# MOD-$p$ ISOGENY CLASSES ON SHIMURA VARIETIES WITH PARAHORIC LEVEL STRUCTURE

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Abstract. We study the special fiber of the integral model for Shimura varieties of Hodge type with parahoric level structure constructed by Kisin and Pappas in [KP]. We show that when the group is residually split, the points in the mod $p$ isogeny classes have the form predicted by the Langlands–Rapoport conjecture in [LR87].

We also verify most of the He–Rapoport axioms for these integral models without the residually split assumption. This allows us to prove that all Newton strata are non-empty for these models. The verification of the axioms in full is reduced to a question on the connected components of affine Deligne–Lusztig varieties.

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## 1. Introduction

An essential part of Langlands' philosophy is that the Hasse–Weil zeta function of an algebraic variety should be a product of automorphic $L$-functions. In [Lan76], [Lan77], [Lan79], Langlands outlined a program to verify this for the case of Shimura varieties for which an essential ingredient was to obtain a description of the mod-$p$ points of a suitable integral model for the Shimura variety. Such a conjectural description first appeared in [Lan76], and was later refined by [LR87], [Kot97] and [Rap05]. To explain it we first introduce some notations.

Let $(G, X)$ be a Shimura datum and $\mathrm{K}_p \subset G(\mathbb{Q}_p)$ and $\mathrm{K}^p \subset G(\mathbb{A}_f^p)$ compact open subgroups where $\mathbb{A}_f^p$ are the finite adeles with trivial component at $p$. We assume $\mathrm{K}_p$ is a parahoric subgroup of $G(\mathbb{Q}_p)$. For $\mathrm{K}^p$ sufficiently small we have the Shimura variety $Sh_{\mathrm{K}_p\mathrm{K}^p}(G, X)$ which is an algebraic variety over a number field $E$ known as the reflex field. We will mostly be considering Shimura varieties of Hodge-type in which case $Sh_{\mathrm{K}_p\mathrm{K}^p}(G, X)$ can be thought of as a moduli space of abelian varieties equipped with some cycles in its Betti cohomology (at least on the level of its complex points). Let $p$ be a prime and $v|p$ a prime of $E$, then conjecturally there should exist an integral model $\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)/\mathcal{O}_{E_{(v)}}$ for $Sh_{\mathrm{K}_p\mathrm{K}^p}(G,X)$ satisfying certain good properties. When the group $G_{\mathbb{Z}_p}$ is unramified and $\mathrm{K}_p$ is hyperspecial there is a characterization of such an integral model using a certain extension property; see [Mil92, Proposion 2.8]. For general parahorics such a characterization is not known. However as long as the integral model has good local properties (more precisely, one desires that its nearby cycles are amenable to computation) and one can obtain some global information about the $\mathbb{F}_q$-rational points, then this is already enough for many applications such as the computation of the (semisimple) local factor of the Hasse–Weil zeta function of the Shimura variety; see for example [Hai05, §11].

We consider also the inverse limit of integral models

$$\mathscr{S}_{\mathrm{K}_p}(G,X) := \lim_{\leftarrow \mathrm{K}^p} \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X).$$

Then conjecturally there should be a bijection (see [LR87], [Rap05], [Hai05]):

$$\mathscr{S}_{\mathrm{K}_p}(G,X)(\overline{\mathbb{F}}_p) \cong \coprod_{\phi} S(\phi)$$

where

$$S(\phi) = \lim_{\leftarrow \mathrm{K}^p} I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi)/\mathrm{K}^p.$$

When $\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)$ arises as a moduli space of abelian varieties, this represents the decomposition of the special fiber into disjoint isogeny classes parametrised by $\phi$. The individual isogeny class $S(\phi)$ breaks up into a prime-to-$p$ part $X^p(\phi)$ and $p$-power part $X_p(\phi)$, and the $I_\phi(\mathbb{Q})$ is the group of self-isogenies of any member of the isogeny class $S(\phi)$. For general $G$, the objects appearing are approriate group-theoretic analogues of the objects described.

The bijection should satisfy compatibility conditions with respect to certain group actions on either side. For example, on $S(\phi)$ one can define an operator $\Phi$, and this should correspond under the above bijection to the action of Frobenius on $\mathscr{S}_{\mathrm{K}_p}(G,X)(\overline{\mathbb{F}}_p)$. Using this, one obtains a completely group theoretic description of the $\mathbb{F}_q$-points of the Shimura variety.

The first major result in this direction was obtained by Kottwitz [Kot97] who gave a description of the $\overline{\mathbb{F}}_p$-points for PEL-type Shimura varieties (more precisely the moduli spaces he considered are actually a union of Shimura varieties, but for the application to computing the zeta function, this was not an issue). In this case, the integral models of Shimura varieties are indeed moduli spaces of abelian varieties with extra structure, so one ends up counting such abelian varieties. Then after constructing good integral models for Shimura varieties of abelian type in [Kis10], Kisin [Kis17] proved the conjecture for these integral models. In that case the Shimura varieties and their integral models are no longer moduli spaces of abelian varieties with any obvious additional structure and many new ideas were needed. In both these works, the authors worked with hyperspecial level structure at $p$; in particular this meant the Shimura varieties had good reduction at $v$, i.e. the integral models $\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)$ were smooth over $\mathcal{O}_{E_{(v)}}$. In constrast, when considering arbitrary parahoric level structure, the integral models will not in general be smooth and this presents many new difficulties in proving such a result. However, if one is to get a complete description of the zeta function of the Shimura variety, then knowledge of the places of bad reduction is still needed. Moreover, understanding the cohomology of these spaces at places of bad reduction has many other important applications such as the local Langlands correspondences; see [HT01].

We assume now that $p > 2$. Let $(G,X)$ be a Shimura datum of Hodge type such that $G_{\mathbb{Q}_p}$ is tamely ramified, $p \nmid |\pi_1(G_{\mathrm{der}})|$ and $\mathrm{K}_p$ is a connected parahoric[1] (we will refer to these assumptions as

[1] A connected parahoric is one which is equal to the Bruhat–Tits stabilizer scheme

(*)). Under these assumptions Kisin and Pappas have constructed good integral models $\mathscr{S}_{\mathrm{K}_p}(G,X)$ for the Shimura varieties associated to the above data. These integral models satisfy the correct local properties in the sense that there exists a local model diagram as in [Hai01, §6]. The main result of this paper is the following.

**Theorem 1.1.** *Let $(G,X)$ be a Shimura datum of Hodge type as above. We assume $G_{\mathbb{Q}_p}$ is residually split at $p$.*

*(i) The isogeny classes in $\mathscr{S}_{\mathrm{K}_p}(G,X)(\overline{\mathbb{F}}_p)$ have the form*

$$\lim_{\leftarrow \mathrm{K}^p} I_\phi(\mathbb{Q})\backslash X_p(\phi)\times X^p(\phi)/\mathrm{K}^p.$$

*(ii) Each isogeny class contains a point $x$ which lifts to a special point in $Sh_{\mathrm{K}_p}(G,X)$.*

Let us first explain what we mean by an isogeny class. We assume for simplicity that $\mathrm{K}_p=\mathcal{G}(\mathbb{Z}_p)$, where $\mathcal{G}$ is an Iwahori group scheme for the rest of the introduction. It follows from the construction of the integral models that to each $x\in\mathscr{S}_{\mathrm{K}_p}(G,X)(\overline{\mathbb{F}}_p)$ one can associate an abelian variety $\mathcal{A}_x$ with $G$-structure. This means $\mathcal{A}_x$ is equipped with certain tensors in its étale and crystalline cohomology whose stabilizer subgroups are related to the group $G$. This leads to a natural notion of the isogeny class of $x$, which breaks up into a prime-to-$p$ part and a $p$-power part. We then obtain a decomposition of the special fiber into disjoint isogeny classes as in the conjecture. To prove the conjecture in full one needs therefore a description of the points in an individual isogeny class and then also an enumeration of the set of all isogeny classes. In this paper we focus on the first problem. The key ingredient needed for the enumeration of the set of all isogeny classes is part (ii) of the above theorem; this allows one to relate the set of isogeny classes to some data on the generic fiber where one has a good description of the points. Note that some results in the direction of part (ii) of the Theorem has also been proved in [KMPS]; here we provide a different proof more along the lines of [Kis17, §2]. To go from the above theorem to the conjecture in full requires some technical computations involving Galois cohomology, which the author intends to return to in a future work. The above then can really be thought of as the arithmetic heart of the conjecture of [LR87].

Let us now give some details about the theorem and its proof. The general strategy follows that of [Kis17]; however there are many obstructions to adapting the proof over directly for the case of general parahorics. As was mentioned above, each isogeny class decomposes into a $p$-power part and a prime-to-$p$ part. Describing the $p$-power part is the most difficult part of the problem.

To an $x$ as above we can associate an $X_p(\phi)$ which is a union of affine Deligne–Lusztig varieties (see §5.2 for the precise definition). By the construction of these integral models, one has a map

$$\mathscr{S}_{\mathrm{K}}(G,X)\to\mathscr{S}_{\mathrm{K}'}(GSp(V),S^{\pm})\otimes\mathcal{O}_{E_{(v)}}$$

where $\mathscr{S}_{\mathrm{K}'}(GSp(V),S^{\pm})$ is an integral model for the Siegel Shimura variety $Sh_{\mathrm{K}'}(GSp(V),S^{\pm})$, defined as a moduli space for abelian varieties with polarization and level structure. Using Dieudonné theory it is possible to define a natural map

$$\tilde{i}_x: X_p(\phi)\to\mathscr{S}_{\mathrm{K}'}(GSp,S^{\pm})(\overline{\mathbb{F}}_p)$$

and one would like to show this lifts to a well-defined map

$$i_x: X_p(\phi)\to\mathscr{S}_{\mathrm{K}}(G,X)(\overline{\mathbb{F}}_p)$$

satisfying good properties. The image of the map will then be the $p$-power part of the isogeny class for $x$. This is carried out in two steps. One can show that $X_p(\phi)$ has a geometric structure as a closed subscheme of the Witt vector affine flag variety of [Zhu17] and [BS17]. In particular there is a notion of connected components for the $X_p(\phi)$. The two steps are as follows.

(1) Show that if $i_x$ is defined at a point of $X_p(\phi)$ then it is defined on the whole connected component containing the point.

(2) Show that every connected component of $X_p(\phi)$ contains a point at which $i_x$ is well-defined by lifting isogenies to characteristic 0.

For part 1), one uses an argument involving deformations of $p$-divisible groups. The analogous argument in [Kis17] uses Grothendieck-Messing theory. In our context this is not possible since the test rings one needs to deform to are no longer smooth. Hence we use a new argument using Zink's theory of displays.

To carry out 2), an essential part is to get a description of (or at least a bound on) the connected components of $X_p(\phi)$. Such a bound is obtained in [HZ]. The bound obtained there is somewhat more complicated than the description for the case of hyperspecial level. This necessitates an improvement in the argument for lifting isogenies to characteristic 0. The main new innovation here is that one can move about through different Levi subgroups of $G$ using characteristic 0 isogenies.

Note that the bound obtained in [HZ] is only good enough to carry out the argument for groups which are residually split at $p$. More generally, we will describe how one can get rid of the residually split condition in the statement of Theorem 1.1 if one assumes a natural conjecture (Conjecture 5.4) on the set of connected components of affine Deligne–Lusztig varieties. Indeed, the bound on the connected components obtained in [HZ] can thought of as a reasonable substitute for this conjecture in the residually split case. However, without assuming $G_{\mathbb{Q}_p}$ residually split, part (1) of the argument goes through unconditionally. This already allows us to deduce the following interesting corollary.

By construction of $\mathscr{S}_{\mathrm{K}_p}(G,X)$, we have a well-defined map

$$\delta : \mathscr{S}_{\mathrm{K}_p}(G,X)(\overline{\mathbb{F}}_p) \to B(G,\mu)$$

which induces the so-called Newton stratification on the special fiber of $\mathscr{S}_{\mathrm{K}_p}(G,X)$. Here $\mu$ is the inverse of the Hodge cocharacter and $B(G,\mu) \subset B(G)$ consists of the set of neutral acceptable $\sigma$-conjugacy classes as in [RV14]; it is the group-theoretic analogue of the set of isomorphism classes of isocrystals satisfying Mazur's inequality. In the case $G = GSp$, the integral model is a moduli space for polarized abelian varieties and this map sends an abelian variety to the isomorphism class of the associated isocrystal. The following result can then be thought of as a generalization of the classical Manin's problem, which asks whether a $p$-divisible with Newton slopes between 0 and 1 symmetric about $\frac{1}{2}$ arises from an abelian variety up to isogeny.

**Theorem 1.2.** *Let $(G,X)$ and $\mathrm{K}_p$ satisfy the assumptions $(*)$. Then $\delta$ is surjective.*

This is proved by verifying some of the He–Rapoport axioms for integral models of Shimura varieties in [HR17]. We cannot yet verify all of the axioms. However, we are able to reduce the verification of the axioms in full to Conjecture 5.4; for the application the non-emptiness of Newton strata, this is not needed.

In recent work [KMPS], the authors have shown surjectivity of this map for groups which are quasi-split at $p$ using a different method. There is an obstruction to their technique working for certain non quasi-split groups. In contrast, our proof works for non quasi-split groups. For this it is essential to be able to work at Iwahori level. The key part is to prove non-emptiness of the minimal Kottwitz–Rapoport stratum at Iwahori level; this shows the surjectivity at Iwahori level. This allows one to deduce the surjectivity statement for all parahoric levels by using suitable comparision maps between models with different levels. However, one major input to the proof is the non-emptiness of the basic locus, which is proved in [KMPS].

Let us give a brief outline of the paper. In section 2 we recall some preliminaries on Bruhat–Tits buildings and Iwahori Weyl groups associated to a $p$-adic group. In section 3 we recall the construction of the local models of Shimura varietes in [PZ13] and prove certain results about their embeddings into Grassmannians. In section 4 we recall the construction in [KP] of the universal $p$-divisible group over the completion of an $\overline{\mathbb{F}}_p$-point of the Shimura variety. We construct in Proposition 4.8 a specific lifting which will be needed in the lifting isogenies argument. Section 5 is the technical heart of the paper. We recall the notion of affine Deligne–Lusztig varieties and state a conjecture (Conjecture 5.4) on their connected components. We then recall the bound on the connected components of affine Deligne-Lusztig varieties obtained in [HZ] and show that for the basic case, or when Conjecture 5.4 holds, enough isogenies lift to characteristic 0. In section 6 we put the results together to deduce the existence of the required map from $X_p(\phi)$ into the integral model when the level $\mathrm{K}_p$ is Iwahori; this is Proposition 6.5. This allows us to deduce part (i) of Theorem 1.1 for the case of Iwahori level at $p$; this is the most pertinent case. In section 7 we deduce the existence of good maps between Shimura varieties of different level which allows us to use the case of Iwahori level to deduce the result for other parahorics. This also verifies one of the He–Rapoport axioms for these integral models. The rest of the axioms are verified in section 8 which allows us to deduce the non-emptiness of Newton strata. Finally in section 9 we prove part (ii) of the main theorem.

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## 2. Preliminaries

2.1. Let $p>2$ be a prime. Let $F$ be a $p$-adic field with ring of integers $\mathcal{O}_F$ and residue field $\mathbb{F}_q$. Let $L$ be the completion of the maximal unramified extension $F^{\mathrm{ur}}$ of $F$ and $\mathcal{O}_L$ its ring of integers. Fix an algebraic closure $\overline{F}$ of $F$ and let $\Gamma := \mathrm{Gal}(\overline{F}/F)$. We write $I$ for the absolute Galois group of $L$ which can be identified with the inertia subgroup $\mathrm{Gal}(\overline{F}/F^{\mathrm{ur}})$ of $\Gamma$. We let $\sigma \in \mathrm{Gal}(F^{\mathrm{ur}}/F)$ denote the Frobenius automorphism which extends by continuity to an automorphism of $L$.

Let $G$ be a connected reductive group over $F$. We assume $G$ splits over a tamely ramified extension of $F$. Let $\mathcal{B}(G,F)$ be the (extended) Bruhat–Tits building of $G(F)$. For any $x\in\mathcal{B}(G,F)$, there is a smooth affine group scheme $\mathcal{G}_x$ over $\mathcal{O}_F$ such that $\mathcal{G}_x(\mathcal{O}_F)$ can be identified with the stabilizer of $x$ in $G(F)$. The connected component $\mathcal{G}_x^\circ$ of $\mathcal{G}_x$ is the parahoric group scheme associated to $x$. We can also consider the corresponding objects over $L$. Then for $x\in\mathcal{B}(G,L)$, we have $\mathcal{G}_x^\circ(\mathcal{O}_L)=\mathcal{G}_x(\mathcal{O}_L)\cap\ker\tilde{\kappa}_G$ where

$$\tilde{\kappa}_G : G(L)\to\pi_1(G)_I$$

is the Kottwitz homomorphism, cf. [HR08, Prop. 3 and Remarks 4 and 11]. Thus if $x\in\mathcal{B}(G,F)$, $\mathcal{G}_x^\circ(\mathcal{O}_F)=\mathcal{G}_x(\mathcal{O}_F)\cap\ker\tilde{\kappa}_G$. We say a parahoric subgroup $\mathcal{G}_x^\circ$ is *connected* if $\mathcal{G}_x^\circ=\mathcal{G}_x$. If $G$ is an unramified group (i.e. $G$ is quasi-split and split over an unramified extension of $F$), then every parahoric subgroup is connected; see [KP, 4.2.14 b)]. For ramified groups there does not seem to be a nice characterization of which parahorics are connected. For some examples of cases when connected parahorics arise in the study of Shimura varieties we refer to [PR09, 1.b.3] for the case of ramified unitary groups and [PR05] for cases of groups associated with restrictions of scalars.

Let $S\subset G$ be a maximal $L$-split torus defined over $F$ and $T$ its centralizer. Since $G$ is quasi-split over $L$ by Steinberg's theorem, $T$ is a maximal torus of $G$. Let $\mathfrak{a}$ denote a $\sigma$-invariant alcove in the apartment $V$ associated to $S$ over $L$. The relative Weyl group $W_0$ and the Iwahori Weyl group are defined as

$$W_0=N(L)/T(L)\quad W=N(L)/\mathcal{T}_0(\mathcal{O}_L)$$

where $N$ is the normalizer of $T$ and $\mathcal{T}_0$ is the connected Néron model for $T$. These are related by an exact sequence

$$0 \to X_*(T)_I \to W \to W_0 \to 0. \tag{2.1.1}$$

For an element $\lambda \in X_*(T)_I$ we write $t_\lambda$ for the corresponding element in $W$; such elements will be called translation elements.

Let $\mathbb{S}$ denote the set of simple reflections in the walls of $\mathfrak{a}$. We let $W_a$ denote the affine Weyl group, it is the subgroup of $W$ generated by the reflections in $\mathbb{S}$. Then $W_a$ fits into an exact sequence

$$0 \to X_*(T_{\mathrm{sc}})_I \to W_a \to W_0 \to 0$$

where $T_{\mathrm{sc}}$ is the preimage of $T$ in the simply connected cover of the derived group of $G$. The Iwahori Weyl group and affine Weyl group are related via the following exact sequence:

$$0 \to W_a \to W \to \pi_1(G)_I \to 0.$$

The choice of $\mathfrak{a}$ induces a splitting of this exact sequence and $\pi_1(G)_I$ can be identified with the subgroup $\Omega \subset W$ consisting of length 0 elements. $W_a$ has the structure of a Coxeter group and hence a notion of length and Bruhat order which extends to $W$ in the natural way using the splitting above. As in [HR08], there is a reduced root system $\Sigma$ such that

$$W_a \cong Q^\vee(\Sigma) \ltimes W(\Sigma)$$

where $Q^\vee(\Sigma)$ is the coroot lattice of $\Sigma$ and $W(\Sigma)$ is its Weyl group. The roots in $\Sigma$ are proportional to the roots in the relative root system of $G$ over $L$, however the root systems themselves may not be proportional. As explained in [HR17], this induces a pairing $\langle\ ,\ \rangle$ between $X_*(T)_I \otimes \mathbb{R}$ and the root lattice of $\Sigma$.

2.2. Now let $\{\mu\}$ be a geometric conjugacy class of homomorphisms of $\mathbb{G}_m$ into $G$. Let $\underline{\mu}$ denote the image in $X_*(T)_I$ of a dominant (with respect to some choice of Borel defined over $L$) representative of $\mu \in X_*(T)$ of $\{\mu\}$. The $\mu$-admissible set (cf. [Rap05, §3]) is defined to be

$$\mathrm{Adm}(\{\mu\}) = \{w \in W | w \leq t_{x(\underline{\mu})} \text{ for some } x \in W_0\}.$$

Note that the $\mu$-admissible set has a unique minimal element denoted $\tau_{\{\mu\}}$; it is the unique element of $\mathrm{Adm}(\{\mu\}) \cap \Omega$.

Now let $K \subset \mathbb{S}$ and let $W_K$ denote the subgroup of $W$ generated by $K$. We write $W^K$ (resp. ${}^K W$) for the set of minimal length elements of the cosets $W/W_K$ (resp. $W_K\backslash W$). If $K' \subset \mathbb{S}$ we write ${}^{K'}W^K$ for the set of minimal length elements in the double cosets $W_{K'}\backslash W/W_K$. If $K \subset \mathbb{S}$ is $\sigma$-stable and $W_K$ is finite, then the fixed points of $K$ determines a parahoric subgroup $\mathcal{G}$ which is defined over $\mathcal{O}_F$. We set $\mathrm{Adm}_K(\{\mu\})$ to be the image of $\mathrm{Adm}(\{\mu\})$ in $W_K\backslash W/W_K$. This subset only depends on the parahoric $\mathcal{G}$ and not on the choice of alcove $\mathfrak{a}$. We sometimes write $\mathrm{Adm}_K^G(\{\mu\})$ if we want to specify the group $G$ we are working with.

We have the Iwahori decomposition; for $w \in W$, we write $\dot{w}$ for a lift of $w$ to $N(L)$. Then the map $w \mapsto \dot{w}$ induces a bijection

$$W_K\backslash W/W_K \cong \mathcal{G}(\mathcal{O}_L)\backslash G(L)/\mathcal{G}(\mathcal{O}_L).$$

Finally we recall the definition and some properties of $\sigma$-straight elements. The Frobenius $\sigma$ induces an action on $W$ and $W_a$ which preserves $\mathbb{S}$.

**Definition 2.1.** We say an element $w$ is *$\sigma$-straight* if $nl(w) = l(w\sigma(w)\dots\sigma^{n-1}(w))$ for all $n \in \mathbb{N}$.

If the action of $\sigma$ is trivial, we simply write *straight* for $\sigma$-straight in this context.

For $w \in W$, there exists $m \in \mathbb{Z} \geq 1$ such that $\sigma^m = 1$ and $w\sigma(w)\ldots\sigma^{m-1}(w) = t_\lambda$ for some $\lambda \in X_*(T)_I$. We define $\nu_w \in X_*(T)_I \otimes \mathbb{Q}$ to be $\frac{\lambda}{m}$ and $\overline{\nu}_w$ to be the dominant representative of $\nu_w$ with respect to some choice of Borel $B$ defined over $L$. Then it is known that $w$ is $\sigma$-straight if and only if $\langle \overline{\nu}_w, 2\rho\rangle = l(w)$ where $\rho$ is the half sum of positive roots in $\Sigma$.

For $w, w' \in W$ and $s \in \mathbb{S}$ we write $w \approx_s w'$ if $w' = sw\sigma(s)$ and $l(w) = l(sw\sigma(s))$. We write $w \approx w'$ if there exists a sequence $w = w_1, \ldots, w_n \in W$, $s_1, \ldots, s_{n-1} \in \mathbb{S}$ and $\tau \in \Omega$ such that $w_i \approx_{s_i} w_{i+1}$ for all $i$ and we have $\tau^{-1}w_n\sigma(\tau) = w'$. The following result is [HN14, Theorem 3.9].

**Theorem 2.2.** *Let $w, w' \in W$ be $\sigma$-straight elements such that there exists $x \in W$ with $x^{-1}w\sigma(x) = w'$. Then $w \approx w'$.*

*Remark* 2.3. By [He14, Theorem 3.7], for $w, w'$ $\sigma$-straight, the condition that there exists $x \in W$ such that $x^{-1}w\sigma(x)$ is equivalent to the existence of $g \in G(L)$ such that $g^{-1}\dot{w}\sigma(g) = \dot{w}'$.

We will also need the following property of the Iwahori double coset corresponding to straight elements.

**Theorem 2.4** ([He14] Proposition 4.5)**.** *Let $w$ be $\sigma$-straight and $\mathcal{I}$ the Iwahori subgroup correspondig to $\mathfrak{a}$. Then for every $g \in \mathcal{I}(\mathcal{O}_L)\dot{w}\mathcal{I}(\mathcal{O}_L)$ there exists $i \in \mathcal{I}(\mathcal{O}_L)$ such that $i^{-1}g\sigma(i) = \dot{w}$.*

## 3. Local models of Shimura varieties

3.1. In this section we recall the construction of the local models of Shimura varieties and prove certain results concerning their embeddings into Grassmannians. Let $F$ denote a finite unramified extension of $\mathbb{Q}_p$ and $L/F$ the completion of the maximal unramified extension of $F$. We write $k$ for the residue field of $\mathcal{O}_L$.

We start with a triple $(G, \mathcal{G}, \{\mu\})$ where:

- $G$ is a connected reductive group over $F$ which splits over a tamely ramified extension of $F$.
- $\mathcal{G}$ is a connected parahoric group scheme associated to a point $x \in \mathcal{B}(G, F)$
- $\{\mu\}$ is a conjugacy class of minuscule geometric cocharacters of $G$.

We assume $\mathcal{G}$ is the parahoric group scheme associated to a subset $K \subset \mathbb{S}$.

Let $E$ be the field of definition of the conjugacy class $\{\mu\}$. In [PZ13] there is a construction of a reductive group scheme $\underline{G}$ over $\mathcal{O}_F[u^\pm] := \mathcal{O}_F[u, u^{-1}]$ which specializes to $G$ under the map $\mathcal{O}_F[u^\pm] \to F$ given by $u \to p$. There is also the construction of a smooth affine group scheme $\underline{\mathcal{G}}$ over $\mathcal{O}_F[u]$ which extends $\underline{G}$ and which specialises to $\mathcal{G}$ under the map $\mathcal{O}_F[u] \to \mathcal{O}_F$ given by $u \to p$. Moreover the specialization of $\underline{\mathcal{G}}_{k[[t]]}$ of $\underline{\mathcal{G}}$ under the map $\mathcal{O}_F[u] \to k[[t]]$ given by $u \mapsto t$ is a parahoric subgroup of $\underline{G}_{k((t))} := \underline{G} \otimes_{\mathcal{O}_F[u^\pm]} k((t))$.

Using these groups, there is a construction of the global affine Grassmanian $Gr_{\mathcal{G},X}$ over $X :=$ Spec $(\mathcal{O}_F[u])$ which, under the base change $\mathcal{O}_F[u] \to F$ given by $u \mapsto p$, can be identified with the affine Grassmanian $Gr_{G,F}$ for $G$. Recall $Gr_{G,F}$ is the ind-scheme which represents the fpqc sheaf associated to the functor on $F$-algebras $R \mapsto G(R((t)))/G(R[[t]])$ (the identification is given by $t = u - p$).

Let $P_{\mu^{-1}}$ be the parabolic corresponding to $\mu^{-1}$ (we use the convention that the parabolic $P_\nu$ defined by a cocharacter $\nu$ has Lie algebra consisting of the subspace of the Lie algebra of $G$ where $\nu$ acts by weights $\geq 0$). The homogeneous space $G_{\overline{\mathbb{Q}}_p}/P_{\mu^{-1}}$ has a canonical model $X_\mu$ defined over $E$. We may consider $\mu$ as a $\overline{\mathbb{Q}}_p((t))$-point $\mu(t)$ of $G$ which gives a $\overline{\mathbb{Q}}_p$ point of $Gr_{G,F}$. As $\mu$ is minuscule, the action of $G(\overline{\mathbb{Q}}_p[[t]])$ on $\mu(t)$ factors through $G(\overline{\mathbb{Q}}_p[[t]]) \to G(\overline{\mathbb{Q}}_p)$ and the image of the stabilizer of $\mu(t)$ in $G(\overline{\mathbb{Q}}_p)$ is equal to $P_{\mu^{-1}}$. Thus the $G(\overline{\mathbb{Q}}_p[[t]])$-orbit of $\mu(t)$ in $Gr_{G,F}$ can be $G_E$-equivariantly identified with $X_\mu$.

**Definition 3.1.** The local model $M^{\text{loc}}_{\mathcal{G},\mu}$ is defined to be the Zariski closure of $X_\mu$ in $Gr_{\mathcal{G},X}\times_X \text{Spec } \mathcal{O}_E$, where the specialization map $X \to \text{Spec } \mathcal{O}_E$ is given by $u \mapsto p$

We will usually write $M^{\text{loc}}_{\mathcal{G}}$ for $M^{\text{loc}}_{\mathcal{G},\mu}$ when it is clear what the cocharacter $\mu$ is. By its construction $M^{\text{loc}}_{\mathcal{G}}$ is a projective scheme over $\mathcal{O}_E$ admitting an action of $\mathcal{G}\otimes_{\mathcal{O}_F}\mathcal{O}_E$. The following is [PZ13, Theorem 8.1].

**Theorem 3.2.** *Suppose $p$ does not divide the order of $\pi_1(G_{der})$. Then the scheme $M^{\text{loc}}_{\mathcal{G}}$ is normal, the geometric special fibre is reduced and admits a stratification by locally closed smooth strata; the closure of each stratum is normal and Cohen–Macaulay.*

This theorem is proved by identifying the geometric special fiber with an explicit subscheme of the partial affine flag variety $\mathcal{FL}_{\underline{\mathcal{G}}_{k[[t]]}}$ for $\underline{\mathcal{G}}_{k[[t]]}$. This is an ind-scheme which represents the fpqc sheaf associated to the functor $R \mapsto \underline{G}_{k((t))}(R((t)))/\underline{\mathcal{G}}_{k[[t]]}(R[[t]])$ for a $k$-algebra $R$. We have an identification

$$\mathcal{FL}_{\underline{\mathcal{G}}_{k[[t]]}}(k) \cong \underline{G}_{k((t))}(k((t)))/\underline{\mathcal{G}}_{k[[t]]}(k[[t]]).$$

By [PZ13, §3.a.1], there is an identification of Iwahori Weyl groups for $G$ and $\underline{G}_{k((t))}$. Thus for $w \in W_K\backslash W/W_K$ we obtain a point $\dot{w}_{k[[t]]} \in \mathcal{FL}_{\underline{\mathcal{G}}_{k[[t]]}}$ corresponding to the image in $\mathcal{FL}_{\underline{\mathcal{G}}_{k[[t]]}}$ of a lift of $w$ to $\underline{G}_{k((t))}(k((t)))$. We let $C_w$ denote the $\underline{\mathcal{G}}_{k[[t]]}$ orbit of $\dot{w}_{k[[t]]}$ in $\mathcal{FL}_{\underline{\mathcal{G}}_{k[[t]]}}$ and $S_w$ its closure, both of which are independent of the lift $\dot{w}$. $C_w$ and $S_w$ are respectively known as the Schubert cell and Schubert variety corresponding to $w$. Then by [PZ13, Theorem 8.3] there is an identification

$$M^{\text{loc}}_{\mathcal{G}}\otimes_{\mathcal{O}_E} k \cong \bigcup_{w\in \text{Adm}_K(\{\mu\})} C_w.$$

**Example 3.3.** Let $G = GL_n$ and let $\mu$ be the cocharacter $a \mapsto \text{diag}(a^{(r)}, 1^{(n-r)})$. Let $e_1,\dots,e_n$ be the standard basis for $F^n$. For a sequence of integers $0 \le m_0 < \ldots < m_k \le n-1$, let $\mathcal{GL}$ be the parahoric subgroup of $GL_n(F)$ stabilizing the lattice chain

$$\Lambda_{m_0} \supset \Lambda_{m_1} \supset \cdots \supset \Lambda_{m_k}$$

where

$$\Lambda_{m_i} := \text{span}\langle pe_1,\dots,pe_{m_i}, e_{m_i+1},\dots,e_n\rangle.$$

The local model in this case agrees with that considered in [RZ96], as mentioned in [PZ13, §6.b.1]. In this case there is the following description. Given an $\mathcal{O}_F$-scheme $\mathcal{S}$, we let $\mathbf{M}^{\text{loc}}_{\mathcal{GL}}(\mathcal{S})$ denote the set of isomorphism classes of commutative diagrams:

$$\begin{tikzcd}
\Lambda_{m_0,\mathcal{S}} & \Lambda_{m_1,\mathcal{S}} \arrow[l] & \ldots \arrow[l] & \Lambda_{m_k,\mathcal{S}} \arrow[l] \\
\mathcal{F}_{m_0} \arrow[u] & \mathcal{F}_{m_1} \arrow[l] \arrow[u] & \ldots \arrow[l] & \mathcal{F}_{m_k} \arrow[l] \arrow[u]
\end{tikzcd}$$

where $\Lambda_{m_i,\mathcal{S}} := \Lambda_{m_i}\otimes_{\mathcal{O}_F}\mathcal{O}_{\mathcal{S}}$ and $\mathcal{F}_{m_i}$ is a locally free $\mathcal{O}_{\mathcal{S}}$-module of rank $r$ and $\mathcal{F}_{m_i}\to\Lambda_{m_i,\mathcal{S}}$ is an inclusion which locally on $\mathcal{S}$ is a direct summand of $\Lambda_{m_i,\mathcal{S}}$. Let us explain how this description is related to the $M^{\text{loc}}_{\mathcal{GL}}$ considered by [PZ13] which was described in the last section.

We use the following convention for a filtration defined by a cocharacter. Let $V$ be a finite dimensional vector space over $F$ or a finite free $\mathcal{O}_F$-module. Then a cocharacter $\mu : \mathbb{G}_m \to GL(V)$ induces a grading $V = \bigoplus_{i\in\mathbb{Z}} V_i$ where $\mathbb{G}_m$ acts on $V_i$ by the character $z \mapsto z^i$. It induces the filtration on $V$

given by $F^i := \bigoplus_{j\geq i} V_j$. The stabilizer of this filtration is given by the parabolic subgroup $P_\mu \subset GL_n$ associated to $\mu$.

From the description of $\mathbf{M}^{\mathrm{loc}}_{\mathcal{GL}}$ above, its generic fiber can be identified with the homogeneous space $GL_n/P_\mu$. Indeed when $p$ is inverted, all the $\Lambda_{m_i}$ coincide and the choice of $\mathcal{F}_{m_0} \subset \Lambda_{m_0,F}$ determines the other $\mathcal{F}_{m_i}$.

This implies that in the construction of the local model $M^{\mathrm{loc}}_{\mathcal{GL}}$, we must therefore take the defining cocharacter to be $\mu^{-1}$. In this case the stabilizer of the point of $GL_n(\overline{\mathbb{Q}}_p((t)))/GL_n(\overline{\mathbb{Q}}_p[[t]])$ corresponding to $\mu^{-1}$ is $P_\mu$, hence the generic fiber of $M^{\mathrm{loc}}_{\mathcal{G},\mu^{-1}}$ is identified with $GL_n/P_\mu$ as above.

The identification of $\mathbf{M}^{\mathrm{loc}}_{\mathcal{GL}}(k)$ and $M^{\mathrm{loc}}_{\mathcal{GL}}(k)$ is given as follows. The group $\underline{\mathcal{GL}}_{k[[t]]}$ is the parahoric subgroup in $GL_n(k((t))$ stabilizing the lattice chain

$$\Lambda'_{m_0} \supset \cdots \supset \Lambda'_{m_k}$$

in $k((t))^n$ where $\Lambda'_{m_i} := \mathrm{span}\langle te_1, \ldots, te_{m_i}, e_{m_i+1}, \ldots, e_n\rangle$. We may identify the special fibers of $\Lambda_{m_i}$ and $\Lambda'_{m_i}$ by the choice of standard basis. Given a point of $\mathbf{M}^{\mathrm{loc}}_{\mathcal{GL}}(k)$, we obtain a subspace $F^{m_i} \subset \Lambda'_{m_i} \otimes k$ via the above identification, where $F^{m_i}$ is of dimension $r$. The preimages $\mathcal{L}_{m_i}$ of $F^{m_i}$ in $\Lambda'_{m_i}$ corresponds to a lattice chain of the same type as $\Lambda'_{m_0} \supset \cdots \supset \Lambda'_{m_k}$. As in [Gör01, 3.1.3], there exists an element $g \in GL_n(k((t)))/\underline{\mathcal{GL}}(k[[t]])$ such that $\mathcal{L}_{m_i} = g\Lambda'_{m_i}$ for all $m_i$. The corresponding point of $M^{\mathrm{loc}}_{\mathcal{GL}}(k)$ is given by $xt^{-1}$ (here we consider $k((t))^\times \subset GL_n(k((t)))$ via the scalar matrices).

3.2. We recall the construction of certain lattice chains of $\mathcal{O}_F[u]$-modules from [PZ13, §4.b.1]. Let $W = \mathcal{O}_F[u]^n$ and $\overline{W} = W \otimes_{\mathcal{O}_F[u], u\mapsto 0} \mathcal{O}_F \cong \mathcal{O}_F^n$. Write $\overline{W} = \oplus_{i=0}^r V_i$ and let $U_i = \oplus_{j\geq i} V_j$. For $i = 0, \ldots, r-1$ we let $W_i \subset W$ denote the inverse image of $U_i$ under $W \to \overline{W}$; the sequence $W_i$ satisfies

$$uW \subset W_{r-1} \subset \ldots \subset W_0 = W.$$

We extend the sequence to $\mathbb{Z}$ by letting $W_{i+kr} = u^k W_i$ and we write $W_\bullet$ for the resulting chain indexed by $\mathbb{Z}$.

Now let $\rho : G \to GL_{2n}$ be a closed group scheme immersion over $F$ such that $\rho \circ \mu$ is in the conjugacy class of the minuscule cocharacter $a \mapsto \mathrm{diag}(1^{(n)}, (a^{-1})^{(n)})$. Suppose also that $\rho$ satisfies the following conditions:

• $\rho$ extends to a closed group scheme immersion $\underline{\mathcal{G}} \to GL(W_\bullet)$, where $W_\bullet$ is a lattice chain in $\mathcal{O}_F[u]^{2n}$ as in [PZ13, 4.b.1] and which was recalled above.

• The Zariski closure of $\underline{G}_{k((u))}$ in $GL(W_\bullet \otimes_{\mathcal{O}_F[u]} k[[u]])$ is a smooth group scheme $\mathcal{P}'$ whose identity component can be identified with $\underline{\mathcal{G}}_{k[[u]]}$.

Then it is shown in [PZ13, Proposition 7.1] that extending torsors along $\underline{\mathcal{G}} \to GL(W_\bullet)$ induces a closed immersion:

$$\iota : M^{\mathrm{loc}}_{\mathcal{G},\mu} \to M^{\mathrm{loc}}_{\mathcal{GL},\rho\circ\mu} \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

where $\mathcal{GL}$ is the parahoric subgroup of $GL_{2n}$ corresponding to the lattice chain $W_\bullet \otimes_{\mathbb{Z}_p[u]} \mathcal{O}_F$. We will need a more explicit description of this map on the level of $k$-points which we now explain.

3.3. For the rest of this section we assume $G$ is quasi-split. By Steinberg's theorem, this can be achieved upon making a finite unramified base extension $F'/F$. For later applications it will suffice to make such a base change which is why we are allowed to make this assumption. We remind the reader that $S$ is a maximal $L$-split torus of $G$ defined over $F$.

Let $\rho : G \to GSp(V)$ be a local Hodge embedding in the sense of [KP, §2.3], in particular we assume $G$ contains the scalars in $GL(V)$. As explained in [KP, Proposition 2.3.7], there is an embedding $\underline{\mathcal{G}} \to GL(W_\bullet)$ satisfying the above conditions and hence an embedding of local models. Base changing to $\mathcal{O}_L[u^{\pm}]$ we obtain an embedding $\underline{G} \to GL(\underline{\Lambda})$ where $\underline{\Lambda}$ is a free module over $\mathcal{O}_L[u^{\pm}]$ and is the common generic fibre of $W_\bullet$. The fiber over $L$ of this embedding is given by $\rho$ and we denote the fibers over $\kappa((u))$, where $\kappa = k, L$, by $\rho_{\kappa((u))}$. As shown in [KP, §1.2], these maps induce embeddings of buildings

$$\mathcal{B}(G, L) \to \mathcal{B}(GL_{2n}, L) \tag{3.3.1}$$

$$\mathcal{B}(G, \kappa((u))) \to \mathcal{B}(GL_{2n}, \kappa((u))). \tag{3.3.2}$$

These embeddings satisfy the following property: There is a choice of a maximal $\mathcal{O}_L[u^{\pm}]$-split torus $\underline{S}$ of $\underline{G}$ and a choice of basis $\underline{b}$ for $\underline{\Lambda}$ such that the embeddings of buildings (3.3.1) and (3.3.2) induce embeddings of the corresponding apartments

$$\mathcal{A}(G, S, L) \to \mathcal{A}(GL_{2n}, S', L) \tag{3.3.3}$$

$$\mathcal{A}(\underline{G}_{\kappa((u))}, \underline{S}_{\kappa((u))}, \kappa((u))) \to \mathcal{A}(GL_{2n}, S', \kappa((u))); \tag{3.3.4}$$

here the choice of $\underline{b}$ determines an isomorphism of $GL(\underline{\Lambda})$ with $GL_{2n}$ and $S'$ is the diagonal torus of $GL_{2n}$. Indeed taking $\underline{S}$ to be the torus associated to $S$ by the construction in [PZ13, 2.c] and taking the choice of $\underline{b}$ as in [PZ13, Proof of Proposition 2.3.7, Step 1], we see that the embedding $\underline{G} \to GL(\underline{\Lambda})$ takes $\underline{S}$ to the standard torus. By [PZ13, 3.a.2], the base change $\underline{S}_{\kappa((u))}$ is a maximal torus of $\underline{G}_{\kappa((u))}$. This gives the embedding of apartments by the construction of the corresponding embedding of buildings.

