# CONSTRUCTING THE COTANGENT COMPLEX VIA HOMOTOPICAL ALGEBRA

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Contents



Abstract. We given an exposition of the theory of model categories following [6] and we will use it to construct the cotangent complex of a morphism of rings. We then give some applications of the cotangent complex in characterizing complete intersection and smooth morphisms.

## 1. Introduction

The application of ideas from algebraic topology to help solve problems in algebraic geometry and number theory has been a recurring theme in mathematics in the last century. From Grothendieck's etale cohomology to the application of homological methods in algebra, this interplay between the different areas of mathematics has given algebraists a vast toolkit for tackling many problems. In this paper we give a brief description of one small area of where this interaction has proved particularly fruitful, that of homotopical methods to studying problems in algebra.

Homological algebra is one way of using techniques from topology to study algebra, however this case is limited only when things (such as our base category) are abelian. Non-abelian situation arise very naturally even in basic algebraic concepts, for example the category of rings is a highly non abelian category. In algebraic topology, one thinks of homotopy theory as a sort of non-abelian homology theory, so it is natural to wonder if homotopic techniques can be applied in non-abelian situations.

This question was first answered by Quillen who gave an axiomatic treatment of homotopy theorem in [6]. Using intuition from the case of topilogical spaces, it can be shown that one can define a good notion of homotopy theory on an arbitrary category. A category endowed with this structure is known as a model category.

One of the first applications of this theory was the construction of the cotangent complex of a morphism of rings. Recall if $R \to S$ is a morphism of rings, then the Kahler differentials $\Omega_{S/R}$ is the $S$ module which represents the functor on $S$-modules

$$N \mapsto \mathrm{Der}_R(S, N)$$

where $\mathrm{Der}_R(S, N)$ is the set of $R$-linear derivations of $S$ with coefficients in $N$. Given a composition of ring homomorphisms $R \to S \to T$ one can show there is an exact sequence of $S$ modules

$$\Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R} \longrightarrow \Omega_{R/S} \longrightarrow 0 \tag{1.1}$$

This is reminiscent of the short exact obtained of a left exact functor applied to an exact sequence in an abelian category, so a natural question to ask is whether there is a continuation of (1.1) to a long exact sequence of $S$ modules. Of course in the abelian situation, one achieves this by taking left derived functors, but the category of $R$ algebras being non-abelian, a new construction is needed.

It turns out that in model categories there is good notion of a left derived functors, this allows the construction of the cotangent complex $\mathcal{L}_{S/R}$ of a morphism of rings which indeed does extend the sequence (1.1). The model category we work in will be the category of simplicial algebras over a ring. This category has a simplicial structure (see chapter 9) compatible with the model category structure, which allows computation of left derived functors via simplicial resolutions. Simplicial resolutions are a direct generalization of projective resolutions in the abelian case and gives a powerful tool for proving properties of the cotangent complex.

We now give a brief description of the contents and organization of the paper. The first half of the paper is concerned with with definition and properties of model categories in full generality. In chapter 2 and 3, we define model categories and show the existence of the associated homotopy category. Chapter 4 gives some examples of model categories, in general it is non-trivial to show that certain categories have a structure of a model category, we show this for the case of topological spaces to demonstrate some of the techniques involved. We also include the case of non-negatively graded chain complexes to show the generality of the theory, this example will persist throughout the paper in order to demonstrate how model categories generalize all the constructions in homological algebra. Chapter 5 and 6 will be concerned with definitions of loop and suspension functors, and the derived functor of a functor model categories. These constructions are used in the definition of homology and cohomology for model categories, and ultimately in the definition of the cotangent complex. Chapter 7 introduces the concept of simplicial model categories, most of the examples of model categories we encounter will be of this form and some are given in Chapter 8. In Chapter 9 we give a brief treatment of homology and cohomology in an arbitrary category. The idea here is that since homotopy theorem can be thought of as a non abelian version of homology, one would hope that taking abelianizations would give a good notion homology and cohomology. Indeed this is case, and under certain conditions this definition can be related to other definitions of homology and cohomology.

In the second half, we show how the theory developed in the first half allows us to construct the cotangent complex. Chapter 10 is devoted to the definition of the cotangent complex and proofs of its basic properties. In Chapter 11 we will see some applications of the cotangent in characterizing complete intersections and smooth homomorphisms.

The paper is aimed at algebraicists with a familiarity of the basic constructions in algebraic topology. The material that we cover on model categories is far more general that what one needs to construct to cotangent complex. However the author feels that, much in the same way that the constructions of the $Tor$ and $Ext$ functors become much more clear in the general setting of derived functors, being able to work in general model categories will make the later constructions clearer as well putting them in the correct context. Indeed there many more model categories that may be of interest to algebraicists, which one would be able to access once one has got to grips with the abstract formulations.

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1.1. **Notations and Conventions.** The pushout of two morphisms $A \to B$ and $A \to C$ in a category will be denoted $B \vee_A C$, and the canonical morphisms $B \to B \vee_A C$ and $C \to B \vee_A C$ will denoted $in_1$ and $in_2$ respectively. Similarly the pullback of two morphisms $B \to A$ and $C \to A$ will be denoted $B \times_A C$ and the canonical maps $B \times_A C \to B$ and $B \times_A C \to C$ will be denoted $pr_1$ and $pr_2$ respectively. When $A$ is the initial (resp. terminal object in a category) we drop the subscript $A$ in $\vee$ (resp. $\times$).

## 2. Definition of a Model Category

In this section we give the definition of a model category and prove the basic results about them. The idea behind the definition of a model category is that it should allow us to have a notion of homotopy theory in a more general setting. This allows us to use intuition from classical homotopy theory to prove powerful theorems about any such category, and moreover the definitions are sufficiently general so as to allow various applications.

In the classical example of topological spaces, there are three distinguished classes of maps, the fibrations, cofibrations and weak equivalences. Here the fibrations are in the sense of Serre, i.e. maps which have the homotopy lifting property with respect to $CW$ complexes, and weak equivalences are maps which induce isomorphisms on all homotopy groups. Cofibrations are then defined to be maps which satisfy the following lifting property:

Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

where $p$ is both a fibration and a weak equivalence, there exists a map $h : B \to X$ such that $ih = f$ and $ph = g$.

A morphism $f : X \to Y$ in this category satisfies the property that it can be factored into$f = pi$ and $f = p'i'$ where $i$ and $i'$ are cofibrations, $p$ and $p'$ are fibrations and $i$ and $p'$ are also weak equivalences. This is proved in Chapter 4, Propositionr 4.5.

These are the two most important properties that model categories should satisfy, in the presence of other axioms one can many prove results about such categories. However since the axioms we assume are quite strong, the hard work comes in later in showing that certain categories are indeed model categories. We will begin with the axiomatic treatment of homotopy theory following [6]. The notations we use are very suggestive of the origins of the theory in algebraic topology, and the reader is encouraged to use the connection as intuition while reading through the proofs.

We first make a couple of preliminary definitions:

**Definition 2.1.** Suppose we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

then we say $i$ has the left lifting property with respect to $p$ and $p$ has the right lifting property with respect to $i$ if there exists a map $h : B \to X$ such that $ih = f$ and $ph = g$.

**Definition 2.2.** We say $f : A \to B$ is a retract of $g : X \to Y$ if it is so in the arrow category of $\mathcal{C}$. In other words, there exists a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

where the horizontal composites are the identity

We can now define the concept of a model structure on category

**Definition 2.3.** A model structure on a category $\mathcal{C}$ consists of three distinguished classes of morphisms in $\mathcal{C}$, the fibrations, cofibrations and weak equivalences such that the following conditions are satisfied:

**M1)** $\mathcal{C}$ is closed under finite projective and inductive limits

**M2)** (Lifting Property) Define a trivial fibration (resp. cofibration) to be a fibration (resp. cofibration) which is also a weak equivalence. Then the cofibrations have the left lifting property with respect to trivial fibrations and the fibrations have the right lifting property with respect to trivial cofibrations.

**M3)** (Factorization) Any morphism $f$ in $\mathcal{C}$ can be factored as $f = pi$ an $f = p'i'$ where $i$ and $i'$ are cofibrations, $p$ and $p'$ are cofibrations and $i$ and $p'$ are trivial.

**M4)** Fibrations are stable under composition, base change and any isomorphism is a fibration. Cofibrations are stable under composition, cobase change and any isomorphism is a cofibration.

**M5)** The (co)base change of a trivial (co)fibration is a weak equivalence.

**M6)** (2 out of 3) given morphisms $f = gh$, if two out of the three morphisms are weak equivalences, then so is the third. Any isomorphism is a weak equivalence.

By a model category we mean a category with together with a model structure, usually this model structure will be implicit and so will not be mentioned.

*Remark* 2.4. The definitions given above satisfy a certain compatibility with duality, in the sense that if $\mathcal{C}$ is a model category, then $\mathcal{C}^{op}$ is also a model category where we switch the cofibrations and fibrations. Thus to every statement that we prove, there will also be a dual statement whose formulation we leave to the reader.

*Remark* 2.5. There can be more than one model structure on a category. Moreover there can be more than one model structure on the same category which gives the same homotopy theory in a sense to be made more precise later.

Most model structures that we come across will satisfy the following additional property:

**Definition 2.6.** A closed model category is one whose model structure satisfies in addition to M1)-M6), the following two equivalent conditions

**M7a)** If $f$ and $g$ are morphisms and $f$ is a retract of $g$ and $g$ is either a fibration, cofibration or weak equivalence, then so is $f$.

**M7b)** i) A morphism is fibration if and only it has the right lifting property with respect to all trivial cofibrations

ii) A morphism is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations

A proof of the equivalence of the above conditions can be found [6] Chapter I, Section 6.

*Remark* 2.7. Most of the model categories that we encounter are actually closed, so some authors such as [3] drop the adjective closed and define a model category to be one which also satisfies $M7a$). Although we will not need the extra generality of non-closed model categories in the construction of the cotangent complex, the author feels a little extra generality never hurt anyone so we follow Quillen's original definition.

Having given this rather abstract definition of a model category, it is tempting, if for no other reason than motivation, to give some (more general) examples of model categories. However this will have to wait until after the next section. The main reason for this is that in general it is rather non-trivial to show that a category has the structure of a model category, and the author does not feel it worth delving into the details of these proofs before one can see why it would be interesting to have a model structure on your category. Thus in the next section we will jump straight into the the proofs about the basic properties of model categories, the pay-off will then come in Chapter 4 when we show how this axiomatic treatment of homotopy theory allows for a direct generalization of homological algebra, hence provide a natural framework for applying techniques from algebraic topology over categories which are not necessarily abelian.

## 3. The Homotopy Category

In this chapter we describe some of the basic properties and constructions associated to an arbitrary model category. We will see how the axioms allow use to define a good notion of homotopy between morphisms in a model category, generalizing the case of topological spaces. The main result of this chapter is then the existence of a certain homotopy category Ho$\mathcal{C}$ associated to a model category $\mathcal{C}$, where Ho$\mathcal{C}$ is obtained by formally inverting weak equivalences. Moreover this homotopy category is equivalent to a certain category $\pi\mathcal{C}_{cf}$ whose morphisms are homotopy classes of maps in a subcategory $\mathcal{C}_{cf}$ of $\mathcal{C}$. The proofs in this section are largely category theoretic, however they are based on the corresponding proofs in algebraic topology. Readers with basic knowledge of algebraic topology, but who are not necessarily familiar with model categories are advised to read through the proofs carefully at least once, using the analogy with usual homotopy as a source of intuition.

