m}_{U_Q}}$, is acted on by $\mathbb{T}_Q := \tilde{\mathbb{T}}_{U_Q, \tilde{m}_{U,Q}}$. (Note $Q = \emptyset, \mathbb{T}_Q = \mathbb{T}, \tilde{\mathfrak{m}}_{U,Q} = \mathfrak{m}_{U,Q} = \mathfrak{m}$ etc.)

As before we construct a representation

$$\rho^{mod}_{\tilde{m}_{U,Q}} : G_F \to GL_2(\mathbb{T}_Q)$$

which is a lift of ρ of type $\mathcal{S}_Q$. This allows us to define a map $R_Q \to \mathbb{T}_Q$.

We have two Δ_Q actions.

(a) $\Delta_Q = U_{Q,0}/U_Q$ acts of S_Q by right translation, automorphic at Q.

(b) $\mathcal{O}[\Delta_Q] \to R_Q \to \mathbb{T}_Q$ acts on S_Q the Galois action transferred via the Global Langlands correspondence.

Lemma 21.2. 1) These two actions coincide

2) S_Q is a finite free $\mathcal{O}[\Delta_Q]$ module.

Proof. 1) Use Local Global compatibility and same choices $(\overline{\alpha}_v)$ made on both sides.

2) [4] Prop 5.3, need to use $[F(\zeta_l : F] > 2$ □

Recall $\rho^{mod}_{\mathfrak{m}} : G_F \to GL_2(\mathbb{T})$ unramified outside T ($Q = \emptyset$ case)

$v \in Q$,

$$\mathrm{char}\rho^{mod}_{\mathfrak{m}}(Frob_v) \equiv (x - \overline{\alpha}_v)(x - \overline{\beta}_v) \mod \mathfrak{m}$$

By Hensel's Lemma we can choose $A_v, B_v \in \mathbb{T}$ reducing to $\overline{\alpha}_v, \overline{\beta}_v \mod \mathfrak{m}$ such that

$$\mathrm{char}\rho^{mod}_{\mathfrak{m}}(Frob_v) = (x - A_v)(x - B_v)$$

Lemma 21.3. 1) $S(U_{Q,0}, \mathcal{O})_{\tilde{m}_Q} \cong (S_Q)_{\Delta_Q}$

2) $\prod_{v \in Q}(U_{\varpi_v} - B_v) : S \cong S(U_{Q,0}, \mathcal{O})_{\tilde{m}_{U,Q}}$

Proof. Why it's plausible- "Same away from Q". For $v \in Q$ U_{ϖ_v} acts on the LHS buy A_v, B_v and on the RHS by A_v. □

Now lets finish step 2. Recall $\mathfrak{a}_Q \subset J[\Delta_Q] \to R_Q^{\square}$.

$$R_Q^{\square}/\mathfrak{a}_Q \cong (R_Q)_{\Delta_Q} \cong R$$

$$R_Q^{\square} \cong R_Q \otimes J \to R_Q^{\square}/\mathfrak{a}_Q \cong (R_Q)_{\Delta_Q} \cong R$$

This acts on

$$S_Q^{\square} = S_Q \otimes_{R_Q} R_Q^{\square} \to S_Q \otimes (R_Q^{\square}/\mathfrak{a}_Q) \cong (S_Q)_{\Delta_Q} \cong S$$

Observe $J[\Delta_Q] \cong J \otimes_{\mathcal{O}} \mathcal{O}[\Delta_Q]$, these acts on $S_Q^{\square} \cong J \otimes_{\mathcal{O}} S_Q$. The latter is finite free by the previous lemma, and its rank is $\mathrm{rank}_{\mathcal{O}} S$.

Summary: $J[\Delta_Q] \to R_Q^{\square} \to R$, these last two groups act on $S_Q^{\square}$ and S respectively, and $S_Q^{\square}$ is finite free over $J[\Delta_Q]$ of rank $rk_{\mathcal{O}} S$.

21.3. **Patching.** Step 3. Presenting $R_Q^{\square}$

$R^{loc} := \hat{\bigotimes}_{v\in T} R^{\square}_{\overline{\rho}_v,\chi_v}/I(\mathcal{D}_v)$, $v \in T_r, I(\mathcal{D}_v) = 0$.

We showed in section 3, that $R_Q^{\square}$ is a quotient of $R^{loc}[[x_1,...,x_{h_Q}]]$, (the quotient is unnecessary if $H^2_{\mathcal{S},T,Q}(G_{F,T\cup Q}, \mathrm{ad}^{\circ}\overline{\rho}) = 0$, where $h_Q = \dim H^1_{\mathcal{S},T}(\mathrm{ad}^{\circ}\overline{\rho})$.

$$h_Q = \#T - 1 - \sum_{v|\infty} \dim_{\mathbb{F}} H^0(G_{F_v}, \mathrm{ad}^{\circ}\overline{\rho}) + \sum_{v\in Q}(\dim_{\mathbb{F}} L(\mathcal{D}_v) - \dim_{\mathbb{F}} H^0(G_{F_v}, \mathrm{ad}^{\circ}\overline{\rho})$$

$$+ \dim_{\mathbb{F}} H^1_{\mathcal{S}_Q T}(G_{F,S}\mathrm{ad}^{\circ}\overline{\rho}(1)) - \dim_{\mathbb{F}} H^0(G_{F,S}\mathrm{ad}^{\circ}\overline{\rho}(1))$$