The choice of $\underline{S}$ and $\underline{b}$ also give identifications

$$\mathcal{A}(G, S, L) \cong \mathcal{A}(\underline{G}_{\kappa((u))}, \underline{S}_{\kappa((u))}, \kappa((u))) \tag{3.3.5}$$

and

$$\mathcal{A}(GL_{2n}, S', L) \cong \mathcal{A}(GL_{2n}, S', \kappa((u))). \tag{3.3.6}$$

Moreover there is an identification of Iwahori Weyl groups for the different groups over the fields $L$ and $\kappa((u))$ and the identification of apartments is compatible with the actions of these groups, see [PZ13, §3.a.1]. The maps (3.3.1) and (3.3.2) are compatible with these identifications.

Let $x \in \mathcal{A}(G, S, L)$ be a point corresponding to the parahoric and let

$$x_{\kappa((u))} \in \mathcal{A}(\underline{G}_{\kappa((u))}, \underline{S}_{\kappa((u))}, \kappa((u)))$$

be the image of $x$ under the identification (3.3.5). Then the group scheme $\underline{\mathcal{G}}_{\kappa[[u]]}$ is identified with the parahoric group scheme corresponding to $x_{\kappa((u))}$. The images of $x$ (resp. $x_{\kappa((u))}$) under the embeddings (3.3.1) and (3.3.2) give points $y$ (resp. $y_{\kappa((u))}$) whose corresponding parahoric $\mathcal{GL}$ (resp. $\underline{\mathcal{GL}}_{\kappa[[u]]}$) is the stabilizer of the base change of $W_\bullet$ to $L$ (resp. $\kappa((u))$).

3.4. The image of the alcove $\mathfrak{a}$ under the embedding of apartments $\mathcal{A}(G, S, L) \to \mathcal{A}(GL_{2n}, S', L)$ is contained in (the closure of) an alcove in the apartment for $GL_{2n}$. We fix such an alcove and let $\mathbb{S}'$ denote the corresponding set of simple reflections. Then $\mathcal{GL}$ corresponds to a subset $K' \subset \mathbb{S}'$ and we may apply the constructions in §2 to $\mathcal{GL}$ to obtain $\mathrm{Adm}_{K'}^{GL_{2n}}(\{\mu\}_{GL_{2n}})$, this is a subset of $W'_{K'} \backslash W' / W'_{K'}$ where $W'$ denotes the Iwahori subgroup for $GL_n$ and $\{\mu\}_{GL_{2n}}$ denotes the $GL_{2n}$-conjugacy class of cocharacters induced by $\{\mu\}$.

We identify

$$M_{\mathcal{G}}^{\mathrm{loc}}(k) \subset \underline{G}(k((u)))/\underline{\mathcal{G}}_{k[[t]]}(k[[u]])$$

with the union over the Schubert varieties $S_w$ where $w \in \mathrm{Adm}_K(\{\mu\})$. Similarly

$$M^{\mathrm{loc}}_{\mathcal{GL}}(k) \subset GL_{2n}(k((u)))/\mathcal{GL}(k[[u]])$$

is the union of the Schubert varieties $S^{GL_{2n}}_{w'}$ in $GL_{2n}$ for $w' \in \mathrm{Adm}^{GL_{2n}}_{K'}(\{\mu\}_{GL_{2n}})$.

On the level of $k$ points the embeddings $M^{\mathrm{loc}}_{\mathcal{G}}(k) \hookrightarrow M^{\mathrm{loc}}_{\mathcal{GL}_{2n}}(k)$ is induced by the map $\underline{G}(k((u))) \to GL_{2n}(k((u)))$. On the other hand, the choice of basis $\underline{b}$ gives an embedding

$$M^{\mathrm{loc}}_{\mathcal{GL}}(k) \subset GL_{2n}(L)/\mathcal{GL}(\mathcal{O}_L). \tag{3.4.1}$$

Indeed the choice of basis gives an identification between the special fibers of the lattice chains $W_\bullet \otimes k[[u]]$ and $W_\bullet \otimes \mathcal{O}_L$. Then as in Example 3.3 a $k$-point of $M^{\mathrm{loc}}_{\mathcal{GL}}$ corresponds to a filtration on each of the $k$ vector spaces $W_\bullet \otimes k$. If $g' \in GL_n(k((u)))/\mathcal{GL}(k[[u]])$ lies in $M^{\mathrm{loc}}_{\mathcal{GL}}(k)$, the filtration is induced by reducing the image of the lattice chain $ug'W_\bullet \otimes k$ modulo $u$. Taking the preimage of this filtration in $W_\bullet \otimes \mathcal{O}_L$, we obtain a lattice chain of type $W_\bullet \otimes \mathcal{O}_L$ which is given by $p^{-1}gW_\bullet \otimes \mathcal{O}_L$ for some $g \in GL_{2n}(L)/GL_{2n}(\mathcal{O}_L)$. The embedding is then given by $g' \mapsto g$. We may thus use (3.4.1) to identify $M^{\mathrm{loc}}_{\mathcal{G}}(k)$ with a subset of $GL_n(L)/GL_n(\mathcal{O}_L)$.

Note that the embedding (3.4.1) identifies $M^{\mathrm{loc}}_{\mathcal{GL}}(k)$ with

$$\bigcup_{w \in \mathrm{Adm}^{GL_{2n}}_{K'}(\{\mu\})} \mathcal{GL}(\mathcal{O}_L)\dot{w}\mathcal{GL}(\mathcal{O}_L)/\mathcal{GL}(\mathcal{O}_L) \tag{3.4.2}$$

where $\dot{w}$ denotes a representative of $w$ in $G(L)$ (see for example, [Hai05, §11]). Then this identification is equivariant for the action of $\mathcal{GL}(k)$, where $\mathcal{GL}(k)$ acts on (3.4.2) by left multiplication. Indeed since $\mu$ is minuscule, left multiplication by $\mathcal{GL}(\mathcal{O}_L)$ factors through $\mathcal{GL}(k)$.

3.5. Recall we have assumed $G \subset GL(V)$ contains the scalars. We let $\lambda : \mathbb{G}_m \to G$ denote the cocharacter giving scalar multiplication.

**Proposition 3.4.** *Let $g \in G(L)$ with*

$$g \in \mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O}_L)$$

*for some $w \in W_K\backslash W/W_K$. Then the image $\overline{\rho}(g)$ of $\rho(g)$ in $GL_{2n}(L)/\mathcal{GL}(\mathcal{O}_L)$ lies in $M^{\mathrm{loc}}_{\mathcal{G}}(k)$ if and if and only if $w \in \mathrm{Adm}_K(\{\mu\})$.*

*Proof.* Let $g_1\dot{w}g_2$ in the Bruhat decomposition $\mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O}_L)$. Since $\mathcal{G}(\mathcal{O}_L)$ maps to $\mathcal{GL}(\mathcal{O}_L)$, we may assume $g = g_1\dot{w}$.

Now $M^{\mathrm{loc}}_{\mathcal{G}}$ is equipped with an action by $\mathcal{G} \times_{\mathcal{O}_F} \mathcal{O}_E$ and $M^{\mathrm{loc}}_{\mathcal{GL}}$ with an action of $\mathcal{GL}$. Over the special fiber this action is identified with the one given by left multiplication by $\mathcal{G}(\mathcal{O}_L)$ on $M^{\mathrm{loc}}_{\mathcal{G}}(k) \subset GL_n(L)/\mathcal{GL}(\mathcal{O}_L)$, which as above, factors through $\mathcal{G}(k)$. Thus, modifying $g$ by $g_1$ on the left, we may assume $g = \dot{w}$.

Since the embedding of apartments (3.1) and (3.2) over $L$ and $k((u))$ is compatible with the identification of apartments

$$\mathcal{A}(G, S, L) \cong \mathcal{A}(\underline{G}_{k((u))}, \underline{S}_{k((u))}, k((u)))$$

respecting the action of Iwahori Weyl groups, we see that $\overline{\rho}(g)$ corresponds to the point $\rho_{k((u))}(\dot{w}_{k[[u]]}) \in GL_{2n}(k((u)))/\mathcal{GL}(k[[u]])$. Thus by the description of $M^{\mathrm{loc}}_{\mathcal{G}}(k)$ above, we see that $\overline{\rho}(g) \in M^{\mathrm{loc}}_{\mathcal{G}}(k)$ if and only if $w$ lies in $\mathrm{Adm}_K(\{\mu\})$. $\square$

**Corollary 3.5.** *Let $g \in \mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O}_L)$ with $w \in \mathrm{Adm}_K(\{\mu\})$, then*

$$\rho(g) \in \mathcal{GL}(\mathcal{O}_L)\dot{w}'\mathcal{GL}(\mathcal{O}_L)$$

*for some $\dot{w}' \in \mathrm{Adm}_K^{GL_{2n}}(\{\mu\}_{GL_{2n}})$.*

*Proof.* This follows from Proposition 3.4 and the description in (3.4.2) of $M^{\mathrm{loc}}_{\mathcal{GL}}(k)$ as a subset of $GL_{2n}(L)/\mathcal{GL}(\mathcal{O}_L)$. □

3.6. As explained in [KP, 2.3.15], we may compose $\rho : G \to GSp(V, \Psi)$ with a diagonal embedding to obtain a new minuscule Hodge embedding $\rho' : GSp(V', \Psi')$ with $\dim V' = 2n'$ such that there is a self-dual lattice $V'_{\mathbb{Z}_p} \subset V'$ and the above embedding of buildings takes $x$ to the hyperspecial point $y \in \mathcal{B}(GL(V'), L)$ corresponding to $V'_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. $V'_{\mathbb{Z}_p}$ is constructed by taking the direct sum of the lattices in the lattice chain corresponding to $\mathcal{GL}$. Then $\rho'$ factors through a diagonal embedding $GL(V) \to GL(V')$. In this case we obtain an embedding of local models:

$$M^{\mathrm{loc}}_{\mathcal{G}} \to Gr(V'_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$$

where $Gr(V'_{\mathbb{Z}_p})$ is the smooth grassmannian parametrizing dimension $n'$ sub-bundles $\mathcal{F} \subset V'_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{S}}$ for any $\mathbb{Z}_p$-scheme $\mathcal{S}$.

Choosing a basis $\underline{b}$ as above, we obtain an embedding

$$M^{\mathrm{loc}}_{\mathcal{G}}(k) \hookrightarrow GL_{2n'}(L)/\mathcal{GL}'(\mathcal{O}_L)$$

where $\mathcal{GL}'$ is the hyperspecial subgroup stabilising $V'_{\mathbb{Z}_p}$. Let $T' \subset GL(V')$ denote a maximal torus whose apartment contains the hyperspecial vertex corresponding to $\mathcal{GL}'$ and such that $\rho$ maps $S$ to $T'$.

**Corollary 3.6.** *Let $g \in \mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O}_L) \subset G(L)$ with $w \in \mathrm{Adm}_K(\{\mu\})$, then*

$$\rho(g) \in \mathcal{GL}'(\mathcal{O}_L)\mu'_{GL}(p)\mathcal{GL}'(\mathcal{O}_L)$$

*where $\mu'_{GL}$ is a representative of $\{\mu\}_{GL(V')}$.*

*Proof.* Under the diagonal embedding $GL(V) \to GL(V')$, we have that $\mathrm{Adm}_{K'}^{GL(V)}(\{\mu\}_{GL(V)})$ maps to $\mathrm{Adm}_{K''}^{GL(V')}(\{\mu\}_{GL(V')})$. This follows by the equality $\mathrm{Adm}(\{\mu\}) = \mathrm{Perm}(\{\mu\})$ for general linear groups, see [HC02]. Since $\mathcal{GL}'$ is hyperspecial, $\mathrm{Adm}_{K''}^{GL(V')}(\{\mu\}_{GL(V')})$ is just the single coset corresponding to $t_\mu$, hence the result followsw from Corollary 3.5. □

## 4. $p$-divisible groups

In this section we review the theory of $\mathfrak{S}$-modules and their applications to deformation theory of $p$-divisible groups equipped with a collection of crystalline tensors. The main result is the construction of a certain deformation of such a $p$-divisible group in Proposition 4.8 which is needed in the arguments of §5.

4.1. We now let $F = \mathbb{Q}_p$ so that $L = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$. For $K/L$ a finite totally ramified extension, let $\Gamma_K$ be the absolute Galois group of $K$. Let $\mathrm{Rep}_{\mathrm{cris}}$ denote the category of crystalline $\Gamma_K$-representations, and $\mathrm{Rep}^{\circ}_{\mathrm{cris}}$ the category of $\Gamma_K$-representations in finite free $\mathbb{Z}_p$-modules which are lattices in some crystalline representation of $\Gamma_K$. For $V$ a crystalline representation of $\Gamma_K$, recall Fontaine's functors $D_{\mathrm{cris}}, D_{\mathrm{dR}}$:

$$D_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \qquad D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}.$$

Fix a uniformizer $\pi$ for $K$ and let $E(u)$ be the Eisenstein polynomial which is the minimal polynomial of $\pi$. Let $\mathfrak{S} = \mathcal{O}_L[[u]]$, we equip this with a lift $\varphi$ of Frobenius given by the usual Frobenius on $\mathcal{O}_L$ and $u \mapsto u^p$. We write $D^\times$ for the scheme Spec $\mathfrak{S}$ with its closed point removed. Let $\mathrm{Mod}^\varphi_{\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a $\varphi$-linear isomorphism:

$$1 \otimes \varphi : \mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[1/E(u)] \to \mathfrak{M}[1/E(u)].$$

Let $\mathrm{BT}^\varphi$ denote the subcategory of $\mathrm{Mod}^\varphi_{\mathfrak{S}}$ consisting of $\mathfrak{M}$, such that $1 \otimes \varphi$ maps $\varphi^*(\mathfrak{M})$ into $\mathfrak{M}$ and whose cokernel is killed by $E(u)$.

Given $\mathfrak{M} \in \mathrm{Mod}^\varphi_{\mathfrak{S}}$ we equip $\varphi^*(\mathfrak{M})$ with the filtration

$$\mathrm{Fil}^i \varphi^*(\mathfrak{M}) = (1 \otimes \varphi)^{-1}(E(u)^i \mathfrak{M}) \cap \varphi^*(\mathfrak{M}).$$

Let $\mathcal{O}_{\mathcal{E}}$ denote the $p$-adic completion of $\mathfrak{S}_{(p)}$; it is a discrete valuation ring with uniformizer $p$ and residue field $k((u))$ and let $\mathcal{E}$ denote its fraction field. We equip $\mathcal{O}_{\mathcal{E}}$ with the unique Frobenius $\varphi$ which extends that on $\mathfrak{S}$, and let $\mathrm{Mod}^\varphi_{\mathcal{O}_{\mathcal{E}}}$ denote the category of finite free $\mathcal{O}_{\mathcal{E}}$-modules $M$ equipped with a Frobenius semilinear isomorphism

$$1 \otimes \varphi : \varphi^*(M) \to M$$

.

There is a functor $\mathrm{Mod}^\varphi_{\mathfrak{S}} \to \mathrm{Mod}^\varphi_{\mathcal{O}_{\mathcal{E}}}$ given by

$$\mathfrak{M} \mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}.$$

with the Frobenius on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ induced by that on $\mathfrak{M}$.

Let $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ denote the $p$-adic completion of a strict Henselization of $\mathcal{O}_{\mathcal{E}}$. The following is contained in [Kis17, Theorem 1.1.2] and [KP, Theorem 3.3.2]:

**Proposition 4.1.** *There is a fully faithful functor*

$$\mathfrak{M} : \mathrm{Rep}^\circ_{\mathrm{cris}} \to \mathrm{Mod}^\varphi_{\mathfrak{S}}$$

*which is compatible with the formation of symmetric and exterior products and is such that* $\Lambda \mapsto \mathfrak{M}(\Lambda)|_{D^\times}$ *is exact. If* $\Lambda$ *is in* $\mathrm{Rep}^\circ_{\mathrm{cris}}$, $V := \Lambda \otimes \mathbb{Q}_p$ *and* $\mathfrak{M} = \mathfrak{M}(\Lambda)$

*(i) There are canonical isomorphisms*

$$D_{\mathrm{cris}}(V) \cong \mathfrak{M}/u\mathfrak{M}[\frac{1}{p}] \textit{ and } D_{\mathrm{dR}}(V) \cong \varphi^*(\mathfrak{M}) \otimes_{\mathfrak{S}} K$$

*the first being compatible with* $\varphi$ *and the second being compatible with filtrations.*

*(ii) There is a canonical isomorphism*

$$\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cong \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}.$$

4.2. For an $R$-module $M$, we let $M^\otimes$ denote the direct sum of all $R$-modules obtained from $M$ by taking duals, tensor products, symmetric and exterior products.

Let $\Lambda \in \mathrm{Rep}^\circ_{\mathrm{cris}}$ and suppose $s_{\alpha,\text{ét}} \in \Lambda^\otimes$ are a collection of $\Gamma_K$-invariant tensors whose stabilizer is a smooth group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ with reductive generic fiber $G$. Since the $s_{\alpha,\text{ét}}$ are $\Gamma_K$-invariant, we obtain a representation

$$\rho : \Gamma_K \to \mathcal{G}(\mathbb{Z}_p).$$

We may think of each $s_{\alpha,\text{ét}}$ as a morphism in $\mathrm{Rep}^\circ_{\mathrm{cris}}$ from the trivial representation $\mathbb{Z}_p$ to $\Lambda^\otimes$. Applying the functor $\mathfrak{M}$ to these morphisms gives us $\varphi$-invariant tensors $\tilde{s}_\alpha \in \mathfrak{M}(\Lambda)^\otimes$.

**Proposition 4.2.** *Suppose that the special fiber of $\mathcal{G}$ is connected and $H^1(\mathcal{G}, D^\times) = 1$. Then there exists an isomorphism.*

$$\Lambda \otimes_{\mathbb{Z}_p} \mathfrak{S} \cong \mathfrak{M}(\Lambda)$$

*taking $s_{\alpha,\text{ét}}$ to $\tilde{s}_\alpha$.*

*Proof.* This is a special case of [KP, 3.3.5], indeed with our assumptions $\mathcal{G} = \mathcal{G}^\circ$. □

4.3. For a $p$-divisible group $\mathscr{G}$ over a scheme where $p$ is locally nilpotent we write $\mathbb{D}(\mathscr{G})$ for its contravariant Dieudonné crystal. For $\mathscr{G}$ a $p$-divisible group over $\mathcal{O}_K$, we let $T_p\mathscr{G}$ be the Tate module of $\mathscr{G}$ and $T_p\mathscr{G}^\vee$ the linear dual of $T_p\mathscr{G}$. We will apply the above to $\Lambda = T_p\mathscr{G}^\vee$.

Let $R$ be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field $k$. We let $W(R)$ denote the Witt vectors of $R$. Recall [Zin01] we have a subring $\widehat{W}(R) = W(k) \oplus \mathbb{W}(\mathfrak{m}) \subset W(R)$, where $\mathbb{W}(\mathfrak{m}) \subset W(R)$ consists of Witt vectors $(w_i)_{i\geq 1}$ with $w_i \in \mathfrak{m}$ and $w_i \to 0$ in the $\mathfrak{m}$-adic topology. Then $\widehat{W}(R)$ is preserved by both the Frobenius $\varphi$ and Verschiebung $V$ on $W(R)$. We have $I_R := V\widehat{W}(R)$ is the kernel of the projection map $\widehat{W}(R) \to R$. Fix a uniformizer $\pi$ of $K$ and write $[\pi] \in \widehat{W}(\mathcal{O}_K)$ for its Teichmuller representative. Recall the following definition from [Zin01].

**Definition 4.3.** A Dieudonné display over $R$ is a tuple $(M, M_1, \Phi, \Phi_1)$ where

(i) $M$ is a free $\widehat{W}(R)$-module.

(ii) $M_1 \subset M$ is a $\widehat{W}(R)$-submodule such that

$$I_R M \subset M_1 \subset M$$

and $M/M_1$ is a projective $R$-module.

(iii) $\Phi : M \to M$ is a $\varphi$-semilinear map.

(iv) $\Phi_1 : M_1 \to M$ is a $\varphi$-semilinear map whose image generates $M$ as a $\widehat{W}(R)$ module and which satisfies

$$\Phi_1(V(w)m) = w\Phi(m), \text{ for } w \in \widehat{W}(R), m \in M.$$

Let $\mathscr{G}$ be a $p$-divisible group over $R$. Then $\mathbb{D}(\mathscr{G})(\widehat{W}(R))$ naturally has the structure of a Dieudonné display, and by the main result of [Zin01] the functor $\mathscr{G} \mapsto \mathbb{D}(\mathscr{G})(\widehat{W}(R))$ is an anti-equivalence of categories between $p$-divisible groups over $R$ and Dieudonné displays over $R$.

4.4. If $\mathscr{G}$ is a $p$-divisible group over $\mathcal{O}_K$, then by [KP, Theorem 3.3.2] there is a canonical isomorphism

$$\mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K)) \cong \mathfrak{M} \otimes_{\mathfrak{S},\varphi} \widehat{W}(\mathcal{O}_K)$$

where $\mathfrak{M} = \mathfrak{M}(T_p\mathscr{G}^\vee)$ and the tensor product is over the map given by composing the map $\mathfrak{S} \to \widehat{W}(\mathcal{O}_K), u \mapsto [\pi]$ with $\varphi$. Moreover the induced map

$$\mathbb{D}(\mathscr{G})(\mathcal{O}_K) \cong \varphi^*(\mathfrak{M}) \otimes_{\mathfrak{S}} \mathcal{O}_K \to D_{\mathrm{dR}}(T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

respects filtrations and we have a canonical identification

$$\mathbb{D}(\mathscr{G}_0)(\mathcal{O}_L) \cong \varphi^*(\mathfrak{M}/u\mathfrak{M})$$

where $\mathscr{G}_0 := \mathscr{G} \otimes_{\mathcal{O}_K} k$.

If $s_{\alpha,\text{ét}} \in T_p\mathscr{G}^{\vee,\otimes}$ are a collection of $\Gamma_K$ invariant tensors, we let

$$s_{\alpha,0} \in D_{\mathrm{cris}}(T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

denote the $\varphi$-invariant tensors corresponding to $s_{\alpha,\text{ét}}$ under the $p$-adic comparison isomorphism. We assume from now on that the stabilizer of $s_{\alpha,\text{ét}}$ is of the form $\mathcal{G}_x$ for $x \in \mathcal{B}(G, \mathbb{Q}_p)$, where $G$ is a tamely ramified reductive group containing no factors of type $E_8$. The following is [KP, 3.3.8]

**Proposition 4.4.** *$s_{\alpha,0} \in \mathbb{D}(\mathscr{G}_0)(\mathcal{O}_L)^{\otimes}$ where we view $\mathbb{D}(\mathscr{G}_0)(\mathcal{O}_L)^{\otimes}$ as an $\mathcal{O}_L$-submodule of the $L$ vector space $D_{\mathrm{cris}}(T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\otimes}$. Moreover the $s_{\alpha,0}$ lift to $\varphi$-invariant tensors $\tilde{s}_\alpha \in \mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K))^{\otimes}$ which map to $\mathrm{Fil}^0\mathbb{D}(\mathscr{G})(\mathcal{O}_K)^{\otimes}$, and there exists an isomorphism:*

$$\mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K)) \cong T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \widehat{W}(\mathcal{O}_K)$$

*taking $\tilde{s}_\alpha$ to $s_{\alpha,\text{ét}}$. In particular, there is an isomorphism*

$$\mathbb{D}(\mathscr{G}_0)(\mathcal{O}_L) \cong T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_L$$

*taking $s_{\alpha,0}$ to $s_{\alpha,\text{ét}}$.*

4.5. Now let $\mathscr{G}_0$ be a $p$-divisible group over $k$ and suppose $(s_{\alpha,0}) \in \mathbb{D}^{\otimes}$, where $\mathbb{D} := \mathbb{D}(\mathscr{G}_0)(\mathcal{O}_L)$, are a collection of $\varphi$-invariant tensors whose image in $\mathbb{D}(\mathscr{G}_0)(k)$ lie in $\mathrm{Fil}^0$. We assume that the stabilizer $\mathcal{G}_{\mathcal{O}_L}$ of the $s_{\alpha,0}$ is a connected Bruhat–Tits parahoric group scheme, i.e. $\mathcal{G}_{\mathcal{O}_L} = \mathcal{G}_x = \mathcal{G}_x^\circ$ for some $x \in \mathcal{B}(G, L)$ as above and also that $G$ contains the scalars.

Let $P \subset GL(\mathbb{D})$ be a parabolic subgroup lifting the parabolic $P_0$ corresponding to the filtration on $\mathbb{D}(\mathscr{G}_0)(k)$. Write $M^{\mathrm{loc}} = GL(\mathbb{D})/P$ and $\mathrm{Spf}A = \widehat{M}^{\mathrm{loc}}$ the completion at the identity. We write $\overline{M}_1$ for the universal filtration on $\mathbb{D} \otimes_{\mathcal{O}_L} A$. Let $K'/L$ be a finite extension and $y : A \to K'$ be a map such that $s_{\alpha,0} \in \mathrm{Fil}^0\mathbb{D} \otimes_{\mathcal{O}_L} K'$ for the filtration induced by $y$ on $\mathbb{D} \otimes_{\mathcal{O}_L} K'$. By [Kis10, Lemma 1.4.5], the filtration corresponding to $y$ is induced by a $G$-valued cocharacter $\mu_y$.

Let $G.y$ be the orbit of $y$ in $M^{\mathrm{loc}} \otimes_{\mathcal{O}_L} K'$ which is defined over the field of definition $E$ of the $G$-conjugacy class of cocharacters $\{\mu_y\}$, and we write $M^{\mathrm{loc}}_{\mathcal{G}_{\mathcal{O}_L}}$ for the closure of this orbit in $M^{\mathrm{loc}}$. By [KP, Proposition 2.3.16], $M^{\mathrm{loc}}_{\mathcal{G}_{\mathcal{O}_L}}$ can be identified with the local model for $\mathcal{G}_{\mathcal{O}_L}$ and the conjugacy class of cocharacters $\{\mu_y^{-1}\}$ considered in §3.

**Definition 4.5.** Let $\mathscr{G}$ be a $p$-divisible group over $\mathcal{O}_K$ whose special fiber is isomorphic to $\mathscr{G}_0$. We say $\mathscr{G}$ is $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted if the tensors $s_{\alpha,0}$ lift to Frobenius invariant tensors $\tilde{s}_\alpha \in \mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K))^{\otimes}$ such that the following two conditions hold:

(1) There is an isomorphism $\mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K)) \cong \mathbb{D} \otimes_{\mathcal{O}_L} \widehat{W}(\mathcal{O}_K)$ taking $\tilde{s}_\alpha$ to $s_{\alpha,0}$.

(2) Under the canonical identification $\mathbb{D}(\mathscr{G})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \cong \mathbb{D} \otimes_{\mathcal{O}_L} K$ coming from [KP, Lemma 3.1.17], the filtration on $\mathbb{D} \otimes_{\mathcal{O}_L} K$ is induced by a $G$-valued cocharacter conjugate to $\mu_y$.

*Remark* 4.6. It can be checked from the construction in [KP], that the notion of $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted liftings only depends on the $G$-conjugacy class of $\mu_y$ and the specialization of the filtration induced by $\mu_y$.

**Proposition 4.7.** *Let SpfA denote the versal deformation space of $\mathscr{G}_0$. Then there is a versal quotient $A_{\mathcal{G}}$ of $A \otimes_{\mathcal{O}_L} \mathcal{O}_E$ such that for any $K$ as above, a map $\varpi : A \otimes_{\mathcal{O}_L} \mathcal{O}_E \to \mathcal{O}_K$ factors through $A_{\mathcal{G}}$ if and only if the p-divisible group $\mathscr{G}_\varpi$ induced is $(\mathcal{G}, \mu_y)$-adapted.*

*Proof.* This is essentially [KP, Prop. 3.2.17]. It follows from the construction that the $p$-divisible group $\mathscr{G}_\varpi$ induced by a map $\varpi : A_{\mathcal{G}} \to \mathcal{O}_K$ is $(\mathcal{G}, \mu_y)$-adapted. Indeed by the construction of the versal $p$-divisible group in [KP, §3.2.12], the Dieudonné display $M_{A_{\mathcal{G}}}$ of the versal $p$-divisible group base changed to $A_{\mathcal{G}}$ is equipped with Frobenius invariant tensors $s_{\alpha,0,A_{\mathcal{G}}} \in M_{A_{\mathcal{G}}}^{\otimes}$ and there is an identification

$$\beta : M_{A_{\mathcal{G}}} \xrightarrow{\sim} \mathbb{D} \otimes_{\mathcal{O}_L} \widehat{W}(A_{\mathcal{G}})$$

taking $s_{\alpha,0,A_{\mathcal{G}}}$ to $s_{\alpha,0}$. Base changing to $\widehat{W}(\mathcal{O}_K)$, we obtain $\tilde{s}_\alpha$ and the identification $\beta$ gives condition (1) of Definition 4.5. For the second condition, by [KP, Lemma 3.2.13], the canonical map

$$\gamma : \mathbb{D} \otimes_{\mathcal{O}_L} K \cong \mathbb{D}(\mathscr{G}_{\mathcal{O}_K})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \tag{4.5.1}$$

induced by [KP, Lemma 3.1.17] sends $s_{\alpha,0}$ to $s_{\alpha,0,\mathcal{O}_K}$, where $s_{\alpha,0,\mathcal{O}_K}$ denotes the base change $s_{\alpha,0,A_{\mathcal{G}}}$ to $\mathcal{O}_K$. Note that this map is *not* necessarily the same map induced by the identification $\beta^{-1}|_K$. By [Kis10, Lemma 1.4.5], the filtration on $\mathbb{D}(\mathscr{G}_{\mathcal{O}_K})(\mathcal{O}_K)\otimes_{\mathcal{O}_K} K$ is induced by a $G$-valued cocharacter conjugate to $\mu_y$ under the identification with $\mathbb{D}\otimes_{\mathcal{O}_L} K$ coming from $\beta|_K$, see also [KP, §3.2.5]. Since the map $\gamma$ respects $s_{\alpha,0}$ the map $\beta|_K\circ\gamma : \mathbb{D}\otimes_{\mathcal{O}_L} K \xrightarrow{\sim} \mathbb{D}\otimes_{\mathcal{O}_L} K$ is induced by an element of $G(K)$ and hence the filtration on $D\otimes_{\mathcal{O}_L} K$ coming from the map $\gamma$ is induced by a $G$-valued cocharacter conjugate to $\mu_y$. This proves condition (2).

The converse is [KP, Prop. 3.2.17]. $\square$

4.6. Now assume there is a $\mathbb{Z}_p$-module $U$ and an isomorphism $U\otimes_{\mathbb{Z}_p}\mathcal{O}_L\cong\mathbb{D}$ such that $s_{\alpha,0}\in U^{\otimes}$. Then the stabilizer of $s_{\alpha,0}$ in $U^{\otimes}$ is a group $\mathcal{G}$ over $\mathbb{Z}_p$ such that $\mathcal{G}\otimes_{\mathbb{Z}_p}\mathcal{O}_L\cong\mathcal{G}_{\mathcal{O}_L}$. We assume $\mathcal{G}$ is of the form $\mathcal{G}_x$ for some $x\in B(G,\mathbb{Q}_p)$. Since the $s_{\alpha,0}$ are $\varphi$-invariant, $\varphi$ is of the form $b\sigma$ for some $b\in G(L)$.

Under these assumptions one can make the following construction of a certain $(\mathcal{G}_{\mathcal{O}_L},\mu_y)$-adapted lift, which will be needed in §5 for the reduction to Levi subgroups argument.

**Proposition 4.8.** *There exists a $(\mathcal{G}_{\mathcal{O}_L},\mu_y)$-adapted deformation of $\mathscr{G}_0$ such that $s_{\alpha,0}\in\mathbb{D}^{\otimes}$ correspond to tensors $s_{\alpha,\text{ét}}\in T_p\mathscr{G}^{\vee}$ under the $p$-adic comparison isomorphism and such that there exists an isomorphism:*

$$T_p\mathscr{G}^{\vee}\otimes_{\mathbb{Z}_p}\mathcal{O}_L\cong\mathbb{D}$$

*taking $s_{\alpha,\text{ét}}$ to $s_{\alpha,0}$.*

*Proof.* Let $\mathfrak{M}:=\mathbb{D}\otimes_{\sigma^{-1},\mathcal{O}_L}\mathfrak{S}$, then $\sigma^*(\mathfrak{M})\cong\mathbb{D}\otimes_{\mathcal{O}_L}\mathfrak{S}$. Note that the map $y: A\to K'$ necessarily factors through $\mathcal{O}_{K'}$ since $A$ is a power series ring over $\mathcal{O}_L$; we abuse notation and also write $y$ for the map $A\to\mathcal{O}_{K'}$. Let $y^*(\overline{M}_1)\subset\mathbb{D}\otimes_{\mathcal{O}_L}\mathcal{O}_{K'}$ denote the filtration induced by $y: A\to\mathcal{O}_{K'}$ and let $F\subset\sigma^*(\mathfrak{M})$ denote be the premiage of $y^*(\overline{M}_1)$. By [KP, Lemma 3.2.6], $F$ is a free $\mathfrak{S}$-module and $s_{\alpha,0}\in F$, moreover the scheme $\underline{\mathrm{Isom}}_{s_{\alpha,0},\mathfrak{S}}(F,\sigma^*(\mathfrak{M}))$ of $\mathfrak{S}$-isomorphisms which respect the $s_{\alpha,0}$ is a $\mathcal{G}$ torsor. The Frobenius $\varphi$ on $\mathbb{D}$ induces a map

$$\mathbb{D}_1\xrightarrow{\simeq}\mathbb{D}\xrightarrow{\cong}\sigma^{-1*}\mathbb{D}.$$

Here $\mathbb{D}_1$ is the preimage of the filtration on $\mathbb{D}(\mathscr{G}_0)(k)$; the first arrow is given by $\sigma^{-1}(b/p)$ and the second isomorphism is induced by the identity on $U$. The specialization of $F$ at $u=0$ is identified with $\mathbb{D}_1$. Then since $G$ contains the scalars, $\sigma^{-1}(b/p)$ preserves $s_{\alpha,0}$ and hence corresponds to a point $\underline{\mathrm{Isom}}_{s_{\alpha,0},\mathfrak{S}}(F,\sigma^*(\mathfrak{M}))(W)$. By smoothness of $\mathcal{G}$, this lifts to an isomorphism

$$\Theta: F\xrightarrow{\sim}\sigma^*(\mathfrak{M})$$

respecting $s_{\alpha,0}$. Let $c=p\frac{E(u)}{E(0)}$. The morphism

$$\varphi:\sigma^*(\mathfrak{M})\xrightarrow{\times c}F\xrightarrow{\Theta}\sigma^*(\mathfrak{M})\to\mathfrak{M}$$

where the last map is induced from the identity on $U$, gives $\mathfrak{M}$ the structure of an element of $BT^{\varphi}$ such that $\varphi^*(\mathfrak{M}/u\mathfrak{M})$ is identified with $\mathbb{D}$, and hence corresponds to a $p$-divisible group $\mathscr{G}$ over $\mathcal{O}_K$ deforming $\mathscr{G}_0$.

Since $\varphi$ preserves $s_{\alpha,0}$, these give rise to Frobenius invariant tensors in $\tilde{s}_\alpha\in\mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K))^{\otimes}$, and by construction there is an isomorphism

$$\beta':\mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K))\cong\mathbb{D}\otimes_{\mathcal{O}_L}\widehat{W}(\mathcal{O}_K)$$

taking $\tilde{s}_\alpha$ to $s_{\alpha,0}$. Moreover under this isomorphism, the filtration on $\mathbb{D}\otimes_{\mathcal{O}_L} K$ is given by $\mu_y$. The canonical isomorphism

$$\mathbb{D}\otimes_{\mathcal{O}_L} K \cong \mathbb{D}(\mathscr{G})(\mathcal{O}_K)\otimes_{\mathcal{O}_K} K \tag{4.6.1}$$

induced by [KP, Lemma 3.1.17] takes $\tilde{s}_\alpha$ to $s_{\alpha,0}$ by [KP, Lemma 3.2.13]. As in the proof of Proposition 4.7, the map in (4.6.1) is not necessarily identified with $\beta'^{-1}|_K$. However both the maps respect tensors and hence by the same argument of 4.7 the filtration on $\mathbb{D}\otimes_{\mathcal{O}_L} K$ coming from (4.6.1) is induced by a $G$-valued cocharacter conjugate to $\mu_y$. Thus $\mathscr{G}$ is a $(\mathcal{G},\mu_y)$-adapted deformation, and since $\tilde{s}_\alpha\in\mathfrak{M}^\otimes$, we have $s_{\alpha,\text{ét}}\in T_p\mathscr{G}^\vee$ by the fully faithfulness of $\mathfrak{M}$ in Proposition 4.1.

We now show that there exists an isomorphism $T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_L\cong\mathbb{D}$ respecting tensors. It will suffice to show that $\mathcal{P}$ is a trivial $\mathcal{G}$-torsor. Let

$$\mathcal{P}\subset \underline{\mathrm{Isom}}(T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_L,\mathbb{D})$$

be the isomorphism scheme taking $s_{\alpha,\text{ét}}$ to $s_{\alpha,0}$. By construction there is an isomorphism

$$\mathfrak{M}(T_p\mathscr{G}^\vee)\cong\mathbb{D}\otimes_{\sigma^{-1},\mathcal{O}_L}\mathfrak{S}\xrightarrow{\sim}\mathbb{D}\otimes_{\mathcal{O}_L}\mathfrak{S}$$

taking $\tilde{s}_\alpha$ to $s_{\alpha,0}$. By Proposition 4.1 there is a canonical isomorphism

$$T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\cong\mathfrak{M}(T_p\mathscr{G}^\vee)\otimes_{\mathfrak{S}}\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$$

and this isomorphism takes $s_{\alpha,\text{ét}}$ to $\tilde{s}_\alpha$. Thus there is an isomorphism

$$T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}\cong\mathbb{D}\otimes_{\mathcal{O}_L}\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$$

taking $s_{\alpha,\text{ét}}$ to $s_{\alpha,0}$, i.e. $\mathcal{P}\otimes_W\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is a trivial $\mathcal{G}$-torsor. Since $\mathcal{O}_L\to\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is faithfully flat, $\mathcal{P}$ is a $\mathcal{G}$-torsor which is necessarily trivial since $\mathcal{G}$ is smooth and $\mathcal{O}_L$ is strictly henselian.

□

## 5. Affine Deligne–Lusztig varieties

This section forms the main part of the local argument for the description of the isogeny classes. It is used for the argument in §6 for lifting isogenies to characteristic 0. An essential part is a bound on the connected components of affine Deligne-Lusztig varieties obtained in [HZ], which is recalled here. We also state a conjecture on the connected components of affine Deligne–Lusztig varieties of which the results of [HZ] can be thought of as a special case.

5.1. Let $G$ be a reductive group over $\mathbb{Q}_p$ which splits over a tamely ramified extension. Recall $S$ is a maximal $L$-split torus defined over $\mathbb{Q}_p$ and $T$ its centralizer. We have fixed a $\sigma$-invariant alcove $\mathfrak{a}$ in the apartment corresponding to $S$ which induces a length function and ordering on the affine Weyl group $W_a$ and hence on the Iwahori Weyl group $W$. We also fix a special vertex $\mathfrak{s}$ (not necessarily $\sigma$-invariant) contained in the closure of $\mathfrak{a}$. This induces an identification $\mathcal{A}(G,S,L)\cong X_*(T)_I\otimes\mathbb{R}$ by sending $\mathfrak{s}$ to 0. This also determines a dominant chamber $C_+$ in $X_*(T)_I\otimes\mathbb{R}$ by taking the one containing the alcove $\mathfrak{a}$. We let $B$ denote the Borel subgroup over $L$ corresponding to this choice of dominant chamber. The choice also determines a splitting of the exact sequence (2.1.1) so we may think of $W_0\subset W$. It is generated by the simple reflections $\mathbb{S}_0$ corresponding to the special vertex $\mathfrak{s}$.