We first give a more precise definition of the homotopy category associated to a model category $\mathcal{C}$. The objects of Ho$\mathcal{C}$ are the same as the objects of the $\mathcal{C}$, and morphisms in Ho$\mathcal{C}$ are finite composible strings $(f_0, ..., f_n)$ where the $f_i$ is a morphism in $\mathcal{C}$ or $w^{-1}$ for $w$ a weak equivalence. The identity is the empty string and composition is given by concatenation. We also make the identifications between morphisms $(f, g) = (g \circ f)$ for all composable arrows $f, g$, $(w, w^{-1}) = 1_{dom(w)}$ for all weak equivalences $w$ and $() = (1_A)$ for all objects $A$.

There is a foundational issue here in that the arrows between two objects in Ho$\mathcal{C}$ may not form a set. However in Theorem 3.8 we will show that Ho$\mathcal{C}$ is in fact equivalent to a much more concrete category $\Pi\mathcal{C}_{cf}$ so we set aside any set theoretic difficulties for now and consider Ho$\mathcal{C}$ as a genuine category.

As in most constructions in algebra, the best way to think about the above construction is in terms of the following universal property. If we let $\gamma : \mathcal{C} \to$ Ho$\mathcal{C}$ denote the natural functor, then (Ho$\mathcal{C}, \gamma$) satisfies the following:

**Proposition 3.1.** *Given a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(w)$ is an isomorphism for all weak equivalences $w$, there there is a unique functor $HoF : Ho\mathcal{C} \to \mathcal{D}$ such that $HoF \circ \gamma = F$*

*Remark* 3.2. Said more concretely, any functor on $\mathcal{C}$ which inverts weak equivalences factors through Ho$cat$. In this sense one can of Ho$\mathcal{C}$ as a localization of the category $\mathcal{C}$.

It is clear that the above universal property determines Ho$\mathcal{C}$ up to equivalence.

From now fix a model category $\mathcal{C}$. Since $\mathcal{C}$ has finite limits and colimits, there exists an initial object 0 and a terminal object $*$. An object $A \in ob\mathcal{C}$ is then said to be fibrant (resp. cofibrant) if $A \to *$ is a fibration (resp. $0 \to A$ is a cofibration).

The next definition gives a good notion of a homotopy between morphisms in the category $\mathcal{C}$.

**Definition 3.3.** Let $A$ and $B$ be objects of $\mathcal{C}$

1) A cylinder object for $A \in ob\mathcal{C}$ is an object $A \times I$ which fits in a triangle

$$\begin{tikzcd} A \vee A \arrow[d, "\nabla"'] \arrow[dr, "j_0+j_1"] & \\ A \arrow[r, "\sigma"] & A \times I \end{tikzcd}$$

where $A \vee A$ is the coproduct of two copies of $A$, $\nabla$ is the codiagonal, $j_0 + j_1$ is a cofibration and $\sigma$ is weak equivalence.

2) Dually a path object for $B$ is an object $B^I$ fitting in the triangle

$$\begin{tikzcd} B^I \arrow[r, "s"] \arrow[dr, "i_0+i_1"'] & B \arrow[d, "\Delta"] \\ & B \times B \end{tikzcd}$$

where $B \times B$ is the product of two copies of $B$, $\Delta$ the diagonal, $(i_0, i_1)$ a fibration and $s$ is a weak equivalence.

3) Let $f, g : A \to B$ be two morphisms. By a left homotopy between $f$ and $g$ we mean a morphism $h : A \times I \to B$ where $A \times I$ is an path object for $A$ such that $hj_0 = f$ and $hj_1 = g$. A good picture to keep in mind is the following:

$$\begin{tikzcd} A \vee A \arrow[r, "f+g"] \arrow[d, "\nabla"'] \arrow[dr, "j_0+j_1"] & B \\ A & A \times I \arrow[l, "\sigma"'] \arrow[u, "h"'] \end{tikzcd}$$

We write $f \sim^l g$ if there exists a left homotopy between $f$ and $g$.

By M3) cylinder objects exist for any object in a model category.

The dual notion to that of 3) is that of a right homotopy and is defined using a path object for $b$, we write $f \sim^r g$ if there exist a right homotopy between $f$ and $g$.

When we consider path or cylinder objects for more than one object, we will sometimes abuse notation and use the $(i_0, i_1)$ and $s$, $j_0 + j_1$ and $\sigma$ for both the objects, this should not cause any confusion.

In the example of topological spaces, taking $I$ to be the unit interval and $A \times I$ and $B^I$ to have the obvious meanings, we find that if $f$ and $g$ are homotopic, then they are left and right homotopic. We also have the implication right homotopic implies left homotopic by the dual of Prop. however in general these implications may be strict.

We have the following properties of these homotopy relations. As usual, to each result there is a dual which won't mention:

**Proposition 3.4.** *Let $A, B \in ob\mathcal{C}$ and $f, g : A \to B$*

*i) If $u : X \to A$, then $f \sim^r g$ implies $fu \sim^r gu$.*

*ii) Let $A$ be cofibrant and $v : X \to A$ then $vf \sim^r vg$*

*iii)If $A$ is cofibrant, then left homotopy is an equivalence relation on $Hom_{\mathcal{C}}(A, B)$.*

*iv) If $A$ is cofibrant the $f \sim^l g$ implies $f \sim^r g$.*

*Proof.* i) This is obvious from definitions, indeed if $h : A \to B^I$ is a right homotopy between $f$ and $g$, then $hu$ is a right homotopy between $uf$ and $ug$.

ii) Let $h : A \to B^I$ be a right homotopy between $f$ and $g$. We claim that since $A$ is cofibrant we may take $s : B \to B^I$ to be a trivial cofibration. Indeed given $s : B \to B^I$ in the definition of a path object, we may factorize $s = is'$ where $s'$ is a trivial cofibration $i$ is a fibration. Letting $\tilde{B}$ be the target of $s'$, we have the triangle

$$\begin{tikzcd} \tilde{B} \arrow[dr, "(i_0+i_1)\circ i"'] & B \arrow[l, "s'"'] \arrow[d, "\Delta"] \\ & B \times B \end{tikzcd}$$

which tuns $\tilde{B}$ into a path object for $B$. By M6), $i$ is a weak equivalence hence a trivial fibration, so that given the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \tilde{B} \arrow[d, "i"] \\ A \arrow[r, "h"] \arrow[ur, dashed, "k"] & B^I \end{tikzcd}$$

there exists a lift $k : A \to \tilde{B}$, which is a right homotopy from $f$ to $g$ with the path object $\tilde{B}$.

Thus assume $s : B \to B^I$ is a trivial cofibration, and let $C^I$ be a path object for $C$, then we have the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "sv"] \arrow[d, "s"'] & C^I \arrow[d, "i_0+i_1"] \\ B^I \arrow[r, "(i_0 v, i_1 v)"'] \arrow[ur, dashed, "H"] & C \times C \end{tikzcd}$$

and we obtain a lift $H : B^I \to C^I$. It can then be checked that the composition $Hh : A \to C^I$ is a homotopy between $vf$ and $vg$.

iii) The reflexivity is obvious. For symmetry let $h : A \times I \to B$ be a left homotopy from $f$ to $g$. Letting denote $A \times I'$ be the cylinder object for $A$ where $A \times I = A \times I'$ and we switch $j_0$ and $j_1$, then $h \to A^{I'}$ is a left homotopy from $g$ to $f$.

Now let $h : A \times I \to B$ is a left homotopy from $f_0$ to $f_1$ and $h' : A \times I' \to B$ a left homotopy from $f_1$ to $f_2$. Let $A'$ be the pushout of $A \times I$ and $A \times I'$ with respect to $A$:

$$\begin{tikzcd} A \arrow[r, "j_0"] \arrow[d, "j_0'"'] & A \times I \arrow[d, "d"] \\ A \times I' \arrow[r, "d'"'] & A' \end{tikzcd}$$

The motivation here is gluing unit intervals end to end. Let $j_0'' + j_1'' : A \vee A \to A'$ be the map where $j_0'' : A \to A'$ is the map $dj_0$ and $j_1'' : A \to A'$ is the map $d'j_1'$. We obtain a factorization of the codiagonal $\nabla : A \vee A \to A$ given by $t \circ (j_0'' + j_1'')$ where $t$ is induced by $\sigma : A \to A \times I$ and $\sigma' : A \to A \times I'$.

The maps $h$ and $h'$ then induce a map $k : A' \to B$ such that $kj_0'' = f_0$ and $kj_1'' = f_2$. This is not quite a left homotopy between $f_0$ and $g_0$ since $j_0'' + j_1''$ might not be a cofibration, so that $A'$ might not be a cylinder object for $A$. However we can factorize $j_0'' + j_1'' = pi$ where $i$ is a cofibration and $p$ is a trivial fibration. Then letting $A \times I''$ denote the source of $p$, $A \times I''$ is a cylinder object for $A$ and $kp$ gives a left homotopy from $f_0$ to $f_2$.

iv) Let $h : A \times I \to B$ be a left homotopy from $f$ to $g$, we first show that $j_0 : A \to A \times I$ is a trivial cofibration, which we will need to apply a lifting argument. Indeed since $\sigma j_0 = id_A$, it follows that $j_0$ is a weak equivalence. Also since $A$ is cofibrant, we have by M5) that $in_1 : A \to A \vee A$ is a cofibration being the pushout of $0 \to A$ by $0 \to A$, and since $i_0 = (i_0 + i_1) \circ in_1$ it follows that $i_0$ is a cofibration.

Now let $B^I$ be a path object for $B$ and consider the diagram:

$$\begin{tikzcd} A \arrow[r, "sf"] \arrow[d, "j_0"'] & B^I \arrow[d, "(i_0,i_1)"] \\ A \times I \arrow[r, "(f\sigma,h)"] & B \times B \end{tikzcd}$$

We obtain a lift $H : A \times I \to B^I$, then and one checks that $Hj_1 : A \to B^I$ is a right homotopy from $f$ to $g$. □

Given $A, B \in ob\mathcal{C}$, we let $\pi^l(A,B)$ (resp. $\pi^r(A,B)$) be the set of equivalence classes of $\mathrm{Hom}(A,B)$ with respect to the equivalence relation generated by $\sim^l$ (resp. $\sim^r$), we refer to these as the left (resp. right) homotopy classes. When $A$ is cofibrant (resp. $B$ is fibrant), $\sim^l$ (resp. $\sim^r$) is already an equivalence relation by Proposition 3.4 iii) and its dual. When $A$ is cofibrant and $B$ is fibrant part *iv*) of Proposition 3.4 and its dual shows that the relations $\sim^l$ and $\sim^r$ coincide, in this case we write $\pi(A,B)$ for the equivalence classes. We say $f : A \to B$ and $g : B \to A$ are a pair of (left) homotopy equivalences if the $fg \sim^l id_B$ and $gf \sim^l id_A$.

We would like to define a category whose morphisms are homotopy classes of maps, the following shows that the homotopy classes behave well with respect to composition.

**Corollary 3.5.** *Given $A, B, C \in ob\mathcal{C}$ with $A$ cofibrant, then composition of morphisms induces well-defined map $\pi^r(A,B) \times \pi^r(B,C) \to \pi^r(A,C)$*

*Proof.* This follows immediately from Prop. 3.4 i) and ii), using the fact that $A$ is cofibrant. □

We will also need the following result in what follows

**Proposition 3.6.** *Let $A$ be cofibrant and $p : X \to Y$ a trivial fibration, then $p$ induces a bijection $p_* : \pi^l(A,X) \to \pi^l(A,Y)$*

*Proof.* The map is well defined by (the dual of) part i) of Proposition 3.4. Given $f : A \to Y$ we may by M2) take a lifting in the diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "f"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

Thus the map is in fact surjective without taking homotopy classes.