Everything in this is computable, in fact we find:

$$\sum_{v|\infty} \dim_{\mathbb{F}} H^0(G_{F_v}, \mathrm{ad}^{\circ}\overline{\rho}) = [F:\mathbb{Q}]$$

by totally odd.

$$\sum_{v\in Q}(\dim_{\mathbb{F}} L(\mathcal{D}_v) - \dim_{\mathbb{F}} H^0(G_{F_v}, \mathrm{ad}^{\circ}\overline{\rho})) = \#Q$$

$$\dim_{\mathbb{F}} H^0(G_{F,S}, \mathrm{ad}^{\circ}\overline{\rho}(1)) = 0$$

The only mysterious term is then $\dim_{\mathbb{F}} H^1_{\mathcal{S}_Q,T}(G_{F,S}, \mathrm{ad}^{\circ}\overline{\rho}(1))) := d_Q$.

Thus we obtain $h_Q = \#T - 1 - [F:\mathbb{Q}] + \#Q + d_Q$

Lemma 21.4. $\dim R^{loc} = 1 + 3\#T + [F:\mathbb{Q}]$

t

Proof. From section 3, $\dim R^{\square}_{\overline{\rho}_v,\chi_v}/I(\mathcal{D}_v) = \begin{cases} 4 & v \in T_r \\ 4 + [F_v : \mathbb{Q}_l] & v \in T_l \end{cases}$ □

Proposition 21.5. *(Taylor Wiles primes)* $r := max(d_{\emptyset}, [F:\mathbb{Q}] - \#T + 1)$.

$\forall N \geq 1$, $\exists \mathbb{Q}_N$, $|\mathbb{Q}_N| = r$ such that

1) $Q_N \cup T = \emptyset$

2) $\forall v \in Q_N$, $\overline{\rho}(Frob_v)$ has distinct eigenvalues, and $q_v \equiv 1 \mod l^N$

3) $d_{Q_N} = 0$

Proof. [4] Proposition 5.9. □

Define

$$R_\infty := R^{loc}[[x_1, ..., x_h]]$$

where $h := \#T - 1 - [F : \mathbb{Q}] + r$ and

$$J_\infty := J[[y_1, ..., y_r]]$$

Lemma 21.6. $\dim R^\infty = \dim J_\infty = r + 4\#T$

Proof. $\dim J_\infty = \dim J + r = \#T + r$
$\dim R_\infty = \dim R^{loc} + h =$ same. $\square$

$\forall N \geq 1$, fix $Q_N = \{v_1, ..., v_r\}$ fix ordering. Choose a surjection $R_\infty \to R^\square_{Q_N}$. Choose also a surjection $J_\infty \to J[\Delta_{Q_N}]$ given by sending y_i to $\gamma_i - 1$ where γ_i is a generator of v_i. Since J_∞ is formally smooth, we may lift the composite

$$J_\infty \to J[\Delta_{Q_N}] \to R_{Q_N} \otimes J \cong R^\square_{Q_N}$$

to a map $J_\infty \to R_\infty$. Writing $\mathfrak{a}_\infty$ to be the ideal $(\mathfrak{a}, y_1, ..., y_r)$ of J_∞, we have $R^\square_{Q_N}/\mathfrak{a}_\infty = R$, $S^\square_{Q_N}/\mathfrak{a}_\infty = S$. Let $\mathfrak{b}_N$ denote the kernel of $J_\infty \to J[\Delta_{Q_N}]$, so that $S^\square_{Q_N}/\mathfrak{b}_N$ is free over $J[\Delta_{Q_N}]$. Since $q_v \equiv 1 \mod l^N$ for every $v \in Q_N$, it follows that $\mathfrak{b}_N \subseteq ((1 + y_1)^{l^N} - 1, ..., (1 + y_r)^{l^N} - 1)$.

Let us choose open ideals $\mathfrak{c}_N$ of J_∞ such that:

- $\mathfrak{c}_N \cap \mathcal{O} = (\lambda^N)$
- $\mathfrak{c}_N \supseteq \mathfrak{b}_N$
- $\mathfrak{c}_N \supseteq \mathfrak{c}_{N+1}$
- $\cap_{N\geq 1}\mathfrak{c}_N = 0$

One can take $\mathfrak{c}_N = ((1 + X^{p^N}_{v,i,j} - 1, (1 + y_i)^{l^N} - 1, \lambda^N)$. Since $\mathfrak{c}_N \supseteq \mathfrak{b}_N$, we have $S^\square_{\mathbb{Q}_N}/\mathfrak{c}_N$ is free over $J_\infty/\mathfrak{c}_N$. Choose also open ideals $\mathfrak{d}_N$ such that:

- $\mathfrak{d}_N \subseteq \ker(R \to S/\lambda^N)$.
- $\mathfrak{d}_N \supseteq \mathfrak{d}_{N+1}$.
- $\cap_N \mathfrak{d}_N = 0$