Under the identification $\mathcal{A}(G,S,L)\cong X_*(T)_I\otimes\mathbb{R}$ induced by $\mathfrak{s}$, $\sigma$ acts by affine transformations on $X_*(T)_I\otimes\mathbb{R}$ and we write $\varsigma$ for the linear part of this action. We write $\sigma_0$ for the automorphism of $X_*(T)_I\otimes\mathbb{R}$ defined by $\sigma_0:=w\circ\varsigma$ where $w\in W_0$ is the unique element such that $w\circ\varsigma(C_+)=C_+$. We call this the $L$-action on $X_*(T)_I\otimes\mathbb{R}$; it preserves $C_+$.

Let $b \in G(L)$, we denote by $[b] = \{g^{-1}b\sigma(g) | g \in G(L)\}$ its $\sigma$-conjugacy class. Let $B(G)$ be the set of $\sigma$-conjugacy classes of $G(L)$. We let $\overline{\nu}$ be the Newton map

$$\overline{\nu} : B(G) \to (X_*(T)_{I,\mathbb{Q}}^+)^{\sigma_0}$$

where $X_*(T)_{I,\mathbb{Q}} := X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X_*(T)_{I,\mathbb{Q}}^+$ is its intersection with the dominant chamber $C_+$. Following convention, we denote by $\overline{\nu}_b$ the image of the $\sigma$-conjugacy class of $b$ under the map $\overline{\nu}$. We let

$$\kappa_G : B(G) \to \pi_1(G)_\Gamma$$

denote the map induced by composition of the Kottwitz map $\tilde{\kappa}_G : G(L) \to \pi_1(G)_I$ with the projection $\pi_1(G)_I \twoheadrightarrow \pi_1(G)_\Gamma$.

By [Kot97, §4.13] the map

$$(\overline{\nu}, \kappa_G) : B(G) \to (X_*(T)_{I,\mathbb{Q}}^+)^{\sigma_0} \times \pi_1(G)_\Gamma$$

is injective.

We say a $\sigma$-conjugacy class $[b] \in B(G)$ is basic if $\overline{\nu}_b$ is central.

5.2. Let $K \subset \mathbb{S}$ be a $\sigma$-invariant subset and $W_K$ the group generated by the reflections in $K$. Let $\mathcal{G}$ denote the associated parahoric group scheme over $\mathbb{Z}_p$. For $b \in G(L)$ and $w \in W_K \backslash W / W_K$ the affine Deligne–Lusztig variety is defined to be

$$X_{K,w}(b) := \{g \in G(L)/\mathcal{G}(\mathcal{O}_L) | g^{-1}b\sigma(g) \in \mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O}_L)\}.$$

It is known that $X_{K,w}(b)$ arises as the $k$-points of a perfect scheme over $k$, for example by [BS17] (see also [Zhu17]). When $K = \emptyset$ and $\mathcal{G} = \mathcal{I}$ is an Iwahori subgroup, we write $X_w(b)$ for the corresponding affine Deligne–Lusztig variety.

Let $\{\mu\}$ be a geometric conjugacy class of cocharacters for $G$ and let $\underline{\mu}$ be the image in $X_*(T)_I$ of a dominant representative $\mu$ in $X_*(T)$. Recall we have associated to this data the $\mu$-admissible set $\mathrm{Adm}_K(\{\mu\}) \subset W_K \backslash W / W_K$.

Let

$$\begin{aligned} X(\{\mu\}, b)_K : &= \{g \in G(L)/\mathcal{G}(\mathcal{O}_L) | g^{-1}b\sigma(g) \in \bigcup_{w \in \mathrm{Adm}_K(\{\mu\})} \mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O}_L)\} \\ &= \bigcup_{w \in \mathrm{Adm}_K(\{\mu\})} X_{K,w}(b). \end{aligned}$$

As before, when $\mathcal{G}$ is the Iwahori subgroup we write $X(\{\mu\}, b)$ for this union of affine Deligne–Lusztig varieties. For notational convenience we will also consider the unions

$$X(\sigma(\{\mu\}), b)_K := \bigcup_{w \in \mathrm{Adm}_K(\{\mu\})} X_{K,\sigma(w)}(b).$$

It can be identified with $X(\{\sigma'(\mu)\}, b)$ where $\sigma' \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a lift of Frobenius. The map $g \mapsto b\sigma(g)$ defines an isomorphism from $X(\{\mu\}, b)$ to $X(\sigma(\{\mu\}), b)$.

We recall the definition of the neutral acceptable set $B(G, \{\mu\})$ in [RV14]. For $\lambda, \lambda' \in X_*(T)_{I,\mathbb{Q}}^+$, we write $\lambda \leq \lambda'$ if $\lambda' - \lambda$ is a non-negative rational linear combination of positive relative coroots. Set

$$B(G, \{\mu\}) = \{[b] \in B(G) : \kappa_G([b]) = \mu^\natural, \overline{\nu}_b \leq \mu^\diamond\}$$

where $\mu^\natural$ is the common image of $\mu \in \{\mu\}$ in $\pi_1(G)_\Gamma$, and $\mu^\diamond \in X_*(T)_{I,\mathbb{Q}}$ denotes the Galois average of $\underline{\mu} \in X_*(T)_I$ with respect to the action of $\sigma_0$.

The following result on the non-emptiness pattern of the $X(\{\mu\}, b)_K$ was conjectured by Kottwitz and Rapoport in [KR03] and proved by He in [He16].

**Theorem 5.1** ([He16, Theorem 1.1] ). *(i) The set $X(\{\mu\}, b)_K \neq \emptyset$ if and only if $[b] \in B(G, \{\mu\})$.*

*(ii) Let $K' \subset K$ with $K'$ also $\sigma$-invariant and let $\mathcal{G}'$ be the associated parahoric. The natural projection $G(L)/\mathcal{G}'(\mathcal{O}_L) \to G(L)/\mathcal{G}(\mathcal{O}_L)$ induces a surjection*

$$X(\{\mu\}, b)_{K'} \twoheadrightarrow X(\{\mu\}, b)_K.$$

We write $J_b$ for the reductive group over $\mathbb{Q}_p$ whose points in a $\mathbb{Q}_p$-algebra $R$ is defined to be

$$J_b(R) := \{g \in G(L \otimes_{\mathbb{Q}_p} R) | g^{-1} b\sigma(g) = b\}.$$

Then $J_b(\mathbb{Q}_p)$ acts on $X_{K,w}(b)$ and $X(\{\mu\}, b)$.

5.3. We now state a conjecture on the connected components of affine Deligne–Lusztig varieties. For $[b] \in B(G)$, we let $M_{\overline{\nu}_b}$ denote the standard Levi subgroup of $G$ corresponding to $\overline{\nu}_b$. A standard Levi subgroup $M \subset G$ defined over $L$ is said to $\sigma_0$-stable if it is the centralizer of a cocharacter $\nu \in X_*(T)_{I,\mathbb{Q}} \cong X_*(T)^I \otimes_{\mathbb{Z}} \mathbb{Q}$ which is stable under $\sigma_0$. Such a Levi determines a subroot system $\Phi_M \subset \Sigma$.

**Definition 5.2** ([GHN, Definition 2.1 (1)]). Let $M \subset G$ be a proper $\sigma_0$-stable standard Levi subgroup. We say that $[b] \in B(G, \{\mu\})$ is Hodge–Newton decomposable with respect to $M$ if $M_{\overline{\nu}_b} \subset M$ and $\mu^\diamond - \overline{\nu}_b \in \mathbb{R}_{\geq 0}\Phi_M^\vee$.

It is Hodge-Newton indecomposable if no such $M$ exists.

**Lemma 5.3.** *Let $[b] \in B(G, \{\mu\})$ be Hodge–Newton indecomposable and assume $G_{\mathrm{ad}}$ is $\mathbb{Q}_p$-simple. Then either $b$ is $\sigma$-conjugate to $\dot{t}_{\underline{\mu}}$ and $\underline{\mu}$ is central (meaning $\langle \underline{\mu}, \alpha \rangle = 0$ for all $\alpha \in \Sigma$), or the coefficient of each simple coroot of $\Sigma^\vee$ in $\mu^\diamond - \overline{\nu}_b$ is strictly positive.*

*Proof.* The proof is the same as [CKV15, Theorem 2.5.6]. Suppose the coefficient of some simple coroot $\alpha_0^\vee \in \Sigma^\vee$ vanishes in $\mu^\diamond - \overline{\nu}_b$, we claim that this implies $\mu^\diamond = \overline{\nu}_b$. Assuming this we can prove the Lemma as follows. Let $M = M_{\overline{\nu}_b}$, then $M$ is $\sigma_0$-stable and if it is proper, $[b]$ is Hodge–Newton decomposable with respect to $M$. Therefore $M_{\overline{\nu}_b} = G$, i.e. $\overline{\nu}_b$ is central in $G$. Since $\kappa_G(\dot{t}_{\underline{\mu}}) = \kappa_G(b)$ and $\overline{\nu}_{\dot{t}_{\underline{\mu}}} = \underline{\mu}^\diamond = \overline{\nu}_b$, $b$ is $\sigma$-conjugate to $\dot{t}_{\underline{\mu}}$. Let $\alpha \in \Sigma$ be a positive root. We have

$$0 = \langle \mu^\diamond, \alpha \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \langle \sigma_0^i(\underline{\mu}), \alpha \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \langle \underline{\mu}, \sigma_0^i(\alpha) \rangle \tag{5.3.1}$$

where $n$ is the order of $\sigma_0$ acting on $\Sigma$. Since $\sigma_0$ preserves the positive roots in $\Sigma$, and $\underline{\mu}$ is dominant, $\langle \underline{\mu}, \sigma_0^i(\alpha) \rangle \geq 0$ for all $i$, and hence $\langle \underline{\mu}, \sigma_0^i(\alpha) \rangle = 0$ for all $i$ by (5.3.1).

It suffices therefore to prove the claim. Let $\alpha^\vee$ be a simple coroot in $\Sigma^\vee$; we show by induction on the distance between $\alpha^\vee$ and the $\sigma_0$-orbit of $\alpha_0^\vee$ in the Dynkin diagram that the coefficient of $\alpha^\vee$ in $\mu^\diamond - \overline{\nu}_b$ also vanishes. Suppose this is known for some simple coroot $\alpha^\vee$, then it is known for any element of the $\sigma_0$-orbit of $\alpha^\vee$ since $\mu^\diamond - \overline{\nu}_b$ is $\sigma_0$-invariant. If $\langle \alpha, \overline{\nu}_b \rangle \neq 0$ then the standard $\sigma_0$-stable Levi subgroup $M$ corresponding to the $\sigma_0$-orbit of $\alpha$ satisfies $M_{\overline{\nu}_b} \subset M$ contradicting the Hodge–Newton indecomposability assumption. Here $M$ is generated by $T$ and all the relative root subgroups $U_\beta$ for $\beta$ not proportional to an element of the $\sigma_0$-orbit of $\alpha$. Therefore $\langle \alpha, \overline{\nu}_b \rangle = 0$. It follows that

$$\langle \mu^\diamond, \alpha \rangle = \langle \mu^\diamond - \overline{\nu}_b, \alpha \rangle = \sum_\beta \lambda_\beta \langle \beta^\vee, \alpha \rangle$$

where $\lambda_\beta$ is the coefficient of $\beta^\vee$ in $\mu^\diamond - \overline{\nu}_b$ and the sum runs over neighbors $\beta$ of $\alpha$ in the Dynkin diagram of $\sigma$. Since $\langle \mu^\diamond, \alpha\rangle \geq 0$ and $\lambda_\beta \geq 0$, this implies $\lambda_\beta = 0$ for all neighbors $\beta$ of $\alpha$. This proves the induction step and the lemma. □

If $[b] \in B(G, \{\mu\})$ is Hodge–Newton indecomposable and the coefficient of each simple coroot of $\Sigma^\vee$ in $\mu^\diamond - \overline{\nu}_b$ is strictly positive, then we say that $[b]$ is *Hodge–Newton irreducible.*

For $b \in G(L)$ with $[b] \in B(G, \{\mu\})$, we let $c_{b,\mu} \in \pi_1(G)_I$ be an element such that $(\sigma - 1)(c_{b,\mu}) = [\mu] - \tilde{\kappa}_G(b)$. Here $[\mu]$ is the common image of $\{\mu\}$ in $\pi_1(G)_I$, and the existence of $c_{b,\mu}$ is guaranteed by the property $\kappa_G([b]) = \mu^\natural$ defining $B(G, \{\mu\})$.

**Conjecture 5.4.** *Suppose $[b] \in B(G, \{\mu\})$ is Hodge–Newton irreducible. Then the map $\tilde{\kappa}_G : G(L) \to \pi_1(G)_I$ induces an isomorphism*

$$\pi_0(X(\{\mu\}, b)) \to c_{b,\mu}\pi_1(G)_I^\sigma,$$

*Remark* 5.5. (i) There is also a version of this conjecture for arbitrary parahoric subgroups. Suppose $K \subset \mathbb{S}$ is a $\sigma$-stable subset with $W_K$ finite. Then we may conjecture that $\tilde{\kappa}_G : G(L) \to \pi_1(G)_I$ induces an isomorphism

$$\pi_0(X(\{\mu\}, b)_K) \to c_{b,\mu}\pi_1(G)_I^\sigma.$$

By the surjectivity in Theorem 5.1, the conjecture for arbitrary parahorics follows from Conjecture 5.4.

(ii) Suppose $G_{\mathrm{ad}}$ is $\mathbb{Q}_p$-simple. If $[b]$ is Hodge–Newton indecomposable but not Hodge–Newton irreducible, then it is easy to see that $X(\{\mu\}), b)$ is discrete and there is an identification

$$G(\mathbb{Q}_p)/\mathcal{I}(\mathbb{Z}_p) \cong X(\{\mu\}, b).$$

Thus together with the above conjecture, this gives a description of $\pi_0(X(\{\mu\}, b))$ in the Hodge–Newton indecomposable case in this case. The description of $\pi_0(X(\{\mu\}, b))$ when $G_{\mathrm{ad}}$ is not assumed simple can easily be deduced from this using standard reductions; see for example [HZ, §6.1].

In the case the group $G$ is unramified and when $K$ corresponds to a hyperspecial subgroup of $G$, this conjecture has been proved in [CKV15]. Recently, the case when $G$ is split and $K$ corresponds to any parahoric has been settled in [CN17].

5.4. The motivation for the above definitions comes from the following Theorem which is proved in [GHN].

**Theorem 5.6** ([GHN, Theorem 3.16])**.** *Let $[b] \in B(G, \{\mu\})$ and suppose that $[b]$ is Hodge–Newton decomposable with respect to $M \subset G$. Then*

$$X(\{\mu\}, b) \cong \coprod_{M'N' = P' \in \mathfrak{P}^\sigma} X^{M'}(\{\mu_{P'}\}, b_{P'}). \tag{5.4.1}$$

*Moreover the terms in the union are open and closed.*

The notation in the Theorem is as follows. We write $P$ for the parabolic subgroup of $G$ containing $M$ and the Borel $B$. $\mathfrak{P}^\sigma$ denotes the set of $\sigma$-stable parabolic subgroups of $G$ which contain $T$ and are conjugate to $P$. For $P' \in \mathfrak{P}^\sigma$, $N'$ denotes the unipotent radical of $P'$ and $M'$ the semistandard Levi subgroup contained in $P'$ (here semistandard means containing the maximal torus $T$). $[b_{P'}]$ denotes the $\sigma$-conjugacy class of $M'(L)$ constructed in [GHN, Proposition 3.8]. To define the $M$ conjugacy class of cocharacters $\{\mu_{P'}\}$, we write $J \subset \mathbb{S}_0$ for the $\sigma_0$-stable subset corresponding to the standard Levi subgroup $M$. We write $W_0^J$ for the set of minimal length representatives of the cosets $W_0/W_J$. As $P$ is conjugate to $P'$ there is a unique $z_{P'} \in W_0^J$ such that $z_{P'}(P) := \dot{z}_{P'} P \dot{z}_{P'}^{-1} = P'$. we take

$\mu \in X_*(T)$ a dominant representative of $\{\mu\}$, then $\{\mu_{P'}\}$ is the $M'$ conjugacy class of cocharacters containing $\mu_{P'} := z_{P'}(\mu)$.

Since $M'$ is $\sigma$-stable, it is defined over $\mathbb{Q}_p$, and $M'(L) \cap \mathcal{I}(\mathcal{O}_L)$ is the $\mathcal{O}_L$-points of an Iwahori subgroup $\mathcal{I}_{M'}$ of $M'(L)$ defined over $\mathbb{Z}_p$. The choice of Iwahori $\mathcal{I}_{M'}$ induces a Bruhat order on the Iwahori Weyl group $W_{M'}$ (defined in terms of the torus $T$) assocated to $M'$ and hence there is a notion of $\mu_{P'}$-admissible set $\mathrm{Adm}^{M'}(\{\mu_{P'}\})$. We then define $X^{M'}(\{\mu_{P'}\}, b_{P'})$ to be the affine Deligne–Lusztig variety for the data $(M', \mathcal{I}_{M'}, \{\mu_{P'}\}, b_{P'})$.

**Lemma 5.7.** *Let $[b] \in B(G, \{\mu\})$ then there exists a unique $\sigma_0$-stable $M \subset G$ such that for each $P'$ appearing in the decomposition (5.4.1), $[b_{P'}] \in B(M', \{\mu_{P'}\})$ is Hodge–Newton indecomposable.*

*Proof.* Let $M$ denote the smallest $\sigma_0$-stable standard Levi subgroup of $G$ such that $M_{\overline{b}} \subset M$ and such that $\mu^\diamond - \overline{\nu}_b \in \Phi_M^\vee$. Indeed a smallest such Levi exists since if $M_1$ and $M_2$ are such, then so is $M_1 \cap M_2$.

Let $P' \in \mathfrak{P}^\sigma$ and $M'$ its semistandard Levi subgroup. Then $\overline{\nu}_{b_{P'}}^{M'} = z_{P'}(\overline{\nu}_b)$ (see [GHN, Proposition 3.8]) and $\underline{\mu}_{P'} = z_{P'}(\underline{\mu})$. We write $\sigma_0^{M'}$ for the $L$-action on $X_*(T)_{I,\mathbb{Q}}$ corresponding to $M'$. Then it can be checked that $\sigma_0^{M'} = z_{P'}\sigma_0 z_{P'}^{-1}$. Suppose $H \subset M'$ is a proper $\sigma_0^{M'}$-stable Levi subgroup of $M'$ such that $[b_{P'}] \in B(M', \{\mu_{P'}\})$ is Hodge–Newton decomposable with respect to $H$. We will show that $[b]$ is Hodge–Newton decomposable with respect to $z_{P'}^{-1}(H)$, which contradicts our choice of $M$.

First note that $z_{P'}^{-1}(H)$ is a $\sigma_0$-stable Levi subgroup of $G$ which is (properly) contained in $M$. Then $\overline{\nu}_{b_{P'}}^{M'} - \mu_{P'}^\diamond \in \Phi_H^\vee$ implies that $\overline{\nu}_b - \mu^\diamond \in \Phi^\vee_{z_{P'}^{-1}(H)}$. This is a contradication.

□

Theorem 5.6 and Lemma 5.7 shows that in order to understand $X(\{\mu\}, b)$ is suffices to understand the Hodge–Newton indecomposable case. Using Lemma 5.3, the understanding of connected components is essentially reduced to Conjecture 5.4.

For later arguments we will need the following Lemma.

**Lemma 5.8.** *Let $P', P'' \in \mathfrak{P}^\sigma$. Then there exists $v \in W^\sigma$ (i.e. $\sigma(v) = v$) such that following properties are satisfied:*

*(i)* $v\mathrm{Adm}^{M'}(\{\mu_{P'}\})v^{-1} = \mathrm{Adm}^{M''}(\{\mu_{P''}\})$

*(ii)* $\dot{v}\mathcal{I}_{M'}(\mathcal{O}_L)\dot{v}^{-1} = \mathcal{I}_{M''}(\mathcal{O}_L)$.

*Proof.* Let $u = z_{P''}z_{P'}^{-1}$; then $\dot{u}P'\dot{u}^{-1} = P''$. By definition $u(\mu_{P'}) = \mu_{P''}$, so $u$ maps the $M'$ conjugacy class of cocharacters $\{\mu_{P'}\}$ into the $M''$ conjugacy class of cocharacters $\{\mu_{P''}\}$.

The Iwahori weyl group $W_{M'}$ for $M'$ is a reflection subgroup of $W$. Therefore by [Dye90, Corollary 3.4], each coset $W/W_{M'}$ has a unique element of minimal length; we write $W^{M'}$ for the set of such elements. Since $W_{M'}$ is $\sigma$-stable, so is $W^{M'}$. We let $v$ denote the element of $W^{M'}$ corresponding to the coset containing $u$. Since $\dot{u}P'\dot{u}^{-1} = P''$ and $P'$ is normalized by $W_{M'}$, it follows that $\dot{v}P'\dot{v}^{-1} = P''$. Since $P'$ and $P''$ are $\sigma$-stable, it follows that $\dot{v}^{-1}\sigma(\dot{v})P'\sigma(\dot{v})^{-1}\dot{v} = P'$ and hence $v^{-1}\sigma(v) \in W_{M'}$. Since $v, \sigma(v) \in W^{M'}$, it follows that $\sigma(v) = v$.

Since $v \in W^{M'}$, we have $\dot{v}\mathcal{I}_{M'}(\mathcal{O}_L)\dot{v}^{-1} = \mathcal{I}'_{M'}(\mathcal{O}_L)$, and hence the isomorphism $W_{M'} \xrightarrow{\sim} W_{M''}$ induced by conjugation by $v$ preserves Bruhat order. Moreover, since conjugation by $v$ takes $\{\mu_{P'}\}$ to $\{\mu_{P''}\}$ (since $v \in uW_{M'}$), it follows that $v\mathrm{Adm}^{M'}(\{\mu_{P'}\})v^{-1} = \mathrm{Adm}^{M''}(\{\mu_{P''}\})$. □

5.5. As a first step towards Conjecture 5.4 we have the following two Theorems which are proved in [HZ].

**Theorem 5.9** ([HZ, Theorem 0.1, Theorem 4.1] )**.** *(i) Let $Y \subset X(\{\mu\}, b)$ be a connected component, then $Y \cap X_w(b) \neq \emptyset$ for some $\sigma$-straight element $w \in W$.*

*(ii) Assume $G_{\mathrm{ad}}$ is $\mathbb{Q}_p$-simple and that $b$ corresponds to a basic $\sigma$-conjugacy class in $B(G)$. If $\mu$ is not central, then the Kottwitz homomorphism induces an isomorphism:*

$$\pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})) \xrightarrow{\sim} \pi_1(G)_I^\sigma$$

*and if $\mu$ is central, $X(\{\mu\}, b)$ is discrete and there is a bijection*

$$X(\{\mu\}, b) \simeq G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p).$$

*Remark* 5.10. (i) The condition $[b] \in B(G, \{\mu\})$ basic in part (ii) of the Theorem implies that $[b]$ is Hodge–Newton indecompsable. So one can think of the Theorem as a special case of Conjecture 5.4.

(ii) A version of part (ii) of this Theorem can also be proved for arbitrary parahorics; see [HZ, Theorem 8.1].

Now assume $G$ is residually split; we recall that this means $G$ and $G \otimes_{\mathbb{Q}_p} L$ have the same split rank, or equivalently that $\sigma$ acts trivially on $W$. In this case, the results of [HZ] give a bound on $\pi_0(X(\{\mu\}, b))$ in terms of affine Deligne–Lusztig varieties for Levi subgroups. This bound will be a sufficient substitute for Conjecture 5.4 needed for later applications.

To any element $w \in W$ one may associate a vector $\nu_w \in X_*(T)_{I,\mathbb{Q}}$ as in 2.2, its non-dominant Newton vector. We write $M_{\nu_w}$ for the associated semistandard Levi subgroup which is generated by $T$ and the root subgroups $U_a$, where $a$ is a relative root such that $\langle \nu_w, a \rangle = 0$. Alternatively, if we consider $\nu_w \in X_*(T)_{I,\mathbb{Q}} \cong X_*(T)_{\mathbb{Q}}^I$ as a fractional cocharacter of $G$, then $M_{\nu_w}$ is just its centralizer.

We now assume $w$ is a straight (equivalently $\sigma$-straight since $\sigma$ is trivial) element. As in §5.4, the Iwahori $\mathcal{I}$ determines an Iwahori subgroup of $M_{\nu_w}$, namely $\mathcal{I}_{M_{\nu_w}}(\mathcal{O}_L) = \mathcal{I}(\mathcal{O}_L) \cap M_{\nu_w}(L)$. This induces a Bruhat order and length function on the Iwahori Weyl group $W_{M_{\nu_w}}$ for $M_{\nu_w}$. Then it is known that $w$ lies in $W_{M_{\nu_w}}$ and is a length 0 element, i.e. $\dot{w}\mathcal{I}_{M_{\nu_w}}(\mathcal{O}_L)\dot{w}^{-1} = \mathcal{I}_{M_{\nu_w}}(\mathcal{O}_L)$, cf. [Nie15, Theorem 1.3] for the case of unramified groups and [HZ, Theorem 5.2] in general. Hence $\dot{w}$ is a basic element in $M_{\nu_w}(L)$. The following is [HZ, Theorem 7.1].

**Theorem 5.11.** *There is a map*

$$\coprod_{w \in W, w \text{ a straight element with } \dot{w} \in [b]} X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w}) \to X(\{\mu\}, b)$$

*which induces a surjection*

$$\coprod_{w \in W, w \text{ a straight element with } \dot{w} \in [b]} \pi_0(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \to \pi_0(X(\{\mu\}, b)).$$

Here $\{\lambda_w\}_{M_{\nu_w}}$ is a certain $M_{\nu_w}$ conjugacy class of cocharacters of $M_{\nu_w}$ which maps to $\{\mu\}$. Since $\dot{w}$ is basic in $M_{\nu_w}$, we may use Theorem 5.9 to describe the connected components of the terms on the left (again using [HZ, §6.1] to reduce the general case to when $G_{\mathrm{ad}}$ is $\mathbb{Q}_p$-simple).

5.6. Now let $\mathscr{G}_0$ be a $p$-divisible group over $k = \overline{\mathbb{F}}_p$ and write $\mathbb{D}(\mathscr{G}_0)$ for $\mathbb{D}(\mathscr{G}_0)(\mathcal{O}_L)$. Let $s_{\alpha,0} \in \mathbb{D}(\mathscr{G}_0)^\otimes$ be a collection of $\varphi$-invariant tensors such that $s_{\alpha,0}$ lie in $\mathrm{Fil}^0\mathbb{D}(\mathscr{G}_0)(k)$ and let $\mathcal{G}_{\mathcal{O}_L} \subset GL(\mathbb{D}(\mathscr{G}_0))$ denote their stabilizer. Assume that there is a free $\mathbb{Z}_p$-module $U$ together with an isomorphism $U \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_0)$ such that $s_{\alpha,0} \in U^\otimes$. Assume also that the stabilizer $\mathcal{G} \subset GL(U)$ of these tensors has generic fiber $G$ and is a connected parahoric group scheme corresponding to $K \subset \mathbb{S}$. Then we have an isomorphism $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \simeq \mathcal{G}_{\mathcal{O}_L}$ so that $\mathcal{G}_{\mathcal{O}_L}$ is also a parahoric group scheme over $\mathcal{O}_L$. If $U'$ is another such $\mathbb{Z}_p$-module, the scheme of isomorphisms $U' \xrightarrow{\sim} U$ taking $s_{\alpha,0}$ to $s_{\alpha,0}$ is a $\mathcal{G}$-torsor which is necessarily trivial since $\mathcal{G}$ is smooth and has connected special fiber. Let $G$ denote the generic fibre of $\mathcal{G}$ which is reductive group over $\mathbb{Q}_p$.

Since the $s_{\alpha,0}$ are $\varphi$ invariant, we can write $\varphi = b\sigma$ for some $b \in G(L)$, and the choice is independent of the choice of $U$ up to $\sigma$-conjugation by an element of $\mathcal{G}(\mathcal{O}_L)$. Let $K'/L$ be a finite extension. We pick a filtration on $\mathbb{D}(\mathscr{G}_0) \otimes_{\mathcal{O}_L} K'$ lifting the one on $\mathbb{D}(\mathscr{G}_0)(k)$ as in §4.1 (this means that $\mathrm{Fil}^0\mathbb{D} \otimes_{\mathcal{O}_L} K' \cap \mathbb{D} \otimes_{\mathcal{O}_L} \mathcal{O}_{K'}$ lifts the filtration on $\mathbb{D}(\mathscr{G}_0)(k)$) so that $s_{\alpha,0} \in \mathrm{Fil}^0\mathbb{D} \otimes_{\mathcal{O}_L} K'$. This filtration is defined by a $G$-valued cocharacter $\mu_y$ and we have the embedding of local models

$$M^{\mathrm{loc}}_{\mathcal{G}} \hookrightarrow M^{\mathrm{loc}}_{\mathcal{GL}} \otimes_{\mathcal{O}_L} \mathcal{O}_E$$

where the defining cocharacter for $M^{\mathrm{loc}}_{\mathcal{G}}$ is $\mu_y^{-1}$.

The filtration on $\mathbb{D}(\mathscr{G}_0) \otimes k = \mathbb{D}(\mathscr{G}_0)(k)$ is by definition the kernel of $\varphi$, thus the preimage of the filtration in $\mathbb{D}(\mathscr{G}_0)$ is given by

$$\{v \in \mathbb{D}(\mathscr{G}_0) | b\sigma(v) \in p\mathbb{D}(\mathscr{G}_0)\}.$$

This is precisely the sub $\mathcal{O}_L$-lattice in $\mathbb{D}(\mathscr{G}_0)$ corresponding to $\sigma^{-1}(b^{-1})p\mathbb{D}(\mathscr{G}_0)$. By Proposition 3.4 we have

$$\sigma^{-1}(b^{-1}) \in \bigcup_{w \in \mathrm{Adm}_K(\{\mu_y^{-1}\})} \mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O}_L)$$

i.e., $1 \in X(\{\sigma(\mu_y)\}, b)_K$.

We let $\mathscr{G}$ be a $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted lifting to $\mathcal{O}_K$ for some $K/L$ finite such that if $s_{\alpha,\text{ét}} \in T_p\mathscr{G}^{\vee,\otimes} \otimes \mathbb{Q}_p$ denotes the tensors corresponding to $s_{\alpha,0}$ under the $p$-adic comparison isomorphism, then we have $s_{\alpha,\text{ét}} \in T_p\mathscr{G}^{\vee,\otimes}$. By Proposition 4.8, such a $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted lifting $\mathscr{G}$ always exists. Then we have an isomorphism

$$T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathfrak{S} \cong \mathfrak{M}(T_p\mathscr{G}^\vee)$$

taking $s_{\alpha,\text{ét}}$ to $\tilde{s}_\alpha$. This induces an isomorphism

$$T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_L \simeq \mathbb{D}(\mathscr{G}_0)$$

taking $s_{\alpha,\text{ét}}$ to $s_{\alpha,0}$. As in §4, $\tilde{s}_\alpha$ denotes the functor $\mathfrak{M}$ applied to $s_{\alpha,\text{ét}}$.

5.7. Now suppose $M \subset G$ is a closed reductive subgroup defined over $\mathbb{Q}_p$ such that $b \in M(L)$. Suppose that $M(L) \cap \mathcal{G}(\mathcal{O}_L)$ is the set of $\mathcal{O}_L$-points of a parahoric subgroup $\mathcal{M}$ of $M$, then $\mathcal{M}$ is a defined over $\mathbb{Z}_p$. Since $b \in M(L)$, we may extend the tensors $s_{\alpha,0} \in U^\otimes$ to $\varphi$-invariant tensors $t_{\beta,0}$ whose stabilizer is $\mathcal{M}$. We make the following assumption:

**Assumption 5.12.** The filtration on $\mathbb{D}(\mathscr{G}_0) \otimes_{\mathcal{O}_L} k$ lifts to a filtration on $\mathbb{D}(\mathscr{G}_0) \otimes_{\mathcal{O}_L} K$ which is induced by an $M$-valued cocharacter $\mu'_y$ which is conjugate $\mu_y$ in $G$.

We have the local model $M^{\mathrm{loc}}_{\mathcal{M},\mu_{y'}^{-1}}$ which is defined over $\mathcal{O}_{E'}$ where $E'$ is the local reflex field for $\{\mu'_y\}$. Then there is an embedding

$$M^{\mathrm{loc}}_{\mathcal{M},\mu_y'^{-1}} \hookrightarrow M^{\mathrm{loc}}_{\mathcal{GL}} \otimes_{\mathcal{O}_F} \mathcal{O}_{E'}$$

which factors through a closed embedding $M^{\mathrm{loc}}_{\mathcal{M},\mu_y'^{-1}} \hookrightarrow M^{\mathrm{loc}}_{\mathcal{G},\mu_y^{-1}} \otimes \mathcal{O}_{E'}$.

Now we let $\mathscr{G}$ be a $(\mathcal{M}, \mu'_y)$-adapted lifting such that if $t_{\alpha,\text{ét}} \in T_p\mathscr{G}^{\vee,\otimes} \otimes \mathbb{Q}_p$ denotes the tensors corresponding to $t_{\alpha,0}$ under the $p$-adic comparison isomorphism, then $t_{\alpha,\text{ét}} \in T_p\mathscr{G}^{\vee,\otimes}$. By Proposition 4.8, such a lifting exists. Then there is an isomorphism

$$T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathfrak{S} \cong \mathfrak{M}(T_p\mathscr{G}^\vee) \tag{5.7.1}$$

taking $t_{\alpha,\text{ét}}$ to $\tilde{t}_\alpha$. In particular since $t_{\alpha,0}$ extends $s_{\alpha,0}$, it takes $s_{\alpha,\text{ét}}$ to $\tilde{s}_\alpha$. Note that since $\mu'_y$ is $G$-conjugate to $\mu_y$, any $(\mathcal{M},\mu'_y)$-adapted lifitng is also a $(\mathcal{G},\mu_y)$-adapted lifting of $\mathscr{G}_0$. Fixing an isomorphism (5.7.1), we may take $U$ to be $T_p\mathscr{G}^\vee$. Since the notion of $(\mathcal{G},\mu_y)$-adapted lifting only depends on the $G$-conjugacy class of $\mu_y$ and its specialization, we may replace $\mu_y$ with $\mu'_y$ (see Remark 4.6). We relabel this $\mu_y$; thus $\mu_y$ is an $M$-valued cocharacter inducing the filtration on $\mathbb{D}(\mathscr{G}_0)\otimes K$.

5.8. Let $g\in G(\mathbb{Q}_p)$, then there is a finite extension $K'/K$ for which $g^{-1}T_p\mathscr{G}$ is stable by $\Gamma_{K'}$ in $T_p\mathscr{G}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ hence corresponds to a $p$-divisible group $\mathscr{G}'$ over $K'$ which is isogenous to $\mathscr{G}$. Let $\mathfrak{M}':=\mathfrak{M}(T_p\mathscr{G}'^\vee)$ and $\mathfrak{M}:=\mathfrak{M}(T_p\mathscr{G}^\vee)$, then the quasi-isogeny $\theta:\mathscr{G}\to\mathscr{G}'$ induces an identification

$$\tilde{\theta}:\mathfrak{M}(T_p\mathscr{G}'^\vee)[1/p]\xrightarrow{\sim}\mathfrak{M}(T_p\mathscr{G}^\vee)[1/p]$$

so that $\mathfrak{M}'=\tilde{g}\mathfrak{M}$ for some $\tilde{g}\in GL(\mathfrak{M}[1/p])$.

**Proposition 5.13.** *(i) $\tilde{g}$ can be taken to be in $G(\mathfrak{S}[1/p])$ under the above identification (5.7.1)*

$$T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathfrak{S}\xrightarrow{\sim}\mathfrak{M}.$$

*(ii) We have $\tilde{g}\in\mathcal{M}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})g\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})$.*

*Proof.* (i) Since $s_{\alpha,\text{ét}}$ are fixed by $G(\mathbb{Q}_p)$, we have $s_{\alpha,\text{ét}}\in T_p\mathscr{G}'^{\vee\otimes}$, and the stabilizer of $s_{\alpha,\text{ét}}$ in $T_p\mathscr{G}'^{\vee\otimes}$ is a parahoric subgroup of $G$. Thus by Proposition 3.2. there is an isomorphism:

$$T_p\mathscr{G}'^\vee\otimes_{\mathbb{Z}_p}\mathfrak{S}\cong\mathfrak{M}' \tag{5.8.1}$$

taking $s_{\alpha,\text{ét}}$ to $\tilde{s}_\alpha$. Under the identification $T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathfrak{S}\cong\mathfrak{M}$, $\tilde{g}$ is given by the composition:

$$T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathfrak{S}\xrightarrow{g}T_p\mathscr{G}'^\vee\otimes_{\mathbb{Z}_p}\mathfrak{S}\to\mathfrak{M}'\xrightarrow{\tilde{\theta}}\mathfrak{M}[1/p]\to T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathfrak{S}[1/p]. \tag{5.8.2}$$

Here, the second map preserves tensors of type $s_\alpha$ and the fourth map preserves tensors of type $t_\alpha$. Thus the composition preserves $\tilde{s}_\alpha$ and so $\tilde{g}\in G(\mathfrak{S}[1/p])$.

(ii) Over $\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}}$ there are canonical identifications

$$T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}}\cong\mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}} \tag{5.8.3}$$

$$T_p\mathscr{G}'^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}}\cong\mathfrak{M}'\otimes_{\mathfrak{S}}\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}} \tag{5.8.4}$$

the first one taking $t_{\alpha,\text{ét}}$ to $\tilde{t}_\alpha$ and the second taking $s_{\alpha,\text{ét}}$ to $\tilde{s}_\alpha$. Thus, if we identify $T_p\mathscr{G}^\vee$ with $T_p\mathscr{G}'^\vee$ via $g$, these isomorphisms differ from the isomorphisms (5.7.1) and (5.8.1) by elements of $\mathcal{M}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})$ and $\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})$ respectively. Since $\tilde{g}$ is identified with the map

$$T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}}\xrightarrow{g}T_p\mathscr{G}'^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}}\xrightarrow{(5.8.4)}\mathfrak{M}'\otimes_{\mathfrak{S}}\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}}$$

$$\xrightarrow{\tilde{\theta}_{\widehat{\mathscr{E}^{\text{ur}}}}}\mathfrak{M}\otimes_{\mathfrak{S}[1/p]}\widehat{\mathscr{E}^{\text{ur}}}\xrightarrow{(5.8.3)}T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\widehat{\mathscr{E}^{\text{ur}}},$$

we obtain $\tilde{g}\in\mathcal{M}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})g\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})$.

□

We will apply the above Proposition in the cases when $M\subset G$ is a Levi subgroup or if we are in the situation of Proposition 5.17 below.

5.9. Using the canonical identification $\mathbb{D}(\mathscr{G}_0)$ with $\varphi^*(\mathfrak{M}/u\mathfrak{M})$, we see that $\tilde{\theta}^{-1}$ induces an isomorphism

$$\mathbb{D}(\mathscr{G}_0)[1/p] \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_0')[1/p].$$

Then $\mathbb{D}(\mathscr{G}_0')$ can be identified with $g_0\mathbb{D}(\mathscr{G}_0)$ for $g_0 = \sigma^{-1}(\tilde{g})|_{u=0} \in G(L)$.

**Proposition 5.14.** *The association $g \mapsto g_0$ induces a well-defined map.*

$$G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to X(\sigma(\{\mu_y\}), b)_K$$

*and we have $\tilde{\kappa}(g) = \tilde{\kappa}(g_0) \in \pi_1(G)_I$.*

*Remark* 5.15. $g_0$ and $b \in G(L)$ both depend on the choice of lifting $\mathscr{G}$ as well as on the choice of the isomorphism (5.7.1). However if we fix the lift $\mathscr{G}$, then modifying the isomorphism (5.7.1) by an $\tilde{h} \in \mathcal{G}(\mathfrak{S})$ which lifts $\sigma(h) \in \mathcal{G}(\mathcal{O}_L)$ conjugates $g_0$ by $h$ and $\sigma$-conjugates $b$ by $h$. We then obtain a map $G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to X(\sigma(\{\mu_y\}), b')_K$ where $b' = h^{-1}b\sigma(h)$, which fits into a commutative diagram:

$$\begin{tikzcd} & X(\sigma(\{\mu_y\}), b)_K \arrow[dd, "\sim"'] \\ G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \arrow[ur] \arrow[dr] & \\ & X(\sigma(\{\mu_y\}), b')_K \end{tikzcd}$$

Here the vertical isomorphism is given by $g_0 \mapsto h^{-1}g_0$.

We will need the following Lemma.