To show injectivity let $f, g \in \mathrm{Hom}(A,X)$ and suppose $h : A \times I \to Y$ is a left homotopy between $pf$ and $pg$. Then since $p$ is a trivial fibration, we may take a lifting in the following diagram:

$$\begin{tikzcd} A \vee A \arrow[r, "f+g"] \arrow[d, "i_0+i_1"'] & X \arrow[d, "p"] \\ A \times I \arrow[r, "h"] \arrow[ur, dashed, "k"] & Y \end{tikzcd}$$

Then $k$ is a homotopy between $f$ and $g$, hence the map $p_*$ is injective. □

Now let $\mathcal{C}_c, \mathcal{C}_f$ and $\mathcal{C}_{cf}$ be the full subcategories of $\mathcal{C}$ consisting of the objects which are cofibrant, fibrant and both cofibrant and fibrant respectively. By Corollary 2.11, we may define a category $\pi\mathcal{C}_c$ whose objects are the same $\mathcal{C}_c$ and where $\mathrm{Hom}_{\pi_c\mathcal{C}}(A, B)$ is given by $\pi^r(A, B)$. There is a natural functor $\mathcal{C} \to \pi\mathcal{C}_c$ which is given by the identity on objects and takes $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ to it's homotopy class $\overline{f} \in \pi^r(A, B)$. Similarly we obtain categories $\pi\mathcal{C}_f$ and $\pi\mathcal{C}_{cf}$ together with functors $\mathcal{C}_f \to \pi\mathcal{C}_f$ and $\mathcal{C}_{cf} \to \pi\mathcal{C}_{cf}$.

The above projection functors inverts the (left/right) homotopy equivalences. The next result shows that any functor from $\mathcal{C}$ which inverts weak equivalences also inverts homotopy equivalences. This allows us to define functors from the categories $\pi\mathcal{C}_c, \pi\mathcal{C}_f, \pi\mathcal{C}_{cf}$ to the homotopy category $\mathrm{Ho}\mathcal{C}$.

**Proposition 3.7.** *i) Let $F : \mathcal{C} \to \mathcal{B}$ be a functor such that $F(w)$ is an isomorphism for all weak equivalences $f$, then $f \sim^l g$ or $f \sim^r g$ implies $F(f) = F(g)$.*

*ii)Let $F : \mathcal{C}_c \to \mathcal{B}$ be a functor such that $F(w)$ is an isomorphism for all weak equivalences $w$, then $f \sim^r g$ implies $F(f) = F(g)$*

*Proof.* i) Let $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and suppose $h : A \times I$ be a left homotopy between $f$ and $g$. Since $j_0\sigma = id_A$ and $j_1\sigma = id_A$ and $\sigma$ is weak equivalence, we that $F(j_0) = F(j_1) = F(\sigma)^{-1}$. Thus $F(f) = F(h)F(j_0) = F(h)F(j_1) = F(g)$. The case of $\sim^r$ is dual.

ii) The proof is the same as part i), there is a slight subtlety here in that the path object $B^I$ might not be in $\mathcal{C}_c$. However the proof of part ii) of Proposition shows that we can assume $s : B \to B^I$ is a cofibration, and hence that $B^I$ is cofibrant.

□

*Remark* 3.8. The key ingredient in the proof of Prop 3.6 part ii) is the following analogue of the homotopy lifting theorem. Given a fibration $p : X \to Y$, a left homotopy $h : A \times I \to X$ and a map $\alpha : A \to B$ such that $hj_0 = p\alpha$, then we may take a lifting in the following diagram.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "j_0"'] & X \arrow[d, "p"] \\ A \times I \arrow[r, "h"] \arrow[ur, "H"] & Y \end{tikzcd}$$

Indeed this result is used to show that $s : B \to B^I$ can be taken to be a trivial cofibration.

Similarly the key to the proof of (the injectivity part of) Proposition 3.6 is the lifting property for the diagram:

$$\begin{tikzcd} A \vee A \arrow[r, "f+g"] \arrow[d, "i_0+i_1"'] & X \arrow[d, "p"] \\ A \times I \arrow[r, "h"] \arrow[ur, dashed, "k"] & Y \end{tikzcd}$$

where $p$ is a trivial cofibration. We will see that when we move to the simplicial setting, analogues of these results hold for simplicial homotopies and the proofs of the previous results and the next Theorem will go through.

Recall the homotopy category Ho$\mathcal{C}$ and the projection functor $\gamma : \mathcal{C} \to \text{Ho}\mathcal{C}$ constructed above by inverting weak equivalences. We may apply the same construction (i.e. invert weak equivalences) to the categories $\mathcal{C}_c, \mathcal{C}_f$ to obtain the localized categories Ho$\mathcal{C}_c$, Ho$\mathcal{C}_f$ together with functors $\gamma_c : \mathcal{C}_c \to \pi\mathcal{C}_c, \gamma_f : \mathcal{C}_f \to \pi\mathcal{C}_f$. By Proposition 3.7 the functors $\gamma_c, \gamma_f$ and $\gamma$ induce functors $\overline{\gamma}_c : \pi\mathcal{C}_c \to \text{Ho}\mathcal{C}$, $\overline{\gamma}_f : \pi\mathcal{C}_f \to \text{Ho}\mathcal{C}_f$ and $\pi\mathcal{C}_{cf} \to \text{Ho}\mathcal{C}$. We can now state out first main result.

**Theorem 3.9.** *The category Ho$\mathcal{C}$ exists and $\overline{\gamma} : \pi\mathcal{C}_{cf} \to$ Ho$\mathcal{C}$ is an equivalence of categories.*

*Remark* 3.10. The theorem says that $\pi\mathcal{C}_{cf}$ and Ho$\mathcal{C}$ are only equivalent, they are *not* necessarily the same category.

The theorem proved in [6] is slightly stronger, we state it here for completeness, but we refer to loc. cit. for the proof.

**Theorem 3.11.** *Ho$\mathcal{C}_c$, Ho$\mathcal{C}_f$ and Ho$\mathcal{C}$ exists and we have a diagram of functors:*

$$\begin{tikzcd} \pi\mathcal{C}_c \arrow[r, "\overline{\gamma_c}"] & Ho\mathcal{C}_c \arrow[d] \\ \pi\mathcal{C}_{cf} \arrow[u, hook] \arrow[d, hook] \arrow[r, "\overline{\gamma}"] & Ho\mathcal{C} \\ \pi\mathcal{C}_f \arrow[r, "\overline{\gamma_f}"] & Ho\mathcal{C}_f \arrow[u] \end{tikzcd}$$

*Here $\hookrightarrow$ indicates a fully faithful functor, and the two right vertical arrows and $\overline{\gamma}$ are equivalences of categories. Moreover if $\overline{\gamma}^{-1}$ is q quasi inverse for $\overline{\gamma}$, we have that the composition*

$$Ho\mathcal{C}_c \longrightarrow Ho\mathcal{C} \xrightarrow{\overline{\gamma}^{-1}} \pi\mathcal{C}_{cf} \hookrightarrow \pi\mathcal{C}_c$$

*is right adjoint to $\overline{\gamma}_c$, and the composition*

$$Ho\mathcal{C}_f \longrightarrow Ho\mathcal{C} \xrightarrow{\overline{\gamma}^{-1}} \pi\mathcal{C}_{cf} \hookrightarrow \pi\mathcal{C}_f$$

*is left adjoint $\overline{\gamma}_f$*

We will now give the proof of Prop 3.9. We will show the equivalence by first defining an auxiliary category $\mathcal{B}$ together with a natural equivalence of categories $\pi\mathcal{C}_{cf} \to \mathcal{B}$. We will then show that the composite $\mathcal{C} \to \pi\mathcal{C}_{cf} \to \mathcal{B}$ satisfies the universal property of Ho$\mathcal{C}$ which is enough to prove the proposition

*Proof.* For each object $X \in ob\mathcal{C}$ choose a factorization of $0 \to X$ into

$$0 \longrightarrow Q(A) \xrightarrow{i_A} A$$

with $Q(A)$ cofibrant and$p_X$ a trivial fibration. For $A$ cofibrant we choose $Q(A) = A$ and $p_A$ the identity on $A$ and we will call $Q(A)$ a *cofibrant replacement* for the object $A$.

Given a map $f : A \to B$, pick a lifting in the diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Q(B) \arrow[d, "p_B"] \\ Q(A) \arrow[r, "fp_A"] \arrow[ur, dashed, "Q(f)"] & B \end{tikzcd}$$

Since $Q(B) \to B$ is a trivial fibration, it follows that by Proposition 3.6 that $Q(f)$ is unique up to left homotopy. If $g : B \to C$ is another morphism, it follows that $Q(g)Q(f) \sim^l Q(gf)$ and $Q(id_X) \sim^l id_{Q(X)}$. By part (iv) of Proposition 3.4, left homotopic implies right homotopic, thus $Q$ induces a functor $\overline{Q} : \mathcal{C} \to \pi\mathcal{C}_c$.

Similarly, picking a factorization

$$A \xrightarrow{i_A} R(A) \longrightarrow *$$

of $A \to *$ into a trivial cofibration followed by a fibration for each object $A$ and applying the same construction we obtain a functor $R : \mathcal{C} \to \pi\mathcal{C}_f$.

Now suppose $A$ is cofibrant and $f, g : A \to B$ with $f \sim^r g$, then by Prop 2.10 part ii) we have $i_B f \sim^r i_B g$, and hence by dual of Prop 3.6, we have $R(f) \sim^r R(g)$. This implies the restriction of $R$ to $\mathcal{C}_c$ induces a well defined functor $\pi\mathcal{C}_c \to \pi\mathcal{C}_{cf}$ and hence there exists a functor $\overline{RQ} : \mathcal{C} \to \pi\mathcal{C}_{cf}$ which takes to $A$ to $RQ(A)$ and takes a map $f$ to the homotopy class of $RQ(f)$.

Let $\mathcal{B}$ be the category having the same objects as $\mathcal{C}$ with

$$\mathrm{Hom}_{\mathcal{B}}(A, B) = \mathrm{Hom}_{\pi\mathcal{C}_{cf}}(\overline{RQ}(A), \overline{RQ}(B) = \pi(RQ(A), RQ(B))$$

The is a natural functor $\Theta : \mathcal{C} \to \mathcal{B}$ given by the identity on objects and $f \mapsto \overline{RQ}(f)$. Since $RQ(A) = A$ for all $A \in \mathcal{C}_{cf}$, the induced functor $\overline{\Theta} : \pi\mathcal{C}_{cf} \to \mathcal{B}$ is fully faithful.

Now if $p$ is trivial fibration or trivial cofibration in $\mathcal{C}_{cf}$, it follows from Proposition 3.6 applied to $id_Y \in \pi(Y, Y)$, that $p$ is an isomorphism in $\pi\mathcal{C}_{cf}$. Thus given any weak equivalence we may factorize it into a trivial cofibration followed by a fibration which is necessarily trivial by M6), hence any weak equivalence in $\mathcal{C}_{cf}$ is an isomorphism in $\pi\mathcal{C}_{cf}$.

For $f : A \to B$ any weak equivalence in $\mathcal{C}$, it follows from $p_B Q(f) = fp_A$ that $Q(f)$ is a weak equivalence. Similarly $RQ(f)$ is a weak equivalence so that $\Theta(f)$ is an isomorphism. Thus

$$A \xleftarrow{p_A} Q(A) \xrightarrow{i_{Q(A)}} RQ(A)$$

yields an isomorphism in $\mathcal{B}$ between $X$ and $RQ(X) = \overline{\Theta}(X)$, thus $\overline{\Theta}$ is essentially surjective and hence an equivalence of categories.

We now show that $(\mathcal{B}, \Theta)$ satisfies the same universal property as Ho$\mathcal{C}$.