For $M \geq N$, let $S_{M,N} := S_{Q_M}/\mathfrak{c}_N$, then $S_{M,N}$ is finite free over $J_\infty/\mathfrak{c}_N$ of rank equal to the $\mathcal{O}$ rank of S. We obtain a diagram of the form

$$J_\infty \longrightarrow R_\infty \longrightarrow R/\mathfrak{d}_N$$

acts on acts on

$$S_{M,N} \longrightarrow S/\mathfrak{d}_N$$

Since $\mathfrak{c}_N$ and $\mathfrak{d}_N$ are open, $S_{M,N}, S/\mathfrak{d}_N, R/\mathfrak{d}_N$ are all finite, thus one can find a subsequence of pairs M_i, N_i with $M_{i+1} > M_i$ and $N_{i+1} > N_i$ such that the diagram

$$\begin{array}{ccccc} J_\infty & \longrightarrow & R_\infty & \longrightarrow & R/\mathfrak{d}_{N_i} \\ & & \text{acts on} & & \text{acts on} \\ & & S_{M_{i+1},N_{i+1}/\mathfrak{c}_{N_i}} & \longrightarrow & S/\mathfrak{d}_{N_i} \end{array}$$

is isomorphic to the diagram for (M_i, N_i). Taking a projective limit over this subsequence we obtain the diagram:

$$\begin{array}{ccccc} J_\infty & \longrightarrow & R_\infty & \longrightarrow & R \\ & & \text{acts on} & & \text{acts on} \\ & & S_\infty & \longrightarrow & S \end{array}$$

where S_∞ is finite free over J_∞. This is precisely the situation that we needed to construct. From here one can apply the commutative algebra arguments from before to deduce the results of minimal ALT.

22. Lecture

22.1. **Eichler-Shimura theory.** We would like to associate to f a Hecke modular form an abelian variety $A_f/\mathbb{Q}$ a Galois representations $T_l\mathbb{A}_f$ with coefficients in $K_\lambda/\mathbb{Q}_l$.

Definition 22.1. A subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is a congruence subgroup if $\Gamma \supset \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ for some $N \in \mathbb{Z}_{\geq 1}$

Example 22.2.

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}$$

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N\}$$

$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N\}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ acts on $\mathcal{H}$ via $z \mapsto \frac{az+b}{cz+d}$

Definition 22.3. Let $f : \mathcal{H} \to \mathbb{C}$ we say f is a modular form of weight k and level Γ if...

Notation we say f has level N if it has level $\Gamma_0(N)$ in this case $f(z+1) = f(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z) = f(z)$, so f has a fourier expansion

$$f(q) = a_0 + a_1 q + a_2 q^2 ...$$

where $q = e^{2\pi i z}$.

Definition 22.4. The Hecke operator T_p acts on the space of modulars forms by taking f to $T_p f$ where $T_p f$ has fourier coefficients

$$a_k(T_p f) = a_{kp}(f) + p^{k-1} a_{k/p}(f)$$

One needs to check that $T_p f$ is also a modular form.

Definition 22.5. Let $f = q + a_2 q^2 + a_3 q^3 ...$, we say f is a Hecke form if it is an eigenform for each T_p. In this case we have $T_p f = a_p f$

Definition 22.6. Let $\rho : G_{\mathbb{Q}} \to GL_2(K_\lambda)$ be a Galois representation, f a Hecke eigengorm of level N, ρ is associated to f if for each $p \nmid lN$ the characteristic polynomial $\rho(Frob_p) = X^2 - a_p(f)X + p$.

Let f be a Hecke eigenform of level N.

Theorem 22.7. *1) $K_f = \mathbb{Q}(a_1, a_2, a_3 ...)$is a finite extension of $\mathbb{Q}$*

2) For each l place of $\mathbb{Q}$ $\lambda | l$ a place of K_f, there is a representation $\rho_{f,\lambda}$ associated to f with coefficients in $K_{f,\lambda}$

3) There is an abelian variety $A_f/\mathbb{Q}$ such that $V_l(A_f)$ has a $K_f \times G_{\mathbb{Q}}$ action and decomposes as a direct sum of the $\rho_{\lambda,f}$.

Corollary 22.8. $\dim A_f = [K_f : \mathbb{Q}]$

Theorem 22.9. *There is a canonical isomorphism*

$$S_2(\Gamma_0(N)) \oplus \overline{S_2(\Gamma_0(N))} \cong H^1(X_0(N))$$

Proof.

$$S_2(\Gamma_0(N) \oplus \overline{S_2(\Gamma_0(N))} \cong H^0(X_0(N), \Omega^1_{X_o(N)}) \oplus H^1(X_0(N), \Omega^1_{X_0(N)})$$

$$H^1(X_0(N), \mathbb{C})$$

The first isomorphism comes from the interpretation of modular forms as sections of line bundles on our modular curve, explicitly $f \mapsto f(z)dz$. The second isomorphism comes from the Hodge decomposition, this comes from integrating.