**Lemma 5.16.** *Let $\mu$ and $\mu'$ be cocharacters of $G$ which induce the same filtration on $\mathbb{D} \otimes_{\mathcal{O}_L} K$. Then $\mu$ and $\mu'$ are conjugate in $G$.*

*Proof.* The proof is the same as in [Kis17, Lemma 1.1.9]; for the reader's convenience we recall the argument. The cocharacters $\mu$ and $\mu'$ define the same parabolic $P \subset G \otimes_{\mathcal{O}_L} K$ and induce the same $P/U$ valued cocharacter by the proof of [Kis10, Lemma 1.1.5] where $U$ is the unipotent radical of $P$. Note that the proof of this Lemma only uses the property that the group is reductive which is true for $G$. Let $H$ and $H'$ denote the centralizers of $\mu$ and $\mu'$; these are Levi subgroups of $P$ and hence are conjugate by an element of $U$. Hence $\mu'$ is conjugate to a cocharacter $\mu'' : \mathbb{G}_m \to H$ which induces the same $P/U$ valued cocharacter as $\mu$. Since $H \cong P/U$, $\mu'' = \mu$. $\square$

*Proof of Proposition 5.14.* We identify $\mathbb{D}(\mathscr{G}_0') \otimes_{\mathcal{O}_L} L$ with $\mathbb{D}(\mathscr{G}_0) \otimes_{\mathcal{O}_L} L$, so that we consider

$$\mathbb{D}(\mathscr{G}_0') = g_0\mathbb{D}(\mathscr{G}_0) \subset \mathbb{D}(\mathscr{G}_0) \otimes_{\mathcal{O}_L} L.$$

Under this identification, we have $s_{\alpha,0} \in \mathbb{D}(\mathscr{G}_0')^{\otimes}$ and the stabilizer of these tensors in $\mathbb{D}(\mathscr{G}_0')$ can be identified with $g_0\mathcal{G}_{\mathcal{O}_L}g_0^{-1}$.

By [KP, Corollary 3.3.10], there exists a $G$-valued cocharacter $\mu_y'$ defined over a finite extension $K''/L$ such that $\mathscr{G}'$ is a $(g_0\mathcal{G}_{\mathcal{O}_L}g_0^{-1}, \mu_y')$ adapted-lifting of $\mathscr{G}' \otimes_{\mathcal{O}_K} k$. Upon enlarging $K$ (but with $K/L$ still finite) we assume all the cocharacters above are defined over $K$. Then we have three filtrations on $\mathbb{D} \otimes_{\mathcal{O}_L} K$: the one induced by $\mu_y$, the canonical filtration corresponding to the Galois representation $T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong T_p\mathscr{G}'^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the one induced by $\mu_y'$. The second filtration is induced by a $G$-valued cocharacters $\mu$ and $\mu'$ which are conjugate to $\mu_y$ and $\mu_y'$ respectively. By Lemma 5.16, $\mu$ and $\mu'$ are $G$-conjugate.

Thus $g_0^{-1}\mu'_y g_0$ induces a filtration on $\mathbb{D}\otimes_{\mathcal{O}_L} K$ corresponding to a point in $M^{\mathrm{loc}}_{\mathcal{G},\mu_y^{-1}}(K)$. The specialization of this point gives a filtration on $\mathbb{D}(\mathscr{G}_0)\otimes_{\mathcal{O}_K} k$ which lies in $M^{\mathrm{loc}}_{\mathcal{G},\mu_y^{-1}}(k)$. This filtration is given by the reduction of $g_0^{-1}\sigma^{-1}(b^{-1}g_0)p \mod p$, hence by Propositon 3.4 we have

$$g_0^{-1}b\sigma(g_0)\in\mathcal{G}(\mathcal{O}_L)\sigma(\dot{w})\mathcal{G}(\mathcal{O}_L)$$

where $w\in\mathrm{Adm}_K(\{\mu_y\})$, i.e. $g_0\in X(\sigma(\{\mu_y\}),b)$.

To show $\kappa(g)=\kappa_G(g_0)$, note that $\tilde{g}$ gives a $k[[u]]^{\mathrm{perf}}$-point of $Gr_{\mathcal{G}}$, where $Gr_{\mathcal{G}}$ is the Witt vector affine flag variety of [Zhu17], [BS17], and the superscript perf denotes the perfection of a ring in characteristic $p$. For $\mathbf{k}$ a perfect field of characterstic $p$, the Kottwitz homomorphism induces a map

$$\tilde{\kappa}_G: Gr_{\mathcal{G}}(\mathbf{k})\to\pi_1(G)_I$$

and this induces an isomorphism $\pi_0(Gr_{\mathcal{G}})\cong\pi_1(G)_I$. In particular, $\tilde{\kappa}_G$ is a locally constant function. Let $h\in Gr_{\mathcal{G}}(\overline{k((u))})$ be the generic point of $\tilde{g}$, then by Proposition 5.13 (ii) we have $\tilde{\kappa}_G(h)=\tilde{\kappa}_G(g)$, hence $\tilde{\kappa}_G(\sigma^{-1}(g_0))=\tilde{\kappa}_G(g)$. Since $g$ is $\sigma$-invariant, we hav $\tilde{\kappa}_G(g_0)=\tilde{\kappa}_G(g)$.

□

**Proposition 5.17.** *Let $\mathcal{H}$ be a parahoric subgroup scheme of some reductive group $H$ over $\mathbb{Q}_p$. Suppose $f:\mathcal{G}\to\mathcal{H}$ is a surjection such that the following conditions hold:*

*(i) The composition of $f$ with the map $\Gamma_K\to\mathcal{G}(\mathbb{Z}_p)$ coming from the action of the Galois group $\Gamma_K$ on $T_p\mathscr{G}$ factors through the center $Z_{\mathcal{H}}$ of $\mathcal{H}$.*

*(ii) The connected component $\mathcal{G}'$ of the identity of $f^{-1}(Z_{\mathcal{H}})$ has reductive generic fiber $G'$ and $\mathcal{G}'$ is a parahoric subgroup of $G'$.*

*(iii) The kernel of $f$ is a smooth group scheme over $\mathbb{Z}_p$.*

*Then we may choose the isomorphism (5.7.1) such that for every $g\in G(\mathbb{Q}_p)$, we have*

$$f(g)=f(g_0)\in H(L)/\mathcal{H}(\mathcal{O}_L),$$

*Proof.* By assumption $\mathcal{G}'$ is a parahoric subgroup of its reductive generic fibre $G'$. Upon replacing $K$ by a finite extension, we may assume $\Gamma_K\to\mathcal{G}(\mathbb{Z}_p)$ factors through $\mathcal{G}'(\mathbb{Z}_p)$, and we may extend $s_{\alpha,\acute{e}t}\in T_p\mathscr{G}^{\vee,\otimes}$ to a set $t_{\beta,\acute{e}t}$ of $\Gamma_K$-invariant tensors whose stabilizer is $\mathcal{G}'$. By Proposition 4.4, we obtain tensors $t_{\beta,0}\in\mathbb{D}(\mathscr{G}_0)^{\otimes}$ whose stabilizer $\mathcal{G}'_{\mathcal{O}_L}$ can be identified with $\mathcal{G}'\otimes_{\mathbb{Z}_p}\mathcal{O}_L$. By [KP, Corollary 3.3.10] there is a $G'$-valued cocharacter $\mu'_y$ such that the filtration it induces lifts the one on $\mathbb{D}(\mathscr{G}_0)\otimes_{\mathcal{O}_L} k$ and is $G'$ conjugate to a $G'$-valued cocharacter $\mu'$ inducing the filtration on $\mathbb{D}(\mathscr{G}_0)\otimes K$. There is a cocharacter $\mu$ of $G$ which induces the filtration on $\mathbb{D}(\mathscr{G}_0)\otimes_{\mathcal{O}_L} K$, which is $G$-conjugate to $\mu_y$. Since $\mu$ and $\mu'$ are $G$-conjugate by Lemma 5.16, $\mu_y$ and $\mu'_y$ are conjugate in $G$. In other words, $\mu'_y$ satisfies the conditions in Assumption 5.12, hence we may apply the construction in 5.13. We fix the $\mathfrak{S}$-linear bijection (5.7.1)

$$T_p\mathscr{G}^{\vee}\otimes_{\mathbb{Z}_p}\mathfrak{S}\cong\mathfrak{M}(T_p\mathscr{G}^{\vee})$$

so that it takes $t_{\beta,\acute{e}t}$ to $\tilde{t}_\beta$.

Let $g\in G(\mathbb{Q}_p)$, applying the previous construction we obtain $\tilde{g}\in G(\mathfrak{S}[1/p])$ and by Proposition 5.13 we have $\tilde{g}=hgi$ where $h\in\mathcal{G}'(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})$ and $i\in\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})$. Since $\mathcal{G}'\subset f^{-1}(Z_{\mathcal{H}})$, we have $f(g^{-1}hg)=f(h)$, so

$$p:=g^{-1}hgh^{-1}\in\mathcal{P}(\widehat{\mathscr{E}^{\mathrm{ur}}})$$

where $\mathcal{P}:=\ker(f:\mathcal{G}\to\mathcal{H})$ is a smooth group scheme over $\mathbb{Z}_p$ by assumption. Thus

$$f(\tilde{g})=f(gphi)=f(g)f(hi)$$

and we obtain

$$f(hi) \in \mathcal{H}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}}) \cap \mathcal{H}(\mathfrak{S}[1/p]) = \mathcal{H}(\mathfrak{S}).$$

The induced map on points $f : \mathcal{G}(\mathfrak{S}) \to \mathcal{H}(\mathfrak{S})$ is surjective. Indeed the cokernel injects into $H^1(\mathfrak{S}, \mathcal{P}) = 0$ since $\mathcal{P}$ is smooth and $\mathfrak{S}$ is strictly Henselian. We let $m \in \mathcal{G}(\mathfrak{S})$ such that $f(m) = f(hi)$. Thus $f(phim^{-1}) = 1$, and we have

$$phim^{-1} \in \mathcal{P}(\widehat{\mathscr{E}^{\mathrm{ur}}}) \cap \mathcal{G}(\mathfrak{S}[1/p]).$$

But this last group is just $\mathcal{P}(\mathfrak{S}[1/p])$. Thus $\tilde{g} = g(phim^{-1})m \in g\mathcal{P}(\mathfrak{S}[1/p])\mathcal{G}(\mathfrak{S})$, and hence

$$g_0 = \sigma^{-1}(\tilde{g})|_{u=0} \in g\mathcal{P}(L)\mathcal{G}(\mathcal{O}_L)$$

so that $f(g_0) = f(\sigma(g)) = f(g) \in H(L)/\mathcal{H}(\mathcal{O}_L)$. □

**Lemma 5.18.** *(i) The map $\tilde{\kappa}_G|_{G(\mathbb{Q}_p)} : G(\mathbb{Q}_p) \to \pi_1(G)_I^\sigma$ is surjective.*

*(ii) Let $g_{\mathrm{ad}} \in G_{\mathrm{ad}}(\mathbb{Q}_p)$. Suppose the image of $g_{\mathrm{ad}}$ under $\tilde{\kappa}_{G_{\mathrm{ad}}}$ lifts to an element of $\pi_1(G)_I^\sigma$, then the image of $g_{\mathrm{ad}}$ in $G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p)$ is in the image of*

$$G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to G_{\mathrm{ad}}(\mathbb{Q}_p)/\mathcal{G}_{\mathrm{ad}}(\mathbb{Z}_p)$$

*Proof.* (i) By [HR08, Lemma 5], the Kottwitz homomorphism induces an exact sequence:

$$0 \to \mathcal{T}^\circ(\mathcal{O}_L) \to T(L) \xrightarrow{\tilde{\kappa}_G} \pi_1(G)_I \to 0$$

where $\mathcal{T}^\circ$ is the connected Neron model of $T$. Since $H^1(\mathbb{Z}_p, \mathcal{T}^\circ) = 0$ we have

$$\tilde{\kappa}_G|_{T(\mathbb{Q}_p)} : T(\mathbb{Q}_p) \to \pi_1(G)_I^\sigma$$

is surjective, hence $\tilde{\kappa}_G|_{G(\mathbb{Q}_p)}$ is surjective.

(ii) By part (i), there exists $g \in G(\mathbb{Q}_p)$ such that $\tilde{\kappa}_G(g) \in \pi_1(G)_I^\sigma$ lifts $\tilde{\kappa}_{G_{\mathrm{ad}}}(g_{\mathrm{ad}})$. Replacing $g_{\mathrm{ad}}$ with $g_{\mathrm{ad}}g^{-1}$, we may assume $\tilde{\kappa}_{G_{\mathrm{ad}}}(g_{\mathrm{ad}})$ is trivial.

By the Iwahori decomposition there exists $w_{\mathrm{ad}} \in W_{\mathrm{ad}}^\sigma$, $g_1, g_2 \in \mathcal{G}_{\mathrm{ad}}(\mathbb{Z}_p)$ such that $g_{\mathrm{ad}} = g_1\dot{w}_{\mathrm{ad}}g_2$, with $\dot{w}_{\mathrm{ad}} \in G_{\mathrm{ad}}(\mathbb{Q}_p)$ a lift of $w_{\mathrm{ad}}$. Changing our choice of torus $T$ to $T' := g_1Tg_1^{-1}$ we may assume $g_{\mathrm{ad}} = \dot{w}_{\mathrm{ad}}g_2$. Since $\tilde{\kappa}_{G_{\mathrm{ad}}}(g)$ is trivial, $w_{\mathrm{ad}}$ lies in the affine Weyl group $W_{a,\mathrm{ad}}$ of $G_{\mathrm{ad}}$. But the natural projection induces an isomorphism $W_a \cong W_{a,\mathrm{ad}}$ hence, $w_{\mathrm{ad}}$ lifts to an element $w$ of $W_a^\sigma$. Thus we may take $\dot{w} \in G_{\mathrm{der}}(\mathbb{Q}_p)$ lifting $w$, and the image of $\dot{w}$ in $G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p)$ gives the required lifting. □

With the above notations we have the following.

**Proposition 5.19.** *Assume $b = \sigma(\dot{\tau}_{\{\mu_y\}})$. Then for any $(\mathcal{G}, \mu_y)$-adapted lifting $\mathscr{G}$ such that $s_{\alpha,\text{ét}} \in T_p\mathscr{G}^{\vee,\otimes} \subset T_p\mathscr{G}^{\vee,\otimes} \otimes \mathbb{Q}_p$, the map*

$$G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to \pi_0(X(\sigma(\{\mu_y\}), \sigma(\dot{\tau}_{\{\mu_y\}}))_K), \quad g \mapsto g_0$$

*defined above is surjective.*

*Proof.* We let $\mu_y^{\mathrm{ad}}$ denote the cocharacter of $G_{\mathrm{ad}}$ induced by $\mu_y$. Let $G_{\mathrm{ad}} = G_1 \times G_2$ where $\mu_y^{\mathrm{ad}}$ induces the trivial cocharacter of $G_1$ and induces a non-trivial cocharacter in every $\mathbb{Q}_p$-factor of $G_2$. We write $\mathcal{G}_{\mathrm{ad}}$ for the parahoric in $G_{\mathrm{ad}}$ corresponding to $\mathcal{G}$; it decomposes as $\mathcal{G}_1 \times \mathcal{G}_2$ where $\mathcal{G}_1$ and $\mathcal{G}_2$ are parahorics of $G_1$ and $G_2$ respectively. By Theorem 5.9 (see also [HZ, Theorem 8.1] for the case of general parahorics) and using [HZ, §6.1], there is an isomorphism

$$\pi_0(X(\sigma(\{\mu_y^{\mathrm{ad}}\}), \sigma(\dot{\tau}_{\{\mu_y^{\mathrm{ad}}\}}))_K) \cong G_1(\mathbb{Q}_p)/\mathcal{G}_1(\mathbb{Z}_p) \times \pi_1(G_2)_I^\sigma. \tag{5.9.1}$$

We pick the isomorphism (5.7.1) so that the conclusion of Proposition 5.17 holds for the projection $\mathcal{G} \to \mathcal{G}_1$. Note that condition (iii) in Proposition 5.17 holds since the kernel is an extension of $\mathcal{G}_2$ by

$\mathcal{Z}_G$, the center of $\mathcal{G}$ which is smooth, by [KP, Proposition 1.1.4]. Morever, its generic fiber is reductive since it is a central extension of a reductive group hence condition (ii) holds.

Let $h \in X(\sigma(\{\mu_y\}), \sigma(\dot{\tau}_{\{\mu_y\}}))_K$ and $h_{\mathrm{ad}}$ the image of $h_{\mathrm{ad}}$ in $X(\sigma(\{\mu_y^{\mathrm{ad}}\}), \sigma(\dot{\tau}_{\{\mu_y^{\mathrm{ad}}\}}))_{K_{\mathrm{ad}}}$, where we write $K_{\mathrm{ad}}$ for the corresponding set of simple reflections for $G_{\mathrm{ad}}$. Then by Lemma 5.18 (i), there exists $g_{\mathrm{ad}} \in G_{\mathrm{ad}}(\mathbb{Q}_p)/\mathcal{G}_{\mathrm{ad}}(\mathbb{Z}_p)$ mapping to the image of $h_{\mathrm{ad}}$ on the right hand side of (5.9.1). Since $\tilde{\kappa}_{G_{\mathrm{ad}}}(g_{\mathrm{ad}})$ lifts to the element $\tilde{\kappa}_G(h) \in \pi_1(G)_I^\sigma$, Lemma 5.18 (ii) implies that $g_{\mathrm{ad}}$ lifts to an element $g \in G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p)$. By Proposition 5.14 and 5.17, the image of $g_0$ is equal to the image of $h$ in $\pi_0(X(\sigma(\{\mu_y^{\mathrm{ad}}\}), \sigma(\dot{\tau}_{\{\mu_y^{\mathrm{ad}}\}}))_{K_{\mathrm{ad}}})$. By [HZ, Corollary 4.4], there exists $z \in Z(\mathbb{Q}_p)$ such that $g_0 z = h$ in $\pi_0(X(\sigma(\{\mu_y\}), \sigma(\dot{\tau}_{\{\mu_y\}})_K)$. By the functoriality of the construction, $(gz)_0 = g_0 z_0 = g_0 z = h$ in $\pi_0(X(\sigma(\{\mu_y\}), \sigma(\dot{\tau}_{\{\mu_y\}}))_K)$. ☐

More generally, there is the following.

**Proposition 5.20.** *Assume Conjecture 5.4 holds and that $[b] \in B(G, \sigma(\{\mu\}))$ is Hodge–Newton indecomposable. Then for any $(\mathcal{G}, \mu_y)$-adapted lifting $\mathscr{G}$ such that $s_{\alpha,\text{ét}} \in T_p\mathscr{G}^{\vee,\otimes} \subset T_p\mathscr{G}^{\vee,\otimes} \otimes \mathbb{Q}_p$, the map*

$$G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to \pi_0(X(\sigma(\{\mu_y\}), b)_K), \quad g \mapsto g_0$$

*is surjective.*

*Proof.* As in the proof of Proposition 5.19, $\mu_y^{\mathrm{ad}}$ will denote the cocharacter of $G_{\mathrm{ad}}$ induced by $\mu_y$ and $G_{\mathrm{ad}} = G_1 \times G_2$ where $\mu_y^{\mathrm{ad}}$ induces the trivial cocharacter of $G_1$ and is non-trivial in every simple factor of $G_2$; we write $\{\mu_1\}$ and $\{\mu_2\}$ for the induced conjugacy class of cocharacters for $G_1$ and $G_2$ respectively. Similarly, writing $\mathcal{G}_{\mathrm{ad}}$ for the parahoric in $G_{\mathrm{ad}}$ corresponding to $\mathcal{G}$, $\mathcal{G}_{\mathrm{ad}}$ decomposes as $\mathcal{G}_1 \times \mathcal{G}_2$. We write $b_{\mathrm{ad}} = b_1 \times b_2$ in the decomposition $G_{\mathrm{ad}}(L) = G_1(L) \times G_2(L)$. It is easy to see from the definition that $[b_{\mathrm{ad}}] \in B(G_{\mathrm{ad}}, \{\mu_y^{\mathrm{ad}}\})$ is Hodge–Newton indecomposable, and it follows from 5.3 that $b_2$ is Hodge–Newton irreducible in $G_2$. Therefore assuming Conjecture 5.4, there is an identification

$$\pi_0(X(\sigma(\{\mu_y^{\mathrm{ad}}\}), b_{\mathrm{ad}})_{K_{\mathrm{ad}}}) \cong G_1(\mathbb{Q}_p)/\mathcal{G}_1(\mathbb{Z}_p) \times c_{b_2,\mu_2}\pi_1(G_2)_I^\sigma.$$

The same argument as in Proposition 5.19 then shows that the map $g \mapsto g_0$ is surjective. ☐

## 6. Shimura varieties

6.1. We recall the construction of the integral models of Shimura varieties of Hodge type in [KP].

Let $G$ be a reductive group over $\mathbb{Q}$ and $X$ a conjugacy class of homomorphisms

$$h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$$

such that $(G, X)$ is a Shimura datum in the sense of [Del71].

Let $c$ be the complex conjugation. Then $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}) \cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \cong \mathbb{C}^\times \times c^*(\mathbb{C}^\times)$ and we write $\mu_h$ for the cocharacter given by

$$\mathbb{C}^\times \to \mathbb{C}^\times \times c^*(\mathbb{C}^\times) \xrightarrow{h} G(\mathbb{C})$$

We set $w_h := \mu_h^{-1}(\mu_h^c)^{-1}$.

Let $\mathbb{A}_f$ denote the ring of finite adeles and $\mathbb{A}_f^p$ the subring of $\mathbb{A}_f$ with trivial $p$-component. Let $\mathrm{K}_p \subset G(\mathbb{Q}_p)$ and $\mathrm{K}^p \subset G(\mathbb{A}_f^p)$ be compact open subgroups and write $\mathrm{K} := \mathrm{K}_p\mathrm{K}^p$. Then for $\mathrm{K}^p$ sufficiently small

$$Sh_{\mathrm{K}}(G, X)_{\mathbb{C}} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\mathrm{K} \tag{6.1.1}$$

arises as the complex points of an algebraic variety over $\mathbb{C}$, which has a model over the reflex field $\mathbf{E} := E(G, X)$; this is a number field and is the field of definition of the conjugacy class of $\mu_h$.

We will also consider the pro-varieties

$$Sh(G,X) := \lim_{\leftarrow \mathrm{K}} Sh_{\mathrm{K}}(G,X)$$

$$Sh_{\mathrm{K}_p}(G,X) := \lim_{\leftarrow \mathrm{K}^p} Sh_{\mathrm{K}_p\mathrm{K}^p}(G,X)$$

6.2. Let $V$ be a vector space over $\mathbb{Q}$ of dimension $2g$ equipped with an alternating bilinear form $\psi$; we write $V_R = V \otimes_{\mathbb{Q}} R$ for a $\mathbb{Q}$-algebra $R$. Let $GSp = GSp(V,\psi)$ denote the corresponding group of symplectic similitudes. The Siegel half-space is defined to be the set of homomorphisms

$$h : \mathbb{S} \to GSp_{\mathbb{R}}$$

such that:

(1) The Hodge structure on $V_{\mathbb{C}}$ induced by $h$ is of type $(-1,0),(0,-1)$, i.e.

$$V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$$

(2) $(x,y) \mapsto \psi(x,h(i)y)$ is positive or negative definite on $V_{\mathbb{R}}$.

For the rest of this section we assume there is an embedding of Shimura data

$$\rho : (G,X) \to (GSp, S^{\pm})$$

We sometimes write $G$ for $G_{\mathbb{Q}_p}$ when there is no risk of confusion. For the rest of the paper we will assume the following condition holds

(6.2.1) $\qquad G$ splits over a tamely ramified extension of $\mathbb{Q}_p$ and $p \nmid |\pi_1(G_{der})|$.

Let $\mathcal{G}$ be a connected parahoric subgroup of $G$, i.e. $\mathcal{G} = \mathcal{G}_x = \mathcal{G}_x^\circ$ for some $x \in B(G,\mathbb{Q}_p)$. We assume that the compact open $K_p \subset G(\mathbb{Q}_p)$ is identified with $\mathcal{G}(\mathbb{Z}_p)$. By [KP, 2.3.15], upon replacing $\rho$ by another symplectic embedding, there is a closed immersion $\mathcal{G} \to \mathcal{GSP}$, where $\mathcal{GSP}$ is a parahoric group scheme of $GSp$ corresponding to the stabilizer of a lattice $V_{\mathbb{Z}_p} \subset V$. Upon scaling $V_{\mathbb{Z}_p}$, we may assume $V_{\mathbb{Z}_p}^\vee \subset V_{\mathbb{Z}_p}$ and we let $p^d = |V_{\mathbb{Z}_p}/V_{\mathbb{Z}_p}^\vee|$. This induces a closed immersion of local models

$$M^{\mathrm{loc}}_{\mathcal{G},\mu_h} \to M^{\mathrm{loc}}_{\mathcal{GSP},\mu_h} \otimes \mathcal{O}_E$$

where $E$ is the local reflex of $\mu_h$ in $G_{\mathbb{Q}_p}$.

6.3. Let $V_{\mathbb{Z}_{(p)}} = V_{\mathbb{Z}_p} \cap V$, we write $G_{\mathbb{Z}_{(p)}}$ for the Zariski closure of $G$ in $GL(V_{\mathbb{Z}_{(p)}})$, then $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \cong \mathcal{G}$. The choice of $V_{\mathbb{Z}_{(p)}}$ gives rise to an interpretation of $Sh_{\mathrm{K}'}(GSp, S^{\pm})$ as a moduli space of abelian varieties and hence an integral model over $\mathbb{Z}_{(p)}$ which we now describe. We let $\mathrm{K}' = \mathrm{K}'_p \mathrm{K}'^p$ where $\mathrm{K}'_p = \mathcal{GSP}(\mathbb{Z}_p)$ and $\mathrm{K}'^p \subset GSp(\mathbb{A}_f^p)$ is a compact open.

Let $\mathcal{A}$ be an abelian scheme of dimension $g$ over a scheme $T$. We write

$$\widehat{V}(\mathcal{A}) = \lim_{\leftarrow p\nmid n} \mathcal{A}[n]$$

Consider the category obtained from the category of abelian varieties by tensoring the Hom groups by $\mathbb{Z}_{(p)}$. An object in this category will be called an *abelian variety up to prime to p isogeny* and an isomorphism in this category will be called a prime to $p$ isogeny.

Let $\mathcal{A}$ be an abelian variety up to prime to $p$ isogeny and let $\mathcal{A}^*$ be the dual abelian variety. By a weak polarization we mean an equivalence class of quasi-isogenies $\lambda : \mathcal{A} \to \mathcal{A}^*$ such that $p^d$ exactly divides $\deg \lambda$ and some multiple of $\lambda$ is a polarization. Two such quasi-isogenies are equivalent if they differ by a multiple of $\mathbb{Z}_{(p)}^\times$.

Let $(\mathcal{A}, \lambda)$ be a pair as above. We write $\underline{\mathrm{Isom}}_{\lambda,\psi}(\widehat{V}(\mathcal{A}), V_{\mathbb{A}_f^p})$ for the (pro)-étale sheaf of isomorphisms $\widehat{V}(\mathcal{A}) \cong V_{\mathbb{A}_f^p}$ which preserves the pairings induced $\lambda$ and $\psi$ up to a $\mathbb{A}_f^{p\times}$-scalar.

We write $\mathscr{A}_{g,d,\mathrm{K}'}(T)$ for the set of triples $(\mathcal{A}, \lambda, \epsilon^p_{\mathrm{K}'})$ consisting of an abelian variety up to prime to $p$ isogeny $\mathcal{A}$ over $T$ together with a weak polarization $\lambda : \mathcal{A} \to \mathcal{A}^*$ and a global section

$$\epsilon^p_{\mathrm{K}'} \in \Gamma(T, \underline{\mathrm{Isom}}_{\lambda,\psi}(\widehat{V}(\mathcal{A}), V_{\mathbb{A}_f^p})/\mathrm{K}'^p)$$

For $\mathrm{K}'^p$ sufficiently small, $\mathscr{A}_{g,d,\mathrm{K}'}$ is representable by a quasi-projective scheme over $\mathbb{Z}_{(p)}$ which we denote by $\mathscr{S}_{\mathrm{K}'}(GSp, S^{\pm})$.

6.4. For the rest of this paper we fix an algebraic closure $\overline{\mathbb{Q}}$, and for each place $v$ of $\mathbb{Q}$ an algebraic closure $\overline{\mathbb{Q}}_v$ together with an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$.

By [Kis10, Lemma 2.1.2], we can choose $\mathrm{K}'$ such that $\iota$ induces a closed immesion:

$$Sh_{\mathrm{K}}(G, X) \hookrightarrow Sh_{\mathrm{K}'}(GSp, S^{\pm})_{\mathbf{E}}$$

defined over $\mathbf{E}$. The choice of embedding $\mathbf{E} \to \overline{\mathbb{Q}}_p$ determines a place $v$ of $\mathbf{E}$. Write $\mathcal{O}_{\mathbf{E},(v)}$ for the localisation of $\mathcal{O}_{\mathbf{E}}$ at $v$, $E$ the completion of $\mathbf{E}$ at $v$ and $\mathcal{O}_E$ the ring of intgers of $E$. We assume the residue field has $q = p^r$ elements and as before $k$ will denote an algebraic closure of $\mathbb{F}_q$. We define $\mathscr{S}_{\mathrm{K}}(G, X)^-$ to be the Zariski closure of $Sh_{\mathrm{K}}(G, X)$ inside $\mathscr{S}_{\mathrm{K}'}(GSp, S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{\mathbf{E},(v)}$, and $\mathscr{S}_{\mathrm{K}}(G, X)$ to be the normalization of $\mathscr{S}_{\mathrm{K}}(G, X)^-$. By construction, for $\mathrm{K}_1^p \subset \mathrm{K}_2^p$ compact open subgroups of $G(\mathbb{A}_f^p)$, there are well defined maps $\mathscr{S}_{\mathrm{K}_p\mathrm{K}_1^p}(G, X) \to \mathscr{S}_{\mathrm{K}_p\mathrm{K}_2^p}(G, X)$ and we write

$$\mathscr{S}_{\mathrm{K}_p}(G, X) := \varprojlim_{\mathrm{K}^p} \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G, X).$$

Under these assumptions we have the following.

**Theorem 6.1** ([KP] Theorem 4.2.2, Theorem 4.2.7). *(i) The $\mathcal{O}_{\mathbf{E},(v)}$ scheme $\mathscr{S}_{\mathrm{K}_p}(G, X)$ is a flat $G(\mathbb{A}_f^p)$-equivariant extension of $Sh_{\mathrm{K}_p}(G, X)$.*

*(ii) Let $\widehat{U}_{\overline{x}}$ be the completion of $\mathscr{S}_{\mathrm{K}}(G, X)^-$ at some $k$-point $\overline{x}$. There exists a point $\overline{x}' \in M^{\mathrm{loc}}_{\mathcal{G},\mu_h}(k)$ such that the irreducible components of $\widehat{U}_{\overline{x}}$ are isomorphic to the completion $\widehat{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}$ of $M^{\mathrm{loc}}_{\mathcal{G},\mu_h}$ at $\overline{x}'$. Moreover $\mathscr{S}_{\mathrm{K}}(G, X)$ fits in a local model diagram:*

$$\begin{tikzcd}
 & \widetilde{\mathscr{S}}_{\mathrm{K}}(G, X)_{\mathcal{O}_E} \arrow[dl, "q"'] \arrow[dr, "\pi"] & \\
\mathscr{S}_{\mathrm{K}}(G, X)_{\mathcal{O}_E} & & M^{\mathrm{loc}}_{\mathcal{G},\mu_h}
\end{tikzcd}$$

*where $q$ is a $\mathcal{G}$-torsor and $\pi$ is smooth of relative dimension* $\dim G$.

6.5. We will need a more explicit description of $\widehat{U}_{\overline{x}}$ and this local model diagram for the next section. To do this we will need to introduce Hodge cycles.

By [Kis10, 1.3.2], the subgroup $G_{\mathbb{Z}_{(p)}}$ is the stabilizer of a collection of tensors $s_\alpha \in V^{\otimes}_{\mathbb{Z}_{(p)}}$. Let $h : \mathcal{A} \to \mathscr{S}_{\mathrm{K}}(G, X)$ denote the pullback of the universal abelian variety on $\mathscr{S}_{\mathrm{K}'}(GSp, S^{\pm})$ and let $V_{\mathrm{B}} := R^1 h_{\mathrm{an},*}\mathbb{Z}_{(p)}$, where $h_{\mathrm{an}}$ is the map of complex analytic spaces associated to $h$. We also let $\mathcal{V} = R^1 h_* \Omega^\bullet$ be the relative de Rham cohomology of $\mathcal{A}$. Using the de Rham isomorphism, the $s_\alpha$ give rise to a collection of Hodge cycles $s_{\alpha,\mathrm{dR}} \in \mathcal{V}^{\otimes}_{\mathbb{C}}$, where $\mathcal{V}_{\mathbb{C}}$ is the complex analytic vector bundle associated to $\mathcal{V}$. By [Kis06, §2.2.], these tensors are defined over $E$, and in fact over $\mathcal{O}_{E,(v)}$ by [KP, Proposition 4.2.6].

Similarly for a finite prime $l \neq p$, we let $\mathcal{V}_l = R^1 h_{\text{ét}*}\mathbb{Q}_l$ and $\mathcal{V}_p = R^1 h_{\eta,\text{ét}*}\mathbb{Z}_p$ where $h_\eta$ is the generic fibre of $h$. Using the étale-Betti comparison isomorphism, we obtain tensors $s_{\alpha,l} \in \mathcal{V}_l^{\otimes}$ and $s_{\alpha,\text{ét}} \in \mathcal{V}_p^{\otimes}$ which are Galois invariant by the same argument as in [Kis10, Lemma 2.2.1].

For $T$ an $\mathcal{O}_{\mathbf{E}_{(p)}}$-scheme (resp $\mathbf{E}$-scheme, resp. $\mathbb{C}$-scheme), $* = l$ or dR (resp. ét, resp. B) and $x \in \mathscr{S}_{\mathrm{K}_p}(G,X)(T)$, we write $\mathcal{A}_x$ for the pullback of $\mathcal{A}$ to $x$ and $s_{\alpha,*,x}$ for the pullback of $s_{\alpha,*}$ to $x$.

As in [Kis10, 3.4.2.], if $x \in \mathscr{S}_{K_p}(G,X)(T)$ corresponds to a triple $(\mathcal{A}_x, \lambda, \epsilon^p_{\mathrm{K}'})$, then $\epsilon^p_{K'}$ can be promoted to a section:

$$\epsilon^p_{\mathrm{K}} \in \Gamma(T, \underline{\mathrm{Isom}}_{\lambda,\psi}(\widehat{V}^p(\mathcal{A}_x), V_{\mathbb{A}_f^p})/K^p)$$

which takes $s_{\alpha,l,x}$ to $s_\alpha$ $(l \neq p)$.

6.6. Recall $k$ is an algebraic closure of $\mathbb{F}_q$ and $L = W(k)[1/p]$. Let $\overline{x} \in \mathscr{S}_{\mathrm{K}}(G,X)(k)$ and $x \in \mathscr{S}_{\mathrm{K}}(G,X)(\mathcal{O}_K)$ a point lifting $\overline{x}$, where $K/L$ is a finite extension.

Let $\mathscr{G}_x$ denote the $p$-divisible group associated to $\mathcal{A}_x$ and $\mathscr{G}_{x,0}$ its special fiber. Then $T_p\mathscr{G}_x^\vee$ is identified with $H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Z}_p)$ and we obtain $\Gamma_K$-invariant tensors $s_{\alpha,\text{ét},x} \in T_p\mathscr{G}^{\vee\otimes}$ whose stabilizer can be identified with $\mathcal{G}$. We may thus apply the constructions of §4 and we obtain $\varphi$-invariant tensors $s_{\alpha,0,x} \in \mathbb{D}(\mathscr{G}_{x,0})$ whose stabilizer group $\mathcal{G}_{\mathcal{O}_L}$ can be identified with $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. The filtration on $\mathbb{D} \otimes_{\mathcal{O}_L} K$ corresponding to $\mathscr{G}_x$ is induced by a $G$-valued cocharacter conjugate to $\mu_h^{-1}$. By [KP, Corollary 3.3.10], there is an isomorphism:

$$\mathbb{D}(\mathscr{G}_x)(\mathcal{O}_K) \cong \mathbb{D}(\mathscr{G}_{x,0}) \otimes_{\mathcal{O}_L} \mathcal{O}_K$$

taking $s_{\alpha,\mathcal{O}_K}$ to $s_{\alpha,0,x}$ lifting the identity mod$\pi$, where $s_{\alpha,\mathcal{O}_K} \in \mathbb{D}(\mathscr{G}_x)(\mathcal{O}_K)^{\otimes}$ denotes the image of $\tilde{s}_\alpha \in \mathfrak{M}(T_p\mathscr{G}_x^\vee)^{\otimes}$ (in fact the $s_{\alpha,\mathcal{O}_K}$ are equal to $s_{\alpha,\mathrm{dR},x}$ under the identification $\mathbb{D}(\mathscr{G}_x)(\mathcal{O}_K) \cong \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_x)$ by the compatibility of $\mathfrak{M}$ with comparison isomorphisms). Moreover, upon enlarging $K$, there is a $G$-valued cocharacter $\mu_y$ which is $G$-conjugate to $\mu_h^{-1}$ and induces a filtration on $\mathbb{D}(\mathscr{G}_{x,0}) \otimes_{\mathcal{O}_L} \mathcal{O}_K$ lifting the filtration on $\mathbb{D}(\mathscr{G}_{x,0}) \otimes k$. Thus we have a notion of $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted liftings of $\mathscr{G}_{x,0}$ as in section §4 and by definition $\mathscr{G}_x$ is a $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted lifting.

As before we let $P \subset GL(\mathbb{D})$ be a parabolic lifting $P_0$. We obtain formal local models $\widehat{M}^{\mathrm{loc}}_{\mu_y^{-1}} = \mathrm{Spf}A$ and $\widehat{M}^{\mathrm{loc}}_{\mathcal{G},\mu_y^{-1}} = \mathrm{Spf}A_{\mathcal{G}}$, and the filtration corresponding to $\mu_y$ is given by a point $y : A_{\mathcal{G}} \to \mathcal{O}_K$.

**Proposition 6.2.** *Let $\widehat{U}_{\overline{x}}$ be the completion of $\mathscr{S}^-_{\mathrm{K}}(G,X)$ at $\overline{x}$.*

*(i) $\widehat{U}_{\overline{x}}$ can be identified with a closed subspace of SpfA containing Spf$A_{\mathcal{G}}$. Moreover Spf$A_{\mathcal{G}}$ is an irreducible component of $\widehat{U}_{\overline{x}}$.*

*(ii) Let $x' \in \mathscr{S}_{\mathrm{K}}(G,X)(\mathcal{O}_{K'})$ whose special fibre $\overline{x}'$ maps to the image of $\overline{x}$ in $\mathscr{S}^-_{\mathrm{K}}(G,X)$. Then $s_{\alpha,0,x'} = s_{\alpha,0,x} \in \mathbb{D}(\mathscr{G}_{x,0})$ if and only if $x$ and $x'$ lie on the same irreducible component of $\widehat{U}_{\overline{x}}$.*

*(iii) A deformation $\mathscr{G}$ of $\mathscr{G}_{x,0}$ to $\mathcal{O}_{K'}$ corresponds to a point on the same irreducible component of $\widehat{U}_{\overline{x}}$ as $x$ if and only if $\mathscr{G}$ is $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted.*

*Proof.* This is effectively [KP, Proposition 4.2.2] we recall the argument for the reader's convenience.

Recall we assumed that $G$ splits over a tamely ramified extension of $\mathbb{Q}_p$. Moreover $G_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} L \subset GL(\mathbb{D}(\mathscr{G}_{x,0}))$ contains the scalars, since it contains the image of $w_h$. Thus we may apply the construction of §4 to the tensors $s_{\alpha,0,x}$; we may equip Spf$A$ with the structure of a versal deformation space for $\mathscr{G}_{x,0}$ and the subspace Spf$A_{\mathcal{G}}$ is such that $\varpi : A \otimes_{\mathbb{Z}_p} \mathcal{O}_E \to K$ factors through $A_{\mathcal{G}}$ if and only if the induced $p$-divisible group $\mathscr{G}_\varpi$ is $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted, where $\mathcal{G}_{\mathcal{O}_L}$ is the stabilizer of $s_{\alpha,0,x}$ and $\mu_y$ is $G$-conjugate to $\mu_{h^{-1}}$.

The $p$-divisible group over $\widehat{U}_{\overline{x}}$ is induced by pullback from a map $\widehat{U}_{\overline{x}} \to \mathrm{Spf}A$ which is a closed immersion by the Serre-Tate theorem. Let $Z \subset U_{\overline{x}}$ be the irreducible component containing $x$. Let $x' \in Z(K')$. As in the proof of [KP, Proposition 4.2.2], $s_{\alpha,\text{ét},x'}$ corresponds to $s_{\alpha,0}$ under the $p$-adic comparison isomorphism for the $p$-divisible group $\mathscr{G}_x$. Hence we obtain one direction in (ii) and $x' \in \mathrm{Spf}A_{\mathcal{G}}$ since the filtration on $\mathbb{D} \otimes_{\mathcal{O}_L} K'$ corresponding to $\mathscr{G}_{x'}$ is given by a $G$-valued cocharacter conjugate to $\mu_h^{-1}$. Since this holds for all $x' \in Z(K')$, we have $Z \subset \mathrm{Spf}A_{\mathcal{G}}$ and hence they are equal since they have the same dimension. We thus obtain (iii) and the other implication in (ii) as well as the moreover part in (i).