Given $F : \mathcal{C} \to \mathcal{D}$ be a functor such that $F(w)$ is an isomorphism for all weak equivalences $w$. Define $G : \mathcal{B} \to \mathcal{D}$ by setting $G(A) = F(A)$ and for $\overline{f} \in$

$\mathrm{Hom}_{\mathcal{B}}(A,B) = \pi(RQ(A), RQ(B))$, we pick a representative $f \in \mathrm{Hom}(RQ(A), RQ(B))$ of $\overline{f}$, and define $G(\overline{f}) : F(A) \to F(B)$ to be the composite:

$$\begin{tikzcd}
F(A) \arrow[r, dashed, "G(\overline{f})"] \arrow[d, "F(p_A)^{-1}"'] & F(B) \\
F(Q(A)) \arrow[d, "F(i_{Q(A)})"'] & F(Q(B)) \arrow[u, "F(p_B)"'] \\
F(RQ(A)) \arrow[r, "F(f)"'] & F(RQ(B)) \arrow[u, "F(i_{Q(B)})^{-1}"']
\end{tikzcd}$$

By Proposition 3.7 part i), this is independent of the choice of $f$ and it is clear that $G$ is the unique functor such that $G \circ \Theta = F$. This concludes the proof. $\square$

As a corollary we obtain the following simple description of the set $\mathrm{Hom}_{\mathrm{Ho}\mathcal{C}}(A,B)$ when $A$ is cofibrant and $B$ fibrant.

**Corollary 3.12.** *Let $A$ be cofibrant and $B$ fibrant, then $Hom_{Ho\mathcal{C}}(A,B) = \pi(A,B)$*

*Proof.* Indeed $\mathrm{Hom}_{\mathrm{Ho}\mathcal{C}}(A,B) = \pi(RQ(A), RQ(B)) = \pi(R(A), Q(B)) = \pi(A, Q(B)) = \pi(A,B)$ by Proposition 3.6 and its dual. $\square$

When the $\mathcal{C}$ is a closed model category, the situation is slightly nicer as we have the following result.

**Proposition 3.13.** *A morphism $f$ in $\mathcal{C}$ is a weak equivalence if and only if $\gamma(f)$ is an isomorphism in $Ho\mathcal{C}$.*

*Proof.* The implication $\Rightarrow$ follows by definition of $\mathrm{Ho}\mathcal{C}$. Suppose $f : A \to B$ satisfies $\gamma(f)$ is an isomorphism in $\mathrm{Ho}\mathcal{C}$, then it follows from the proof of Theorem 3.9 that $RQ(f)$ is an isomorphism in $\pi\mathcal{C}_{cf}$.

Factoring $RQ(f)$ as:

$$RQ(A) \xrightarrow{i} C \xrightarrow{p} RQ(B)$$

where $i$ is trivial cofibration and $p$ is a fibration, it suffices to show that $p$ is a weak equivalence. As $C$ is cofibrant and fibrant, it thus suffices to prove the implication when $f$ is a fibration in $\mathcal{C}_{cf}$.

By Proposition 3.6, $f$ is a homotopy equivalence (left or right, it does not matter since we are in $\mathcal{C}_{cf}$) so let $g$ be a homotopy inverse for $f$ and $h : B \times I \to B$ is a left homotopy between $fg$ and $id_B$. We may take a lifting $h'$ in the following diagram

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "j_0"'] & A \arrow[d, "f"] \\
B \times I \arrow[r, "h"'] \arrow[ur, dashed, "h'"] & B
\end{tikzcd}$$

so that letting $q = h'i_1$, we have $fq = id_B$ and $h'$ is a left homotopy between $g$ and $q$. Now $qf \sim^l gf \sim^l id_A$, so $q$ is also a homotopy inverse for $f$ and we may take $k : A \times I \to A$ be homotopy between $qf$ and $id_A$. We have $j_1 k = id_A$ and hence $k$ is a weak equivalence by M6), so that $j_0 k = qf$ is also a weak equivalence. The following diagram then presents $f$ as a retract of $qf$:

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "f"'] & A \arrow[r, "id_A"] \arrow[d, "qf"'] & \mathcal{A} \arrow[d, "f"'] \\ \mathcal{B} \arrow[r, "q"] & A \arrow[r, "f"] & B \end{tikzcd}$$

Hence it follows from M7a) that $f$ is also a weak equivalence. □

*Remark* 3.14. The careful reader should have observed that during the course of the above proofs, we tacitly proved Whiteheads theorem for the category $\mathcal{C}_{cf}$. When $\mathcal{C}$ is a closed model category, the converse also holds for $\mathcal{C}_{cf}$.

## 4. Examples: Topological spaces and chain complexes

In this section we give the two fundamental (at least for the purposes of this paper) examples of model categories. Since the whole idea behind model categories stemmed from that of topological spaces, we will show that indeed this category does have the natural structure of a model category. It turns out even for this basic case, the proofs of some of the axioms are quite non-trivial and already demonstrates techniques which can easily be generalized to other contexts. The two hardest axioms to prove are usually the lifting the proper M2) and the factorization property M3). We will omit the proof of the lifting properties as these can be found in any algebraic topology textbook, the proof of M3) however is important and gives a good concrete demonstration of the small object object which can be used to prove M3) in much more general circumstances.

The second example that we discuss is that of non-negatively graded chain complexes. In this case we will not prove that the model structure we define satisfies all the axiom, rather we will content ourselves with a discussion of what the constructions of the previous chapter look like in this context. What we are find should be familiar to anyone who has come into contact with any sort of homological algebra. This example not only demonstrates the generality of model categories but will be useful throughout the paper as good source of intuition for later applications.

Let **Top** be the category of topological spaces and continuous maps. We define fibrations in $\mathcal{C}$ to be Serre fibrations, i.e. have the homotopy lift in property with respect to $CW$-complexes. A weak equivalence is a weak homotopy equivalence, that is, a map which induces isomorphisms on all homotopy groups and a cofibration will be a map which has the left lifting property with respect to a trivial fibrations.

Let $|\Delta(n)|$ denote the standard $n$-simplex and $|\partial\Delta(n)|$ its boundary, so that the inclusion $|\partial\Delta(n)| \hookrightarrow |\Delta(n)|$ is homeomorphic to $S^{n-1} \hookrightarrow D^{n-1}$. We also define the $|V(n,k)|$ to be the boundary of $|\Delta(n)|$ without the $k^{th}$ face, in this case the inclusion $|V(n,k)| \hookrightarrow |\Delta(n)|$ is homeomorphic to $D^{n-1} \to D^{n-1} \times I$. The following two results are proved in [3].

**Lemma 4.1.** $f$ is a fibration if and only if $f$ has the right lifting property with respect to $|V(n,k)| \hookrightarrow ||\Delta(n)||$ for $0 \leq k \leq n$.

Similarly $f$ is a trivial fibration if and only if $f$ has the right lifting property with respect to $|\partial\Delta(n)| \hookrightarrow \Delta(n)$.

**Lemma 4.2.** $f$ is fibration (resp. trivial fibration ) if and only if it it has the right lifting property with respect to all trivial cobfrations (resp. cofibrations)

It follows that with these definitions, axioms M1), M2), M6) M7b) become clear and M4) and M5) follow by an easy diagram chase.

We will now give a proof of M3). The argument, known as the small argument, works whenever the fibrations can be characterized by the fact that they satisfy the left lifting property with respect to morphisms $A \to B$ where $A$ satisfies the property that the natural map

$$\lim_{\to n} \mathrm{Hom}_{\mathcal{C}}(A, Z_n) \to \mathrm{Hom}_{\mathcal{C}}(A, \lim_{\to n} Z_n) \tag{4.1}$$

is a bijection where $Z_0 \to Z_1 \to ...$ is a sequence of objects indexed by $\mathbb{N}$. We say $A$ is small if (4.1) is satisfied. We will only treat the special case but should give the reader the flavor of the argument.

We begin by showing (4.1) is satisfied when $A = |\partial\Delta(n)|$ and $Z_i \to Z_{i+1}$ is a $T_1$ inclusion in the sense below.

**Definition 4.3.** An inclusion $i : A \hookrightarrow B$ of spaces is said to be a $T_1$ inclusion if it is a closed inclusion and $B \backslash i(A)$ consists of closed points.

**Lemma 4.4.** Suppose $Z_0 \to Z_1 \to ...$ is a sequence of $T_1$ inclusions, and $A$ is compact, then

$$\lim_{\to n} \mathrm{Hom}_{\mathcal{C}}(A, Z_n) \to \mathrm{Hom}_{\mathcal{C}}(A, \lim_{\to n} Z_n)$$

is a bijection.

*Proof.* The map is clearly an injection and by definition of the inductive limit topology on $\lim_{\to n} Z_n$, if $f : A \to \lim_{\to n} Z_n$ factors through some $Z_n$, the map $A \to Z_n$ is automatically continuous. Thus it suffices to show that any such $f$ factors through some $Z_n$. Suppose not, then we can find an increasing sequence of integers $(i_n)_{n\geq 0}$ with $i_0 = 0$ and points $\alpha_n \in f(A)$ such that $\alpha_n \in Z_{i_n} \backslash Z_{i_{m-1}}$. For any subset of the $\{\alpha_n\}$, the intersection with any $Z_i$ is finite and not contained in $Z_0$, hence is closed. Thus the image the $\alpha_n$ in the compact space $f(A)$ is discrete which is a contradiction. $\square$

**Proposition 4.5.** *Any map $f : A \to B$ can be factored as $f = pi$ where $i$ is a cofibration and $p$ is a trivial fibration*

*Proof.* We will construct a diagram

$$\begin{tikzcd} A \arrow[r, "j_0"] \arrow[d, "f"'] & Z_0 \arrow[r, "j_1"] \arrow[dl, "p_0"] & Z_1 \arrow[r] \arrow[dll, "p_1"] & ... \\ B & & & \end{tikzcd}$$

Let $Z_{-1} = A$ and suppose we have constructed $Z_{n-1}$ and $p_{n-1}$ and let $\underline{D}$ be the set of diagrams $D$:

$$\begin{tikzcd} \partial\Delta(n_D) \arrow[r, "\alpha_D"] \arrow[d, hook] & Z_{n-1} \arrow[d, "p_{n-1}"] \\ \Delta(n_D) \arrow[r, "\beta_D"] & B \end{tikzcd}$$

We define $Z_n$ to be the pushout

$$\begin{tikzcd} \bigvee_{D\in\underline{D}} \partial\Delta(n_D) \arrow[r] \arrow[d, "\Sigma\alpha_D"'] & \bigvee_{D\in\underline{D}} \Delta(n_D) \arrow[d, "in_2"] \\ Z_{n-1} \arrow[r, "j_{n-1}=in_1"] & Z_n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

and $p_n : Z_n \to B$ is defined to be the map such that $p_n in_1 = p_{n-1}$ and $p_n in_2 = \Sigma\beta_D$.

Let us define $Z$ to be the inductive limit of the spaces $Z_i$, $p$ the inductive limit of the $p_i$'s and $i = ...j_2 \circ j_1 \circ j_0$. We will show that $i$ is cofibration and that $p$ is a trivial fibration thus obtaining the desired factorization.

We claim that $i$ is a cofibration, indeed since $|\partial\Delta(n)| \hookrightarrow |\Delta(n)|$ has the right lifting property with respect to all trivial fibrations by Lemma 4.1, it follows that

$$\bigvee_{D\in\underline{D}} \partial\Delta(n_D) \to \bigvee_{D\in\underline{D}} \Delta(n_D)$$

also has the right lifting property with respect to all trivial fibrations and hence is a cofibration. Since $j_{n-1}$ is the cobase change of this morhpism, we have $j_n$ is a cofibration for all $n$ and it follows that $i$ satisfies the left lifting property with respect to all trivial fibrations since each $j_n$ does. Therefore $i$ is cofibration.