□

Note we can integrate over cusps since f is a cusp form, this is good because we can now apply T_p to $H^1(X_0(N), \mathbb{Z})$

$Y_0(N)^{an}$ classifies elliptic curves together with a cyclic subgroup of order N. We can also define $Y_0(N,p)$ triples (E, P, C) with $(E, P) \in Y_0(N)$, and C a cyclic subgroup of E of order N which intersects P trivially.

We obtain a diagram

$$\begin{tikzcd} & Y_0(N,p) \arrow[dl, "(E,P,C)\mapsto(E,P)"', "\pi_1"] \arrow[dr, "(E/C,P \mod C)", "\pi_2"'] & \\ Y_0(N) & & Y_0(N) \end{tikzcd}$$

Since both maps are finite we have pull back and push forwards on $H^1(X_0(N), \mathbb{Z})$.

$$T_p : H^1(X_0(N), \mathbb{Z}) \xrightarrow{\pi_1^*} H^1(X_0(N,p), \mathbb{Z}) \xrightarrow{\pi_2^*} H^1(X_0(N), \mathbb{Z})$$

Important fact: Under the the isomorphism

$$S_2(\Gamma_0(N)) \oplus \overline{S_2(\Gamma_0(N))} \to H^1(X_0(N), \mathbb{Z}) \otimes \mathbb{C}$$

$T_p \oplus \overline{T_p}$ corresponds to T_p.

We can also extend this to an operator on the Picard group $H^1(X_0(N), \mathcal{O}_X^\times)$. In fact we can define such an operator on $H^1(X_0(N), \mathcal{O}_X)$ using the same method, then the long exact sequence applied to

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

gives an isomorphism $H^1(X_0(N), \mathcal{O}_X)/H^1(X_0(N), \mathbb{Z}) \cong Pic^0(X_0(N))$. Thus we get a Hecke operator on $V_l Pic^0(X_0(N)) \cong H^1(X_0(N), \mathbb{Z}) \otimes \mathbb{Q}_l$.

So if we define $T_0(N)$ to be the image of $\mathbb{Z}[\{T_p\}]$ in $End(H^1(X_0(N), \mathbb{Z}))$ then $T_0(N)$ acts on $V_l(Pic^0(X_0(N))$ and we have a morphism $T_0(N) \to K_f$ given by $t \mapsto tf/f$. Thus we have proven $[K_f : \mathbb{Q}] < \infty$ since $T_0(N)$ is finite over $\mathbb{Z}$.

We would like to translate all of this into the algebraic world, to this end we consider the functor from schemes over $\mathbb{Z}[1/N]$ which takes S to pairs $(\mathcal{E}, P)$ where $\mathcal{E}$ is and elliptic curve over S and $P \subset \mathcal{E}(S)$ is a cyclic subgroup of order N.

22.2. **The weight 2 Shimura isomorphism.** Fact: This functor is representable as a scheme (called $Y_0(N)$) over $\mathbb{Z}[1/N]$, and has a canonical copmactification $X_0(N)$. This allows us to carry over the picture above into the algebraic setting so we obtain operators T_p on $Pic^0(X_0(N))_{\mathbb{Z}[1/Np]}$. One can extend this to an operator over the original base $\mathbb{Z}[1/N]$.

Now we study the action of $T_0(N)$ and $G_{\mathbb{Q}}$ on the Tate module of the Picard group.

Definition 22.10. Let p_f be the kernel of $T_0(N) \to K_f$ $T \mapsto tf/f$ and define A_f to be $Pic^0(X_0(N)) \otimes_{T_0(N)} K_f$

$T_0(N)$ acts on A_f via K_f, and hence it acts on its Tate module.

$V_l(A_f)$ has a $\mathbb{Q}_l \otimes_{\mathbb{Q}} K_f$ vector space structure and decomposes into $K_{f,\lambda}$ vector spaces $V_{f,\lambda}$

Proposition 22.11. $\dim_{K_{f,\lambda}} V_{f,\lambda} = 2$

This follows from the fact that $V_l Pic^0$ is free module of rank 2 over $T_0(N)$.

Now we've at least constructed these $\rho_{f,\lambda} = (V_l A_f) \otimes_{K_f \otimes \mathbb{Q}_l} K_{f,\lambda}$. The previous fact says that the dimension of this is 2. We now need to show that the characteristic polynomial of $Frob_p$ is $X^2 - a_p X + p$.

Let $J_p = Pic^0(X_0(N)) \otimes \mathbb{F}_p$.

Then we have the formula formula due to Eichler and Shimura:

Theorem 22.12. *In $End(J_p)$ we have*

$$T_p = F + F^\vee = F + pF$$

Now we have that $F^2 = T_p F - p$, so that F satisfies $X^2 - T_p X + p$ on J_p and hence F satisfies $X^2 - a_p X + p$ on $A_f \otimes \mathbb{F}_p$.

This proves $\rho_{f,\lambda}$ is associated to f.

22.3. **Extension to $\Gamma_1(N)$.** Define $Y_1(N)$ to be $\mathcal{H}/\Gamma_1(N)$, moduli theoretically this classifies pairs (E, P) where E is an elliptic curve over $\mathbb{C}$ and P is a point of order N.