□

The previous proposition shows that the tensors $s_{\alpha,0,x}$ are independent of the choice of $x \in \mathscr{S}_{\mathrm{K}_p}(G,X)$ lifting $\overline{x}$, thus we denote them by $s_{\alpha,0,\overline{x}}$. The following is then immediate.

**Corollary 6.3.** *Let $\overline{x}, \overline{x}' \in \mathscr{S}_{\mathrm{K}}(G,X)(k)$ be points whose image in $\mathscr{S}_{\mathrm{K}}(G,X)^{-}(k)$ coincide. Then $\overline{x} = \overline{x}'$ if and only if $s_{\alpha,0,\overline{x}} = s_{\alpha,0,\overline{x}'}$.*

6.7. We would like to show the isogeny classes in $\mathscr{S}_{\mathrm{K}}(G,X)(k)$ admit maps from $X(\sigma(\{\mu_y\}),b)$. We will show this when $G$ is residually split at $p$ and in general for the basic case. More generally, we show the existence of such a map without the residually split condition (but still under the assumptions in (6.2.1)) upon assuming Conjecture 5.4. In the rest of this section we will prove the case when $\mathcal{G}$ is an Iwahori subgroup of $G$; the general case will be deduced from this in §7. **We thus assume $\mathcal{G}$ is an Iwahori subgroup for the rest of the section and that the assumptions in (6.2.1) hold.**

Let $\overline{x} \in \mathscr{S}_{\mathrm{K}}(G,X)(k)$ and $x \in \mathscr{S}_{\mathrm{K}}(G,X)(K)$ a point lifting $\overline{x}$. Let $\mathcal{G}_{\mathcal{O}_L}$ denote the stabilizer $s_{\alpha,0,\overline{x}}$. By the above $\mathscr{G}_x$ is a $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted lifting of $x$ and there is an $\mathcal{O}_L$-linear bijection

$$T_p\mathscr{G}_x^{\vee} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}}) \tag{6.7.1}$$

taking $s_{\alpha,\text{ét},x}$ to $s_{\alpha,0,\overline{x}}$. We fix an isomorphism $V_{\mathbb{Z}_p}^* \cong T_p\mathscr{G}_x^{\vee}$ taking $s_{\alpha,0}$ to $s_{\alpha,\text{ét},x}$; this identifies the stabilizer $\mathcal{G}_{\mathcal{O}_L}$ of $s_{\alpha,\text{ét},x}$ with $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$.

Since the $s_{\alpha,0,\overline{x}}$ are $\varphi$-invariant, we may write $\varphi = b\sigma$ for some $b \in G(L)$ which is independent of the above choices up to $\sigma$-conjugation by elements of $\mathcal{G}(\mathcal{O}_L)$.

Fix $S$ a maximal $L$-split torus in $G$ with centralizer $T$ as in §5 so that $\mathcal{G}$ corresponds to an alcove in the apartment $\mathcal{A}(G,S,\mathbb{Q}_p)$. As in §5.6, we have

$$b \in \bigcup_{w \in \mathrm{Adm}(\{\mu_y\})} \mathcal{G}(\mathcal{O}_L)\sigma(\dot{w})\mathcal{G}(\mathcal{O}_L).$$

Write $\mu \in X_*(T)$ for the dominant (with respect to a choice of Borel defined over $L$) representative of $\{\mu_y\} = \{\mu_h^{-1}\}$, and $\underline{\mu}$ its image in $X_*(T)_I$. With the notation of §5, we have $1 \in X(\sigma(\{\mu_y\}),b)$.

Recall $X(\sigma(\{\mu_y\}),b)$ is equipped with an action $\Phi$ given by

$$\Phi(g) = (b\sigma)^r(g) = b\sigma(b)\dots\sigma^{r-1}(b)\sigma^r(g)$$

where $r$ is the residue degree of $\mathcal{O}_E/\mathbb{Z}_p$. Then

$$\Phi(g)^{-1}b\sigma(\Phi(g)) = \sigma^r(g^{-1}b\sigma(g)) \in \bigcup_{w \in \mathrm{Adm}(\{\mu_y\})} \mathcal{G}(\mathcal{O}_L)\sigma^{r+1}(\dot{w})\mathcal{G}(\mathcal{O}_L).$$

By [Rap05, Lemma 5.1], $\mathrm{Adm}(\{\mu_y\})$ is stable under $\sigma^r$, hence $\Phi(g) \in X(\sigma(\{\mu_y\}),b)$ and $\Phi$ is well defined.

Pick a basis for $V_{\mathbb{Z}_p}$ compatible with $S$ as in §3.3. In other words, the corresponding maximal split torus $T' \subset GL(V_{\mathbb{Z}_p})$ satisfies the compatiblity conditions in §3.3; in particular $S$ maps to $T'$

and there are embeddings of apartments (3.3.3) and (3.3.4). By Corollary 3.6, for $g \in X(\sigma(\{\mu_y\}), b))$ we have $g^{-1}b\sigma(g) \in \mathcal{GL}(\mathcal{O}_L)v_{GL}(p)\mathcal{GL}(\mathcal{O}_L)$ where $v_{GL}$ is the cocharacter $(1^{(g)}, 0^{(g)})$ and $\mathcal{GL}$ is the hyperspecial subgroup of $GL_{2n}$ over $\mathbb{Z}_p$ which stabilizes the lattice $V_{\mathbb{Z}_p}$. Thus the Hodge polygon of the $F$-crystal $g\mathbb{D}(\mathscr{G}_{\overline{x}})$ has slopes 0,1 hence corresponds to a $p$-divisible group $\mathcal{G}_{g\overline{x}}$ which is isogenous to $\mathscr{G}_{\overline{x}}$ and hence to an abelian variety $\mathcal{A}_{g\overline{x}}$ isogenous to $\mathcal{A}_{\overline{x}}$. $\mathcal{A}_{g\overline{x}}$ is equipped with a prime to $p$ level structure corresponding to the one on $\mathcal{A}_{\overline{x}}$. Since $g(s_{\alpha,0,\overline{x}}) = s_{\alpha,0,\overline{x}}$, we have $s_{\alpha,0,\overline{x}} \in \mathbb{D}(\mathscr{G}_{g\overline{x}})$.

Since $g \in GSp(L)$ the weak polarisation on $\lambda_{\overline{x}}$ induces a weak polarisation on $\mathcal{A}_{g\overline{x}}$. Thus $\mathcal{A}_{g\overline{x}}$ together with the extra structure gives a point of $\mathscr{S}_{\mathrm{K}'}(GSp(V), S^{\pm})(k)$ and we obtain a map

$$i'_{\overline{x}} : X(\sigma(\{\mu_y\}), b) \to \mathscr{S}_{\mathrm{K}'}(GSp(V), S^{\pm})(k).$$

In fact this map is non other than the one induced by the Rapoport-Zink uniformization [RZ96, §6] for $\mathscr{S}_{\mathrm{K}'}(GSp(V), S^{\pm})$. Note in this paper, we will only consider such a map on the level of points and not as a map of perfect schemes.

*Remark* 6.4. Note that if we modify the trivialization (6.7.1) by an element $h \in \mathcal{G}(\mathcal{O}_L)$, the Frobenius is given by the element $b' = h^{-1}b\sigma(h)$. The map

$$i''_{\overline{x}} : X(\sigma(\{\mu_y\}), b') \to \mathscr{S}_{\mathrm{K}'}(GSp(V), S^{\pm})(k)$$

obtained using this trivialization fits into the commutative diagram

(6.7.2)
$$\begin{tikzcd}
X(\sigma(\{\mu_y\}), b) \arrow[dr, "i'_{\overline{x}}"] \arrow[dd, "\sim"'] & \\
& \mathscr{S}_{\mathrm{K}'}(GSp(V), S^{\pm})(k) \\
X(\sigma(\{\mu_y\}), b') \arrow[ur, "i''_{\overline{x}}"] &
\end{tikzcd}.$$

where the vertical map is the isomorphism induced by $g \mapsto h^{-1}g$. Therefore the map $i'_{\overline{x}}$ is essentially independent of the trivialization (6.7.1).

The main result of this section is the following:

**Proposition 6.5.** *Suppose either of the following assumptions hold:*
*(i) $b$ is basic in $B(G)$.*
*(ii) $G_{\mathbb{Q}_p}$ is residually split.*
*(iii) Conjecture 5.4 holds.*
*Then there exists a unique map*

$$i_{\overline{x}} : X(\sigma(\{\mu_y\}), b) \to \mathscr{S}_{\mathrm{K}}(G, X)(k)$$

*lifting $i'_{\overline{x}}$ such that $s_{\alpha,0,i_{\overline{x}}(g)} = s_{\alpha,0,\overline{x}}$. Moreover we have*

$$\Phi \circ i_{\overline{x}} = i_{\overline{x}} \circ \Phi$$

*where $\Phi$ acts on $\mathscr{S}_{\mathrm{K}}(G, X)(k)$ via the geometric $r$-Frobenius.*

*Remark* 6.6. To be more specific, to deduce the existence of the lifting in part (iii) we only need to assume the Conjecture 5.4 holds for the following specific case. We let $M$ denote standard $\sigma_0$-stable Levi constructed in Lemma 5.7. Then for any $M'$ in the decomposition 5.4.1, we need the conjecture holds for $X^{M'}(\{\mu_{P'}\}, b_{P'})$, or more precisely it holds for each simple factor of $M'_{\mathrm{ad}}$ in which $b_{P'}$ is Hodge–Newton irreducible.

The rest of this section will be devoted to the proof of Proposition 6.5.

6.8. The uniqueness follows from Corollary 6.3. The same proof as in [Kis06, §1.4.4] shows the compatibility with $\Phi$. Thus it remains to show the existence of $i_{\overline{x}}$. The strategy follows [Kis06, §1.4]; the first step is to show that if $g \in X(\sigma(\{\mu_y\}), b)$ can be lifted, then every point on the connected component of $X(\sigma(\{\mu_y\}), b)$ containing $g$ also lifts. This first step can be carried out without any of the extra hypothesis in Proposition 6.5. The second step is to show that every connected component of $X(\sigma(\{\mu_y\}), b)$ contains a point which lifts; this is done by showing the quasi-isogeny $\mathcal{A}_{g\overline{x}} \to \mathcal{A}_{\overline{x}}$ lifts to characteristic 0.

We recall some definitions from [HZ, Appendix A], see also [CKV15].

**Definition 6.7.** Let $R$ be $k$-algebra. A frame for $R$ is a $p$ torsion free, $p$-adically complete and separated $\mathcal{O}_L$-algebra $\mathscr{R}$ equipped with an isomorphism $R \cong \mathscr{R}/p\mathscr{R}$ and a lift (again denoted $\sigma$) of the Frobenius $\sigma$ on $R$.

Let $R$ be as above and fix $\mathscr{R}$ a frame for $R$. We write $\mathscr{R}_L$ for $\mathscr{R}[\frac{1}{p}]$. If $\kappa$ is any perfect field of characteristic $p$ and $s : R \to \kappa$ is a map, then there is a unique $\sigma$-equivariant map $\mathscr{R} \to W(\kappa)$, also denoted $s$. Indeed this follows from the universal property of Witt vectors once one notes that any map $\mathscr{R} \to W(\kappa)$ factors through the natural map $\mathscr{R} \to \widehat{\mathscr{R}^\infty}$; here $\widehat{\mathscr{R}^\infty}$ is the $p$-adic completion $\mathscr{R}^\infty := \varinjlim_{\sigma} \mathscr{R}$ and is identified with the Witt vectors of the perfection of $R$. If $R \to R'$ is an étale map, then there exists a canonical frame $\mathscr{R}'$ of $R'$ equipped with a canonical $\sigma$-invariant map $\mathscr{R} \to \mathscr{R}'$ lifting $R \to R'$, see [CKV15, Lemma 2.1.4].

Let $g \in G(\mathscr{R}_L)$. For $C \subset W$, we write

$$S_C(g) = \bigcup_{w \in C} \{s \in \text{Spec } R | s(g^{-1} b \sigma(g)) \in \mathcal{G}(W(\overline{\kappa}(s))) \dot{w} \mathcal{G}(W(\overline{\kappa}(s)))\}$$

where $\overline{\kappa}(s)$ is an algebraic closure of residue field $k(s)$ of $s$. Note that this only depends on the image of $g \in G(\mathscr{R}_L)/\mathcal{G}(\mathscr{R})$, hence we can define $S_C(g)$ for any element of $g \in G(\mathscr{R}_L)/\mathcal{G}(\mathscr{R})$. For $b \in G(L)$, we define the set

$$X_C(b)(\mathscr{R}) = \{g \in G(\mathscr{R}_L)/\mathcal{G}(\mathscr{R}) | S_C(g) = \text{Spec } R\}.$$

When $C = \text{Adm}(\{\mu\})$ we write $X(\{\mu\}, b)(\mathscr{R})$ for $X_C(b)(\mathscr{R})$. Similarly when $C = \sigma(\text{Adm}(\{\mu\}))$ we write $X(\sigma(\{\mu\}), b)(\mathscr{R})$.

**Definition 6.8.** For $g_0, g_1 \in X(\{\mu\}, b)$ and $R$ a smooth $k$-algebra with connected spectrum and frame $\mathscr{R}$, we say $g_0$ is connected to $g_1$ via $R$ if there exists $g \in X(\{\mu\}, b)(\mathscr{R})$ and two $k$-points $s_0, s_1$ of Spec $R$ such that $s_0(g) = g_0$ and $s_1(g) = g_1$.

We write $\sim$ for the equivalence relation on $X(\{\mu\}, b)$ generated by the relation $g_0 \sim g_1$ if $g_0$ is connected to $g_1$ via some $R$ as above, and we write $\pi_0'(X(\{\mu\}, b))$ for the set of equivalence classes.

By [HZ, Theorem A.4], we have

$$\pi_0'(X(\{\mu\}, b)) = \pi_0(X(\{\mu\}, b)). \tag{6.8.1}$$

6.9. Returning to the situation of Proposition 6.5, let $g \in G(\mathscr{R}_L)$ be a lift of some element of $X(\sigma(\{\mu_y\}), b)(\mathscr{R})$. By Corollary 3.6, for all $s \in \text{Spec } R$, we have $g(s) \in \mathcal{GL}(\mathcal{O}_L) v_{GL}(p) \mathcal{GL}(\mathcal{O}_L)$. Since $\mathcal{GL}$ is hyperspecial, it follows from [CKV15, Lemma 2.1.14] that there is an étale covering $R \to R'$ with canonical frame $\mathscr{R} \to \mathscr{R}'$ such that

$$g \in \mathcal{GL}(\mathscr{R}') \mu_{GL}(p) \mathcal{GL}(\mathscr{R}'),$$

For $n \geq 1$ we write $\mathscr{R}_n$ for the ring $\mathscr{R}$ considered as a $\mathscr{R}$-algebra via $\sigma^n : \mathscr{R} \to \mathscr{R}$ and we define $R_n$ similarly. By [Kis17, Lemma 1.4.6], there exists $n \geq 1$ and a $p$-divisible group $\mathscr{G}_{g\overline{x}}$ over $R'_n$ together with a quasi-isogeny $\mathscr{G}_{g\overline{x}} \to \mathscr{G}_{\overline{x}} \otimes R'_n$ which identifies $\mathbb{D}(\mathscr{G}_{g\overline{x}})(\mathscr{R}'_n)$ with $g\mathbb{D}(\mathscr{G}_{\overline{x}}) \subset \mathbb{D}(\mathscr{G}_{\overline{x}}) \otimes_{\mathcal{O}_L} \mathscr{R}'_{nL}$. Upon relabelling $R'_n$ as $R$ and $\mathscr{R}'_n$ as $\mathscr{R}$, we obtain an abelian variety $\mathcal{A}_{g\overline{x}}$ over Spec $R$.

Since $g \in GSp(\mathscr{R}_L)$, $\lambda_{\overline{x}}$ induces a weak polarization $\lambda_{g\overline{x}}$ on $\mathcal{A}_{g\overline{x}}$, and $\mathcal{A}_{g\overline{x}}$ is also equipped with a prime to $p$ level structure. Hence $g$ gives a map

$$\text{Spec } R \to \mathscr{S}_{\mathrm{K}'}(GSp, S^{\pm}). \tag{6.9.1}$$

Since $g \in G(\mathscr{R}_L)$, we have $s_{\alpha,0,\overline{x}} = g(s_{\alpha,0,\overline{x}}) \in \mathbb{D}(\mathscr{G}_{g\overline{x}})(\mathscr{R})$.

**Proposition 6.9.** *Suppose there is a $k$-point $x_R : R \to k$ of* Spec $R$ *such that $x_R^*(g) = 1$. Then there is a unique lifting $i_R :$* Spec $R \to \mathscr{S}_{\mathrm{K}}(G, X)$ *of (6.9.1) such that*

$$i_R^*(s_{\alpha,0}) = s_{\alpha,0,\overline{x}}$$

*Proof.* The uniqueness can be checked on $k$ points, hence this follows from Corollary 6.3.

To show existence, we first claim (6.9.1) factors through $\mathscr{S}_{\mathrm{K}}(G, X)^-$. Note that in the process of replacing $R$ by $R'_n$, $R$ may no longer be irreducible. However, by induction we may apply the argument to each irreducible component successively. Therefore we may assume $R$ is integral. Let $\widehat{R}$ denote the completion of $R$ at $x_R$. Since $R$ is integral, it suffices to prove the claim for $\widehat{R}$.

Note that the filtration induced by $g^{-1}b\sigma(g)$ gives an $R$-point of the local model $M^{\mathrm{loc}}_{\mathcal{GL}}$. For all $k$ points $s : R \to k$, we have

$$g(s)^{-1}b\sigma(g(s)) \in \bigcup_{w \in \mathrm{Adm}(\{\mu_y\})} \mathcal{G}(\mathcal{O}_L)\sigma(\dot{w})\mathcal{G}(\mathcal{O}_L)$$

By Corollary 3.6, the map Spec $R \to M^{\mathrm{loc}}_{\mathcal{GL}}$ factors through $M^{\mathrm{loc}}_{\mathcal{G}}$. Taking completions at the image of $x_R$, we obtain a map

$$\psi : A_{\mathcal{G}} \to \widehat{R}$$

By smoothness of $R$, $\widehat{R}$ is power series ring over $k$. We may choose coordinates so that $\widehat{R} \cong k[[t_1, \ldots, t_m]]$ for some $m \geq 1$.

We have a $p$-divisible group over $k[[t_1, \ldots, t_m]]$; we would like to use the map $\psi$ to deform this $p$-divisible group to $\tilde{\mathscr{G}}$ over a ring in characteristic 0, such that the pullback to every $\mathcal{O}_K$-point satisfies the condition in Definition 4.5, i.e. it is $(\mathcal{G}_{\mathcal{O}_L}, \mu_y)$-adapted. The ring we will deform to is $A_{\mathcal{G}}[[t_1, \ldots, t_m]]$.

We have a map

$$A_{\mathcal{G}}[[t_1, \ldots, t_m]] \twoheadrightarrow k[[t_1, \ldots, t_m]]$$

induced by $\psi : A_{\mathcal{G}} \to k[[t_1, \ldots, t_m]]$ and $t_i \mapsto t_i$. This induces a surjection

$$\widehat{W}(A_{\mathcal{G}}[[t_1, \ldots, t_m]]) \twoheadrightarrow \widehat{W}(k[[t_1, \ldots, t_m]]).$$

Let $\widehat{\mathscr{R}}$ denote the completion of $\mathscr{R}$ at the image of the point $x_R$. We write $\widehat{g}$ for the image of $g$ in $G(\widehat{\mathscr{R}})$ and $\mathscr{G}_{\widehat{g}\overline{x}}$ for the induced $p$-divisible group, then $\mathbb{D}(\mathscr{G}_{\widehat{g}\overline{x}})(\widehat{\mathscr{R}})$ can be identified with $\widehat{g}\mathbb{D}(\mathscr{G}_{\overline{x}}) \subset \mathbb{D}(\mathscr{G}_{\overline{x}}) \otimes \widehat{\mathscr{R}}_L$. We may use $\widehat{g}^{-1}$ to identify $\mathbb{D}(\mathscr{G}_{\widehat{g}\overline{x}})(\widehat{\mathscr{R}})$ with $\mathbb{D}(\mathscr{G}_{\overline{x}}) \otimes_{\mathcal{O}_L} \widehat{\mathscr{R}}$ as an $\widehat{\mathscr{R}}$-module. Under this identification the Frobenius is given by $\widehat{g}^{-1}b\sigma(\widehat{g})$. It follows that the Dieudonné display $\mathbb{D}(\mathscr{G}_{\widehat{g}\overline{x}})(\widehat{W}(k[[t_1, \ldots, t_m]]))$ can be identified with $\mathbb{D} \otimes_{\mathcal{O}_L} \widehat{W}(k[[t_1, \ldots, t_m]])$ and the Frobenius $\Phi$ preserves $s_{\alpha,0,\overline{x}}$.

Let $\mathrm{Spf}A$ be the completion of $M^{\mathrm{loc}}_{\mathcal{GL},\mu_h^{-1}}$ at the image of $x_R$, then $\mathbb{D}\otimes_{\mathcal{O}_L} A$ is equipped with a universal filtration $\overline{M}_1\subset \mathbb{D}\otimes_{\mathcal{O}_L} A$. We let $M_1$ denote the preimage of $\overline{M}_1$ in $M:=\mathbb{D}\otimes_{\mathcal{O}_L}\widehat{W}(A)$. Let $\widetilde{M}_1$ denote the image of the map $\varphi^*M_1\to\varphi^*M$.

By construction, the pushforward of $\overline{M}_1$ along $A\to A_{\mathcal{G}}\to k[[t_1,\dots,t_m]]$ is the filtration on $\mathbb{D}\otimes_{\mathcal{O}_L} k[[t_1,\dots,t_m]]$ induced by $\widehat{g}^{-1}b\sigma(\widehat{g})$. Therefore by [KP, Lemma 3.1.5] the structure of a display on $\mathbb{D}\otimes_{\mathcal{O}_L}\widehat{W}(k[[t_1,\dots,t_m]])$ corresponding to $\mathscr{G}_{\widehat{g}\overline{x}}$ is given by an isomorphism

$$\Psi_{k[[t_1,\dots,t_m]]}:\widetilde{M}_{1,k[[t_1,\dots,t_m]]}\to \mathbb{D}\otimes_{\mathcal{O}_L}\widehat{W}(k[[t_1,\dots,t_m]])$$

where for any ring $B$ with $A\to B$, we write $\widetilde{M}_{1,B}$ for the base change $\widetilde{M}_1\otimes_{\widehat{W}(A)}\widehat{W}(B)$. Since $A\to k[[t_1,\dots,t_m]]$ factors through $A_{\mathcal{G}}$ it follows from [KP, Corollary 3.2.11] that $s_{\alpha,0,\overline{x}}\in \widetilde{M}^{\otimes}_{1,k[[t_1,\dots,t_m]]}$, and since $\widehat{g}^{-1}b\sigma(\widehat{g})$ preserves $s_{\alpha,0,\overline{x}}$ we have $\Psi_{k[[t_1,\dots,t_m]]}(s_{\alpha,0,\overline{x}})=s_{\alpha,0,\overline{x}}$.

By [KP, Corollary 3.2.11], the scheme

$$\mathcal{T}=\underline{\mathrm{Isom}}_{s_{\alpha,0,\overline{x}}}(\widetilde{M}_{1,A_{\mathcal{G}}},M\otimes_{\widehat{W}(A)}\widehat{W}(A_{\mathcal{G}}))$$

of tensor preserving isomorphisms is a $\mathcal{G}$-torsor. Base changing to $A_{\mathcal{G}}[[t_1,\dots,t_m]]$ we obtain a $\mathcal{G}$-torsor

$$\mathcal{T}_{A_{\mathcal{G}}[[t_1,\dots,t_m]]}=\underline{\mathrm{Isom}}_{s_{\alpha,0,\overline{x}}}(\widetilde{M}_{1,A_{\mathcal{G}}[[t_1,\dots,t_m]]},M\otimes_{\widehat{W}(A)}\widehat{W}(A_{\mathcal{G}}[[t_1,\dots,t_m]])).$$

By smoothness of $\mathcal{G}$, $\Psi_{k[[t_1,\dots,t_m]]}$ lifts to an isomorphism

$$\Psi:\widetilde{M}_{1,A_{\mathcal{G}}[[t_1,\dots,t_m]]}\xrightarrow{\sim} M\otimes_{\widehat{W}(A)}\widehat{W}(A_{\mathcal{G}}[[t_1,\dots,t_m]])).$$

Again, by [KP, Lemma 3.1.5], this corresponds to a display over $A_{\mathcal{G}}[[t_1,\dots,t_m]]$. Since $\Psi$ lifts $\Psi_{k[[t_1,\dots,t_m]]}$, this display deforms $\mathbb{D}(\mathscr{G}_{\widehat{g}\overline{x}})(\widehat{W}(k[[t_1,\dots,t_m]]))$ by the discussion in [KP, 3.2.6], and hence corresponds to a $p$-divisible group $\tilde{\mathscr{G}}$ over $A_{\mathcal{G}}[[t_1,\dots,t_m]]$ deforming $\mathscr{G}_{\widehat{g}\overline{x}}$.

Let $\varpi: A_{\mathcal{G}}[[t_1,\dots,t_m]]\to\mathcal{O}_K$ be any map and $\tilde{\mathscr{G}}_\varpi$ the $p$-divisible group over $\mathcal{O}_K$ obtained by pullback. By construction we have an isomorphism

$$\iota:\mathbb{D}(\tilde{\mathscr{G}}_\varpi)(\widehat{W}(\mathcal{O}_K))\cong\mathbb{D}\otimes_{\mathcal{O}_L}\widehat{W}(\mathcal{O}_K).$$

Thus $s_{\alpha,0,\overline{x}}$ give rise to $\Phi$-invariant tensors in $\mathbb{D}(\tilde{\mathscr{G}})(\widehat{W}(\mathcal{O}_K))^{\otimes}$. Moreover, under the canonical identification $\mathbb{D}(\tilde{\mathscr{G}}_\varpi)(\mathcal{O}_K)\otimes_{\mathcal{O}_K}K\cong\mathbb{D}\otimes_{\mathcal{O}_L}K$, the filtration is induced by a $G$-valued cocharacter conjugate to $\mu_y$. Indeed the composition

$$\mathbb{D}\otimes_{\mathcal{O}_L}\mathcal{O}_K\xrightarrow{\iota^{-1}}\mathbb{D}(\tilde{\mathscr{G}}_\varpi)(\mathcal{O}_K)\otimes_{\mathcal{O}_K}K\xrightarrow{\sim}\mathbb{D}\otimes_{\mathcal{O}_L}K$$

where the second map is the canonical isomorphism as in [KP, Lemma 3.1.17] takes $s_{\alpha,0,\overline{x}}$ to itself. Since the filtration on the left is induced by the map $A\to A_{\mathcal{G}}\to A_{\mathcal{G}}[[t_1,\dots,t_m]]\to\mathcal{O}_K$, it corresponds to a point of the local model $M^{\mathrm{loc}}_{\mathcal{G}}$ hence is induced by a $G$-valued cocharacter conjugate to $\mu_y$. Thus $\tilde{\mathscr{G}}_\varpi$ is $(\mathcal{G}_{\mathcal{O}_L},\mu_y)$-adapted as desired.

Let $\widehat{U}'_{\overline{x}}$ (resp. $\widehat{U}_{\overline{x}}$) denote the completion of $\mathscr{S}_{\mathrm{K}'}(GSp,S^{\pm})$ (resp. $\mathscr{S}_{\mathrm{K}}(G,X)^-$) at the image of $\overline{x}$. Then the $p$-divisible group $\tilde{\mathscr{G}}$ corresponds to a map

$$\epsilon^-:\mathrm{Spf}A_{\mathcal{G}}[[t_1,\dots,t_m]]\to\widehat{U}'_{\overline{x}}.$$

We let $\widehat{Z}\subset\widehat{U}_x$ denote the irreducible component containing $x$ (recall $x\in\mathscr{S}_{\mathrm{K}}(G,X)(K)$ was a point lifting $\overline{x}$). By the previous paragraph, for any $\varpi: A_{\mathcal{G}}[[t_1,\dots,t_m]]\to\mathcal{O}_K$, the induced point of $\widehat{U}'_x$ lies in $\widehat{Z}$ by Proposition 6.2 (iii). Since this is true for any $\mathcal{O}_K$-point with $K/L$ finite and $A_{\mathcal{G}}[[t_1,\dots.t_m]]$ is flat over $\mathbb{Z}_p$, $\epsilon^-$ factors though $\widehat{Z}$. Thus $i_R$ factors through $\mathscr{S}_{\mathrm{K}}(G,X)^-$.

We have shown that (6.9.1) factors through $\mathscr{S}_{\mathrm{K}}(G,X)^-$. We let

$$i_R^- : \mathrm{Spec}\ R \to \mathscr{S}_{\mathrm{K}}(G,X)^-$$

be the induced map. We now show that $i_R^-$ lifts to $\mathscr{S}_{\mathrm{K}}(G,X)$. We first show there exists an open subscheme $U \subset \mathrm{Spec}\ R$ containing $x_R$ which lifts.

We let $\mathcal{O}_{\mathscr{S},\overline{x}}$ (resp. $\mathcal{O}_{\mathscr{S}^-,\overline{x}}$) denote the local ring of $\mathscr{S}_{\mathrm{K}}(G,X)$ (resp. $\mathscr{S}_{\mathrm{K}}(G,X)^-$) at the (image of the) point $\overline{x}$. The irreducible component $Z_{\overline{x}}$ of Spec $\mathcal{O}_{\mathscr{S}^-,\overline{x}}$ containing $x$ has completion which is normal by Theorem 6.1 (i), hence $Z_{\overline{x}}$ is normal. Therefore the map

$$\mathrm{Spec}\ \mathcal{O}_{\mathscr{S},\overline{x}} \to \mathrm{Spec}\ \mathcal{O}_{\mathscr{S}^-,\overline{x}}$$

induces an isomorphism Spec $\mathcal{O}_{\mathscr{S},\overline{x}} \cong Z_{\overline{x}}$.

Let $R_{x_R}$ be the localization of $R$ at the point $x_R$. The proof above showing that (6.9.1) factors through $\mathscr{S}_{\mathrm{K}}(G,X)^-$ also shows that the induced map Spec $R_{x_R} \to$ Spec $\mathcal{O}_{\mathscr{S}^-,\overline{x}}$ actually factors through $Z_{\overline{x}}$. Hence we obtain a unique lifting

$$\mathrm{Spec}\ R_{x_R} \to \mathscr{S}_{\mathrm{K}}(G,X).$$

It follows that there exists an open subscheme $U \subset$ Spec $R$ containing $\overline{x}$ such that $i_R^-|_U$ lifts to a unique map

$$U \to \mathscr{S}_{\mathrm{K}}(G,X).$$

The existence and uniqueness of the lifting $i_R$ then follows from Lemma 6.10 below.

To show the compatibility of this map with the tensors $s_{\alpha,0,\overline{x}}$, we let $\mathbb{M}$ denote the Dieudonné $F$-crystal over $\mathscr{S}_{\mathrm{K}_p}(G,X)_k$ associated to the universal $p$-divisible group, and $\mathbb{M}[\frac{1}{p}]$ the corresponding $F$-isocrystal. By [KMPS, Corollary A.7], there exist sections

$$\mathbf{s}_{\alpha,0} : 1 \to \mathbb{M}[\frac{1}{p}]^{\otimes}$$

such that for all $\overline{x}' \in \mathscr{S}_{\mathrm{K}}(G,X)(k)$, $\mathbf{s}_{\alpha,0}$ pulls back to $s_{\alpha,0,\overline{x}'} \in \mathbb{D}(\mathscr{G}_{\overline{x}'})[\frac{1}{p}]^{\otimes}$.

Thus pulling back to Spec $R$, we obtain $\mathbf{s}_{\alpha,0,R} \in \mathbb{D}(\mathscr{G}_{g\overline{x}})(\mathscr{R})[\frac{1}{p}]^{\otimes}$ such that for all $z : R \to k$, the pullback coincides with $s_{\alpha,0,\iota_R(z)}$. Now by construction $s_{\alpha,0,\overline{x}} \in \mathbb{D}(\mathscr{G}_{g\overline{x}})(\mathscr{R})^{\otimes}$ are parallel for the connection coming from the crystal structure of $\mathbb{D}(\mathscr{G}_{g\overline{x}})$; indeed this connection is induced by the trivial connection $1 \otimes d$ on $\mathbb{D}(\mathscr{G}_{\overline{x}}) \otimes_{\mathcal{O}_L} \mathscr{R}$, see [Kis17, Lemma 1.4.6]. Moreover $s_{\alpha,0,\overline{x}}$ coincide with $\mathbf{s}_{\alpha,0,R}$ at the point $x_R$. Since $R$ is integral, $\mathbf{s}_{\alpha,0,R} = s_{\alpha,0,\overline{x}}$.

□

**Lemma 6.10.** *Let $Y$ be a reduced scheme and $Y^n$ its normalization. Let $X$ be a normal integral scheme and $U \subset X$ an open subscheme. Suppose we have a diagram:*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Y^n \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

*Then $f$ lifts to a unique map $f' : X \to Y^n$*

*Proof.* Since $X$ is irreducible, so is $U$. As $Y^n$ is a disjoint union of integral schemes, the image of $U$ is contained in a single irreducible component, which is the normalization of a single irreducible component in $Y$. Replacing $Y$ by this component, we may assume that $Y$ is reduced and irreducible, hence integral. By uniqueness we may assume $X =$ Spec $R$ and $Y =$ Spec $S$ are affine. Then $Y^n =$ Spec $S^{\mathrm{int}}$ where $S^{\mathrm{int}}$ is the integral closure of $S$ in $K(Y) := \mathrm{Frac}(S)$. Thus it suffices to show the induced map $S^{\mathrm{int}} \to K(X)$ factors through $R$. But this follows since $R$ is integrally closed in $K(X)$. $\square$

*Proof of Proposition 6.5.* For now we do not make any extra assumptions as in the Proposition 6.5 parts (i), (ii), (iii). By uniqueness, there is a maximal subset $X(\sigma(\{\mu_y\}, b)^\circ \subset X(\sigma(\{\mu_y\}), b)$ which lifts to a map:

$$i_{\overline{x}} : X(\sigma(\{\mu_y\}), b)^\circ \to \mathscr{S}_{\mathrm{K}}(G, X)(k)$$

If $g \in G(L)$ represents an element of $X(\{\mu_y\}, b)^\circ$, then for $\overline{x}' := i_{\overline{x}}(g) \in \mathscr{S}_{\mathrm{K}}(G, X)(k)$, the Frobenius $\varphi$ on $\mathbb{D}(\mathscr{G}_{\overline{x}'})$ is given by $b' := g^{-1}b\sigma(g) \in G(L)$ under a suitable trivialization of $\mathbb{D}(\mathscr{G}_{\overline{x}'})$. The maps $i_{\overline{x}}$ and $i_{\overline{x}'}$ fit into the commutative diagram:

(6.9.2)
$$\begin{tikzcd}
X(\sigma(\{\mu_y\}), b) \arrow[dd, "\sim"'] \arrow[dr, "i'_{\overline{x}}"] & \\
& \mathscr{S}_{\mathrm{K}'}(GSp(V), S^{\pm})(k) \\
X(\sigma(\{\mu_y\}), b') \arrow[ur, "i'_{\overline{x}'}"] &
\end{tikzcd}$$

where the vertical arrow is the isomorphism induced by left multiplication by $g^{-1}$. Therefore

$$X(\sigma(\{\mu_y\}), b)^\circ = X(\sigma(\{\mu_y\}), b')^\circ$$

under this identification. In what follows, we will often change the base-point $\overline{x}$ in the definition of the map $i'_{\overline{x}}$.

**Lemma 6.11.** *The set $X(\sigma(\{\mu_y\}), b)^\circ$ is a union of connected components.*

*Proof.* By (6.8.1), it suffices to show that $X(\sigma(\{\mu_y\}), b)^\circ$ is a union of equivalence classes under $\sim$. By Proposition 6.9, if $g \in X(\sigma(\{\mu_y\}), b)^\circ$ and $g' \in X(\sigma(\{\mu_y\}), b)$ with $g \sim g'$, then there exists a sequence $g = g_0, \dots, g_r = g'$ such that $g_i$ is connected to $g_{i+1}$ via some $R$ as in Definition 6.8. We would like to show $g' \in X(\sigma(\{\mu_y\}), b)^\circ$. By induction, it suffices to show consider the case $g$ is connected to $g'$ via $R$. Upon replacing $\overline{x}$ by $i_{\overline{x}}(g)$ and using (6.9.2), we may assume $g = 1$.

Upon replacing $R$ by $R'_n$, where $R \to R'$ is an étale cover, there exists a map

$$\mathrm{Spec}\ R \to \mathscr{S}_{\mathrm{K}'}(GSp, S^{\pm})$$

as in (6.9.1). Note that the étale covering $R'$ may be disconnected, but applying the argument to each connected component and using induction again, we may assume it is connected. The result then follows from Proposition 6.9. $\square$

*Case (i), $b$ is basic:* By Theorem 5.9 (i) and Lemma 6.11, there exists $g' \in X_{\sigma(\tau_{\{\mu_y\}})}(b) \cap X(\sigma(\{\mu_y\}), b)^\circ$, i.e. $i_{\overline{x}}(g')$ lifts to a point $\overline{x}' \in \mathscr{S}_{\mathrm{K}}(G, X)(k)$. Upon replacing $\overline{x}$ by $\overline{x}'$ and using (6.9.2) we may assume $b \in \mathcal{G}(\mathcal{O}_L)\sigma(\dot{\tau}_{\{\mu\}})\mathcal{G}(\mathcal{O}_L)$. By Theorem 2.4, upon changing the isomorphism

$$V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$$

by an element of $\mathcal{G}(\mathcal{O}_L)$, we may assume $b = \sigma(\dot{\tau}_{\{\mu\}})$.

Let $x \in \mathscr{S}_{\mathrm{K}}(G, X)(\mathcal{O}_K)$ denote a lift of $\overline{x}$ with $K/L$ finite. Fix the isomorphism

$$T_p\mathscr{G}_x^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$$

taking $s_{\alpha,\acute{e}t,x}$ to $s_{\alpha,0,\overline{x}}$ compatibly with the isomorphism $V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$ above. We may now apply the construction of §5.

Let $g \in G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p)$ and $g_0$ the corresponding element in $X(\sigma(\{\mu\}), b)$ constructed in §5.6; upon replacing $K$ by a finite extension, $g^{-1}T_p\mathscr{G}_x$ corresponds to a $p$-divisible group $\mathscr{G}'$ over $\mathcal{O}_K$ together with a quasi-isogeny $\mathscr{G}' \to \mathscr{G}_x$ which identifies $\mathbb{D}(\mathscr{G}'_0)$ with $g_0\mathbb{D}(\mathscr{G}_{\overline{x}})$. This corresponds to a quasi-isogeny $\mathcal{A}' \to \mathcal{A}_x$ hence to a point $gx \in Sh_{\mathrm{K}}(G, X)(K)$ by the moduli interpretation of $Sh_{\mathrm{K}}(G, X)(\mathbb{C})$. Indeed after base changing to $\mathbb{C}$, the quasi- isogeny $\mathcal{A}' \to \mathcal{A}_x$ preserves $s_{\alpha,B,x}$ since it preserves $s_{\alpha,\acute{e}t,x}$. Therefore $g_0 \in X(\sigma(\{\mu_y\}), b)^\circ$ as it is the specialization of $gx$.

By Proposition 5.19, $g \mapsto g_0$ induces a surjection

$$G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \twoheadrightarrow \pi_0(X(\sigma(\{\mu_y\}), b));$$

hence

$$X(\sigma(\{\mu_y\}), b)^\circ = X(\sigma(\{\mu_y\}), b).$$

This proves the Proposition when $b$ is basic in $G$.

*Case (ii), $G_{\mathbb{Q}_p}$ is residually split:* In this case $\sigma$ acts trivially on the Iwahori Weyl group $W$, hence $\mathrm{Adm}(\{\mu_y\}) = \sigma(\mathrm{Adm}(\{\mu_y\}))$. Recall that by Theorem 5.11, there is a map

$$\coprod_{w\in W, w \text{ a straight element with } \dot{w}\in[b]} X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w}) \to X(\{\mu_y\}, b) \tag{6.9.3}$$

which induces a surjection

$$\coprod_{w\in W, w \text{ a straight element with } \dot{w}\in[b]} \pi_0(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \to \pi_0(X(\{\mu_y\}, b)) \tag{6.9.4}$$

Here we write straight instead of $\sigma$-straight since the action is trivial. We note that the map $X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w}) \to X(\{\mu_y\}, b)$ is induced by the map $m \mapsto j_{b,\dot{w}}m$ for $m \in M_{\nu_w}(L)$, and where $j_{b,w}$ is an element in $G(L)$ such that $j_{b,\dot{w}}^{-1}b\sigma(j_{b,\dot{w}}) = \dot{w}$. Such a map depends on the choice of $j_{b,\dot{w}}$, however its image does not. We now fix a choice a of $j_{b,\dot{w}}$ for each $w$ in the decomposition 6.9.3 and we write $\iota_w$ for the induced map $X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w}) \to X(\{\mu_y\}, b)$.