By Lemma 4.2, to show $p$ is a trivial fibration we need to check that it satisfies the right lifting property with respect to $|\partial\Delta(n)| \hookrightarrow |\Delta(n)|$. Suppose we have a diagram:

$$\begin{tikzcd} {|\partial\Delta(n)|} \arrow[r, "\alpha"] \arrow[d, hook] & Z \arrow[d, "p"] \\ {|\Delta(n)|} \arrow[r, "\beta"] & B \end{tikzcd}$$

Since $j_n : Z_{n-1} \to Z_n$ is given as the pushout of the inclusion $\bigvee_{D\in\underline{D}} \partial\Delta(n_D) \to \bigvee_{D\in\underline{D}} \Delta(n_D)$ which is a $T_1$ inclusion, it is clear that $j_n$ itself is a $T_1$ inclusion. Thus since $|\partial\Delta(n)|$ is compact it follows by Lemma 4.4 that $\alpha$ factors through $Z_n$ for some $n$, thus by definition of $Z_{n+1}$ we obtain a map $|\Delta(n)| \to Z_{n+1}$ and hence a lift $|\Delta(n)| \to Z$. This completes the proof. $\square$

Lemma 4.5 gives half of axiom M3). To obtain the other factorization, let $f : A \to B$ be a map and factor it as

$$A \xrightarrow{i} A_f \xrightarrow{p} B$$

where $A_f$ is the homotopy fibre of $f$. This is the set of pairs $(a, \gamma)$ where $a \in A$ and $\gamma$ is a path in $B$ with $\gamma(0) = f(a)$ endowed with the subspace topology of $A \times B^I$. The map $p : A_f \to B$ is given by $(a, \gamma) \mapsto \gamma(1)$ and $i$ is the map $a \mapsto (a, \gamma_a)$ where $\gamma_a$ is the constant path based at $a$. It is a standard exercise to show that $p$ has the homotopy lifting property with respect to $CW$ complexes (see for example 4.64 of [2]), hence is a fibration, and one can show that $i$ is a weak equivalence (in fact a homotopy equivalence) by contracting paths.

Now factor $i = p'i'$ into a cofibration followed by a trivial cofibration using Lemma 4.5. Since $i$ and $p'$ are weak equivalences, $i'$ is a trivial cofibration, and $pp'$ is fibration which gives us the desired factorization of $f$.

Let us now consider the example the example of chain complexes. For an abelian category $\mathcal{A}$, let $Ch_{\geq 0}(\mathcal{A})$ be the category of non-negatively graded chain complexes $A_\bullet = \{d : A_n \to A_{n-1}\}$ of objects of $\mathcal{A}$, i.e. complexes such that $K_n = 0$ for $n \leq 0$. The fibrations are the epimorphisms, cofibrations are injective maps $i$ such that the cokernel of $i$ is a projective object in every degree (note this is not the same as saying the cokernel is a projective object in the category $Ch_{\geq 0}(\mathcal{A})$) and the weak equivalences are maps which induce isomorphisms on homology. To show that this is indeed a model category, all axioms apart from M2) and M3) are easy, for these two axioms we refer to [3] for their proofs.

Taking on faith that the constructions above endows $Ch_{\geq 0}(\mathcal{A})$ with the structure of a model category, one should ask the question what does a cylinder object look in this category?

For simplicity we will assume that $\mathcal{A}$ is the category of modules over some commutative ring $R$, and by abuse of notation we will write $Ch_{\geq 0}(R)$ for this category. The constructions in the general case are completely analogous so we will not lose anything by assuming this.

Recall we already have a notion of homotopy for $Ch_{\geq 0}(R)$: for two maps $f, g : A_\bullet \to B_\bullet$ a homotopy between $f$ and $g$ is a sequence of maps $h_n : A_n \to B_{n+1}$ such that $dh_n - h_{n-1}d = f - g$. So whatever a cylinder object is, it should allow a definition of homotopy which can be related existing one. For general $A_\bullet$ the construction of cylinder objects is difficult; one would have to trace through the proof of the factorization axiom to obtain such a construction, however when $A_\bullet$ is cofibrant there is a much simpler construction. In this case we have that $A_n$ is projective for all $n$. Define $M_\bullet$ to be the object with

$$M_n = A_n \oplus A_{n-1} \oplus A_n$$

and differential given by

$$d(a, b, c) = (da + b, -db, dc - b)$$

The chain map $j_0.j_1 : A_\bullet \to M_\bullet$ are given by the natural inclusions into the first and last component and the map $\sigma : M_\bullet \to A_\bullet$ is given by $(a, b, c) \mapsto a + c$. It is clear that these give a factorization of the codiagonal $\nabla : A \oplus A \to A$.

Define maps $h_n : M_n \to M_{n+1}$ by $h_n(a, b, c) = (0, -c, 0)$. These satisfy

$$(dh_n - h_{n-1}d)(-c, b, c) = id_{M_\bullet} - j_0\sigma$$

i.e. $h$ is a homotopy between $j_0\sigma$ and $id_{M_\bullet}$, then since $\sigma j_0 = id_{A_\bullet}$, we have that $\sigma$ is a weak equivalence.

Let $A[-1]_\bullet$ be the complex with $A[-1]_n = A_{n-1}$, then the map $j_0 + j_1$ is injective and its cokernel is $A[-1]_\bullet$ which is projective in every degree by our assumption on $A$, hence $j_0 + j_1$ is a cofibration and $M_\bullet$ is a cylinder object for $A$.

A homotopy $h$ between two maps $f, g : A \to B$ then determines a left homotopy from $f$ to $g$, given by $h'_n(a, b, c) = f(a) - h(b) + g(c)$ and one checks that this is a map of chain complexes with $j_0h' = f$ and $j_1h' = g$. Conversely one shows that given $h' : M_\bullet \to B$ a map of complexes with $j_0h' = f$ and $j_1h' = g$ that $-h'$ restricted to the middle factor of $M)\bullet$ gives a homotopy between $f$ and $g$, so the two notions coincide.

The homotopy category $\mathrm{Ho}Ch_{\geq 0}(R)$ in this case is then just the derived category of $Ch_{\geq 0}(A)$

## 5. Loop and suspension functors

Given a topological space $X$ with a base point $* \to X$ in the category of base-pointed topological spaces, there is the construction of the loop space $\Omega X$ and the suspension $\Sigma X$. These form a pair of adjoint functors in the homotopy category of topological spaces, more precisely we have:

$$\pi(\Sigma X, Y) = \pi(X, \Omega Y)$$

where $\pi$ denotes homotopy classes of maps. If we let $0 : X \to Y$ be the map taking $X$ to the base point of $Y$, it can be shown that both these sets are groups and actually groups and are isomorphic to $\pi_1(X, Y)$ the group of homotopy classes of homotopies from 0 to itself

Furthermore, given a sequence:

$$F \xrightarrow{i} E \xrightarrow{p} B$$

where $E$ is a Serre fibration and $i$ is the inclusion of the fibre $F$ over the base point $B$, then there is a construction of a sequence

$$... \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B \tag{5.1}$$

which satisfies the property that given an object $A$, the sequence of pointed sets:

$$... \to \pi(A, \Omega^2 B) \to \pi(A, \Omega F) \to \pi(A, \Omega E) \to \pi(A, \Omega B) \to \pi(A, F) \to \pi(A, E) \to \pi(A, B)$$

is exact in a sense to be explained later.

For $n \geq 1$, $\Omega^n X$ is a group object in the homotopy category of pointed topological spaces (which is commutative if $n > 1$) and the above notion of exactness coincides with that for group homomorphisms after the first three terms.

These constructions have corresponding generalizations in an arbitrary (pointed) model category, and are needed to in the definition of homology and cohomology later on. For the purposes of constructing the cotangent complex there are far more concrete constructions that we can do, however these constructions become much easier to understand once placed in this more general context. Our account is based on section 2 of chapter 1 in [6], where we will refer for the more involved proofs.

In the case of topological spaces, to define suspension of loop functors, we need the concept of a base point, the generalization to model categories is provided by the following definition:

**Definition 5.1.** A model category $\mathcal{C}$ is pointed if the the initial and terminal objects are isomorphic, we call such an object a null object.

For any two objects $A, B$ in $\mathcal{C}$ we will denote by $0 \in \mathrm{Hom}(A, B)$ the unique morphism $A \to * \to B$. The fibre of a map $f : A \to B$ is then defined to be the pullback of the two maps $f$ and $0 : * \to B$. The cofibre is defined dually.

From now on we will fix a pointed model category $\mathcal{C}$ with the null object $*$. $f, g : A \to B$ will be two morphisms in $\mathcal{C}$ and we will insist the following holds:

*Assumption: $A$ is cofibrant and $B$ is fibrant*

**Definition 5.2.** Let $A \times I$ be a cylinder object for $A$, the suspension $\Sigma A$ is defined to be the cofibre of the map

$$A \vee A \xrightarrow{(j_0+j_1)} A \times I$$

Although $\Sigma A$ depends on the choice of cylinder object, it can be shown that once we pass to the homotopy category $\mathrm{Ho}\mathcal{C}$, this construction induces a well-defined functor $\Sigma : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{C}$.

Dually the loop space $\Omega B$ of $B$ is defined to be the fibre of the map

$$B^I \longrightarrow B \times B$$

where $B^I$ is a path object for $B$, and this indues a functor $\Omega : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{C}$.

**Example 5.3.** Let $\mathcal{C} = Ch_{\geq 0}(R)$ with the model structure defined in the previous section. Since $A$ is cofibrant we have the cylinder object $M_\bullet$ with $M_n = A_n \oplus A_{n-1} \oplus A_n$. The complex $0_\bullet$ which is 0 in all degrees is a null object for $Ch_{\geq 0}(R)$, it is then clear that the cofibre of $j_0 + j_1 : A_\bullet \oplus A_\bullet \to m_\bullet$ is precisely $A[-1]_\bullet$ where $A[-1]_n = A_{n-1}$ together with the obvious differential $d$. Similarly one can show that $\Omega B_\bullet$ is the complex:

$$\Omega B_n = \begin{cases} B_{n+1} & n \geq 1 \\ \ker(d : B_1 \to B_0) & n = 0 \end{cases}$$

Taking our inspiration from topological spaces we would like an isomorphism $\pi(\Sigma A, B) \cong \pi_1(A, B) \cong \pi(A, \Omega B)$ for an appropriate group $\pi_1(A, B)$. To do this we must define the concept of a homotopy between homotopies.

**Definition 5.4.** Let $h : A \times I \to B, h' : A \times I' \to B$ two left homotopies between $f$ and $g$. By a left homotopy between $h$ and $h'$ we mean a diagram:

$$\begin{tikzcd} A \times I \vee_{A \vee A} A \times I' \arrow[r, "h+h'"] \arrow[d, "\sigma+\sigma'"'] \arrow[dr, "d_0+d_1"] & B \\ A & A \times J \arrow[l, "\tau"'] \arrow[u, "H"'] \end{tikzcd}$$

where $d_0 + d_1$ is a cofibration and $\tau$ is a weak equivalence. $A \times I \vee_{A \vee A} A \times I'$ denotes the pushout of $j_0 + j_1 : A \vee A \to A \times I$ and $j'_0 + j'_1 : A \vee A \to A \times I'$.

The dual notion is that of a right homotopy between right homotopies $k$ and $k'$ which the reader should have no problem formulating.

We now show that the relation "is left homotopic to" is an equivalence relation on the set of homotopies from $f$ to $g$; this follows from the two lemma following the next definition.

**Definition 5.5.** Let $h : A \times I \to B$ be a left homotopy from $f$ to $g$ and $k : A \to B^I$ a right homotopy from $f$ to $g$. By a correspondence between $h$ and $k$ we mean a map $H : A \times I \to B$ such that $Hj_0 = k, Hj_1 = sg, i_0 H = h$ and $i_1 H = g\sigma$.