The Shimura isomorphism holds in this case as well. In this case we also have the diamond operators:

$$\langle d \rangle : Y_1(N) \to Y_1(N)$$

where for $(d, N) = 1$ this is given by $(E, P) \mapsto (E, dP)$. This lifts to an action of $S_2(\Gamma_1(N))$ which acts with finite order and is semi simple.

$$S_2(\Gamma_1(N)) = \bigoplus_{\chi : (\mathbb{Z}/N\mathbb{Z})^0 \to \mathbb{C}} S_2(\Gamma_1(N), \chi)$$

where the subspace corresponding to χ is spanned by those f for which $\langle d \rangle f = \chi(d) f$. Let $T_1(N) = \mathbb{Z}[T_p, \langle d \rangle] \subset End S_2(\Gamma_1(N))$ and let $f \in S_2(\Gamma_1(N), \chi)$ be a Hecke eigenform, still $K_f = \mathbb{Q}[a_p(f) : \chi(d)]$

A_f is still an abelian variety over $\mathbb{C}$, $\rho_{f,\lambda}$ and $Frob_p$ satfies $X^2 - a_p X + p\chi(p)$

23. Lecture 23

23.1. **Fermat's last theorem.** The last two lectures will be devoted to explaining how the results we have proved can be applied to prove Fermat's last theorem.

Theorem 23.1. *The equation*

$$x^n + y^n = z^n$$

has no integer solutions for $n \geq 3$

We may reduce to the case $n \geq 5$ as the cases $n = 3, 4$ have been proved by other means. Note also that it suffices to prove the result for when $n = l$ is prime, and that wlog. we may assume $x \equiv -1 \mod 4$ and that y is even.

The outline of the proof is as follows.

Step 1: Assuming a solution to the equation exists, we will construct an elliptic curve of conductor N (with N even) with remarkable properties. This is due Frey (1985-1986), in fact given a solution $a^l + b^l = c^l$, the elliptic is just the projectivisation of the affine curve defined by the equation

$$E_{a^l,b^l} : y^2 = x(x - a^l)(x - b^l)$$

Step 2: The techniques and results discussed in this course, originally proved by Wiles, allows us to construct a weight 2 cusp form of level $\Gamma_0(N)$ with similar properties.

Step 3: A result of Serre and Ribet then provides a weight 2 cusp form of level $\Gamma_0(2)$.

Step 4: A computation of $\dim S_2(\Gamma_0(2)$ (the space of cusp forms of weight 2 and level $\Gamma_0(2)$) gives a contradiction.

Let us briefly mention the order in which these steps were proved. Step 4 was a classical result, Step 1 came next, and once Ribet proved Serre's ϵ conjecture (Step 3) the door was open to attacking FLT using the Taniyama-Shimura-Weil conjecture. Wiles proved the conjecture for all semistable elliptic curves over $\mathbb{Q}$ which was enough to prove the theorem.

Remark 23.2. The full modularity theorem was proved by Breuil-Conrad-Diamond-Taylor around 1999-2000.

Freitas-Le Hung-Siksek (2013) proved STW for elliptic curves over real quadratic fields.

23.2. **Dimension formula for weight 2 cusp forms.** Let $\mathfrak{h}$ denote the upper half plane, and $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}_1(\mathbb{Q})$. Recall the the objects $\Gamma_0(N), Y_0(N), X_0(N)$ defined in the previous lecture (here we are just considering these objects in the complex analytic context). We have a bijection between $Y_0(N)$ and the set of isomorphism classes elliptic curves $/\mathbb{C}$ together with a cyclic subgroup of order N given by

$$z \mapsto (E_z := \mathbb{C}/\langle 1, z\rangle, \langle 1/N\rangle)$$

We have natural projections

$$\begin{array}{ccc} Y_0(N) & \subset & X_0(N) \\ \downarrow & & \downarrow \\ Y_0(1) & \subset & X_0(1) \end{array}$$

and we may naturally identify $Y_0(1)$ (resp. $X_0(1)$) with $\mathbb{A}^1(\mathbb{C})$ (resp. $\mathbb{P}^1(\mathbb{C})$).

We have an isomorphism

$$S_2(\Gamma_0(N)) \cong H^0(X_0(N), \Omega^1)$$

given by

$$f \mapsto f(z)dz$$

Therefore $\dim S_2(\Gamma_0(N))$ is just the genus of $X_0(N) := g_0(N)$. The idea now is to apply Riemann-Hurwitz to the projection $X_0(N) \to X_0(1)$.

Proposition 23.3. $g_0(2) = 0$

Define the Euler characteristic of a reasonable topological space X to be

$$\chi(X) = \sum_{i \geq)} \dim H^i(X, \mathbb{C})$$

Facts:
•$\chi(pt.) = 1$
•$\chi(X) = 2 - 2\text{genus}(X)$ if X is a compact Riemann surface.
•$\chi(X \coprod Y) = \chi(X) + \chi(Y)$
• Riemann Hurwitz theorem: Let $X \to Y$ be an unramified covering of degree d, then $\chi(X) = d(\chi(Y)$.