We outline the strategy for the proof. By Lemma 6.11 and the surjection (6.9.4), it suffices to show for each $w \in \mathrm{Adm}(\{\mu\})$ straight, that every connected component of $X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})$ contains an element whose image in $X(\{\mu_y\}, b)$ lies in $X(\{\mu_y\}, b)^\circ$. This will follow from Lemmas 6.12, 6.13 and 6.15 below. The first step is to show that if $\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \cap X(\{\mu_y\}, b)^\circ \neq \emptyset$, then $\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w}))$ is contained in $X(\{\mu_y\}, b)^\circ$. This follows from same argument as in the basic case since by [HZ, Theorem 5.2], $\dot{w}$ is basic in $M_{\nu_w}(L)$. It now suffices to show that the image of $X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})$ contains at least one point which lifts. To do this, we use the property of straight elements stated in Theorem 2.2; namely that for any $w, w'$ straight with $[\dot{w}] = [\dot{w}']$ in $B(G)$, we have $w \approx w'$. We show that for such a $w$ and $s \in \mathbb{S}$ such that $w' = sw\sigma(s)$ and $l(sw\sigma(s)) = l(w)$, if

$$\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \cap X(\{\mu_y\}, b)^\circ \neq \emptyset,$$

then

$$\iota_{w'}(X^{M_{\nu_{w'}}}(\{\lambda_{w'}\}_{M_{\nu_{w'}}}, \dot{w}')) \cap X(\{\mu_y\}, b)^\circ \neq \emptyset.$$

Similarly, if $\tau \in \Omega$ and $w' = \tau w \tau^{-1}$, we show that if

$$\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \cap X(\{\mu_y\}, b)^\circ \neq \emptyset$$

then

$$\iota_{w'}(X^{M_{\nu_{w'}}}(\{\lambda_{w'}\}_{M_{\nu_{w'}}}, \dot{w}')) \cap X(\{\mu_y\}, b)^\circ \neq \emptyset.$$

These properties imply that to prove the Proposition, it suffices to show that there exists some $w$ such $\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}},\dot{w}))\cap X(\{\mu_y\},b)^\circ\neq\emptyset$, which again follows from Lemma 6.11 and the surjectivity of (6.9.4).

**Lemma 6.12.** *Let $w$ straight such that $\dot{w}\in[b]$. Suppose there exists*

$$g\in\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}},\dot{w}))\cap X(\{\mu_y\},b)^\circ.$$

*Then $\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}},\dot{w}))\subset X(\{\mu_y\},b)^\circ$.*

*Proof.* By Theorem 5.9 (i) appplied to $X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}},\dot{w})$, there exists

$$g'\in X_w^{M_{\nu_w}}(\dot{w})\subset X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}},\dot{w})$$

such that $g'\sim g$ and hence $\iota_w(g')\in X(\{\mu_y\},b)^\circ$. Indeed in this case, $w\in\mathrm{Adm}^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}})$ is the unique straight element. Upon replacing $\overline{x}$ by $i_{\overline{x}}(\iota_w(g'))$ and using (6.9.2), we may assume $b\in\mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O}_L)$. Since $w$ is straight, upon modifying the isomorphism (6.7.1) and using Theorem 2.4, we may assume $b=\dot{w}$. The inclusion $\iota_w$ is induced by the map $m\mapsto j_{\dot{w},\dot{w}}m$ where $m\in M(L)$ and $j_{\dot{w},\dot{w}}$ is an element of

$$J_{\dot{w}}(\mathbb{Q}_p):=\{h\in G(L)|h^{-1}\dot{w}\sigma(h)=\dot{w}\}.$$

The image of $\iota_w$ is unchanged upon replacing $j_{\dot{w},\dot{w}}$ by 1, hence we may assume $\iota_w$ is the natural inclusion.

Write $M:=M_{\nu_w}$. Then $M(L)\cap\mathcal{G}(\mathcal{O}_L)$ is an Iwahori subgroup of $M$ which is defined over $\mathbb{Z}_p$; we write $\mathcal{M}$ for the associated group scheme. Let $W_M$ denote the Iwahori Weyl group of $M$, then $w\in W_M$ and [HZ, Theorem 5.2] implies $\dot{w}\mathcal{M}(\mathcal{O}_L)\dot{w}^{-1}=\mathcal{M}(\mathcal{O}_L)$. We equip $W_M$ with the Bruhat order $\leq_M$ induced by the Iwahori subgroup $\mathcal{M}$, then $w$ is a length 0 element of $W_M$. The $M$ conjugacy class of cocharacter $\{\lambda_w\}$ may be constructed as follows. We write $\overline{\nu}_w\in X_*(T)_{I,\mathbb{Q}}$ for the dominant representative of $\nu_w$ and $M_{\overline{\nu}_w}$ the corresponding standard Levi. Let $J\subset\mathbb{S}_0$ be the subset corresponding to $M_{\overline{\nu}_b}$. We let $z_w\in W_0^J$ such that $z_w(M_{\overline{\nu}_w})=M$. If $\mu\in X_*(T)$ is a dominant representative of $\{\mu_y\}$, then $\{\lambda_w\}$ is represented by $\lambda_w:=z_w(\mu)$. Therefore by Proposition 3.4, there is an $M$-valued cocharacter conjugate to $\lambda_w$ such that the induced filtration on $\mathbb{D}\otimes_{\mathcal{O}_L}K$ specializes to the one on $\mathbb{D}(\mathscr{G}_{\overline{x}})\otimes_{\mathcal{O}_L}k$ and hence the assumption (5.12) is satisfied for the subgroup $M$ of $G$. We may thus apply the construction of §5.7.

Since $\dot{w}\in M(L)$, we may extend the tensors $s_\alpha$ to tensors $t_\alpha\in V_{\mathbb{Z}_p}^\otimes$ whose stablizer is the Iwahori $\mathcal{M}$. We obtain an embedding of local models $M^{\mathrm{loc}}_{\mathcal{M},\lambda_w^{-1}}\subset M^{\mathrm{loc}}_{\mathcal{GL},\mu_y^{-1}}\otimes\mathcal{O}_{E'}$, where $E'$ is the (local) reflex for the $M$-conjugacy class of $\{\lambda_w\}$. Since $\dot{w}\in\mathrm{Adm}^M(\{\lambda_w\})$ the filtration on $\mathbb{D}(\mathscr{G}_{\overline{x}})\otimes_{\mathcal{O}_L}k$ gives a point in the local model $M^{\mathrm{loc}}_{\mathcal{M}}(k)$. Replacing $\lambda_w$ by an element in its $M$-conjugacy class, we may assume $\lambda_w$ is defined over a finite extension $K/L$ and the induced filtration lifts the filtration on $\mathbb{D}(\mathscr{G}_{\overline{x}})\otimes_{\mathcal{O}_L}k$.

Recall that $\dot{w}$ is basic in $M(L)$; indeed it is the element $\dot{\tau}_{\{\lambda_w\}}$, where $\tau_{\{\lambda_w\}}$ is the minimal element of $\mathrm{Adm}^M(\{\lambda_w\})$. We let $\mathscr{G}$ denote an $(\mathcal{M}_{\mathcal{O}_L},\lambda_w)$-adapted lifting of $\mathscr{G}_{\overline{x}}$ satisfying the conditions in Proposition 4.8. Note that any $(\mathcal{M}_{\mathcal{O}_L},\lambda_w)$-adapted lifting is also $(\mathcal{G}_{\mathcal{O}_L},\mu_y)$-adapted, hence corresponds to a point $x\in\mathscr{S}_{\mathrm{K}}(G,X)(\mathcal{O}_K)$. We apply the construction of §5 to $\mathscr{G}_{\overline{x}}$, and we obtain a map

$$M(\mathbb{Q}_p)/\mathcal{M}(\mathbb{Z}_p)\to X^M(\{\lambda_w\},\dot{w}),\quad m\mapsto m_0$$

such that the image of its composition with $\iota_w$ lies in $X(\{\mu_y\},\dot{w})^\circ$. This follows as in part (i) since $i'_{\overline{x}}(m_0)$ is the specialization of $mx\in\mathscr{S}_{\mathrm{K}}(G,X)(\mathcal{O}_K)$. By Proposition 5.19, it induces a surjection

$$M(\mathbb{Q}_p)/\mathcal{M}(\mathbb{Z}_p)\to\pi_0(X^M(\{\lambda_w\},\dot{w})),$$

and hence $X(\{\mu_y\},b)^\circ$ contains the image of $X^{M_{\nu_w}}(\{\lambda_w\},\dot{w})$. $\square$

**Lemma 6.13.** *Let $w,w'$ straight such that $\dot{w},\dot{w}'\in[b]$ and $w'=sw's$ for some $s\in\mathbb{S}$. Suppose there exists*

$$g\in\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}},\dot{w}))\cap X(\{\mu_y\},b)^\circ.$$

*Then there exists*

$$g'\in\iota_{w'}(X^{M_{\nu_{w'}}}(\{\lambda_{w'}\}_{M_{\nu_{w'}}},\dot{w}'))\cap X(\{\mu_y\},b)^\circ.$$

*Proof.* By Lemma 6.12 $\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}},\dot{w}))\subset X(\{\mu_y\},b)^\circ$; hence upon choosing a different $g$, we may assume $g\in\iota_w(X_w^{M_{\nu_w}}(\dot{w}))$. Upon replacing $\overline{x}$ by $i_{\overline{x}}(g)$ and using (6.9.2), we may assume as in Lemma 6.12 that $b=\dot{w}$ and the $\iota_w$ is the natural inclusion. We note from the proof of [He14, Theorem 4.8] that

$$X_{w'}(\dot{w})\cong\iota_{w'}(X_{w'}^{M_{\nu_{w'}}}(\dot{w}'))\subset\iota_{w'}(X^{M_{\nu_{w'}}}(\{\lambda_{w'}\}_{M_{\nu_{w'}}},\dot{w}')).$$

It therefore suffices to show that there exists $g'\in X_{w'}(\dot{w})\cap X(\{\mu_y\},\dot{w})^\circ$.

We assume that the lift $\dot{s}$ lies in $N(\mathbb{Q}_p)$; such a lift exists since the action of $\sigma$ is trivial under our assumption that $G_{\mathbb{Q}_p}$ is residually split. As in the proof of Lemma 6.12, we may apply the construction of §5.7. We let $\mathscr{G}$ be an $(\mathcal{M}_{\mathcal{O}_L},\lambda_w)$-adapted lifting as in Proposition 4.8. We fix the isomorphism

$$T_p\mathscr{G}^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_L\cong\mathbb{D}(\mathscr{G}_{\overline{x}})$$

taking $t_{\alpha,\text{ét}}$ to $t_{\alpha,0,\overline{x}}$. Upon replacing $K$ by a finite extensions, we have $\dot{s}T_p\mathscr{G}$ corresponds to a $p$-divisible group $\mathscr{G}'$ over $\mathcal{O}_K$ equipped with a quasi isogeny $\mathscr{G}'\to\mathscr{G}$. This identifies $\mathfrak{M}(T_p\mathscr{G}'^\vee)$ with $\tilde{s}\mathfrak{M}(T_p\mathscr{G}^\vee)$ for some $\tilde{s}\in G(\mathfrak{S}[1/p])$. We also obtain a quasi-isogeny $\mathscr{G}'\to\mathscr{G}$ over $k$ which identifies $\mathbb{D}(\mathscr{G}')$ with $s_0\mathbb{D}(\mathscr{G})$ where $s_0=\sigma^{-1}(\tilde{s})|_{u=0}$. By Proposition 5.13 we have $\sigma^{-1}(\tilde{s})=m\dot{s}h$ where $m\in\mathcal{M}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})$ and $h\in\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})$. Using the natural map $L\to\mathfrak{S}[1/p]$, we may consider $\dot{w}\in G(L)$ as an element of $G(\mathfrak{S}[1/p])$. Then we have

$$\sigma^{-1}(\tilde{s}^{-1})\dot{w}\tilde{s}=h^{-1}\dot{s}^{-1}m^{-1}\dot{w}\sigma(m)\dot{s}\sigma(h)$$

We assume $l(sw)=l(w)+1$; the case $l(ws)=l(w)+1$ can be treated in the same way. Since $w$ is basic in $W_M$, we have

$$m':=m^{-1}\dot{w}\sigma(m)\dot{w}^{-1}\in\mathcal{M}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})$$

Since $l(sw)=l(w)+1$, we have $l(sws)=l(ws)+1$ and hence

$$\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})\dot{s}\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})\dot{w}\dot{s}\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})=\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})\dot{s}\dot{w}\dot{s}\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}}).$$

It follows that

$$\sigma^{-1}(\tilde{s}^{-1})\dot{w}\tilde{s}=h^{-1}\dot{s}^{-1}m'\dot{w}\dot{s}\sigma(h)\in\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}})\dot{s}\dot{w}\dot{s}\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\text{ur}}}});$$

note here we use that $s=s^{-1}$. We consider $\sigma^{-1}(\tilde{s}^{-1})\dot{w}\tilde{s}$ as a $k[[u]]^{\text{perf}}$ point of $\mathcal{FL}$. The above calculation shows that the generic fiber of this point lies in the the Schubert variety $S_{w'}\subset\mathcal{FL}$. Since the Schubert variety $S_{w'}$ is closed, the special fiber also lies in $S_{w'}$. Hence we have

$$s_0\dot{w}\sigma(s_0)\in\mathcal{G}(\mathcal{O}_L)\dot{w}''\mathcal{G}(\mathcal{O}_L)$$

for some $w''\leq w'$. By Lemma 6.14 below, we have $w'=w''$ and $s_0\in X_{w'}(\dot{w})$, and moreover it lies in $X(\{\mu_y\},\dot{w})^\circ$ since it is the specialization of $\dot{s}x$, where $x\in\mathscr{S}_{\mathrm{K}}(G,X)(\mathcal{O}_K)$ corresponds to $\mathscr{G}$. $\square$

**Lemma 6.14.** *($G$ not necessarily residually split) Let $w\in W$ be a $\sigma$-straight element and $g\in G(L)$ such that $g^{-1}\dot{w}\sigma(g)\in\mathcal{G}(\mathcal{O}_L)\dot{w}'\mathcal{G}(\mathcal{O}_L)$ with $w'\leq w$. Then $w'=w$.*

*Proof.* Since $w' \leq w$, it suffices to show $l(w) = l(w')$. Let $b = g^{-1}\dot{w}\sigma(g) \in \mathcal{G}(\mathcal{O}_L)\dot{w}'\mathcal{G}(\mathcal{O}_L)$. We claim that there is a $\sigma$-straight element $y \in W$ with $l(y) \leq l(w')$ such that $[\dot{y}] = [b]$ in $B(G)$. We first show by induction on $l(w')$ that there exists $w'' \in W$ with $l(w'') \leq l(w')$ which is minimal length in its $\sigma$-conjugacy class in $W$ and such that $b$ is $\sigma$-conjugate to an element of $\mathcal{G}(\mathcal{O}_L)\dot{w}''\mathcal{G}(\mathcal{O}_L)$. Indeed if $w'$ is minimal length in its $\sigma$-conjugacy class, then there is nothing to show. Otherwise, by [He14, Theorem 2.3] and [He14, Lemma 3.1 (1)], there exists $x \in W$ with $l(x) < l(w')$ such that $b$ is $\sigma$-conjugate to an element of $\mathcal{G}(\mathcal{O}_L)\dot{x}\mathcal{G}(\mathcal{O}_L)$. The result then follows by induction hypothesis applied to $x$.

By [He14, Theorem 2.3], there exists $J \subset \mathbb{S}$ with $W_J$ finite, $u \in W_J$ and $y \in {}^JW^{\sigma(J)}$ $\sigma$-straight with $y\sigma(J) = J$ such that $w'' \approx uy$; here the notation is as in §2. Note that $l(y) \leq l(w'')$. Then by [He14, Lemma 3.1 (2)] and [He14, Lemma 3.2], any element of $\mathcal{G}(\mathcal{O}_L)\dot{w}''\mathcal{G}(\mathcal{O}_L)$ is $\sigma$-conjugate to an element of $\mathcal{G}(\mathcal{O}_L)\dot{y}\mathcal{G}(\mathcal{O}_L)$. Therefore $b$, and hence $\dot{w}$ is $\sigma$-conjugate in $G(L)$ to an element of $\mathcal{G}(\mathcal{O}_L)\dot{y}\mathcal{G}(\mathcal{O}_L)$. By [He14, Theorem 3.5], $b$ is $\sigma$-conjugate to $\dot{y}$ itself.

Therefore $[\dot{w}] = [\dot{y}]$ in $B(G)$ and since they are both $\sigma$-straight, it follows by [He14, Theorem 3.7] that $w$ and $y$ are $\sigma$-conjugate in $W$. Since they are both straight, $l(w) = l(y)$, and hence $l(w) = l(w')$. $\square$

**Lemma 6.15.** *Let $w, w'$ straight such that $\dot{w}, \dot{w}' \in [b]$ and $w' = \tau^{-1}w\tau$ for some $\tau \in \Omega$. Suppose there exists*
$$g \in \iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \cap X(\{\mu_y\}, b)^\circ.$$
*Then there exists*
$$g' \in \iota_{w'}(X^{M_{\nu_{w'}}}(\{\lambda_{w'}\}_{M_{\nu_{w'}}}, \dot{w}')) \cap X(\{\mu_y\}, b)^\circ.$$

*Proof.* As in Lemma 6.13, we may assune $g \in \iota_w(X_w^{M_{\nu_w}}(\dot{w}))$ and upon replacing $\overline{x}$ by $i_{\overline{x}}(g)$ we may assume $b = \dot{w}$. As in the proof of Lemma 6.13, it suffices to show there exists $g' \in X_{w'}(\dot{w}) \cap X(\{\mu_y\}, \dot{w})^\circ$.

Let $\mathscr{G}$ be an $(\mathcal{M}_{\mathcal{O}_L}, \lambda_w)$-adapted lifting of $\mathscr{G}_{\overline{x}}$ to $\mathcal{O}_K$ and apply the construction of Proposition 5.14 to $\dot{\tau} \in N(\mathbb{Q}_p)$; we obtain an element $\tilde{\tau} \in G(\mathfrak{S}[\frac{1}{p}])$. Moreover $\sigma^{-1}(\tilde{\tau}) = m\dot{\tau}g$ with $m \in \mathcal{M}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})$ and $g \in \mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})$. Since $\dot{\tau}\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})\dot{\tau}^{-1} = \mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})$, we have
$$\sigma^{-1}(\tilde{\tau}^{-1})\dot{w}\tilde{\tau} = g^{-1}\dot{\tau}^{-1}m^{-1}\dot{w}\sigma(m)\dot{\tau}\sigma(g) \in \mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})\dot{\tau}^{-1}\dot{w}\dot{\tau}\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}}).$$
By the same argument in Lemma 6.13, $\tau_0 := \sigma^{-1}(\tilde{\tau})|_{u=0} \in X_{w'}(\dot{w}) \cap X(\{\mu_y\}, \dot{w})^\circ$. $\square$

We can now complete the proof of Proposition 6.5 (ii). Recall we are trying to show the map $i'_{\overline{x}}$ lifts to a map
$$i_{\overline{x}} : X(\sigma(\{\mu_y\}), b) \to \mathscr{S}_{\mathrm{K}}(G, X)(k)$$
which is compatible with tensors, or in other words that $X(\{\mu_y\}, b)^\circ = X(\{\mu_y\}, b)$. By the surjectivity of (6.9.4) and the fact that $X(\{\mu_y\}, b)^\circ$ is a union of connected components, it suffices to show $\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \subset X(\{\mu_y\}, b)^\circ$ for all $w$ straight such that $\dot{w} \in [b]$. By Lemma 6.12, it suffices to show $\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \cap X(\{\mu_y\}, b)^\circ$ is non-empty for each such $w$.

By the surjectivity of (6.9.4) and the fact that $X(\{\mu_y\}, b)^\circ$ is a union of components, there exists some $w$ straight with $\dot{w} \in [b]$ such that
$$\iota_w(X^{M_{\nu_w}}(\{\lambda_w\}_{M_{\nu_w}}, \dot{w})) \cap X(\{\mu_y\}, b)^\circ \neq \emptyset.$$
By Theorem 2.2, for any $w'$ straight with $\dot{w}' \in [b]$, there exists $w = w_0, w_1, \ldots, w_m = \tau w' \tau^{-1}$ with $\tau \in \Omega$ such that $w_i \approx_{s_i} w_{i+1}$ for some $s_i \in \mathbb{S}$ and $i = 0, \ldots, m-1$. Applying Lemmas 6.13 and 6.15 inductively, we obtain $\iota_{w'}(X^{M_{\nu_{w'}}}(\{\lambda_{w'}\}_{M_{\nu_{w'}}}, \dot{w}')) \cap X(\{\mu_y\}, b)^\circ \neq \emptyset$.

This completes the proof of Proposition 6.5 part (ii).

*Case (iii), Conjecture 5.4 holds:* The proof in this case follows along the lines of Case (ii). Let $M \subset G$ be a standard $\sigma_0$-stable Levi subgroup as in Lemma 5.7. Recall the decomposition

$$X(\sigma(\{\mu_y\}), b) \cong \coprod_{P'=M'N'} X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})$$

into open and closed subschemes from Theorem 5.4.1. This induces a bijection

$$\pi_0(X(\sigma(\{\mu_y\}), b)) \cong \coprod_{P'=M'N'} \pi_0(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})).$$

We write $\iota_{P'}$ for the map $X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})) \to X(\sigma(\{\mu_y\}), b)$; it is induced by the map $m \mapsto j_{b,b_{P'}} m$ for $m \in M'(L)$, where $j_{b,b_{P'}} \in G(L)$ is a fixed element such that $j_{b,b_{P'}}^{-1} b\sigma(j_{b,b_{P'}}) = b_{P'}$. By our assumption on $M$, $[b_{P'}] \in B(M', \sigma(\{\mu_{P'}\}))$ is Hodge–Newton indecomposable. Using Lemma 6.11, $X(\sigma(\{\mu_y\}), b)^\circ$ is a union of connected components. The Proposition will follow from the next two Lemmas.

**Lemma 6.16.** *Let $P' \in \mathfrak{P}^\sigma$. Suppose there exists*

$$g \in \iota_{P'}(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})) \cap X(\sigma(\{\mu_y\}), b)^\circ.$$

*Then $\iota_{P'}(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})) \subset X(\sigma(\{\mu_y\}), b)^\circ$.*

*Proof.* Let $\mathcal{M}'(\mathcal{O}_L) = \mathcal{M}'(L) \cap \mathcal{G}(\mathcal{O}_L)$. Then $M'(\mathcal{O}_L)$ is preserved by $\sigma$ and arises as the $\mathcal{O}_L$-points of an Iwahori subgroup $\mathcal{M}'$ of $M'$ defined over $\mathbb{Z}_p$. Upon replacing $\overline{x}$ by $i_{\overline{x}}(g)$ and using (6.9.2), we may assume $g = 1$. Upon modifying the isomorphism (6.7.1), we may assume $b \in \mathcal{M}'(\mathcal{O}_L)\dot{w}_{M'}\mathcal{M}'(\mathcal{O}_L)$ where $w_{M'} \in \mathrm{Adm}^{M'}(\sigma(\{\mu_{P'}\}))$. Replacing $b_{P'}$ by another representative of its class in $B(M')$, we assume $b_{P'} = b$.

By Proposition 3.4, upon extending $K$ we may choose a representative $\mu_{P'}$ of $\{\mu_{P'}\}$ which induces a filtration on $\mathbb{D}(\mathscr{G}_{\overline{x}}) \otimes K$ lifting the filtration $\bmod p$. Then since the $\mu_{P'}$ lies in the $G$-conjugacy class of cocharacters $\{\mu_y\}$, the assumption (5.12) is satisfied. We extend the tensors $s_\alpha$ to tensors $t_\alpha \in V_{\mathbb{Z}_p}^{\otimes}$ whose stabilizer is identified with $\mathcal{M}'$. We may thus apply the construction of §5.7.

Let $\mathscr{G}$ be an $(\mathcal{M}', \mu_{P'})$-adapted lifting of $\mathscr{G}_{\overline{x}}$ satisfying the conditions in Proposition 4.8. We fix the isomorphism

$$T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$$

taking $t_{\alpha,\text{ét}}$ to $t_{\alpha,0,\overline{x}}$. Applying the construction of §5.7, we obtain a map

$$M'(\mathbb{Q}_p)/\mathcal{M}'(\mathbb{Z}_p) \to X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})$$

whose image after composition with $\iota_{P'}$ lies in $X(\sigma(\{\mu_y\}, b_{P'}))^\circ$. By Proposition 5.20 and our assumption on $M$ which implies $b_{P'} \in B(M', \sigma(\{\mu_{P'}\}))$ is Hodge–Newton indecomposable, the map

$$M'(\mathbb{Q}_p)/\mathcal{M}'(\mathbb{Z}_p) \to \pi_0(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'}))$$

is surjective. Hence $\iota_{P'}(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})) \subset X(\sigma(\{\mu_y\}), b_{P'})^\circ$. $\square$

**Lemma 6.17.** *For each $P' \in \mathfrak{P}^\sigma$, there exists $g \in \iota_{P'}(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})) \cap X(\sigma(\{\mu_y\}), b)^\circ$.*

*Proof.* The decomposition 5.4.1 implies that there exists $P'' \in \mathfrak{P}^\sigma$ such that

$$1 \in \iota_{P''}(X^{M''}(\sigma(\{\mu_{P''}\}), b_{P''}))$$

Then upon modifying (6.7.1), we may assume $b = b_{P''} \in \mathcal{M}''(\mathcal{O}_L)\dot{w}_{M''}\mathcal{M}''(\mathcal{O}_L)$, where $\mathcal{M}''$ is the Iwahori subgroup scheme over $\mathbb{Z}_p$ of $M''$ such that $\mathcal{M}''(\mathcal{O}_L) = \mathcal{G}(\mathcal{O}_L) \cap M''(L)$ and $w_{M''} \in$

$\mathrm{Adm}^{M''}(\sigma(\{\mu_{P''}\}))$. We write $b_{P''} = m_1\dot{w}_{M''}m_2$ in this decomposition. By [GHN, Theorem 3.6] and [GHN, Theorem 3.12], if $w_{M'} \in \mathrm{Adm}^{M'}(\sigma(\{\mu_{P'}\}))$ is such that $X_{w_{M'}}(b_{P''}) \neq \emptyset$, then $X_{w_{M'}}(b_{P''}) \cong \iota_{P'}(X^{M'}_{w_{M'}}(b_{P'}))$. In particular $X_{w_{M'}}(b_{P''}) \subset \iota_{P'}(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'}))$. Thus it suffices to show that that there exists some $w_{M'} \in \mathrm{Adm}^{M'}(\sigma(\{\mu_{P'}\}))$ and $g \in X_{w_{M'}}(b_{P''}) \cap X(\sigma(\{\mu_y\}), b_{P''})^\circ$.

We write $\mathcal{M}'$ for the Iwahori subgroup scheme of $M'$ defined over $\mathbb{Z}_p$ such that $\mathcal{M}'(\mathcal{O}_L) = M'(L) \cap \mathcal{G}(\mathcal{O}_L)$. By Lemma 5.8, there exists $v \in W^\sigma$ such that $\dot{v}^{-1}\mathcal{M}''(\mathcal{O}_L)\dot{v} = \mathcal{M}'(\mathcal{O}_L)$ and $v^{-1}\mathrm{Adm}^{M''}(\sigma(\{\mu_{P''}\}))v = \mathrm{Adm}^{M'}(\sigma(\{\mu_{P'}\}))$.

As in Lemma 6.16, we may choose a representative $\mu_{P''}$ for $\{\mu_{P''}\}$ which induces a filtration on $\mathbb{D}(\mathscr{G}_{\overline{x}}) \otimes K$ lifting the filtration $\bmod p$. Let $\mathscr{G}$ be an $(\mathcal{M}, \mu_{P''})$-adapted lifting of $\mathscr{G}_{\overline{x}}$ as in Proposition 4.8. We choose the lift $\dot{v}$ of $v$ to lie in $N(\mathbb{Q}_p)$ and we fix the isomorphism

$$T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$$

taking $t_{\alpha,\text{ét}}$ to $t_{\alpha,0,\overline{x}}$. We obtain a map

$$\mathcal{G}(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to X(\sigma(\{\mu_y\}), b_{P''}).$$

Moreover by Proposition 5.13, the image of $v$ under this map is given by $\sigma^{-1}(\tilde{v})|_{u=0}$ where $\tilde{v} \in G(\mathfrak{S}[1/p])$ is an element such that $\sigma^{-1}(\tilde{v}) = m\dot{v}h$ with $m \in \mathcal{M}''(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})$ and $h \in \mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})$. Using the natural map $L \to \mathfrak{S}[1/p]$, we consider $b_{P''} \in G(L)$ as an element of $G(\mathfrak{S}[1/p])$. Then we have

$$\begin{aligned}\sigma^{-1}(\tilde{v})^{-1}b_{P''}\tilde{v} &= h^{-1}\dot{v}^{-1}m^{-1}m_1\dot{w}_{M''}m_2\sigma(m)\dot{v}\sigma(h)\\ &= h^{-1}(\dot{v}^{-1}m^{-1}m_1\dot{v})\dot{v}^{-1}\dot{w}_{M''}\dot{v}(\dot{v}^{-1}m_2\sigma(m)\dot{v})\sigma(h) \in \mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})\dot{w}_{M'}\mathcal{G}(\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}})\end{aligned}$$

where $w_{M'} := v^{-1}w_{M''}v \in \mathrm{Adm}^{M'}(\sigma(\{\mu_{P'}\}))$. Therefore considering $\sigma^{-1}(\tilde{v})$ as a $k[[u]]^{\mathrm{perf}}$-point of $\mathcal{FL}$, it follows that the generic fiber of $\tilde{v}$ lies in $X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})$. Hence since

$$\iota_{P'}(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'})) \subset X(\sigma(\{\mu_y\}), b_{P''})$$

is closed, $v_0 := \sigma^{-1}(\tilde{v})|_{u=0} \in \iota_{P'}(X^{M'}(\sigma(\{\mu_{P'}\}), b_{P'}))$ and $v_0 \in X(\sigma(\{\mu_y\}), b)^\circ$. □

□

In some of the later statements, we will assume the existence of a lifting $i_{\overline{x}}$ as in Proposition 6.5. For notational convenience, we will introduce the following assumption.

**Assumption 6.18.** For each $\overline{x} \in \mathscr{S}_{\mathrm{K}}(G, X)(k)$, there exists a map

$$i_{\overline{x}} : X(\sigma(\{\mu_y\}), b) \to \mathscr{S}_{\mathrm{K}}(G, X)(k)$$

lifting $i'_{\overline{x}}$ such that $s_{\alpha,0,i_{\overline{x}}(g)} = s_{\alpha,0,\overline{x}}$ and $\Phi \circ i_{\overline{x}} = i_{\overline{x}} \circ \Phi$.

Therefore Proposition 6.5 says that this assumption is satisfied if $G_{\mathbb{Q}_p}$ is residually split or if Conjecture 5.4 holds.

6.10. For the rest of this section, we assume Assumption 6.18 is satisfied. Recall the local model diagram from Theorem 6.1:

$$\begin{array}{ccc} & \widetilde{\mathscr{S}_{\mathrm{K}}}(G,X)_{\mathcal{O}_E} & \\ \swarrow^{q} & & \searrow^{\pi} \\ \mathscr{S}_{\mathrm{K}}(G,X)_{\mathcal{O}_E} & & M^{\mathrm{loc}}_{\mathcal{G},\mu_h} \end{array}$$

This induces the Kottwitz Rapoport stratification on the geometric special fiber $\mathscr{S}_{\mathrm{K}}(G,X)_k$. We write $\mu$ for a dominant representative of $\mu_h^{-1}$ in $X_*(T)$. Then we have a map

$$\lambda : \mathscr{S}_{\mathrm{K}}(G,X)(k) \to \mathrm{Adm}(\{\mu\})$$

Since $\mu_y$ and $\mu$ are both conjugate to $\mu_h^{-1}$, the admissible sets $\mathrm{Adm}(\{\mu\})$ and $\mathrm{Adm}(\{\mu_y\})$ coincide; thus we may write $X(\sigma(\{\mu\}),b)$ for $X(\sigma(\{\mu_y\}),b)$.

**Proposition 6.19.** *Let $g \in X_w(b)$ for some $w \in \sigma(\mathrm{Adm}(\{\mu\}))$. Then $\lambda(i_{\overline{x}}(g)) = \sigma^{-1}(w)$.*

*Proof.* Recall how the map $\lambda$ is defined. Let $x \in \mathscr{S}_{\mathrm{K}}(G,X)(\mathcal{O}_K)$ be a point lifting $\overline{x}$. The torsor $\widetilde{\mathscr{S}_{\mathrm{K}}}(G,X)$ is constructed by taking trivializations of the relative de Rham cohomology which respect the cycles $s_{\alpha,\mathrm{dR}}$. We have an isomorphism

$$V_{\mathbb{Z}_p}^* \otimes_{\mathbb{Z}_p} \mathcal{O}_K \cong \mathbb{D}(\mathscr{G}_x)(\mathcal{O}_K)$$

taking $s_\alpha$ to $s_{\alpha,\mathrm{dR},x}$. The local model $M_{\mathcal{G}}^{\mathrm{loc}}$ embeds inside $M_{\mathcal{GL}}^{\mathrm{loc}} \otimes \mathcal{O}_E$, where $M_{\mathcal{GL}}^{\mathrm{loc}}$ classifies sub-modules of $V_{\mathbb{Z}_p}^*$. The pullback of the Hodge filtration on $\mathbb{D}(\mathscr{G}_x)(\mathcal{O}_K)$ gives a point of $M_{\mathcal{GL}}^{\mathrm{loc}} \otimes \mathcal{O}_E$ which lies in the local model $M_{\mathcal{G}}^{\mathrm{loc}}(\mathcal{O}_K)$. We obtain a point $\tilde{x} \in M_{\mathcal{G}}^{\mathrm{loc}}(k)$ which lies in $\underline{\mathcal{G}}(k[[t]])\dot{w}\underline{\mathcal{G}}(k[[t]])/\underline{\mathcal{G}}(k[[t]])$ for some $w \in \mathrm{Adm}(\{\mu\})$. Here $\dot{w}$ denotes a lift of $w$ to $\underline{G}(k((t)))$ via the identification of Iwahori Weyl groups for $G$ and $\underline{G}_{k((t))}$. Then $\lambda(\overline{x}) = \dot{w}$.

There is an isomorphism $\mathbb{D}(\mathscr{G}_x)(\mathcal{O}_K) \cong \mathbb{D}(\mathscr{G}_{\overline{x}}) \otimes \mathcal{O}_K$ lifting the identity mod $p$ and taking $s_{\alpha,\mathrm{dR},x}$ to $s_{\alpha,0,\overline{x}}$. Thus if we fix an isomorphism

$$V_{\mathbb{Z}_p}^* \otimes \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$$

taking $s_\alpha$ to $s_{\alpha,0,\overline{x}}$, then the pullback of the filtration on $\mathbb{D}(\mathscr{G}_{\overline{x}})(k)$ to $V_{\mathbb{Z}_p}^*$ differs from the one above by translation by an element of $\mathcal{G}(\mathcal{O}_K)$. We thus obtain a point on $\tilde{x}' \in M_{\mathcal{G}}^{\mathrm{loc}}(k)$ which lies in the Schubert cell $C_w$. Thus $\lambda(\overline{x})$ can also be computed by a trivialization of $\mathbb{D}(\mathscr{G}_x)$.

Now fix an isomorphism $V_{\mathbb{Z}_p}^* \otimes \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$. Let $g \in X_w(b)$ for some $w \in \mathrm{Adm}(\{\mu\})$; then $\mathbb{D}(\mathscr{G}_{i_{\overline{x}}(g)})$ is identified with $g\mathbb{D}(\mathscr{G}_{\overline{x}})$. We may trivialize $\mathbb{D}(\mathscr{G}_{i_{\overline{x}}(g)}) \cong V_{\mathbb{Z}_p}^* \otimes \mathcal{O}_L$ by composing the trivilization $V_{\mathbb{Z}_p}^* \otimes \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$ with the element $g$. The filtration mod $p$ on $V_{\mathbb{Z}_p}^*$ is then induced by the element $g^{-1}b\sigma(g)$. By the identification of apartments and Iwahori Weyl group in §3.3, this filtration corresponds to a point $M_{\mathcal{G}}^{\mathrm{loc}}(k)$ which lies in $\underline{\mathcal{G}}(k[[t]])\sigma^{-1}(\dot{w})\underline{\mathcal{G}}(k[[t]])/\underline{\mathcal{G}}(k[[t]])$. Hence $\lambda(i_{\overline{x}}(g)) = w$. ☐

6.11. The maps $i_{\overline{x}}$ are compatible with the changing the prime to $p$ level structure $\mathrm{K}^p$. Since $\mathscr{S}_{\mathrm{K}_p}(G,X)$ is equipped with an action of $G(\mathbb{A}_f^p)$, $i_{\overline{x}}$ extends to a map:

$$i_{\overline{x}} : X(\sigma(\{\mu\}),b) \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathrm{K}_p}(G,X)(k)$$

As in [Kis10, Corollary 1.4.13], this map is equivariant for the action of $\Phi \times G(\mathbb{A}_f^p)$.

**Definition 6.20.** Let $\overline{x}, \overline{x}' \in \mathscr{S}_{\mathrm{K}_p}(G,X)(k)$. We say $\overline{x}$ and $\overline{x}'$ are in the same isogeny class if there exists a quasi-isogeny $\mathcal{A}_{\overline{x}} \to \mathcal{A}_{\overline{x}'}$ respecting weak polarizations such that the induced maps $\mathbb{D}(\mathscr{G}_{\overline{x}'}) \to \mathbb{D}(\mathscr{G}_{\overline{x}})$ and $\widehat{V}^p(\mathcal{A}_{\overline{x}}) \to \widehat{V}^p(\mathcal{A}_{\overline{x}'})$ take $s_{\alpha,0,\overline{x}}$ to $s_{\alpha,0,\overline{x}'}$ and $\{s_{\alpha,l,\overline{x}}\}_{l\neq p}$ to $\{s_{\alpha,l,\overline{x}'}\}_{l\neq p}$.

For $\overline{x} \in \mathscr{S}_{\mathrm{K}_p}(G,X)(k)$, we write $\mathscr{I}_{\overline{x}}$ for the isogeny class containing $\overline{x}$. For $\overline{x} \in \mathscr{S}_{\mathrm{K}}(G,X)(k)$, we use the same notation to denote the image of $\mathscr{I}_{\overline{x}_p}$ in $\mathscr{S}_{\mathrm{K}}(G,X)(k)$ where $\overline{x}_p \in \mathscr{S}_{\mathrm{K}_p}(G,X)(k)$ is a point lifting $\overline{x}$.

**Proposition 6.21.** *Let $G_{\mathbb{Q}_p}$ be residually split or $b$ basic, or assume that Conjecture 5.4 holds. Then $\overline{x}, \overline{x}' \in \mathscr{S}_{\mathrm{K}_p}(G,X)(k)$ lie in the same isogeny class if and only if $\overline{x}'$ lies in the image of*

$$i_{\overline{x}} : X(\sigma(\{\mu\}),b) \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathrm{K}_p}(G,X)(k)$$

*Proof.* Suppose $\overline{x}$ and $\overline{x}'$ lie in the same isogeny class. The composition

$$V_{\mathbb{A}_f^p} \xrightarrow[\epsilon_{\overline{x}}]{\sim} \widehat{V}(\mathcal{A}_{\overline{x}}) \xrightarrow{\sim} \widehat{V}(\mathcal{A}_{\overline{x}'}) \xrightarrow[\epsilon_{\overline{x}'}]{\sim} V_{\mathbb{A}_f^p}$$

takes $s_\alpha$ to $s_\alpha$, hence upon replacing $\overline{x}'$ by a translate under $G(\mathbb{A}_f^p)$, we may assume the quasi isogeny $\mathcal{A}_{\overline{x}} \to \mathcal{A}_{\overline{x}'}$ is compatible with $\epsilon_{\overline{x}}$ and $\epsilon_{\overline{x}'}$.