It is convenient to draw the following pictures for a left homotopy, right homotopy and correspondence between $h$ and $k$.

$$
\begin{tikzcd}
 & g & g \arrow[r, dash, "sg"] & g \\
f \arrow[r, dash, "h"'] & g \arrow[u, phantom] & & \\
 & f \arrow[uu, dash, "k"] & f \arrow[uu, dash, "k"] \arrow[r, dash, "h"'] \arrow[uur, phantom, "H"] & g \arrow[uu, dash, "g\sigma"']
\end{tikzcd}
$$

The two vertical lines in the correspondence square represents the composition of $H$ with $j_0$ and $j_1$ and the horizontal squares the compositions with $i_0$ and $i_1$.

**Lemma 5.6.** Given a cylinder object $A \times I$ and a right homotopy $k : A \to B^I$ there is a left homotopy $h : A \times I$ corresponding to $k$.

*Proof.* As $A$ is cofibrant, $j_0$ is a trivial cofibration, hence we may take lifting in the diagram:

$$
\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "j_0"'] & B^I \arrow[d, "i_1"] \\
A \times I \arrow[r, "g\sigma"] \arrow[ur, dashed, "H"] & B
\end{tikzcd}
$$

then $i_0 H$ gives the desired left homotopy and $H$ the correspondence. $\square$

**Lemma 5.7.** Suppose $h : A \times I \to B$ and $h' : A \times I'$ are two left homotopies and $k : A \to B^I$ a right homotopy such that $h$ and $k$ correspond. Then $h'$ is left homotopic to $h$ if and only if $h'$ and $k$ correspond

*Proof.* $\Leftarrow$" Let $H$ (resp .$H'$) be correspondences between $h'$ (resp $h'$) and $k$. Let $A \times J, d_0 + d_1, \tau$ be as in definition 4.3, these exist by M3). Since $B$ is fibrant, $i_1$ is a trivial cofibration and so we may take a lifting in the diagram:

$$
\begin{tikzcd}
A \times I \vee_{A \vee A} A \times I' \arrow[r, "H+H'"] \arrow[d, "d_0+d_1"'] & B^I \arrow[d, "i_1"] \\
A \times J \arrow[r, "g\tau"] \arrow[ur, dashed, "K"] & B
\end{tikzcd}
$$

Then $d_0 K$ is a left homotopy from $h$ to $h'$.

"$\Rightarrow$" Again left $H$ be a correspondence between $h$ and $k$ and let $K : A \times J \to B$ be a left homotopy from $h$ to $h'$. As in the proof of Prop 3.4 part iv), one shows that $d_0$ is a trivial fibration. Thus we may take a lifting in the following diagram:

$$
\begin{tikzcd}
A \times I \arrow[r, "H"] \arrow[d, "d_0"'] & B^I \arrow[d, "(i_0, i_1)"] \\
A \times J \arrow[r, "(K, g\tau)"] \arrow[ur, dashed, "L"] & B \times B
\end{tikzcd}
$$

Then $Ld_1 : A \times I' \to B^I$ gives a correspondence between $h'$ and $k$. $\square$

It follows from Lemma 5.7 that left homotopy is an equivalence on the class of left homotopies $f$ and $g$, and Lemma 5.6 shows that the equivalence classes are are actually sets. We let $\pi_1^l(A, B : f, g)$ denote the set of equivalence classes, similarly we obtain equivalence classes $\pi_1^l(A, B : f, g)$ given by the relation "right homotopic to". Correspondence then yields a bijection $\pi^l(A, B : f, g) \cong \pi_1^r(A, B; f, g)$. When $f$ and $g$ are the zero map we will denote these sets by $\pi_1(A, B)$.

We would like to have a group structure on $\pi_1(A, B)$, in the case of topological spaces, this is given by composing homotopies. In our more abstract setting, we can mimic the construction via the following:

**Lemma 5.8.** Let $A \times I$ and $A \times I'$ be cylinder objects for $A$ and let $A \times I''$ be the pushout of $j_1 : A \to A \times I$ and $j_0 : A \to A \times I'$. Then $A \times I''$ together with $in_1 j_0 + in_2 d'_0 : A \vee A \to A \times I''$ and $\sigma'' : A \times I''$ induced by $\sigma$ and $\sigma'$ is a cylinder object for $A$.

*Proof.* The only thing that needs to be checked is that $in_1 j_0 + in_2 d'_0$ is a cofibration, this is [6] Lemma 3 of Chapter I, Section 1. □

**Definition 5.9.** Given $f_1, f_2, f_3 \in \mathrm{Hom}_{cat}(A, B)$, $h$ a left homotopy between $f_1$ and $f_2$, and $h'$ a left homotopy between $f_2$ and $f_3$. The composition of $h$ and $h'$ is a left homotopy $h'' = h' \circ h : A \times I'' \to B$ given by $h'' in_1 = h, h'' in_2 h'$ where $A \times I''$ is the cylinder object constructed above.

For $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $h$ a homotopy between $f$ and $g$, the inverse of $h$ denoted $h^{-1}$ is the left homotopy between $g$ and $f$, given by $h : A \times I' \to B$ be $A \times I'$ is the cylinder object $A \times I$ with $j_0$ and $j_1$ switched.

The next proposition shows that this process is compatible with taking homotopy classes

**Proposition 5.10.** *Composition of left homotopies induces maps* $\pi_1^l(A, B; f_1, f_2) \times \pi_1^l(A, B; f_2, f_3) \to \pi_1^l(A, B; f_1, f_3)$ *and similarly for right homotopies, indeed these maps are compatible with the correspondence bijection of the previous paragraph. In this way* $Hom_{\mathcal{C}}(A, B)$ *can be endowed with the natural structure of a groupoid, where a morphism between* $f$ *and* $g$ *is a homotopy class of homotopies between* $f$ *and* $g$.

*Proof.* This is Proposition 1 of [6] Chapter I, section 2 □

The main result is then the following:

**Theorem 5.11.** *There is a functor* $A, B \to [A, B]_1$ *from* $Ho\mathcal{C}^0 \times Ho\mathcal{C}$ *to (groups) where* $[A, B]_1 = \pi_1(A, B)$ *when* $A$ *is cofibrant and* $B$ *is fibrant. Moreover we have the canonical isomorphism of functors:*

$$\pi(\Sigma A, B) \cong [A, B]_1 \cong \pi(A, \Omega B)$$

*Proof.* This is Theorem 2 of [6] Chapter I, section 2 □

Now would be a good opportunity to see what the above constructions mean for our toy example $Ch_{\geq 0}(R)$.

**Example 5.12.** Let $\mathcal{C} = Ch_{\geq 0}$ with the model structure defined in the previous section. Let $\Sigma A_\bullet$ and $\Omega B_\bullet$ denote the suspension of $A$ resp. the loop space of $B$ as defined in Example 5.3.

By definition $[A, B]_1$ is the "homotopy classes" of maps between the zero map and itself. But a homotopy between two zero maps is a sequence of maps $h_n : A_n \to B_n + 1$ with $dh_n - h_{n-1} d = 0$, i.e. map of chain complexes $A[-1] \to B$. It is left as an easy exercise for the reader to show that a homotopy between two homotopies $h$ and $h'$ is just a homotopy between $h$ and $h'$ considered as maps of chain complexes, from which we obtain Theorem 5.11 for this special case.

We now are now in a position to construct the fibration and cofibration sequences. Fix a fibration $p : E \to B$ where $B$ is fibrant and let $i : F \to E$ be the fibre of this map. Upon passing to the homotopy category, we would like to extend this sequence to the right as in equation (4.1).

When $\mathcal{C}$ is the category of pointed topological spaces, we can do this as follows. Note that composition of loops endows $\Omega B$ with the structure of a group, we will first define an action of $\Omega B$ on $F$. Let $\gamma$ be an element of $\Omega B$, i.e. a loop in $B$, and $a$ and element of $F$. Since $p : E \to B$ is a fibration we may lift $\gamma$ to a path $\overline{\gamma} : [0,1] \to E$ with $\overline{\gamma}(0) = i(a)$. Then since $p(\overline{\gamma}(1)) = *$, $\overline{\gamma}(1)$ factors through the inclusion $i : F \to E$ and we denote this element by $\gamma.a$. It is easy to show that this defines an action of $\Omega B$ on $F$, the map $\Omega B \to F$ is then given by $\gamma \to \gamma.*$ where $*$ is the base point of $F$.

Using this idea of transporting elements around loops we will define a group action of $\Omega B$ on $F$ in the homotopy category. (Recall $\Omega B$ is a group object in Ho$\mathcal{C}$ by Prop 4.9)

Let $A$ be cofibrant. Since $F \to *$ is the base change of $p$, $F$ is fibrant hence by corollary 2.16 we have $\mathrm{Hom}_{\mathcal{C}}(A, B) = \pi(A, B)$. Suppose $\alpha \in \pi(A, F)$ is represented by $u : A \to F$, and $\lambda \in \pi(A, \Omega B) = [A, B]_1$ is represented by the homotopy $h : A \times I \to B$ with $h(j_0 + j_1) = 0$, and let $h'$ be lifting in the following diagram.

$$\begin{tikzcd} A \arrow[r, "iu"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "h"] \arrow[ur, dashed, "h'"] & B \end{tikzcd}$$

Now as $ph'j_1 = j_1 h = 0$, the map $h'j_1$ factors uniquely through $F$. We will denote this map $i^{-1}h'j_1 : A \to F$ and the homotopy class of this map will be $\lambda.\alpha$. This construction is obviously functorial in the object $A$ of the homotopy category, hence by Yoneda's Lemma this defines a morphism $m : \Omega B \times F \to F$

**Proposition 5.13.** *$m$ defines a group action in Ho$\mathcal{C}$ of $\Omega B$ on $F$.*

*Proof.* Let $\alpha, u, \lambda, hh'$ be as above. Let $\lambda_1 \in [A, B]_1$ represented by the homotopy $h_1 : A \times I' \to B$. Since $i^{-1}h'j_1$ represents $\lambda\alpha$, taking a lifting $h'_1$ in the diagram:

$$\begin{tikzcd} A \arrow[r, "h'j_1"] \arrow[d, "j'_0"'] & E \arrow[d, "p"] \\ A \times I' \arrow[r, "h"] \arrow[ur, "h'_1"] & B \end{tikzcd}$$

we have that $i^{-1}h'_1 j'_1$ represent $\lambda_1.(\lambda.\alpha)$.

As the composite homotopy $h.h_1$ represents $\lambda_1\lambda$, let $h'.h'_1$ denote the composite of the two homotopies $h'$ and $h'_1$. These then fit in the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "iu"] \arrow[d, "j''_0"'] & E \arrow[d, "p"] \\ A \times I'' \arrow[r, "h.h_1"] \arrow[ur, "h'.h'_1"] & B \end{tikzcd}$$

where $A \times I''$ is the path object in Lemma 4.7. Thus $i^{-1}h'.h_1'j_1''$ represents $\lambda_1\lambda.\alpha$. However $h'.h_1'j_1'' = h_1'j_1'$ by definition of $A \times I''$, hence we see that $\lambda_1\lambda.\alpha = \lambda_1.(\lambda.\alpha)$ □

**Definition 5.14.** A fibration sequence is diagram in Ho$\mathcal{C}$ of the form

$$X \to Y \to Z \quad \Omega Z \times X \to X$$

which is isomorphic to a sequence in $\mathcal{C}_f$

$$F \to E \to B \quad \Omega B \times F \to F$$

as constructed above.

*Remark* 5.15. Dual to the above constructions we have the concept of a cofibration sequence which is constructed from a sequence in $\mathcal{C}_c$:

$$A \to B \to C \quad C \to C \vee \Sigma A$$

Where $A \to B$ is a cofibration and $B \to C$ is its cofibre, and $C \to C \vee \Sigma A$ is a cogroup action.