Consider the projection $\pi_N : X_0(N) \to X_0(1)$ and for $z \in X_0(1)$, define $n_z = \#\pi_N^{-1}(z)$.

Lemma 23.4. If $z \neq \infty$ then

$$n_z = \#\frac{\{C \subset E_z[N] \text{ cyclic order } N\ \}}{\text{Image}(Aut(E_z) \to Aut(E_z[N]))}$$

Furthermore, if $z \neq 0, 1728, \infty$ then

$$n_z = \#\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) = N + 1$$

Proof. $\{C \subset E[N] \text{ cyclic order } N\ \}$ is in bijection with the set of lines in $(\mathbb{Z}/N\mathbb{Z})^2$, i.e. $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. Since $Aut(E_z)$ acts trivially on the set, we obtain the second result. When $z = 0, 1728, \infty$ need to calculate n_z explicitly and we find that

$$n_0(2) = 1, n_2(3) = 2$$

$$n_{1728}(2) = 2, n_{1728}(3) = 2$$

$$n_\infty(2) = 2, n_\infty(3) = 2$$

□

Proof. (of Prop) Let $S = 0, 1728, \infty$, then $X_0(N) - \pi_N^{-1}(S) \to X_0(1) - S$ is an unramfied covering of degree $N + 1$, hence

$$\chi(X_0(N) - \pi_N^{-1}(S)) = (N+1)\chi(X_0(1) - S)$$

$$\Rightarrow 2 - 2g_0(N) - (n_0 + n_{1728} + n_\infty) = (N+1).(-1)$$

$$\Rightarrow g_0(N) = \frac{1}{2}(N + 3 - n_S)$$

Hence from the explicit calculations above we obtain $g_0(2) = g_0(3) = 0$ □

Remark 23.5. $n_0 \approx \frac{1}{2}N$, $n_{1728} \approx \frac{1}{3}N$, $n_\infty \equiv 0 \mod N$, so $g_0(N) \approx \frac{1}{12}N$.

Remark 23.6. The first N such that $g_0(N) \neq 0$ is 11.

23.3. **Local Galois representations arising from elliptic curves.** Let E be an elliptic curve over $k = \mathbb{Q}, \mathbb{Q}_p$.

Define $\overline{r}_{E,l} := E[l](\overline{k})$, which has an action of $Gal(\overline{k}/k)$.
$r_{E,l} := T_l E = \lim_{\leftarrow n} E[l^n](\overline{k})$
$\overline{\rho}_{E,l} = H^1_{\acute{e}t}(E_{\overline{k}}, \mathbb{Z}/l\mathbb{Z}) \cong \overline{r}_{E,l}(-1)$
$\rho_{E,l} = H^1_{\acute{e}t}(E_{\overline{k}}, \mathbb{Z}_l) \cong r_{E,l}(-1)$

Proposition 23.7. *Let $E/\mathbb{Q}_p$.*
1) $\det r_{E,l} = \epsilon_l$ *the l-adic cyclotomic character.*
$\det \rho_{E,l} = \epsilon_l^{-1}$
2) Suppose E has good reduction mod p, then for $p \neq l$ $\rho_{E,l}, \overline{\rho}_{E,l}$ are unramified at l and

$$char(\rho_{E,l}(Frob_p)) = X^2 - (1 + p - \#E(\mathbb{F}_p))X + p$$

and for $p = l$ $\rho_{E,l}$ is crystalline with Hodge Tate weight $0, 1$.

Remark 23.8. In fact E has good reduction if and only if $\rho_{E,l}$ is unramified for $(l \neq p)$, or $\rho_{E,l}$ is crystalline for $l = p$.

There are 4 cases: Either $E/\mathbb{Q}_p$ has good reduction or bad reduction. For the case of good reduction, there two further cases corresponding to ordinary reduction and supersingular reduction. In the ordinary case we have $E(\overline{\mathbb{F}}_p)[p] = p$, whereas in the supersingular case $E(\overline{\mathbb{F}}_p)[p] = 1$. For the case of bad reduction, the reduction can either be multiplicative or additive, the first case corresponds to when the reduction is a nodal cubic, the second when there is a cusp.

Proposition 23.9. *([3] 2.11, 2.12) We have the following for $E/\mathbb{Q}_p$*

	Conductor	$\rho_{E,l}$	$\overline{\rho}_{E,l}$
Ordinary	1	$p \neq l$ *unramified* $p = l$ $\begin{pmatrix} \lambda & * \\ & \epsilon_l^{-1}\lambda^{-1} \end{pmatrix}$	$p = l$ $\begin{pmatrix} \lambda & * \\ & \overline{\epsilon}_l^{-1}\lambda^{-1} \end{pmatrix}$
Supersingular	1		
Mult.	p	$\begin{pmatrix} 1 & * \\ & \epsilon_l^{-1} \end{pmatrix}(\delta)$	$\begin{pmatrix} 1 & * \\ & \overline{\epsilon}_l^{-1} \end{pmatrix}(\delta)$
Add.	$p^n (n \geq 2)$		

Where λ is an unramified character $G_{\mathbb{Q}_p} \to \mathbb{Q}_l^\times$, and δ is trivial if E has split multiplicative reduction and the unique unramified quadratic character if E has non-split multiplicative reduction.