Recall there are isomorphisms

$$\mathbb{D}(\mathscr{G}_{\overline{x}}) \xrightarrow{\sim} V^* \otimes_{\mathbb{Z}} \mathcal{O}_L \xrightarrow{\sim} \mathbb{D}(\mathscr{G}'_{\overline{x}})$$

taking $s_{\alpha,0,\overline{x}}$ to $s_{\alpha,0,\overline{x}'}$. Thus $\mathbb{D}(\mathscr{G}_{\overline{x}'})$ corresponds to $g\mathbb{D}(\mathscr{G}_{\overline{x}})$ for some $g \in G(L)$. By the same proof as Proposition 5.14, we have $g \in X(\sigma(\{\mu\}), b)$. It follows from the definition of $i_{\overline{x}}$ that $\overline{x}'$ and $i_{\overline{x}}(g)$ have the same image in $\mathscr{S}_{\mathrm{K}_p}(G,X)^-(k)$. Since the quasi-isogeny induces a map $\mathbb{D}(\mathscr{G}_{\overline{x}'}) \to \mathbb{D}(\mathscr{G}_{\overline{x}})$ taking $s_{\alpha,0,\overline{x}}$ to $s_{\alpha,0,\overline{x}'}$, we have by Corollary 6.3 that $i_{\overline{x}}(g) = \overline{x}'$. Hence $\overline{x}'$ lies in the image of $i_{\overline{x}}$.

The converse is clear. □

## 7. Maps between Shimura varieties

In this section we show that the Shimura varieties associated to different parahorics levels admit maps between them with good properties. This will allow us to deduce the description of the isogeny classes for general parahorics from the result for Iwahori subgroups proved in the previous section. This also verifies one of the axioms of [HR17] for integral models of Shimura varieties with parahoric level.

7.1. We keep the notations from the previous section, so that $\rho : G \to GSp(V,\psi)$ is a Hodge embedding. Let $\mathrm{K}_p$ be a connected parahoric subgroup of $G(\mathbb{Q}_p)$ and let $\mathcal{G}$ denote the corresponding group scheme over $\mathbb{Z}_p$. Let $\mathrm{K}'_p \subset G(\mathbb{Q}_p)$ be another connected parahoric subgroup with corresponding group scheme $\mathcal{G}'$ such that $\mathrm{K}_p \subset \mathrm{K}'_p$. If $\mathrm{K}_p$ and $\mathrm{K}'_p$ have corresponding facets $\mathfrak{f}$ and $\mathfrak{f}'$, then this is equivalent to $\mathfrak{f}$ lying in the closure of $\mathfrak{f}'$. Note that we are changing the notation from the previous section where $\mathrm{K}'_p$ was a subgroup of $GSp(V_{\mathbb{Q}_p})$.

By the construction in the previous section we have integral models $\mathscr{S}_{\mathrm{K}'}(G,X)$ and $\mathscr{S}_{\mathrm{K}}(G,X)$ over $\mathcal{O}_{E_{(v)}}$, where $\mathrm{K}' = \mathrm{K}'_p\mathrm{K}^p$ and $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p$ for some sufficiently small $\mathrm{K}^p$. Since at this point there is no characterization of these integral models, in what follows we must assume that they are constructed using the same Hodge embedding $\rho$; see §7.3 for the precise details of the construction.

**Theorem 7.1.** *Suppose $\mathscr{S}_{\mathrm{K}'}(G,X)$ and $\mathscr{S}_{\mathrm{K}}(G,X)$ are constructed from the same Hodge embedding. Then we have:*

*(i) For sufficiently small $\mathrm{K}^p$, there exists a map*

$$\pi_{\mathrm{K}_p,\mathrm{K}'_p} : \mathscr{S}_{\mathrm{K}}(G,X) \to \mathscr{S}_{\mathrm{K}'}(G,X).$$

*(ii) The induced map*

$$\mathscr{S}_{\mathrm{K}_p}(G,X)(k) \to \mathscr{S}_{\mathrm{K}'_p}(G,X)(k)$$

*is compatible with isogeny classes.*

7.2. Let $\mathfrak{g}$ be a facet in $\mathcal{B}(GSp(V_{\mathbb{Q}_p}),\mathbb{Q}_p)$ and let $\mathcal{GSP}$ denote the associated parahoric group scheme. Then $\mathfrak{g}$ corresponds to a lattice chain $\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_r$ in $V_{\mathbb{Q}_p}$. Let $V'_{\mathbb{Q}_p} = \oplus_{i=1}^r V_{\mathbb{Q}_p}$, then $V'$ is equipped with an alternating form $\psi'$ given by the direct sum of $\psi$. We have the lattice $\Lambda' = \oplus_{i=1}^r \Lambda_i \subset V'_{\mathbb{Q}_p}$ and we write $\mathcal{GSP}'$ for the associated parahoric of $GSp(V'_{\mathbb{Q}_p})$ stabilizing $\Lambda'$.

We have a map

$$GSp(V,\psi) \to GSp(V',\psi')$$

which factors through the subgroup $H := \prod_{i=1}^{\prime r} GSp(V,\psi)$. Here $\prod'$ denotes the subgroup of the product $\prod_{i=1}^{r} GSp(V,\psi)$ consisting of elements $(g_1,\dots,g_r)$ such that $c(g_1)=\dots=c(g_2)$, where

$$c : GSp(V,\psi) \to \mathbb{G}_m$$

is the multiplier homomorphism.

The conjugacy class of cocharacters $S^{\pm}$ for $GSp(V,\psi)$ gives rise to a $H(\mathbb{R})$ conjugacy class of homomorphisms $T$ from $\mathbb{S}$ into $H_{\mathbb{R}}$ and $(H,T)$ is a Shimura datum. We write $\mathrm{H}_p$ and $\mathrm{J}_p$ for the stabilizer of the lattice $\Lambda'$ in $H(\mathbb{Q}_p)$ and $GSp(V'_{\mathbb{Q}_p})$ respectively. We obtain a map of Shimura varieties

$$\iota : Sh_{\mathrm{H}_p\mathrm{H}^p}(H,T) \to Sh_{\mathrm{J}_p\mathrm{J}^p}(GSp(V'),S'^{\pm}))$$

which is a closed immersion. Here $\mathrm{H}^p \subset H(\mathbb{A}_f^p)$ and $\mathrm{J}^p \subset GSp(V'\otimes\mathbb{A}_f^p)$ are sufficiently small compact open subgroups.

The Shimura variety $Sh_{\mathrm{H}_p\mathrm{H}^p}(H,T)$ admits a moduli interpretation over $\mathbb{Z}_{(p)}$ which we will now explain. For $S$ a $\mathbb{Z}_{(p)}$-scheme, we consider the set of tuples $(\mathcal{A}_i,\lambda_i,\epsilon_i^p)_{i=1,\dots,r}$, where:

(i) $\mathcal{A}_i$ is an abelian variety over $S$ up to prime to isogeny.

(ii) $\lambda_i$ is a weak polarization such that $\deg\lambda_i$ is exactly divisible by $|\Lambda_i/\Lambda_i^*|$.

(iii) $\epsilon_i^p \in \Gamma(S, \mathrm{Isom}_{\lambda_i,\psi}(\widehat{V}(\mathcal{A}_i), V\otimes_{\mathbb{Q}}\mathbb{A}_f^p)/H^p)$, where $\mathrm{Isom}_{\lambda_i,\psi}(\widehat{V}(\mathcal{A}_i), V\otimes_{\mathbb{Q}}\mathbb{A}_f^p)$ is the (pro)-étale sheaf of isomorphisms $\widehat{V}(\mathcal{A}_i)\cong V_{\mathbb{A}_f^p}$ which preserves the pairings induced $\lambda_i$ and $\psi$ up to a $\mathbb{A}_f^{p\times}$-scalar. This multiple is required to be independent of $i$.

We obtain an integral model $\mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H,T)$ of $Sh_{\mathrm{H}_p\mathrm{H}^p}(H,T)$.

**Proposition 7.2.** *For sufficiently small $\mathrm{J}^p$, the embedding $\iota$ extends to a closed embedding*

$$\iota : \mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H,T) \to \mathscr{S}_{\mathrm{J}_p\mathrm{J}^p}(GSp(V'),S'^{\pm}).$$

*Proof.* Using the moduli interpretations, we may define $\iota$ by sending

$$(\mathcal{A}_i,\lambda_i,\epsilon_i^p)_{i=1,\dots,r} \in \mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H,T)(S)$$

to the product $\mathcal{A}_1\times\dots\times\mathcal{A}_r$, together with the product polarization and level structure.

We show that for $\mathrm{J}^p$ sufficiently small, the map $\iota$ is proper and injective on points.

As in [Kis06, 2.1.2] (see also [Del71, 1, 1.5]), to show the injectivity on points it suffices to show the map

$$\iota_p : \mathscr{S}_{\mathrm{H}_p}(H,T) \to \mathscr{S}_{\mathrm{J}_p}(GSp(V'),S'^{\pm})$$

is injective on points. This follows from the moduli interpretations of the integral models.

We now fix $J^p$ sufficiently small such that $\iota$ is injective. To check properness, we apply the valuative criterion. Let $R$ be a discrete valuation ring with fraction field $K$. We must show for any diagram

$$\begin{tikzcd}
\mathrm{Spec}\ K \arrow[r] \arrow[d] & \mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H,T) \arrow[d] \\
\mathrm{Spec}\ R \arrow[r] & \mathscr{S}_{\mathrm{J}_p\mathrm{J}^p}(GSp(V'),S'^{\pm})
\end{tikzcd}$$

there exists a unique lift $\mathrm{Spec}\ R \to \mathscr{S}_{\mathrm{H}_p}(H,T)$. Rephrasing in terms of the moduli interpretation, we must show for a triple $(\mathcal{A},\lambda,\epsilon^p)$ over $R$, such that over $K$ this data decomposes into a product coming from $(\mathcal{A}_i,\lambda_i,\epsilon_i^p)_{i=1,\dots,r}$, then the triple over $R$ decomposes. This follows by well known properties of Neron models.

Indeed if $(\mathcal{A},\lambda,\epsilon^p)$ over $R$ is such a triple. We let

$$\mathcal{A}_K \cong \mathcal{A}_{1,K}\times\ldots\times\mathcal{A}_{r,K} \tag{7.2.1}$$

be the isomorphism on the generic fiber and let $\mathcal{A}_i$ denote the Neron models of $\mathcal{A}_{i,K}$. Then by uniqueness of Neron models, the isomorphism (7.2.1) extends to an isomorphism $\mathcal{A} \cong \mathcal{A}_1 \times \ldots \times \mathcal{A}_r$. Similarly, $\lambda$ is induced by the product of polarizations on each $\mathcal{A}_i$. That $\epsilon^p$ breaks up into a product, follows from the fact that an étale sheaf over $R$ is determined by its generic fiber. $\square$

7.3. By the construction in [KP, §1.2], there is an embedding of buildings,

$$\theta : B(G, \mathbb{Q}_p) \to B(GSp(V_{\mathbb{Q}_p}), \mathbb{Q}_p).$$

Let $\mathfrak{f}$ be a facet in $B(G, \mathbb{Q}_p)$ with associated connected parahoric group scheme $\mathcal{G}$. Let $\theta(\mathfrak{f})$ be contained in a facet $\mathfrak{g}$ of $B(GSp(V_{\mathbb{Q}_p}), \mathbb{Q}_p)$ corresponding to $\Lambda_1 \supset \cdots \supset \Lambda_r$. Let $(H, T)$ and $V'$ be as above, we obtain a new embedding of Shimura datum

$$(G, X) \to (GSp(V', \psi'), S'^{\pm})$$

which factorises as

$$(G, X) \xrightarrow{\rho'} (H, T) \to (GSp(V', \psi'), S'^{\pm})$$

Let $\mathrm{H}_p$ and $\mathrm{J}_p$; for a choice of $\mathrm{H}^p$ we let $\mathrm{J}^p$ be as in Proposition 7.2. For $\mathrm{H}^p$ sufficiently small, we obtain maps of Shimura varieties

$$Sh_{\mathrm{K}_p\mathrm{K}^p}(G, X) \to Sh_{\mathrm{H}_p\mathrm{H}^p}(H, T) \to Sh_{\mathrm{J}_p\mathrm{J}'^p}(GSp(V'), S'^{\pm})_E$$

and each of these maps is a closed immersions. Recall $\mathscr{S}^-_{\mathrm{K}_p\mathrm{K}^p}(G, X)$ was defined to be the closure of $Sh_{\mathrm{K}_p\mathrm{K}^p}(G, X)$ in $\mathscr{S}_{\mathrm{J}_p\mathrm{J}^p}(GSp(V'), S'^{\pm})_{\mathcal{O}_{E_{(v)}}}$.

**Corollary 7.3.** *$\mathscr{S}^-_{\mathrm{K}_p}(G, X)$ is the closure of $Sh_{\mathrm{K}_p\mathrm{K}^p}(G, X)$ in $\mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H, T)$.*

*Proof.* Immediate from Proposition 7.2. $\square$

Now suppose $\mathfrak{f}'$ is a facet of $B(GSp(V_{\mathbb{Q}_p}), \mathbb{Q}_p)$ such that $\mathfrak{f}'$ lies in the closure of $\mathfrak{f}$. Then $\mathfrak{f}'$ corresponds to a lattice chain $\Lambda_{i_1} \subset \cdots \subset \Lambda_{i_s}$, where $\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$. Let $(H', T')$ be the Shimura datum obtained from the above construction applied to $\mathfrak{g}'$, i.e. $H' = \prod'^{s}_{j=1} GSp(\oplus^s_{j=1} V)$, and $\mathrm{H}'_p$ the parahoric of $H'(\mathbb{Q}_p)$ stabilizing $\oplus^s_{j=1}\Lambda_{i_j}$. There is an obvious morphism of Shimura data $(H, T) \to (H', T')$. Hence choosing suitable levels $\mathrm{H}^p \subset H(\mathbb{A}^p_f)$ and $\mathrm{H}'^p \subset H'(\mathbb{A}^p_f)$ away from $p$, we obtain a morphism of Shimura varieties

$$\varpi_{H,H'} : Sh_{\mathrm{H}_p\mathrm{H}^p}(H, T) \to Sh_{\mathrm{H}'_p\mathrm{H}'^p}(H', T')$$

Using the moduli interpretation, this extends in a natural way to a morphism of integral models

$$\varpi_{H,H'} : \mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H, T) \to \mathscr{S}_{\mathrm{H}'_p\mathrm{H}'^p}(H', T')$$

*Proof of Theorem 7.1 (i).* Recall $\mathfrak{f}'$ is a facet of $B(G, \mathbb{Q}_p)$ such that $\mathfrak{f}$ lies in the closure of $\mathfrak{f}'$. Let $z \in \mathfrak{f}$ and let $z' \in \mathfrak{f}'$ be a point sufficiently close to $z$ such that if $\mathfrak{g}$ and $\mathfrak{g}'$ denotes the facets of $B(GSp(V_{\mathbb{Q}_p}), S', \mathbb{Q}_p)$ containing $i(z)$ and $i(z')$, we have $\mathfrak{g}$ lies in the closure of $\mathfrak{g}'$. Applying the above constructions we obtain a diagram:

$$\begin{tikzcd}
\mathscr{S}^-_{\mathrm{K}_p\mathrm{K}^p}(G, X) \arrow[r] & \mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H, T) \arrow[d, "\varpi_{H,H'}"] \\
\mathscr{S}^-_{\mathrm{K}'_p\mathrm{K}^p}(G, X) \arrow[r] & \mathscr{S}_{\mathrm{H}'_p\mathrm{H}'^p}(H', T').
\end{tikzcd}$$

On the generic fiber, this can be completed to a diagram

$$\begin{tikzcd}
Sh_{\mathrm{K}_p\mathrm{K}^p}(G,X) \arrow[r] \arrow[d] & Sh_{\mathrm{H}_p\mathrm{H}^p}(H,T) \arrow[d, "\varpi_{H,H'}"] \\
Sh_{\mathrm{K}'_p\mathrm{K}^p}(G,X) \arrow[r] & Sh_{\mathrm{H}'_p\mathrm{H}'^p}(H',T')
\end{tikzcd}$$

hence by Corollary 7.3, we obtain a map $\mathscr{S}^-_{\mathrm{K}_p\mathrm{K}^p}(G,X) \to \mathscr{S}^-_{\mathrm{K}'_p\mathrm{K}^p}(G,X)$. Taking normalizations and applying Lemma 6.10 we obtain:

$$\pi_{\mathrm{K}_p,\mathrm{K}'_p} : \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X) \to \mathscr{S}_{\mathrm{K}'_p\mathrm{K}^p}(G,X)$$

□

The above maps then induce by passage to the limit, a map between the pro-varieties

$$\pi_{\mathrm{K}_p,\mathrm{K}'_p} : \mathscr{S}_{\mathrm{K}_p}(G,X) \to \mathscr{S}_{\mathrm{K}'_p}(G,X).$$

7.4. Now we relate the isogeny classes on $\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)_k$ and $\mathscr{S}_{\mathrm{K}'_p\mathrm{K}^p}(G,X)_k$. Let $\overline{x} \in \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k)$ and $\overline{y} = \pi_{\mathrm{K}_p,\mathrm{K}'_p}(\overline{x})$. Recall $\mathscr{I}_{\overline{x}}$ and $\mathscr{I}_{\overline{y}}$ are the isogeny classes of $\overline{x}$ and $\overline{y}$. Then $\overline{x}$ corresponds to a collection $(\mathcal{A}_i, \lambda_i, \epsilon_i^p)_{i=1,\dots,r}$ and $\overline{y}$ corresponds to $(\mathcal{A}_{i_j}, \lambda_{i_j}, \epsilon_{i_j}^p)_{j=1,\dots,s}$. We write $\mathbb{D}(\mathscr{G}_i)$ for the $p$-divisible group associated to $\mathcal{A}_i$. We have inclusion and projection maps

$$\iota_0 : \oplus_{j=1}^s \mathbb{D}(\mathscr{G}_{i_j}) \to \oplus_{i=1}^r \mathbb{D}(\mathscr{G}_i) \qquad \pi_0 : \oplus_{i=1}^r \mathbb{D}(\mathscr{G}_i) \to \oplus_{j=1}^s \mathbb{D}(\mathscr{G}_{i_j})$$

and for $l \neq p$

$$\iota_l : \oplus_{j=1}^s (T_l\mathcal{A}_{i_j})^* \to \oplus_{i=1}^r (T_l\mathcal{A}_i)^* \qquad \pi_l : \oplus_{i=1}^r (T_l\mathcal{A}_i)^* \to \oplus_{j=1}^s (T_l\mathcal{A}_{i_j})^*$$

where $^*$ denotes the linear dual.

We write $G'_{\mathbb{Z}_{(p)}}$ and $G_{\mathbb{Z}_{(p)}}$ the groups over $\mathbb{Z}_{(p)}$ given by the Zariski closures of $G$ in $GL(\oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}})$ and $GL(\oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}})$ respectively. Here we write $\Lambda_{i,\mathbb{Z}_{(p)}}$ for the $\mathbb{Z}_{(p)}$ module $V \cap \Lambda_i$. Then $G'_{\mathbb{Z}_{(p)}}$ is the stabilizer of a collection of tensors $s_\alpha \in (\oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}})^\otimes$. We have the two maps

$$\iota : \oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}} \to \oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}} \text{ and } \pi : \oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}} \to \oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}}$$

given by the inclusion and projection. These induce maps

$$\iota^\otimes : (\oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}})^\otimes \to (\oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}})^\otimes \text{ and } \pi^\otimes : (\oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}})^\otimes \to (\oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}})^\otimes$$

such that $\pi^\otimes \circ \iota^\otimes$ is the identity. Note that since $(\oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}})^\otimes$ involves taking duals, one needs to use $\pi$ in the definition of the map $\iota^\otimes$. These maps exhibit $(\oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}})^\otimes$ as a direct summand of $(\oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}})^\otimes$

**Lemma 7.4.** *The $\iota^\otimes(s_\alpha)$ are fixed by $G_{\mathbb{Z}_{(p)}}$*

*Proof.* It suffices to check this after inverting $p$. Then $\iota$ and $\pi$ are both equivariant for the action of $G(\mathbb{Q})$, hence so is $\iota^\otimes$. Thus $\iota^\otimes(s_\alpha)$ is preserved by $G(\mathbb{Q})$. □

We may extend $\iota^\otimes(s_\alpha)$ to a collection of tensors $t_\beta \in (\oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}})^\otimes$ whose stabilizer is $G_{\mathbb{Z}_{(p)}}$. We fix an isomorphism

$$(\oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}})^* \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \oplus_{i=1}^r \mathbb{D}(\mathscr{G}_i) \tag{7.4.1}$$

taking $t_\beta$ to $t_{\beta,0}$.

**Lemma 7.5.** *Any isomorphism as in (7.4.1) preserves the product decomposition on either side and induces an isomorphism*

$$(\oplus_{j=1}^{s}\Lambda_{i_j,\mathbb{Z}_{(p)}})^* \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \oplus_{j=1}^{s}\mathbb{D}(\mathscr{G}_{i_j})$$

*taking $s_\alpha$ to $s_{\alpha,0}$.*

*Proof.* We may assume that among the collection of tensors $t_\beta$ there are tensors

$$t_{\beta_k} \in (\oplus_{i=1}^{r}\Lambda_{i,\mathbb{Z}_{(p)}})^* \otimes (\oplus_{i=1}^{r}\Lambda_{i,\mathbb{Z}_{(p)}})$$

corresponding to the projection $(\oplus_{i=1}^{r}\Lambda_{i,\mathbb{Z}_{(p)}})^* \to \Lambda_{k,\mathbb{Z}_{(p)}}^*$ for $k = 1, \ldots, r$. Indeed this follows since

$$G_{\mathbb{Z}_{(p)}} \subset \prod_{i=1}^{r} GL(\Lambda_{i,\mathbb{Z}_{(p)}})$$

and the latter group fixes these tensors. By the functoriality of the constructions, we have

$$t_{\beta_k,0} \in (\oplus_{i=1}^{r}\mathbb{D}(\mathscr{G}_i)) \otimes (\oplus_{i=1}^{r}\mathbb{D}(\mathscr{G}_i))^*$$

corresponds the to projection $\oplus_{i=1}^{r}\mathbb{D}(\mathscr{G}_i) \to \mathbb{D}(\mathscr{G}_k)$ and that the isomorphism takes $t_{\beta_k}$ to $t_{\beta_k,0}$ precisely says that the isomorphism is compatible with the product decompositions.

That the induced isomorphism takes $s_\alpha$ to $s_{\alpha,0}$ follows from the fact that $\pi^{\otimes} \circ \iota^{\otimes} = \mathrm{id}$. □

**Proposition 7.6.** *Let $\overline{y} = \pi_{\mathrm{K}_p,\mathrm{K}_p'}(\overline{x})$. The map $\pi_{\mathrm{K}_p,\mathrm{K}_p'}$ takes $\mathscr{I}_{\overline{x}} \subset \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k)$ to $\mathscr{I}_{\overline{y}} \subset \mathscr{S}_{\mathrm{K}_p'\mathrm{K}^p}(G,X)(k)$.*

*Proof.* It suffices to prove this result for the induced map of inverse limits

$$\pi_{\mathrm{K}_p,\mathrm{K}_p'} : \mathscr{S}_{\mathrm{K}_p}(G,X) \to \mathscr{S}_{\mathrm{K}_p'\mathrm{K}^p}(G,X).$$

We thus assume $\overline{x} \in \mathscr{S}_{\mathrm{K}_p}(G,X)(k)$. Suppose $\overline{x}' \in \mathscr{I}_{\overline{x}}$, then we have the triple $(\mathcal{A}_{\overline{x}'}, \lambda_{\overline{x}'}, \epsilon_{\overline{x}'}^p)$ corresponding to $\overline{x}'$ and there exists a quasi-isogeny $\theta : \mathcal{A}_{\overline{x}} \to \mathcal{A}_{\overline{x}'}$ taking $t_{\beta,l,\overline{x}}$ to $t_{\beta,l,\overline{x}'}$ for $l \neq p$ and $t_{\beta,0,\overline{x}}$ to $t_{\beta,0,\overline{x}'}$.

$\mathcal{A}_{\overline{x}}$ and $\mathcal{A}_{\overline{x}'}$ arise as products $\prod_{i=1}^{r}\mathcal{A}_{\overline{x},,i}$ and $\prod_{i=1}^{r}\mathcal{A}_{\overline{x}',,i}$, and as in Lemma 7.5 we may assume that there exist tensors $t_{\beta_k}$ which correspond to the projections to the $k$-th component. The tensors $t_{\beta_k,l,*}$ and $t_{\beta_k,0,*}$ then correspond to the projections on the Tate module and Dieudonné modules and $\theta$ therefore respects these projections. It follows that $\theta$ decomposes decomposes as a product of quasi-isogenies $\theta_i : \mathcal{A}_{\overline{x},i} \to \mathcal{A}_{\overline{x}',i}$.

By construction $\pi_0^{\otimes} \circ \iota_0^{\otimes}(s_{\alpha,0,\overline{x}}) = s_{\alpha,0,\overline{x}}$ and similarly for $s_{\alpha,0,\overline{x}'}$. Thus

$$\prod_{j=1}^{s}\theta_{i_j} : \prod_{j=1}^{s}\mathcal{A}_{\overline{x},i_j} \to \mathcal{A}_{\overline{x},i_j}$$

takes $s_{\alpha,0,\overline{x}}$ to $s_{\alpha,0,\overline{x}'}$. By a similar argument, it also takes $s_{\alpha,l,\overline{x}}$ to $s_{\alpha,l,\overline{x}'}$ for $l \neq p$, hence $\pi_{\mathrm{K}_p,\mathrm{K}_p'}(\overline{x}')$ lies in $\mathscr{I}_{\overline{y}}$. □

7.5. We now use the description of the isogeny classes on the Shimura variety with Iwahori level to deduce the description for arbitrary parahoric level. The projection map $\oplus_{i=1}^{r}\Lambda_i \to \oplus_{j=1}^{s}\Lambda_{i_j}$ induces a map $\mathrm{K}_p \to \mathrm{K}_p'$ which is the natural inclusion. If $\overline{x} \in \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k)$ and $\overline{y} = \pi_{\mathrm{K}_p,\mathrm{K}_p'}(\overline{x})$, the choice of trivialization

$$\mathbb{D}(\mathscr{G}_{\overline{x}}) \cong \oplus_{i=1}^{r}\Lambda_i \otimes \mathcal{O}_L$$

taking $t_{\beta,0,\overline{x}}$ to $t_\beta$ determines a trivialization

$$\mathbb{D}(\mathscr{G}_{\overline{y}}) \cong \oplus_{i=1}^{s}\Lambda_i \otimes \mathcal{O}_L$$

taking $s_{\alpha,0,\overline{y}}$ to $s_\alpha$. If we fix such a choice of trivialization, then we obtain an element $b \in G(L)$ giving the Frobenius on both $\mathbb{D}(\mathscr{G}_{\overline{x}})$ and $\mathbb{D}(\mathscr{G}_{\overline{y}})$.

Now suppose that $\mathrm{K}_p$ is an Iwahori subgroup and $\mathrm{K}'_p$ is a parahoric whose corresponding facet lies in the closure of the alcove corresponding to $\mathrm{K}_p$. Fix the choice of maximal $L$-split torus $S$ which is compatible with the choice of parahorics. We assume $\mathrm{K}'_p$ corresponds to the subset $K' \subset \mathbb{S}$ of simple reflections. We have the affine Deligne-Lusztig varieties $X(\sigma(\{\mu\}), b)$ and $X(\sigma(\{\mu\}), b)_{K'}$ associated to the parahorics $\mathrm{K}_p$ and $\mathrm{K}'_p$ respectively. As in §6.7, there is an operator $\Phi$ which acts on both groups $X(\sigma(\{\mu\}), b)$ and $X(\sigma(\{\mu\}), b)_{K'}$; it is induced by the map $g \mapsto b\sigma(g)$. The natural projection $G(L)/\mathcal{G}(\mathcal{O}_L) \to G(L)/\mathcal{G}'(\mathcal{O}_L)$ induces a surjection

$$X(\sigma(\{\mu\}), b) \to X(\sigma(\{\mu\}), b)_{K'}$$

by Theorem 5.1 and this map is equivariant for $\Phi$.

Recall from §6.7 we have a map

$$i'_{\overline{x}} : X(\sigma(\{\mu\}), b) \to \mathscr{S}_{\mathrm{K}''}(GSp(V'), S^{\pm})(k)$$

which is easily seen to factor through $\mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H, T)(k)$; by abuse of notation we still use

$$i'_{\overline{x}} : X(\sigma(\{\mu\}), b) \to \mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H, T)(k)$$

to denote the induced map. Similarly we obtain a map

$$i'_{\overline{y}} : X(\sigma(\{\mu\}), b)_{K'} \to \mathscr{S}_{\mathrm{H}'_p\mathrm{H}'^p}(H', T')(k)$$

which fits into a commutative diagram:

(7.5.1)
$$\begin{tikzcd} X(\sigma(\{\mu\}), b) \arrow[r, "i'_{\overline{x}}"] \arrow[d] & \mathscr{S}_{\mathrm{H}_p H^p}(H, T)(k) \arrow[d] \\ X(\sigma(\{\mu\}), b)_{K'} \arrow[r, "i'_{\overline{y}}"] & \mathscr{S}_{\mathrm{H}'_p H'^p}(H', T')(k) \end{tikzcd}$$

**Proposition 7.7.** *Suppose Assumption 6.18 is satisfied. In other words, assume $i'_{\overline{x}}$ lifts to*

$$i_{\overline{x}} : X(\sigma(\{\mu\}), b) \to \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G, X)(k)$$

*such that $\Phi \circ i_{\overline{x}} = i_{\overline{x}} \circ \Phi$ and $t_{\beta,0,i_{\overline{x}}(g)} = t_{\beta,0,\overline{x}}$ for any $g \in X(\sigma(\{\mu\}), b)$. Then the map $i'_{\overline{y}}$ lifts to a unique map*

$$i_{\overline{y}} : X(\sigma(\{\mu\}), b)_{K'} \to \mathscr{S}_{\mathrm{K}'_p\mathrm{K}^p}(G, X)(k)$$

*such that $\Phi \circ i_{\overline{y}} = i_{\overline{y}} \circ \Phi$ and $s_{\alpha,0,i_{\overline{y}}(g')} = s_{\alpha,0,\overline{y}}$ for any $g' \in X(\sigma(\{\mu\}), b)_{K'}$.*

*Remark* 7.8. Recall in particular the Assumption 6.18 is satisfied if $G_{\mathbb{Q}_p}$ is residually split or if Conjecture 5.4 holds.

*Proof.* Let $g' \in X(\sigma(\{\mu\}), b)_{K'}$ and $g \in X(\sigma(\{\mu\}), b)$ an element which projects to $g'$. We define

$$i_{\overline{y}}(g') := \pi_{\mathrm{K}_p,\mathrm{K}'_p}(i_{\overline{x}}(g)).$$

Then $i_{\overline{y}}$ is a lifting of $i'_{\overline{y}}$ by commutativity of the diagram (7.5.1). It remains to show that $s_{\alpha,0,i_{\overline{y}}(g')} = s_{\alpha,0,\overline{y}}$ for any $g' \in X(\sigma(\{\mu\}), b)_{K'}$. This follows from the fact that $t_\beta$ extends $\iota^{\otimes} s_\alpha$ and the equality $t_{\beta,0,\overline{x}} = t_{\beta,0,i_{\overline{x}}(g)}$ for any $g \in X(\sigma(\{\mu\}), b)$. $\square$

As before, $i_{\overline{y}}$ extends to a map

$$i_{\overline{y}} : X(\sigma(\{\mu\}), b)_{K'} \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathrm{K}'_p}(G, X)(k)$$

The same proof as in Proposition 6.21 shows that $\mathscr{I}_{\overline{y}}$ can be identified with the image of $i_{\overline{y}}$, and this map is equivariant for the action of $\Phi \times G(\mathbb{A}_f^p)$.

## 8. He-Rapoport axioms

8.1. In this section, we verify the axioms of He-Rapoport in [HR17]. We keep the assumptions of (6.2.1), so that $(G, X)$ is Hodge type and $G_{\mathbb{Q}_p}$ splits over a tame extension. We have fixed a base alcove $\mathfrak{a}$ in the apartment corresponding to some $\mathbb{Q}_p$-split torus $S$. We write $\mathcal{I}$ for the Iwahori group scheme and $\mathrm{I}_p = \mathcal{I}(\mathbb{Z}_p)$. For $K \subset \mathbb{S}$ a $\sigma$-invariant subset, we write $\mathcal{G}$ for the parahoric group scheme over $\mathbb{Z}_p$ and $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$. As before all parahorics will be assumed connected and we will assume all the integral models are constructed using the same Hodge embedding.

**Theorem 8.1.** *(i) If $G_{\mathbb{Q}_p}$ is residually split, all axioms in* [HR17] *hold, otherwise every axiom apart from the surjectivity in Axiom 4 (c) holds.*

*(ii) If Assumption 6.18 is satisfied (in particular this is the case if Conjecture 5.4 holds), then all the axioms hold.*

*Remark* 8.2. (i) For main application to non-emptiness of Newton strata, we do not need the Axiom 4 (c) (see the remark in [HR17] after 3.7) and hence Theorem 8.3 holds without Conjecture 5.4.

(ii) As in §7 we must assume the integral models are constructed using the same Hodge embedding.

**Axiom 1**: (Compatibility with change in parahoric) The compatibility with the change in parahorics follows from Theorem 7.1. To show the map $\pi_{\mathrm{K}_p, \mathrm{K}'_p}$ is proper, we may apply a similar argument to the one in Theorem 7.1 to reduce to the case of $GSp(V)$ considered in [HZ]. Indeed let $\mathfrak{f}, \mathfrak{f}'$ denote the facets corresponding to $\mathrm{K}_p$ and $\mathrm{K}'_p$ respectively and let $\mathfrak{g}$ and $\mathfrak{g}'$ the facets in $B(GSp(V), \mathbb{Q}_p)$ containing the images of $\mathfrak{f}$ and $\mathfrak{f}'$ constructed in the proof of Theorem 7.1. Then $\mathfrak{g}$ and $\mathfrak{g}'$ correspond to the lattice chains in $V_{\mathbb{Q}_p}$

$$\mathcal{L} := \{\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_r\}$$
$$\mathcal{L}' := \{\Lambda_{i_1} \subset \Lambda_{i_2} \subset \cdots \subset \Lambda_{i_s}\}$$

We write $\mathrm{M}_p$ and $\mathrm{M}'_p$ for the stabilizer of these lattice chains in $GSp(V_{\mathbb{Q}_p})$ and fix a sufficiently small compact open $M^p \subset GSp(V \otimes \mathbb{A}_f^p)$. As in [HR17], we may consider the moduli problem which associates to a $\mathbb{Z}_{(p)}$ scheme $S$ the triple:

(i) An $\mathcal{L}$-set of abelian varieties

(ii) A polarization of the $\mathcal{L}$-set of abelian varieties as in [HR17, §7].

(iii) A prime to $p$ level structure on the common rational Tate module away from $p$ of the $\mathcal{L}$-set

$$\epsilon^p \in \underline{\mathrm{Isom}}(\widehat{V}(\mathcal{A}_i), V \otimes \mathbb{A}_f^p)/M^p$$

compatible with the Riemann form on $\widehat{V}(\mathcal{A}_i)$ and $\psi$ on $V \otimes \mathbb{A}_f^p$.

We refer to [HR17, §7] for the precise definitions of an $\mathcal{L}$-set and a polarization. This moduli functor is representable by a scheme $\mathscr{S}_{\mathrm{M}_p\mathrm{M}^p}(GSp(V), S^{\pm})$ and we have a natural map

$$\mathscr{S}_{\mathrm{M}_p\mathrm{M}^p}(GSp(V), S^{\pm}) \to \mathscr{S}_{\mathrm{H}_p\mathrm{H}^p}(H, T)$$

where $H$ is the group considered in §7. The same proof as in Theorem 7.1 shows that for sufficently small $\mathrm{H}^p$, the above map is a closed immersion. Indeed in this case the valuative criterion follows from the fact that if $R$ is a discrete valuation ring with fraction field $K$, an isogeny of abelian varieties

$\mathcal{A} \to \mathcal{A}'$ extends uniquely to an isogeny of Neron models over $R$. It follows as in Corollary 7.3 that if we take take $\mathrm{M}^p$ sufficiently small so that $Sh_{\mathrm{K}_p\mathrm{K}^p}(G,X) \to Sh_{\mathrm{M}_p\mathrm{M}^p}(GSp(V),S^\pm)$ is a closed immersion, we have a closed immersion

$$\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)^- \to \mathscr{S}_{\mathrm{M}_p\mathrm{M}^p}(GSp(V),S^\pm)$$

and hence a finite map

$$\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X) \to \mathscr{S}_{\mathrm{M}_p\mathrm{M}^p}(GSp(V),S^\pm).$$

We may apply the same considerations to $\mathfrak{g}'$ and $\mathrm{M}'$. We obtain a commutative diagram:

$$\begin{tikzcd} \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X) \arrow[r] \arrow[d, "\pi_{\mathrm{K}_p,\mathrm{K}'_p}"'] & \mathscr{S}_{\mathrm{M}_p\mathrm{M}^p}(GSp(V),S^\pm) \arrow[d] \\ \mathscr{S}_{\mathrm{K}'_p\mathrm{K}^p}(G,X) \arrow[r] & \mathscr{S}_{\mathrm{M}'_p\mathrm{M}^p}(GSp(V),S^\pm) \end{tikzcd} \tag{8.1.1}$$

The horizontal maps are finite hence proper. The vertical map on the right is proper by [HR17, §7]; hence $\pi_{\mathrm{K}_p,\mathrm{K}'_p}$ is proper. Since the map $\pi_{\mathrm{K}_p,\mathrm{K}'_p}$ is surjective on the generic fiber, $\pi_{\mathrm{K}_p,\mathrm{K}'_p}$ is also surjective by properness.

**Axiom 2**: (Local Model diagram) As in [KP, Theorem 4.2.7] we have a diagram:

$$\begin{tikzcd} & \widetilde{\mathscr{S}}_{\mathrm{K}_p\mathrm{K}^p}(G,X) \arrow[dl, "\pi"'] \arrow[dr, "q"] & \\ \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X) & & M^{\mathrm{loc}}_{\mathcal{G}} \end{tikzcd}$$

where $\pi$ is a $\mathcal{G}$-torsor and $q$ is a smooth map equivariant for the action of $\mathcal{G}$. By [PZ13, Theorem 8.3], the special fiber of $M^{\mathrm{loc}}_{\mathcal{G}}$ has a stratification by $\mathrm{Adm}(\{\mu\})_K$, and this diagram induces a stratification on $\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)_k$ as well as a map

$$\lambda_{K_p} : \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k) \to \mathrm{Adm}(\{\mu\})_K.$$

To show the compatibility with change in parahorics, we assume $\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)$, $\mathscr{S}_{\mathrm{K}'_p\mathrm{K}^p}(G,X)$ and the map $\pi_{\mathrm{K}_p,\mathrm{K}'_p}$ are constructed as in §7.3. The torsor $\widetilde{\mathscr{S}}_{\mathrm{K}_p\mathrm{K}^p}(G,X)$ classifies trivializations

$$\mathcal{V} \cong (\oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}}) \tag{8.1.2}$$

taking $t_{\beta,\mathrm{dR}}$ to $t_\beta$, where $\mathcal{V}$ is the relative de Rham cohomology of the abelian variety over $\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)$ as in §6.5. The sheaf $\mathcal{V}$ breaks up into a direct sum over the sheaves associated to each $\mathcal{A}_i$ and as in §7.5, any trivialization (8.1.2) induces a trivialization

$$\mathcal{V}' \cong (\oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}})$$

where $\mathcal{V}'$ is relative de Rham cohomology of the abelian variety over $\mathscr{S}_{\mathrm{K}'_p\mathrm{K}^p}(G,X)$. We thus obtain a map $\widetilde{\pi}_{\mathrm{K}_p,\mathrm{K}'_p} : \widetilde{\mathscr{S}}_{\mathrm{K}_p\mathrm{K}^p}(G,X) \to \widetilde{\mathscr{S}}_{\mathrm{K}_p\mathrm{K}^p}(G,X)$.

The projection map $(\oplus_{i=1}^r \Lambda_{i,\mathbb{Z}_{(p)}}) \to (\oplus_{j=1}^s \Lambda_{i_j,\mathbb{Z}_{(p)}})$ induces a map

$$\pi^{\mathrm{loc}}_{\mathrm{K}_p,\mathrm{K}'_p} : M^{\mathrm{loc}}_{\mathcal{G}} \to M^{\mathrm{loc}}_{\mathcal{G}'},$$

since it induces such a map on the generic fiber. Here we write $\mathcal{G}'$ for the parahoric group scheme associated to $\mathrm{K}'_p$. By construction the map $\widetilde{\pi}_{\mathrm{K}_p,\mathrm{K}'_p}$ is compatible with $\pi^{\mathrm{loc}}_{\mathrm{K}_p,\mathrm{K}'_p}$.