Given a fibration sequence $X \to Y \to Z$, we have a morphism $\delta : \Omega Z \to X$ given by

$$\Omega Z \xrightarrow{(0,id)} \Omega Z \times X \longrightarrow X$$

In Chapter 4 of [2], there is a proof of the topological analogue of the following theorem.

**Proposition 5.16.** *Given a fibration sequence*

$$X \xrightarrow{i} Y \xrightarrow{p} Z \qquad \Omega Z \times X \xrightarrow{m} X$$

*then the diagram*

$$\Omega Z \xrightarrow{\delta} X \xrightarrow{i} Y \qquad \Omega Y \times \Omega Z \longrightarrow \Omega Z$$

*is also a fibration sequence*

*Proof.* This is Proposition 4 in [6] Chapter I, section 3. □

*Remark* 5.17. For topological spaces given a sequence

$$F \xrightarrow{p} E \xrightarrow{i} B$$

where $i$ is a fibration and $p$ is the fibre of $i$, then it is not necessarily true that the sequence

$$\Omega Z \xrightarrow{\delta} X \xrightarrow{i} Y$$

is of the the same form, eg. $i$ might not be a fibration. The proposition states that one can find a sequence which is isomorphic to this in the homotopy category. In the topological case this is achieved by taking homotopy fibers and Quillen's proof is a based on this process.

Proposition 5.16 implies that any fibration sequence may be extended as in equation (5.1). Moreover some diagram chasing shows that this sequence is exact in the following sense.

**Proposition 5.18.** *Let*

$$F \xrightarrow{i} E \xrightarrow{p} B \qquad\qquad \Omega B \times F \longrightarrow F$$

*be a fibration sequence and $A$ any element of $Ho\mathcal{C}$. Then the following sequence is exact*

$$\ldots \xrightarrow{(\Omega^{q+1}p)_*} [A, \Omega^{q+1}B] \xrightarrow{(\Omega^q\delta)_*} [A, \Omega^q F] \xrightarrow{(\Omega^q i)_*} [A, \Omega^q E] \xrightarrow{(\Omega^q p)_*} \ldots$$
$$\ldots \xrightarrow{(\Omega p)_*} [A, \Omega B] \xrightarrow{\delta_*} [A, F] \xrightarrow{i_*} [A, E] \xrightarrow{p_*} [A, B]$$

where for two objects $A, B$ of Ho$\mathcal{C}$ we write $[A, B]$ as shorthand for $\mathrm{Hom}_{\mathrm{Ho}\mathcal{C}}(A, B)$ and we take exact to mean:

i) $p_*^{-1}(0) = \mathrm{im}\ i_*$

ii) $i_*\delta_* = 0$ and $i_*\alpha_1 = \imath_*\alpha_2$, if and only if$\exists\lambda \in [A, \Omega B]$, such that $\lambda.\alpha_1 = \alpha_2$.

iii) $\delta_*(\Omega p)_* = 0$ and $\delta_*\lambda_1 = \delta_*\lambda_2$ if and only if $\lambda_2 = (\Omega p) * \mu.\lambda_1$ for some $\mu \in [A, \Omega E]$

iv) The sequence of group homomorphisms to the left of $[A, \Omega E]$ is exact in the usual sense.

*Proof.* This is Proposition 4 in [6], Chapter I, section 3. □

To finish of this section let's see what fibration sequences look like in the category $Ch_{\geq 0}(R)$

**Example 5.19.** In the category $Ch_{\geq 0}$, a fibration $E \to B$ is an epimorphism, and the fibre of this map is precisely its kernel, hence a fibration sequence is just an exact sequence of complexes (or at least homotopy equivalent to one).

If we let $I$ denote the complex

$$\ldots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow \ldots$$

where the $R$ is in degree 0, then it follows by from definitions that $\pi(I, A) = H^0(A_\bullet)$. More generally we have that $\pi(\Sigma^n I, A) = \pi(I, \Omega^n A)$ and an easy check verifies that the fibration exact sequence is just the long exact sequence arising rom the exact sequence of complexes. A similar statement holds for the cofibration exact sequence.

## 6. Derived functors and equivalences of Model categories

In this section, we define for a functor $F : \mathcal{C} \to \mathcal{B}$ where $\mathcal{C}$ is a model category, its left and derived functors the existence which is guaranteed by a simple criterion on $F$. We will see how these constructions directly generalize left and right functors in the abelian case with our example of $Ch_{\geq 0}(R)$. When the target of $F$ is equipped with a model structure, one can define the concept of the total derived functor of $F$ which is a functor between the homotopy categories and proof a criterion for when this is an equivalence of categories. The constructions of this section are not only essential in the construction of the cotangent complex, but is an important facet of the theory of model categories, so everything in this section will be proved in detail.

The concept of derived functors exists in greater generality than just that of model categories, they are defined as follows

**Definition 6.1.** Let $\gamma : \mathcal{A} \to \mathcal{A}'$ and $F : \mathcal{A} \to \mathcal{B}$ be two functors. The left derived functor of $F$ with respect to $\gamma$ is a functor $L^\gamma F : \mathcal{A}' \to \mathcal{B}$ together with a natural transformation $\epsilon : L^\gamma F \circ \gamma \to F$ such that given any other functor $G : \mathcal{A}' \to \mathcal{B}$ with a natural transformation $\zeta : G \circ \gamma \to F$, there is a unique natural transformation $\Theta : G \to L^\gamma F$ such that the following diagram commutes:

$$\begin{tikzcd} G \circ \gamma \arrow[d, "\Theta\circ\gamma"'] \arrow[dr, "\zeta"] & \\ L^\gamma F \circ \gamma \arrow[r, "\epsilon"] & F \end{tikzcd}$$

In this sense, $L^\gamma F$ is the closest functor to $F$ from the left, similarly we can define the right derived functor $R^\gamma F$ together with the natural transformation $\eta : F \to R^\gamma F$ which is the closest to $F$ from the right.

We will only be interested in the when $\mathcal{A} = \mathcal{C}$ is a model category and $\gamma$ is the localization functor $\gamma$ in which case we omit the $\gamma : \mathcal{C} \to \mathrm{Ho}\mathcal{C}$ from the notation and write $LF$ (resp. $RF$) for the left (resp. right) derived functor of $F$ with respect to $\gamma$.

**Example 6.2.** If the functor $F : \mathcal{C} \to \mathcal{B}$ takes weak equivalences to isomorphisms, then $F$ factors uniquely through $\mathrm{Ho}\mathcal{C}$. It is then clear that $LF : \mathrm{Ho}\mathcal{C} \to C$ is the unique functor such that $LF \circ \gamma = F$.

From now on fix a model category $\mathcal{C}$.

**Proposition 6.3.** *Suppose $F : \mathcal{C} \to B$ takes all weak equivalences in $\mathcal{C}_c$ to isomorphisms in $\mathcal{B}$, then $LF$ exists and furthermore $\epsilon : LF \circ \gamma(X) \to F(X)$ is an isomorphism when $X$ is cofibrant*

*Proof.* Recall the cofibrant replacement $Q(A)$ of an object $A$. We have that $A \mapsto Q(A), f \mapsto Q(f)$ determines a functor $\overline{Q} : \mathcal{C} \to \pi\mathcal{C}_c$. By Proposition 3.7 part ii) we have for any two morphism $f, g : A \to B$ in $\mathcal{C}_c$, $f \sim^r g$ implies $F(f) = F(g)$, hence $A \to FQ(A)$ and $f \to FQ(f)$ induce a well defined functor $\mathcal{C}_c \to \mathcal{B}$. Since this functor inverts all weak equivalences (since $Q(f)$ is a weak equivalence if $f$ is), we obtain a functor $LF : \mathrm{Ho}\mathcal{C} = \mathrm{Ho}\mathcal{C}_c \to \mathcal{B}$. The natural transformation $\epsilon$ is given by $\epsilon(A) : F(p_A) : LF(A) = FQ(A) \to A$, where $p_A$ is the trivial fibration $Q(A) \to A$. We will now show that $(LF, \epsilon)$ has the required universal property.

For a functor $G : \mathrm{Ho}\mathcal{C} \to \mathcal{B}$ with a natural transformation $\zeta : G \circ \gamma \to F$, define a natural transformation $\Theta : G \circ \gamma \to F$ by the following:

$$G \circ \gamma(A) \xrightarrow{G(\gamma(p_A))^{-1}} GQ(A) \xrightarrow{\zeta_{Q(A)}} FQ(A) = LF(A)$$

It is clear that $\Theta$ is a natural transformation and as every morphism in $\mathrm{Ho}\mathcal{C}$ is a composition of finitely many morphisms of the form $\gamma(f)$ or $\gamma(w)^{-1}$ for $w$ a weak equivalence, it follows that $\Theta$ can be extended to a natural transformation $\Theta : G \to LF$. The following diagram then shows that $\epsilon \circ \Theta = \zeta$

$$\begin{tikzcd} G(A) \arrow[r, "G(\gamma(p_A))^{-1}"] \arrow[dr, "id_{G(A)}"'] & GQ(A) \arrow[r, "\zeta_{Q(A)}"] \arrow[d, "G(\gamma(p_A))"'] & FQ(A) \arrow[r, equal] \arrow[d, "F(\gamma(p_A))"'] & LF(A) \arrow[dl, "\epsilon_A"] \\ & G(A) \arrow[r, "\zeta_A"] & F(A) & \end{tikzcd}$$

Uniqueness of $\Theta$ follows since it is determined by what it does on $\mathrm{Ho}\mathcal{C}_c$, and by Theorem 3.11 we have $\mathrm{Ho}\mathcal{C}_c = \mathrm{Ho}\mathcal{C}$. When $A$ is cofibrant, $LF(A) = FQ(A) = FA$ and $\epsilon_A = id_A$ ☐

**Definition 6.4.** Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between two model categories, the total left derived functor denoted $\underline{LF}$ of $F$ is given by the left derived functor of $\gamma' \circ F : \mathcal{C} \to \mathrm{Ho}\mathcal{C}'$

As an immediate corollary of Proposition 6.3, we have that if $F$ is a functor between model categories which takes weak equivalences in $\mathcal{C}_c$ to weak equivalences in $\mathcal{C}'$ then the total left derived functor exists.

Now fix a functor $\mathcal{C} \to \mathcal{C}'$ between model categories. The rest of this section will be devoted to proving a criterion for when the total left derived functor of $F$ is an equivalence of categories $\underline{LF} : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{C}'$.

**Proposition 6.5.** *Let $F : \mathcal{C} :\to \mathcal{C}'$ be a functor between pointed model categories which preserves finite colimits and cofibrations and takes weak equivalences in $\mathcal{C}_c$ to weak equivalences in $\mathcal{C}'$. Then $\underline{LF}$ is compatible with direct sums and if $\Sigma$ and $\Sigma'$ denote the suspension functors on Ho$\mathcal{C}$ and Ho$\mathcal{C}'$ respectively, we have $\underline{LF}\Sigma A \cong \Sigma' \underline{LF} A$.*

*Proof.* By Proposition 6.3, $\underline{LF}$ exists and the proof shows that we may assume $\underline{LF}(A) = F(A)$ if $A$ is cofibrant. It can be shown that the localization map $\mathcal{C}'_c \to \mathrm{Ho}\mathcal{C}'$ preserves direct sums. Since $F(\mathcal{C}_c) = F(\mathcal{C}'_c)$, it follows that for $A, B \in ob\mathcal{C}_c$, we have

$$\underline{LF}(A \vee B) = F(A \vee B) = F(A) \vee F(B) = \underline{LF}(A) \vee \underline{LF}(B)$$

so that $\underline{LF}$ commutes with direct sums.