Proof. For ordinary reduction at $l = p$, let us show $r_{E,l}$ has the form $\begin{pmatrix} \epsilon_l \lambda & * \\ & \lambda^{-1} \end{pmatrix}$.

Let $E/\mathbb{Z}_p$ be the integral model for E. The connected-étale exact sequence gives

$$0 \to E[l^n]^0 \to E[l^n] \to E[l^j]^{\acute{e}t} \to 0$$

Since E is ordinary we have that $\mathrm{rk}E[l^n]^0 = \mathrm{rk}E[l^j]^{\acute{e}t} = l^n$. Taking the inverse limit over the $\overline{\mathbb{Q}}_l$ points gives the result.

For mult. reduction, $p = l$ or $p \neq l$. Use Tate uniformization theorem:

$$E(\overline{\mathbb{Q}}_p) \cong \overline{\mathbb{Q}}_p^\times / q^{\mathbb{Z}}(\delta)$$

as $G_{\mathbb{Q}_p}$ representations with $q \in p\mathbb{Z}_p$ depending on E.

Hence $E[l^n](\overline{\mathbb{Q}}_p) \cong \langle \zeta_{l^n}, q^{1/l^n} \rangle \subset \overline{\mathbb{Q}}_p^\times / q^{\mathbb{Z}}$ so that we obtain the exact sequence:

$$1 \to \langle \zeta_l^n \rangle \to \langle \zeta_l^n, q^{1/l^n} \rangle \to \langle q^{1/l^n} \rangle \to 1$$

with $G_{\mathbb{Q}_p}$ acting on the final term trivially, since $\sigma(q^{1/l^n}) \cong q^{1/l^n} \mod \langle \zeta_{l^n} \rangle$. $\square$

24. Lecture 23

Definition 24.1. E is ordinary at p if E has good ordinary reduction or bad multiplicative reduction, it is semistable if it has good or bad multiplicative reduction. The motivation behind this is that for $p = l$, E is ordinary/ semistable implies its associated Galois representation is ordinary/ semistable.

The following is the key input for level lowering.

Proposition 24.2. *$E/\mathbb{Q}_p$ has bad multi. reduction, let Δ_E^{min} be the minimal discriminant in $p\mathbb{Z}_p$.*

1) $p \neq l$, $l | v_p(\Delta_E^{min}) \Leftrightarrow \overline{\rho}_{E,l}$ is unramified.

2) $p = l$, $l | v_p(\Delta_E^{min}) \Leftrightarrow$, $\overline{\rho}_{E,l}$ lifts to l-adic crystalline representation with HT weight $0, 1$. *(as opposed to* $0, l$*).s*

Proof. Recall there is a $G_{\mathbb{Q}_p}$ equivariant isomorphism

$$E(\mathbb{Q}_p) \cong (\overline{\mathbb{Q}}_p^{\times}/q^{\mathbb{Z}})(\delta)$$

$$q = j^{-1} + 744j^{-2} + ... \in \mathbb{Z}[j^{-1}]$$

where $j = j(E)$ the $j-$invariant.

We have $v_p(q) = -v_p(j) = v_p(\Delta_E^{min})$, since $j\Delta = c_4^3$ and E has multiplicative reduction implies $v_p(c_4) = 0$.

The representation $\overline{\rho}_{E,l}$ is isomorphic to the action of $G_{\mathbb{Q}_p}$ on $E[l]\overline{\mathbb{Q}_p}$.

By Tate's uniformization theorem, this representation is unramified if and only if the extension $\mathbb{Q}_p(\zeta_l, q^{1/l})$ is unramified. This extension is a composite of an unramified cyclotomic extension and the Kummer extension $\mathbb{Q}_p(\zeta_l, q^{1/l})/\mathbb{Q}_p(\zeta_l)$, thus this is unramified if and only if $l | v_p(q) = v_p(\Delta_E^{min})$. This proves 1).

2) Due to Edixhoven (Invent. 92.) $l | v_l(\Delta) \Leftrightarrow \overline{\rho}_{E,l}$ comes from a finite flat group scheme over $\mathbb{Z}_l$. □

24.1. **Frey curve.** (Completing Step 1) Recall we made the following assumptions:

$a^l + b^l = c^l, abc \neq 0, l \geq 5$ prime, $a \equiv -1 \mod 4, b \equiv 0 \mod 2, (a, b) = 1$.

$E := E_{a^l,b^l} : y^2 = x(x - a^l)(x - b^l)$, we obtain $\rho_{E,l}, \overline{\rho}_{E.l}$ Galois representations of $\mathbb{Q}$.

Proposition 24.3. *1)* $\Delta_E^{min} = 2^{-8}(abc)^{2l}$

2) E is semistable (at every prime)

3) $\overline{\rho}_{E,l}$ is unramified outside $2l$, irreducible and $\overline{\rho}_{E,l}|_{G_{\mathbb{Q}_l}}$ lifts to crystalline representation with Hodge Tate weight $0, 1$.