**Axiom 3**: (Newton Stratification) For every $\overline{x} \in \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)$, we may take a trivialization

$$V'^*_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})(\mathcal{O}_L)$$

respecting appropriate tensors. The Frobenius $\varphi$ is given by $b\sigma$; we thus obtain $b \in G(L)$ well defined up to $\sigma$-conjugation by $\mathcal{G}(\mathcal{O}_L)$. The same discussion as §6.7 in the Iwahori case shows that $\sigma^{-1}(b) \in \bigcup_{w\in \mathrm{Adm}(\{\mu\})_K} \mathcal{G}(\mathcal{O}_L)\dot{w}\mathcal{G}(\mathcal{O})$, hence $b \in B(G,\{\mu\})$. The compatibility with change in parahorics follows from the discussion in §7.5. We thus obtain a map

$$\delta : \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k) \to B(G,\{\mu\}).$$

To show that $\delta$ induces a stratification on $\mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)_k$, we must show this map arises from an isocrystal with $G$-structure. This follows from [KMPS, Corollary A7].

**Axiom 4**: (Joint stratification) (a) Let $G(L)/\mathcal{G}(\mathcal{O}_L)_{\cdot\sigma}$ denote the set of $\mathcal{G}(\mathcal{O}_L)$-conjugacy classes of in $G(L)$. We have natural projection maps

$$d_{\mathrm{K}_p} : G(L)/\mathcal{G}(\mathcal{O}_L)_{\cdot\sigma} \to B(G)$$

$$l_{\mathrm{K}_p} : G(L)/\mathcal{G}(\mathcal{O}_L)_{\cdot\sigma} \to \mathcal{G}(\mathcal{O}_L)\backslash G(L)/\mathcal{G}(\mathcal{O}_L).$$

Let $\overline{x} \in \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k)$. The construction in Axiom 3 associates to $\overline{x}$ an element, $b \in G(L)$ well-defined up to $\sigma$-conjugation by $\mathcal{G}(\mathcal{O}_L)$. The map

$$\Upsilon_{\mathrm{K}_p} : \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k) \to G(L)/\mathcal{G}(\mathcal{O}_L)_{\cdot\sigma}$$

is defined by $\Upsilon_{\mathrm{K}_p}(\overline{x}) = [\sigma^{-1}(b)]$. It is clear by definition that $d_{\mathrm{K}_p} \circ \Upsilon_{\mathrm{K}_p} = \delta_{\mathrm{K}_p}$. By Proposition 6.19, $l_{\mathrm{I}_p} \circ \Upsilon_{\mathrm{I}_p} = \lambda_{\mathrm{I}_p}$; the case of general $\mathrm{K}_p$ follows from the same proof. The compatibility of $\Upsilon_{\mathrm{K}_p}$ with change in parahorics follows from the discussion in §7.5.

(b) By [HR17, Lemma 3.11], it suffices to prove this in the case of Iwahori level. We need to show for all $\delta \in l^{-1}_{\mathrm{I}_p}(\mathrm{Adm}(\{\mu\}))$, there exists $\overline{x} \in \mathscr{S}_{\mathrm{I}_p\mathrm{K}^p}(G,X)(k)$ such that $\Upsilon_{\mathrm{I}_p}(\overline{x}) = \delta$. We fix a lift $b \in G(L)$ of $\sigma(\delta)$, then by definition $1 \in X(\sigma(\{\mu\}), b)$. By Theorem 5.9, there exists a $\sigma$-straight element $w \in \sigma(\mathrm{Adm}(\{\mu\}))$ and $h \in X_w(b)$ such that $h$ lies on the same connected component of $X(\sigma(\{\mu\}), b)$ as 1. By Theorem 2.4 we may assume there is a representative $h \in G(L)$ such that $h^{-1}b\sigma(h) = \dot{w}$. We pick $\overline{x}' \in \lambda^{-1}_{\mathrm{I}_p}(\sigma^{-1}(w))$, which exists since $\lambda_{\mathrm{I}_p}$ is surjective (see Theorem 8.2 below), and we may pick the isomorphism $V^*_{\mathbb{Z}_p} \otimes \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}_{\overline{x}})$ so that the Frobenius is given by $\dot{w}\sigma$. Using the isomorphism $X(\sigma(\{\mu\}), b) \cong X(\sigma(\{\mu\}), \dot{w})$ given by $g \mapsto h^{-1}g$, we have $h^{-1} \in X(\sigma(\{\mu\}), \dot{w})$ is connected to $1 \in X_w(\dot{w})$. By Proposition 6.9, $i_{\overline{x}'}(h^{-1}) \in \mathscr{S}_{\mathrm{I}_p\mathrm{K}^p}(G,X)(k)$ is well defined and we have $\Upsilon_{I_p\mathrm{K}^p}(\overline{x}) = \delta$.

(c) For $K_p \subset K'_p$, let $\delta \in \mathrm{Im}(\Upsilon_{\mathrm{K}_p})$ and $\delta'$ its image in $G(L)/\mathcal{G}'(\mathcal{O}_L)_{\cdot\sigma}$. The finiteness of the fibers of $\pi_{\mathrm{K}_p,\mathrm{K}'_p}|_{\Upsilon^{-1}_{\mathrm{K}_p}(\delta)}$ can be deduced from the case of $GSp(V)$ which is proved in [HR17, §7]. Indeed it follows from the diagram 8.1.1, that a fiber of $\pi_{\mathrm{K}_p,\mathrm{K}'_p}|_{\Upsilon^{-1}_{\mathrm{K}_p}(\delta)}$ admits a finite map to a corresponding fiber for the integral model for $GSp$.

For the surjectivity, we need that Assumption 6.18 is satisfied. Let $\overline{x}' \in \Upsilon^{-1}_{\mathrm{K}'_p}(\delta')$. Then by the surjectivity in Axiom 1, there exists $\overline{x} \in \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k)$ such that $\pi_{\mathrm{K}_p,\mathrm{K}'_p}(\overline{x}) = \overline{x}'$. Let $\gamma = \Upsilon_{\mathrm{K}_p}(\overline{x})$, then by compatibility of the map $\Upsilon$ with $\pi_{\mathrm{K}_p,\mathrm{K}'_p}$, the image of $\gamma \in G(L)/\mathcal{G}'(\mathcal{O}_L)_{\cdot\sigma}$ is equal to $\delta'$. Thus $\gamma$ and $\delta$ are sigma-conjugate by an element of $\mathcal{G}'(\mathcal{O}_L)$, and the same is true for $b := \sigma(\gamma)$ and $\sigma(\delta)$. Let $g \in \mathcal{G}'(\mathcal{O}_L)$ such that $g^{-1}b\sigma(g) = \sigma(\delta)$, then $g \in X(\sigma(\{\mu\}), b)_K$ and its image in $X(\sigma(\{\mu\}), b)_{K'}$ is equal to 1. Then by Assumption 6.18 $i_{\overline{x}}(g) \in \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(G,X)(k)$ satisfies $\Upsilon(i_{\overline{x}}(g)) = \delta$, and $\pi_{\mathrm{K}_p,\mathrm{K}'_p}(i_{\overline{x}}(g)) = \overline{x}'$.

**Axiom 5**: (Basic non-emptiness) Recall $\mathrm{I}_p$ was an Iwahori subgroup of $G(\mathbb{Q}_p)$. Let $\tau_{\{\mu\}}$ denote the unique minimal element of $\mathrm{Adm}(\{\mu\})$. We need to show $\lambda_{\mathrm{I}_p}^{-1}(\tau_{\{\mu\}}) \neq \emptyset$.

By [KMPS, Theorem 1.3.13.2] (see also the remark beneath the Theorem), there exists $\overline{x} \in \mathscr{S}_{I_p\mathrm{K}^p}(G,X)(k)$ such that $\delta_{\mathrm{I}_p} = [b]_{\mathrm{basic}}$. Let $g \in X_{\sigma(\tau_{\{\mu\}})}(b) \subset X(\sigma(\{\mu\}), b)$. By Prop 6.5 (i), the map $i_{\overline{x}} : X(\sigma(\{\mu\}), b) \to \mathscr{S}_{\mathrm{I}_p\mathrm{K}^p}(G,X)(k)$ is well defined. Then by Proposition 6.19, $i_{\overline{x}}(g) \in \lambda_{\mathrm{I}_p}^{-1}(\tau_{\{\mu\}})$.

The main application of the above results is the non-emptiness of Newton strata.

**Theorem 8.3.** *$\lambda_{K_p}$ and $\delta_{K_p}$ are surjective.*

*Proof.* Since the $\pi_{\mathrm{K}_p,\mathrm{K}_p'}$ are compatible with the maps $\delta$ and $\lambda$ it suffices to prove the result when $\mathrm{K}_p$ is Iwahori. Since $q : \widetilde{\mathscr{S}}_{\mathrm{I}_p\mathrm{K}^p}(G,X) \to M^{\mathrm{loc}}_{\mathcal{I}}$ is a smooth map, the image of $q$ is open and on the special fiber it is a union of strata. Since the minimal strata $\tau_{\{\mu\}}$ lies in the image, $q \otimes k$ is surjective, in particular $\lambda_{I_p}$ is surjective.

It follows from [HR17, Theorem 5.4] that $\delta_{\mathrm{I}_p}$ is also surjective. Indeed it is proved in loc. cit. that for each $[b] \in B(G,\mu)$, there exists $w \in \mathrm{Adm}(\{\mu\})$ $\sigma$-straight such that $\lambda_{\mathrm{I}_p}^{-1}(w) \subset \delta_{\mathrm{I}_p}^{-1}([b])$. $\square$

*Remark* 8.4. [KMPS, Theorem 1.3.13.2] have proved the surjectivity of $\delta_{K_p}$ under much weaker assumptions on the level structure at $p$ using a different method. In particular, their proof does not rely on the existence of good integral models such as those constructed in [KP]. However they do assume the group $G_{\mathbb{Q}_p}$ is quasi-split whereas our proof does not make this assumption.

## 9. Lifting to special points

9.1. In this section we show, under the Assumption 6.18 and that $G_{\mathbb{Q}_p}$ is quasi-split, that every isogeny class in $\mathscr{S}_{\mathrm{K}}(G,X)$ admits a lift to a special point of $Sh_{\mathrm{K}}(G,X)$. The proof follows ideas from [Kis17, §2], the main new input being a generalization of the so called Langlands–Rapoport lemma, see [Kis17, Lemma 2.2.2]. This allows us to associate a Kottwitz triple to each isogeny class, a key ingredient needed to enumerate the set of isogeny classes. A proof of this result also appears in [KMPS] using a different method.

We first recall some definitions from [Kis17]; these make sense without any reference to the assumption 6.18. As before $(G,X)$ is of Hodge type and $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$ is a connected parahoric subgroup corresponding to $K \subset \mathbb{S}$, but now $k \subset \overline{\mathbb{F}}_p$ will denote a finite extension of the residue field $k_E$ of $\mathcal{O}_{E_{(v)}}$. $\mathrm{K}^p \subset G(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup and $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p$. We write $r$ for the degree of $k$ over $\mathbb{F}_p$, and write $W := W(k)$, and $K_0 = W(k)[\frac{1}{p}]$.

9.2. For $x \in \mathscr{S}_{\mathrm{K}}(G,X)(k)$ we write $\overline{x}$ for the $\overline{\mathbb{F}}_p$-point associated to $x$. Recall we have an associated abelian variety $\mathcal{A}_x$ together with Frobenius invariant tensors $s_{\alpha,l,x} \in H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_l)^{\otimes}$ whose stablizer in $GL(H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_l))$ can be identified with $G_{\mathbb{Q}_l}$ via the level structure $\epsilon^p$. Since the $s_{\alpha,l,x}$ are invariant under the action of the geometric Frobenius $\gamma_l$ on $H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_l)$, we may consider $\gamma_l$ as an element of $G(\mathbb{Q}_l)$. We let $I_{l/k}$ denote the centralizer of $\gamma_l$ in $G(\mathbb{Q}_l)$ and $I_l$ the centralizer of $\gamma_l^n$ for $n$ sufficiently large, cf. [Kis17, §2.1.2].

We also fix an identification

$$\mathbb{D}(\mathscr{G}_x) \otimes_W K_0 \cong V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} K_0 \tag{9.2.1}$$

taking $s_{\alpha,0,x}$ to $s_\alpha$. The existence of such an identification follows from [KP, Proposition 3.3.8] which is valid even when $\mathscr{G}_x$ is defined over a finite field. The Frobenius on $\mathbb{D}(\mathscr{G}_x)$ is of the form $\varphi = \delta\sigma$ for some $\delta \in G(K_0)$ and we write $\gamma_p$ for the element $\delta\sigma(\delta)\dots\sigma^{r-1}(\delta) \in G(K_0)$.

Let $I_{p/k}$ denote the group over $\mathbb{Q}_p$ whose $R$-points are given by

$$I_{p/k}(R) = \{g \in G(K_0 \otimes_{\mathbb{Q}_p} R) | g^{-1}\delta\sigma(g) = \delta\}$$

Clearly $I_{p/k} \subset J_\delta$. We have $\gamma_p \in I_{p/k}(\mathbb{Q}_p)$ and we have $I_{p/k} \otimes_{\mathbb{Q}_p} K_0$ is identified with the centralizer of $\gamma_p$ in $G_{K_0}$.

For $n \in \mathbb{N}$, we write $k_n$ for the degree $n$ extension of $k$, and $I_{p/k_n}$ the group over $\mathbb{Q}_p$ defined as above with $K_0$ replaced with $W(k_n)[\frac{1}{p}]$. $I_p$ will then denote the $I_{p/k_n}$ for sufficiently large $n$.

Finally we let $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ denote the group over $\mathbb{Q}$ defined by

$$\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)(R) = (\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_x) \otimes_{\mathbb{Z}} R)^\times$$

where $\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_x)$ denotes the set of endomorphisms of $\mathcal{A}_x$ viewed as an abelian variety up to isogeny defined over $k$. We write $I_{/k} \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ for the subgroup of elements which preserve the tensors $s_{\alpha,l,x}$ for $l \neq p$ and $s_{\alpha,0,x}$. We obtain maps $I_{/k} \to I_{l/k}$ for all $l$ (including $l = p$).

Similarly we write $I \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x \otimes \overline{\mathbb{F}}_p)$ for the subgroup which fixes $s_{\alpha,l,x}$ for all $l \neq p$ and $s_{\alpha,0,x}$. Again we have maps $I \to I_l$ for all $l$.

9.3. By the argument in Proposition 7.6 the projection maps $\pi_{\mathrm{K}_p,\mathrm{K}'_p}$ are compatible with the construction of $\delta$ and $\gamma_l$. Thus the above definitions are independent of level structure, i.e. for $x \in \mathscr{S}_{\mathrm{K}_p}(G,X)(k)$ and $y = \pi_{\mathrm{K}_p,\mathrm{K}'_p}(x)$, the construction above give rise to the same groups $I_{l/k}, I_{p/k}$ and $I$.

From now on, we assume that Assumption 6.18 is satisfied and moreover we assume that $G_{\mathbb{Q}_p}$ is quasi-split. The same proof as in [Kis17, 2.1.3 and 2.1.5] gives us the following proposition; see also [KMPS, Theorem 6].

**Proposition 9.1.** *(i) The map $i_{\overline{x}}$ of Assumption 6.18 induces an injective map*

$$i_{\overline{x}} : I(\mathbb{Q})\backslash X(\sigma(\{\mu\}),\delta)_K \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathrm{K}_p}(G,X)(\overline{\mathbb{F}}_p).$$

*(ii) Let $H^p = \prod_{l\neq p} I_{l/k}(\mathbb{Q}_l) \cap K^p$ and $H_p = I_{p/k} \cap \mathcal{G}(W(k))$. The map in (i) induces an injective map:*

$$I_{/k}(\mathbb{Q})\backslash \prod_l I_{l/k}(\mathbb{Q}_l)/H_p \times H^p \to \mathscr{S}_{\mathrm{K}}(G,X)(k).$$

*(iii) For some prime $l \neq p$, the connected component of $I_{\mathbb{Q}_l} = I \otimes_{\mathbb{Q}} \mathbb{Q}_l$ contains the connected component of the identity in $I_l$. In particular the ranks of $I$ and $G$ are equal.*

9.4. The next lemma is the key technical ingredient needed for the existence of CM lifts. For this we need to recall some group theoretic preliminaries. If $S$ is a maximal $L$-split torus of $G$ defined over $\mathbb{Q}_p$ and $T$ is its centralizer, then $W$ is the Iwahori Weyl group of $G$ and $\mathbb{S}$ is the set of simple reflections in $W$ corresponding to a choice of base alcove $\mathfrak{a}$ in the apartment for $S$. Since $G$ is quasi-split there exists a $\sigma$-stable special vertex $\mathfrak{s}$ lying in the closure of $\mathfrak{a}$. Let $K := \mathbb{S}_0$ denote the set of simple reflections corresponding to $\mathfrak{s}$ and $\mathcal{G}$ the associated special parahoric subgroup defined over $\mathbb{Z}_p$. Let $W_K$ be the group generated by the reflections in $K$; it is identified with the relative Weyl group $W_0 := N(L)/T(L)$. We have an identification

$$\mathcal{G}(\mathcal{O}_L)\backslash G(L)/\mathcal{G}(\mathcal{O}_L) \cong W_K\backslash W/W_K$$

and this latter set can be identified with $X_*(T)_I/W_K$. The choice of alcove $\mathfrak{a}$ and special vertex $\mathfrak{s}$ determines a positive chamber $V_+$ in $V := X_*(T)_I \otimes \mathbb{R}$ and a Borel subgroup $B$ of $G$ defined over $\mathbb{Q}_p$. We now describe the relationship between $V_+$ and $B$ more explicitly.

Let $( \ , \ ) : X_*(T) \times X^*(T) \to \mathbb{Z}$ be the natural pairing and we use the same symbol to denote the scalar extension to $\mathbb{R}$. We let $\Psi \subset X^*(T)$ denote the set of absolute roots, then $B$ determines a system of positive roots $\Psi_+ \subset \Psi$. Now for $K/L$ a finite Galois extension over which $T$ splits, we have a norm map

$$\mathrm{Nm} : X_*(T)_I \to X_*(T)^I$$

given by

$$\underline{\mu} \mapsto \sum_{\gamma \in \mathrm{Gal}(K/L)} \gamma(\mu)$$

where $\underline{\mu} \in X_*(T)_I$ and $\mu \in X_*(T)$ is a lift. This extends linearly to a map $V \to X_*(T)^I \otimes \mathbb{R}$. Then $V_+$ can be identified with the subset of $V$ consisting of $x$ such that $(\mathrm{Nm}(x), \alpha) \geq 0$ for all $\alpha \in \Psi_+$.

We write $X_*(T)_{I,+}$ for the subset of $X_*(T)_I$ which maps to $V_+$, then $W_K \backslash W / W_K$ can be identified with $X_*(T)_{I,+}$.

Now recall we have the affine Weyl group $W_a \subset W$, and there is reduced root system $\Sigma$ such that $W_a \cong Q^\vee(\Sigma) \ltimes W(\Sigma)$; in particular $Q^\vee(\Sigma)$ is identified with $X_*(T_{\mathrm{sc}})_I$. The choice of Borel $B$, determines an ordering of the roots in $\Sigma$. The length function and Bruhat order $\leq$ on $W$ is determined by $W_a$ and hence by $\Sigma$. For $\underline{\lambda}, \underline{\mu} \in X_*(T)_{I,+}$, we write $\underline{\lambda} \preccurlyeq \underline{\mu}$ if $\underline{\mu} - \underline{\lambda}$ is a positive linear combination of positive coroots in $Q^\vee(\Sigma)$ with integral coefficients. By [Lus83, §2] applied to the root system $\Sigma$, we have for $\underline{\lambda}, \underline{\mu} \in X_*(T)_{I,+}$

$$t_{\underline{\lambda}} \leq t_{\underline{\mu}} \Leftrightarrow \underline{\lambda} \preccurlyeq \underline{\mu}.$$

9.5. The following is a generalization of [Kis17, 2.2.2] to general quasi-split groups. If $\epsilon : \mathbb{G}_m \to G$ is a cocharacter defined over a finite extension of $\mathbb{Q}_p$ which commutes with all its Galois conjugates, we write $\overline{\epsilon}^G$ for the fractional cocharacter given by the average of the Galois conjugates of $\epsilon$. If $\epsilon$ factors through a torus $T$, then we may consider $\overline{\epsilon}^T$ as an element of $X_*(T)_{\mathbb{Q}}$ defined over $\mathbb{Q}_p$. Recall we have the Newton cocharacter $\nu_\delta : \mathbb{D} \to G$, which is central in $J_\delta$ and hence central in $I_p$.

**Lemma 9.2.** *Let $T_p \subset I_p$ be a maximal torus defined over $\mathbb{Q}_p$. Then there exists a cocharacter $\mu_{T_p} \in X_*(T_p)$ such that:*
*(i) Considered as a $G$-valued cocharacter, $\mu_{T_p}$ is conjugate to $\mu$.*
*(ii) $\overline{\mu}_{T_p}^{T_p} = \nu_\delta$.*

*Proof.* First note that the truth of the Lemma is invariant under changing of the isomorphism

$$\mathbb{D}(\mathscr{G}_x) \otimes K_0 \cong V^*_{\mathbb{Z}_p} \otimes K_0.$$

Indeed, since (ii) only depends on the abstract group $T_p \subset I_p$, we need to show the condition (i) does not depend on this choice. But this follows from the fact that if we modify the isomorphism (9.2.1) by $h \in G(K_0)$, the inclusion $I_p(K_0) \subset G(K_0)$ is conjugated by $h$.

Let $T' \subset T_p$ denote the maximal $\mathbb{Q}_p$-split subtorus. The same proof as [Kis17, Lemma 2.2.2] shows upon changing the isomorphism $\mathbb{D}(\mathscr{G}_x) \otimes K_0 \cong V^*_{\mathbb{Z}_p} \otimes K_0$, we may assume that $\delta$ is contained in the centralizer of $T'$. Since the $\mathbb{Q}_p$-structure on the image of $T'_{K_0}$ in $G_{K_0}$ differs by the one on $T'$ by conjugation by $\delta$, we may consider $T'$ as a subtorus of $G$. Let $M$ denote the centralizer of $T'$ in $G$; then we have $\delta \in M(K_0)$.

Let $T'_2$ be a maximal $\mathbb{Q}_p$-split torus in $G$ containing $T'$, and $T_2$ its centralizer; it is a maximal torus since $G$ is quasi-split. We may apply the considerations in §9.4 to $T = T_2$.

Note that $T_2 \subset M$. Let $P$ be a parabolic subgroup containing $M$ with unipotent radical $N$; we suppose it contains a choice of Borel $B$ defined over $\mathbb{Q}_p$ containing $T_2$. Let $g \in X(\sigma(\{\mu\}), \delta)_K$ which exists since $\delta \in B(G, \sigma(\{\mu\}))$. Then there exists $\underline{\mu}_1 \in X_*(T_2)_I$ with $t_{\underline{\mu}_1} \leq t_{\sigma(\underline{\mu})}$ such that

$g^{-1}b\sigma(g) \in \mathcal{G}(\mathcal{O}_L)\dot{t}_{\underline{\mu}_1}\mathcal{G}(\mathcal{O}_L)$. Note that when $G$ splits over an unramified extension, $\underline{\mu}$ is minuscule (for the root system $\Sigma$) and hence $\underline{\mu}_1 = \sigma(\underline{\mu})$; this is not true in general. By the Iwasawa decomposition, we may assume $g = mn$ for $n \in N(L), m \in M(L)$. Let $\mathcal{M}(\mathcal{O}_L) = M(L) \cap \mathcal{G}(\mathcal{O}_L)$ a special parahoric subgroup of $M$ defined over $\mathbb{Z}_p$.

Then we have

$$m^{-1}\delta\sigma(m) \in \mathcal{M}(\mathcal{O}_L)\dot{t}_{\underline{\lambda}}\mathcal{M}(\mathcal{O}_L)$$

for some $\underline{\lambda} \in X_*(T_2)_I$. Let $m^{-1}\delta\sigma(m) = m_1\dot{t}_{\underline{\lambda}}m_2$ in the decomposition above. Now

$$g^{-1}\delta\sigma(g) = n^{-1}m^{-1}\delta\sigma(m)\sigma(n) = \tilde{n}m^{-1}\delta\sigma(m)$$

for some $\tilde{n} \in N(L)$. Thus

$$(m_1^{-1}\tilde{n}m_1)\dot{t}_{\underline{\lambda}}m_2 \in N(L)\dot{t}_{\underline{\lambda}}\mathcal{M}(\mathcal{O}_L) \cap \mathcal{G}(\mathcal{O}_L)\dot{t}_{\underline{\mu}_1}\mathcal{G}(\mathcal{O}_L)$$

hence by [HR10, Lemma 10.2.1] we have $t_{\underline{\lambda}} \leq t_{\underline{\mu}_1} \leq t_{\sigma(\underline{\mu})}$.

By Lemma 9.3 below, there exists a lift of $\underline{\lambda}$ to $v_2' \in X_*(T_2)$ which is conjugate to $\tilde{\sigma}(\mu)$ in $G$, where $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a lift of $\sigma$. We let $v_2 = \tilde{\sigma}^{-1}(v_2')$; then $v_2$ is conjugate to $\mu$ and $\delta \in B(M, \sigma(\{v_2\}_M))$ by [He16, Theorem A]; hence $\overline{v}_2$ has the same image as $\nu_\delta$ in $\pi_1(M)_{\mathbb{Q}}$.

Now let $\mu_{T_p} \in X_*(T_p)$ be a cocharacter conjugate to $v_2$ in $M$. Then since conjugate cocharacters have the same image in $\pi_1$, the images of $\overline{\mu}_{T_p}^M$ and $\nu_\delta$ in $\pi_1(M)_{\mathbb{Q}}$ coincide, where $\overline{\mu}_{T_p}^M$ denotes the Galois average of $\mu_{T_p}$ computed as a cocharacter of $M$. Then as before, since the two $\mathbb{Q}_p$-structures on the image of $T_{K_0}$ differ by conjugation by $\delta$, the images of $\overline{\mu}_{T_p}^{T_p}$ and $\overline{\mu}_{T_p}^M$ in $\pi_1(M)_{\mathbb{Q}}$ coincide.

Now as $\overline{\mu}_{T_p}^{T_p}$ is defined over $\mathbb{Q}_p$, we may consider it as an element of $X_*(T')_{\mathbb{Q}}$. We have $\sigma(\nu_\delta) = \delta^{-1}\nu_\delta\delta$, hence $\nu_\delta$ is defined over $\mathbb{Q}_p$ (as a cocharacter to $T_p$) and we have $\nu_\delta \in X_*(T')_{\mathbb{Q}}$. Since $T' \subset M$ is central, we have $\nu_\delta = \overline{\mu}_{T_p}^{T_p}$.

$\square$

**Lemma 9.3.** *Let $\mu$ be a minuscule cocharacter of $G$ and $\underline{\lambda} \in X_*(T)_I$ whose image in $W_K\backslash W/W_K$ lies in $\mathrm{Adm}(\{\mu\})_K$. Then there exists a cocharacter $v_2 \in X_*(T)$ lifting $\underline{\lambda}$ which is conjugate to $\mu$ in $G$.*

*Proof.* Let $\underline{\lambda}' \in X_*(T)_{I,+}$ denote the dominant representative of $\underline{\lambda}$ for our choice of Borel $B$. By [Lus83, §2], $t_{\underline{\lambda}} \in \mathrm{Adm}(\{\mu\})_K$ implies $t_{\underline{\mu}} - t_{\underline{\lambda}'}$ is a positive linear combination of coroots in $\Sigma$ (Recall $\underline{\mu}$ is the image of a dominant representative of $\{\mu\}$ in $X_*(T)_I$. Note that in general $\mu$ being minuscule in $G$ does not imply $\underline{\mu}$ is minuscule for the root system $\Sigma$, so that it is possible that $t_{\underline{\lambda}} \neq t_{\underline{\mu}}$.

Since $W_0$ is a subgroup of the absolute Weyl group, it suffices to prove the result for $\underline{\lambda}$ replaced by $\underline{\lambda}'$. Upon relabelling, we assume $\underline{\lambda}$ is dominant. By Stembridge's Lemma [Rap00, Lemma 2.3] there exists a sequence of positive coroots $\underline{\alpha}_1^\vee, \ldots, \underline{\alpha}_n^\vee \in \Sigma^\vee$ such that

$$\underline{\mu} - \underline{\alpha}_1^\vee - \ldots \underline{\alpha}_i^\vee \in X_*(T)_I$$

is dominant for all $i$ and $\underline{\mu} - \underline{\alpha}_1^\vee - \ldots. - \underline{\alpha}_n^\vee = \underline{\lambda}$. We prove by induction on $i$ that

$$\underline{\lambda_i} := \underline{\mu} - \underline{\alpha}_1^\vee - \cdots - \underline{\alpha}_i^\vee \in X_*(T)_I$$

admits a lifting $\lambda_i \in X_*(T)$ which is conjugate to $\mu$. The base case is $i = 0$ in which case $\underline{\lambda} = \underline{\mu}$ and the result is obvious.

Suppose we have shown the existence of $\lambda_i$. Let $\alpha_{i+1}^\vee \in X_*(T_{\mathrm{sc}})$ be a positive coroot lifting $\underline{\alpha}_i^\vee$; such a root exists by the construction of $\Sigma$ as in [Bou68, VI, 2.1]. Then since $\underline{\lambda}_{i+1}$ is dominant, we have $\langle \underline{\lambda}_{i+1}, \underline{\alpha}_{i+1} \rangle \geq 0$ and hence $\langle \underline{\lambda}_i, \underline{\alpha}_{i+1} \rangle = \langle \underline{\lambda}_{i+1} + \underline{\alpha}_{i+1}^\vee, \underline{\alpha}_{i+1} \rangle > 0$.

Letting $K/L$ be a finite Galois extension of degree $n$ over which $T$ splits, we have by the definition of $\Sigma$ in [Bou68, VI, 2.1] that

$$\langle \underline{\lambda_i}, \underline{\alpha_{i+1}} \rangle = c \sum_{\gamma \in \mathrm{Gal}(K/L)} (\gamma(\lambda_i), \alpha_{i+1})$$

for some $c \in \mathbb{R}_{>0}$. Since $(\gamma(\lambda_i), \alpha_{i+1}) = (\lambda_i, \gamma(\alpha_{i+1}))$, upon replacing $\alpha_{i+1}$ by some $\gamma(\alpha_{i+1})$, we may assume $\langle \lambda_i, \alpha_{i+1} \rangle > 0$. By assumption $\lambda_i$ is minuscule hence $\langle \lambda_i, \alpha_{i+1} \rangle = 1$. We set $\lambda_{i+1} = s_{\alpha_{i+1}}(\lambda_i) = \lambda_i - \alpha_{i+1}$ where $s_{\alpha_{i+1}}$ is the simple reflection corresponding to $\alpha_{i+1}$. Then $\lambda_{i+1}$ is minuscule since it is the Weyl conjugate of a minuscule cocharacter, and $\lambda_{i+1}$ is a lift of $\underline{\lambda}_{i+1}$. $\square$

**Theorem 9.4.** *Supose that Assumption 6.18 is satisfied. Let $x \in \mathscr{S}_{\mathrm{K}}(G, X)(k)$. The isogeny class of $x$ contains a point which lifts to a special point on $Sh_{\mathrm{K}}(G, X)$.*

*Proof.* Let $T_p \subset I_p$ a maximal torus and $\mu_{T_p}$ the cocharacter constructed in Lemma 9.2. Recall we have fixed the ismorphism $\mathbb{D}(\mathscr{G}_x) \otimes K_0 \cong V^*_{\mathbb{Z}_p} \otimes K_0$ such that $\delta$ commutes with the maximal $\mathbb{Q}_p$-split subtorus $T'$ of $T_p$ and hence $\nu_\delta$ is defined over $\mathbb{Q}_p$. Let $M_{\nu_\delta}$ denote the centralizer of $\nu_\delta$, then there is an inner twisting $M_{\nu_\delta, \overline{\mathbb{Q}}_p} \cong J_{b, \overline{\mathbb{Q}}_p}$. By [Lan89, Lemma 2.1], there is an embedding $j : T_p \hookrightarrow M_{\nu_\delta}$ over $\mathbb{Q}_p$ which is $M_{\nu_\delta}$-conjugate to

$$T_{p, \overline{\mathbb{Q}}_p} \hookrightarrow I_{p, \overline{\mathbb{Q}}_p} \hookrightarrow J_{b, \overline{\mathbb{Q}}_p} \xrightarrow{\sim} M_{\nu_\delta, \overline{\mathbb{Q}}_p} \tag{9.5.1}$$

By Steinberg's theorem, there is an element $m \in M_{\nu_\delta}(L)$ which conjugates $j$ to 9.5.1. Thus upon modifying the isomorphism $\mathbb{D}(\mathscr{G}_x) \otimes L \cong V^*_{\mathbb{Z}_p} \otimes L$ by $m$, we have $\delta$ commutes with $j(T') \subset G$. Let $M$ denote the centralizer of $j(T')$ in $G$, hence $T_p$ is an elliptic maximal torus in $M$ and $\delta \in M(L)$. By [Kot97], we may further modify the isomorphism $\mathbb{D}(\mathscr{G}_x) \otimes L \cong V^*_{\mathbb{Z}_p} \otimes L$ by an element of $M(L)$ so that $\delta \in T_p(L)$.

Let $K/L$ be a finite extension such that $\mu$ is defined over $K$, then by [RZ96], the filtration induced by $j \circ \mu_{T_p}$ is admissible. As $j \circ \mu_{T_p}$ is conjugate to $\mu$, the filtration has weight $0, 1$ hence by [Kis06, 2.2.6], there exists a $p$-divisible group $\tilde{\mathscr{G}}'$ over $\mathcal{O}_K$ with special fiber $\mathscr{G}'$, such that we have an identification $\mathbb{D}(\mathscr{G}') \otimes L \cong \mathbb{D}(\mathscr{G}_x) \otimes L$. This induces a quasi-isogeny $\theta : \mathscr{G}_x \to \mathscr{G}'$.

Let $\tilde{x} \in \mathscr{S}_{\mathrm{K}}(G, X)(\mathcal{O}_K)$ be a point lifting $x$, $s_{\alpha, \text{ét}, \tilde{x}} \in T_p \mathscr{G}_{\tilde{x}}^{\vee, \otimes}$ and $s_{\alpha, 0, x} \in \mathbb{D}(\mathscr{G}_x)^\otimes$ the corresponding crystalline tensors. Let $s'_{\alpha, \text{ét}} \in T_p \tilde{\mathscr{G}}'^{\vee, \otimes}$ the tensors corresponding to the $s_{\alpha, 0, x}$ under the $p$-adic comparison isomorphism. As in [Kis17, 1.1.19], there exists a $\mathbb{Q}_p$-linear isomorphism

$$T_p \mathscr{G}_{\tilde{x}}^\vee \otimes \mathbb{Q}_p \cong T_p \tilde{\mathscr{G}}'^\vee \otimes \mathbb{Q}_p$$

taking $s_{\alpha, \text{ét}, \tilde{x}}$ to $s'_{\alpha, \text{ét}}$. Upon making a finite extension of $K$, we may assume the image of $T_p \mathscr{G}_{\tilde{x}}^\vee$ in $T_p \tilde{\mathscr{G}}'^\vee \otimes \mathbb{Q}_p$ is stable under the Galois action. Upon replacing $\tilde{\mathscr{G}}'$ by an isogenous $p$-divisible group, we may assume there is an isomorphism

$$T_p \mathscr{G}_{\tilde{x}} \cong T_p \tilde{\mathscr{G}}'$$

taking $s_{\alpha, \text{ét}, \tilde{x}}$ to $s'_{\alpha, \text{ét}}$.

By Proposition 4.4, we have $s_{\alpha, 0, x} \in \mathbb{D}(\mathscr{G}')^\otimes$ and we have a sequence of isomorphisms

$$\mathbb{D}(\mathscr{G}_x) \cong T_p \mathscr{G}_{\tilde{x}} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong T_p \tilde{\mathscr{G}}' \otimes_{\mathbb{Z}_p} \mathcal{O}_L \cong \mathbb{D}(\mathscr{G}')$$

which preserve $s_{\alpha, 0, x}$. We may thus identify $\mathbb{D}(\mathscr{G}')$ with $g \mathbb{D}(\mathscr{G}_x)$ for some $g \in G(L)$. As in the proof of Proposition 5.14, the filtration induced by $g^{-1} b \sigma(g)$ is the specialization of a filtration induced by a $G$-valued cocharacter conjugate to $\mu_y$. Hence the filtration corresponds to a point of the local model $M^{\mathrm{loc}}_{\mathscr{G}}(k)$ and we have $g^{-1} b \sigma(g) \in \mathrm{Adm}(\sigma(\{\mu\}))$, i.e. $g \in X(\sigma(\{\mu\}), b)$.

Thus upon replacing $x$ by $i_x(g) \in \mathscr{S}_{\mathrm{K}}(G,X)(k)$, we may assume there is a deformation $\tilde{\mathscr{G}}$ of $\mathscr{G}_x$ to $\mathcal{O}_K$ such that the corresponding filtration on $\mathbb{D}(\mathscr{G}_x) \otimes_{\mathcal{O}_L} K$ is induced by $\mu_{T_p}$. Since $\mu_{T_p}$ is conjugate to $\mu_h^{-1}$ and $T_p\tilde{\mathscr{G}}$ is equipped with Galois invariant tensors corresponding to $s_{\alpha,0,x}$, it follows that $\tilde{\mathscr{G}}$ corresponds to a point $\tilde{x} \in \mathscr{S}_{\mathrm{K}}(G,X)(\mathcal{O}_K)$ by [KP, Proposition 3.3.13] and Proposition 6.2.

That $\tilde{x}$ is a special point of $Sh_{\mathrm{K}}(G,X)$ now follows from the same proof as [Kis17, 2.2.3]. Indeed since $I$ and $I_p$ have the same rank we may assume $T_p$ comes from a maximal torus $T$ in $I$ defined over $\mathbb{Q}$. Then $T \subset I \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ is compatible with filtrations and hence lifts to $\mathcal{A}_{\tilde{x}}$ in the isogeny category. As $T$ fixes $s_{\alpha,0,x}$, it fixes $s_{\alpha,\text{ét},\tilde{x}}$ and hence also $s_{\alpha,B,\tilde{x}}$. Thus $T$ is naturally a subgroup of $G$ and is a maximal torus by Proposition 9.1 (iii). The Mumford-Tate group is a subgroup of $G$ which commutes with $T$, hence is contained in $T$. Hence $\tilde{x}$ is a special point. $\square$

As in [Kis17, §2.3], we may use the above to associate an element $\gamma_0 \in G(\mathbb{Q})$ to each isogeny class such that:

(i) For all $l \neq p$, $\gamma_0$ is $G$-conjugate to $\gamma_l$ in $G(\mathbb{Q}_l)$.

(ii) $\gamma_0$ is stably conjugate to $\gamma_p$ in $G(\overline{\mathbb{Q}}_p)$

(iii) $\gamma_0$ is elliptic in $G(\mathbb{R})$.

In other words, $(\gamma_0, (\gamma_l)_{l\neq p}, \delta)$ form a Kottwitz triple. Indeed using Theorem 9.4, we may assume that $x$ lifts to a special point $\tilde{x} \in \mathscr{S}_{\mathrm{K}}(G,X)(\mathcal{O}_K)$ such that the action of $T \subset \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_x$ lifts to $\mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{\tilde{x}}$. We let $k'/k$ denote the field of definition of $x$ and $K'/K_0$ the fields of definition $x$ and $\tilde{x}$ respectively. Then $k'$ is a finite field $\mathbb{F}_{q^r}$ and $K'$ is a finite extension of $K_0$ since any CM abelian variety is defined over a number field.

Now $\gamma$ lifts to an element $\tilde{\gamma} \in T(\mathbb{Q}) \subset \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{\tilde{x}}$. If we let $\tilde{\gamma}$ act on the Betti cohomology of $\mathcal{A}_{\tilde{x}}$, then $\tilde{\gamma}$ fixes $s_{\alpha,B,\tilde{x}}$ since it fixes $s_{\alpha,\text{ét},\tilde{x}}$. We thus obtain an element $\gamma_0$ in $G(\mathbb{Q})$ which is conjugate to $(\gamma_l)_{l\neq p}$ by the étale Betti comparison. Similarly $\gamma_0$ and $\gamma_p$ are conjugate over $G(\mathbb{C})$ by the comparision isomorphisms between crystalline, de-Rham and Betti cohomology.

By the positivity of the Rosati involution, $T(\mathbb{R})/w_h(\mathbb{R}^\times)$ is compact, and hence $\gamma_0$ is elliptic.

The following version of Tate's theorem, as well as the structural result on the group $I$, can be deduced in the same way as [Kis17, Cor. 2.3.2, 2.3.5].

**Corollary 9.5.** *(i) For every prime $l$ the natural maps*

$$I_{/k,\mathbb{Q}_l} := I_{/k} \otimes_{\mathbb{Q}} \mathbb{Q}_l \to I_{l/k}$$

$$I_{\mathbb{Q}_l} := I \otimes_{\mathbb{Q}} \mathbb{Q}_l \to I_l$$

*are isomorphisms.*

*(ii) Let $I_0$ denote the centralizer of $\gamma_0$. Then $I$ is an inner form of $I$ such that for each place $l$, $I_{\mathbb{Q}_l} \cong I_l$.*

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