Proof. (Sketch, see [3] 2.15) E_{a^l,b^l} has minimal Weierstarss equation

$$y^2 + xy = x^3 + \frac{b^l - a^l - 1}{4}x^2 - \frac{a^l b^l}{16}x$$

From this 1) follows and for 2), check that the equation doesn't have cusps mod each prime.

3) Irreducibility follows from a deep theorem of Mazur (77,78).

The unramifiedness and existence of the crystalline lift follows from the previous proposition since $\forall p \geq 3, l | v_p(\Delta_E^{min})$. □

Remark 24.4. Note that the conditions in 3) are satisfied for Galois representations coming from a cusp form of weight 2, level 2.

Also the existence of crystalline lift is part of Serre's conjecture.

24.2. **Modularity lifting theorem.**

Definition 24.5. ρ (resp. $\overline{\rho}$) is weight 2 modular (of level N) if there exists an eigenform f (level $\Gamma_0(N)$) such that $\rho_{f,l} \cong \rho$ (resp. $\overline{\rho}_{f,l} \cong \overline{\rho}$).

(MLT) $l \geq 3$ Consdier $\rho, \rho_0 : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_l)$. The following hypotheses are commonly imposed in proofs of MLT.

• ρ_0 is weight 2 modular.
• $\rho|_{G_{\mathbb{Q}_l}}, \rho_0|_{G_{\mathbb{Q}_l}}$ semistable, HT weights $0, 1$.
• $\overline{\rho} := \rho \mod l \cong \rho_0 \mod l$ has "big image". (Almost sufficient: $\overline{\rho}|_{G_{\mathbb{Q}(\varsigma_l)}}$ is absolutely irreducible (cf [3] 3.24, let's ignore (harmless for us)).

$E/\mathbb{Q}$ is weight 2 modular if $\rho_{E,l}$ is modular ($\exists l \Leftrightarrow \forall l$).

We use the following version of non-minimal MLT.

Further assume:

(ord) $\rho|_{G_{\mathbb{Q}_l}}, \rho_0|_{G_{\mathbb{Q}_l}}$ are ordinary (crystalline or semistable).

or (BT) $\rho|_{G_{\mathbb{Q}_l}}, \rho_0|_{G_{\mathbb{Q}_l}}$ are crystalline with HT weight $0, 1$ (This is a special case of the Fontaine Laffaile case).

Then ρ is weight 2 modular.

Theorem 24.6. *(Wiles' Theorem) $E/\mathbb{Q}$ is semistable $\Rightarrow$ E is weight 2 modular of some level. (Eg. E_{a^l,b^l}).*

Theorem 24.7. *(Ribet's theorem) $E = E_{a^l,b^l}$, suppose that E is semistable, weight 2 of level N, $\overline{\rho}_{E,l}$ unramified outside $2l$, $\overline{\rho}_{E,l}|_{G_{\mathbb{Q}_l}}$ has crystalline lift with HT weight 0,1. The $\overline{\rho}_{E,l}$ is weight 2 and of level 2.*

24.3. **Proof of Ribet's Theorem.** (Original proof) Used a detailed study of the Jacobian of modular curves, and doesn't generalize to higher dimensions.

There is an alternative method due to Khare and Wintenberger.

Step 1) If $l|N$ then can replace N by N/l.

Step 2) Now $l \nmid N$, replace N by 2.

Khare-Wintenberger: find a Galois representation ρ of weight 2, "lowered level," with prescribed local properties (*) rather than a modular form. Once this is done, apply MLT, to show ρ is weight 2 modular.

Idea: Formulate a global Galois deformation problem $\mathcal{S}$ for $(*)$, and show that $R^{univ}_{\mathcal{S}}(\overline{\mathbb{Q}})l \neq \emptyset$.

To do: $\dim R^{univ}_{\mathcal{S}} \geq 1$ cf. [4] 3.24 or Pset 6, question 1.

$R^{univ}_{\mathcal{S}}$ is finite over $\mathcal{O}$ (this is proved by showing $R^{univ}_{\mathcal{S}}$ is isomorphic to some suitable Hecke algebra $\mathbb{T}$.

24.4. **Proof of Wiles' Theorem.** Case 1: $\overline{\rho}_{E,3}$ irreducible ($\Rightarrow$ abs. irreducible)

Step 1-1 $\overline{\rho}_{E,3}$ is weight 2 modular. (Langlands-Tunnell and Deligne-Serre lifting Lemma)

Step 1-2: Apply MLT implies E is weight 2 modular.

Case 2: $\overline{\rho}_{E,3}$ is reducible $\Rightarrow \overline{\rho}_{E,5}$ is absolutely irreducible.

Step 2-1: (3-5) trick. Find semistable $E'/\mathbb{Q}$ such that $\overline{\rho}_{E',5} \cong \overline{\rho}_{E,5}$, $\overline{\rho}_{E',3}$ abs. irreducible.

Step 2-2: MLT implies $\rho_{E',3}$ is weight 2 modular.

$\Rightarrow \rho_{E',5}$ is weight 2 modular.

$\Rightarrow \overline{\rho}_{E',5} \cong \overline{\rho}_{E,5}$ is weight 2 modular.

$\Rightarrow \rho_{E,5}$ ismodular.

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