Let $A$ be cofibrant and let

$$A \vee A \xrightarrow{j_0 + j_1} A \times I \xrightarrow{\sigma} A$$

be a cylinder object for $A$. Applying $F$ to this sequence and using the fact that $F$ commutes with colimits, we obtain the sequence

$$F(A) \vee F(A) \xrightarrow{F(j_0)+F(j_1)} F(A \times I) \xrightarrow{F(\sigma)} F(A)$$

where $F(j_1) + F(j_1)$ is a cofibration and $F(\sigma)$ is a weak equivalence by our assumptions. Hence $F(A \times I)$ is a cylinder object for $A$, and since $A$ is cofibrant the cofibre of $F(j_0) + F(j_1)$ represent the suspension $\Sigma F(A)$. Since $F$ preserves colimits, this is isomorphic to $F(\Sigma A)$. ☐

We now come to the main result of this section.

**Theorem 6.6.** *Let $F : \mathcal{C} \to \mathcal{C}', G : \mathcal{C}' \to \mathcal{C}$ be two functors between two models categories and where $F$ is left adjoint to $G$. Assume $F$ takes preserves cofibrations and takes weak equivalences in $\mathcal{C}_c$ to weak equivalences in $\mathcal{C}'$, dually suppose $G$ preserves cofibrations and takes weak equivalences in $\mathcal{C}'_c$ to weak equivalences in $\mathcal{C}$. Then the total derived functors $\underline{LF}$ and $\underline{RF}$ are canonically adjoint.*

*Suppose further that for $A \in ob\mathcal{C}_c$ and $B \in ob\mathcal{C}'_f$, we have $LA \to B$ is a weak equivalence if and only if $A \to RB$ is a weak equivalence. Then the unit and count of the adjunction are both natural isomorphisms.*

*Proof.* We will write $u^\flat : A \to GB$ (resp. $v^\# : FA \to B$)be the map which corresponds to $u : FA \to B$ (resp. $v : A \to GB$) under the adjunction. Suppose $A \in ob\mathcal{C}_c$ and $B \in ob\mathcal{C}'_f$, then we saw in the proof of Proposition 6.5 that $F(A)\times I := F(A \times I)$ is a cylinder object for $F(A)$. Thus if $f, g : A \to GB$ are two morphisms in $\mathcal{C}$, and $h : A \times I \to GB$ a left homotopy from $f$ to $g$, then $h^\# : FA \times I \to B$ gives a left homotopy between $f^\#$ and $g^\#$. Since $A$ and $FA$ are cofibrant, and $B$ and $GB$ are fibrant, the left and right homotopy relations coincide, so that $\pi(A, GB) \cong \pi(FA, B)$.

If we let $Q$ denote the cofibrant replacement functor on $\mathcal{C}$ and $R'$ the fibrant replacement functor for $\mathcal{C}'$, then by our construction of the derived functor in Proposition 6.3, it follows that for general $A, B$

$$\mathrm{Hom}_{\mathrm{Ho}\mathcal{C}'}(\underline{LF}(A), B) \cong \pi(FQ(A), R'B) \cong \pi(Q(A), GR'(B)) \cong \mathrm{Hom}_{\mathrm{Ho}\mathcal{C}}(A, \underline{RF}(B))$$

These isormorphisms are clearly functorial and as any morphism in Ho$\mathcal{C}$ is a finite composition of morphisms coming from $\mathcal{C}$, it follows that $\underline{LF}$ and $\underline{RG}$ are adjoint.

Now suppose that $A \in ob\mathcal{C}_c$ and $B \in ob\mathcal{C}'_f$, and $f : A \to GB$ is a weak equivalence if and only if $f^\# : FA \to B$ is a weak equivalence. Thus $i^\flat_{FA} : A \to GR'F(A)$ is a weak equivalence, and hence an isomorphism in the homotopy category. However this tracing through the definitions one finds that the homotopy class of this map is just the unit of the adjunction. Dually we get the counit is also an isomorphism. □

*Remark* 6.7. Quillen also shows that for $F$ and $G$ satisfying the hypothesis of the theorem, then $\underline{LF}$ and $\underline{RG}$ preserve both fibration and cofibration sequences. We will not make use of this result so we omit the proof.

Functors $F$ and $G$ which satisfy the conditions of the first part of Theorem 6.6 will be called a Quillen adjunction.

The next example shows how the above constructions generalize those in the abelian case. In that case one can compute left derived functors by taking projective resolutions and we will see why cofibrant replacement is the appropriate generalization. In the next section, once we have a simplicial structure on the model category, the analogous construction is to take a simplicial resolutions, which gives us a powerful tool for computing derived functors.

**Example 6.8.** Let $R$ be a commutative ring and $N$ an $R$-module. Consider the functor $F : Ch_{\geq 0}(R) \to Ch_{\geq 0}(R)$ induced by the functor $- \otimes_R N$.

We would like to compute the total left derived functor of $F$. To apply prop 6.3 we need to know that $F$ takes weak equivalence in $Ch_{\geq 0}(R)_c$ to weak equivalences in $Ch_{\geq 0}(R)$. Thus suppose $A$ and $B$ are chain complexes which are projective in every degree and $f : A_\bullet \to B_\bullet$ be a map of chain complexes which induces an isomorphism of homology. We may factor $M$ into a trivial cofibration followed by a trivial fibration, thus it suffices to show $F$ takes trivial cofibrations and trivial fibrations to weak equivalences.

Suppose $f$ is a trivial cofibration, then we have an exact sequence of complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

Then since $C_\bullet$ is projective in every degree it follow that upon tensoring by $N$, the short exact sequence of complexes remains short exact. As $f$ is a weak equivalence, it follows by the long exact of homology that $C_\bullet$ is an exact complex, and since it is degreewise projective, it follows that $C \otimes_R N$ is also an exact complex. The long exact sequence of homology groups shows that $F(f)$ induces an isomorphism on homology. The proof for $f$ a trivial fibration is the same. It now follows from Prop 6.3 that $\underline{LF}$ exists.

For an $R$ module $M$, consider the complex $\tilde{M}$ which is $M$ in degree 0 and zero everywhere else. Working through the proof of proposition 6.3, one sees that the image of $\tilde{M}$ under $\underline{LF}$ is the image of $FQ(\tilde{M})$ in the homotopy category. In this context a cobfibrant replacement is given by a projective resolution $P_\bullet$ of $M$, hence the left derived functor is the image of $P_\bullet \otimes_R N$ in the derived category. The $i^{th}$ homology of this complex is well defined and is given by $Tor^i_R(M, N)$.

## 7. The Simplicial Setting

From now on we begin to specialize to the case of interest to us. The definitions and constructions made in the previous chapters are far more general than we need, in fact the model categories that we will come across now are all simplicial model categories. Theses are model categories together such that for any two objects $A, B$ one can attach a simplicial set $\underline{\mathrm{Hom}}(A, B)$ whose zero degree part is just $\mathrm{Hom}(A, B)$. This satisfies certain compatibility conditions with the model category structure, and allows to define a concept of *simplicial homotopy* which coincides with the usual notion in when $A$ is cofibrant and $B$ is fibrant. In certain cases, the simplicial structure provides us with powerful tools for calculations, examples of which the reader will encounter in Chapter 9.

From now we assume all model categories we encounter are closed and pointed with null object $*$.

Let $\Delta$ denote the category with objects $[n] = \{0, 1, ..., n\}$ and $\mathrm{Hom}_\Delta([n], [k])$ the set of weakly order preserving maps between $[n]$ and $[k]$. Let $d_i : [n-1] \to [n]$ be the order preserving map whose image does not include $i$, and $s_i : [n] \to [n-1]$ the surjective map which takes $i$ and $i+1$ to $i$. These morphisms satisfy the identities:

$$\begin{aligned}
1)\quad d_j d_i &= d_i d_{j-1} \qquad (i < j) \\
2)\quad s_j d_i &= d_i s_{j-1} \qquad (i < j) \\
&= \quad id \qquad (i = j, j+1) \qquad\qquad (*) \\
&= d_{i-1} s_j \qquad (i > j+1) \\
3)\quad s_j s_i &= s_{i-1} s_j \qquad (i > j)
\end{aligned}$$

and it can be shown that in fact these morphisms generate all morphisms in $\Delta$. We call the $d_i$ *face maps* and $s_i$ *degeneracy maps*.

**Definition 7.1.** Let $\mathcal{C}$ be an arbitrary category, $A$ simplicial object of $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$. Morphisms between simplicial objects are natural transformations of functors and the we will denote the category of simplicial objects in $\mathcal{C}$ by $s\mathcal{C}$.

When $\mathcal{C}$ is the category of sets, we will denote the category of simplicial sets s**Sets**. Concretely the objects of this category are a chain of sets $A_n$ together with maps $d_i : A_n \to A_{n-1}$ and $s_i : A_{n-1} \to A_n$ satisfying (the dual of) the relations $(*)$.

The simplicial set $\mathrm{Hom}_\Delta(-,[n])$ will be denoted $\Delta(n)$, we will call this the standard $n$ simplex and for any simplicial set $A$ we write $A_n$ for the set $A([n])$ which by Yoneda's Lemma is equal $\mathrm{Hom}_{s\mathbf{Sets}}(\Delta(n),A)$. We call an element of $A_n$ an $n$-simplex of $A$. For any $n$-simplex $\sigma$ of $A$, we call a simplex a face of $\sigma$ if it is the image of $\sigma$ under a composition of face maps, and a degeneracy of $\sigma$ will denote the image of $\sigma$ under a combination of degeneracy maps. We specify that $\sigma$ is both a degeneracy and face of itself. A simplex will be called non-degenerate if it is only a degeneracy of itself. For example the simplicial set $\Delta(n)$ has $\binom{n}{k}$ non-degenerate $k$ simplices.

For the readers who are unfamiliar with simplicial sets, there is a more concrete way to think about them in terms of the geometric realization functor. That is, to any simplicial set $K$ we can associate to it a topological space $|K|$ with a simplicial decomposition such that the $k$ simplifies of $|K|$ are precisely the non-degenerate $k$ simplices of $K$. For example, $|\Delta(n)|$ is precisely the standard $n$ simplex so that the notation of section 4 is consistent with this definition. We refer to [3] for more details.

**Definition 7.2.** A simplicial category is a category $\mathcal{C}$ together with a functor $\mathcal{C}^{op}\times\mathcal{C}\to s\mathbf{Sets}$ denoted $\underline{\mathrm{Hom}}(A,B)$ for $A,B\in ob\mathcal{C}$ and for any $A,B,C\in ob\mathcal{C}$ a "composition" map of simplicial sets

$$\circ:\underline{\mathrm{Hom}}(A,B)\times\underline{\mathrm{Hom}}(B,C)\to\underline{\mathrm{Hom}}(A,C)$$

satisfying the following properties:

i) Composition is associative in that for $f\in\underline{\mathrm{Hom}}(A,B)_n, g\in\underline{\mathrm{Hom}}(B,C)_n\, h\in\underline{\mathrm{Hom}}(C,D)_n$ we have $f\circ(g\circ h)=(f\circ g)\circ h$

ii) There is an isomorphism $\mathrm{Hom}(A,B)\cong\underline{\mathrm{Hom}}(A,B)_0$ denoted $u\mapsto\tilde{u}$. And composition in $\underline{\mathrm{Hom}}_0$ agrees with the usual composition in Hom under the identification above identification.

iii) For $u\in\mathrm{Hom}(A,B)$ and $f\in\underline{\mathrm{Hom}}(B,C)_n$ then $f\circ s_0^n\tilde{u}=\underline{\mathrm{Hom}}(u,B)_n f$. Similarly for $g\in\underline{\mathrm{Hom}}(D,A)_n$ we have $s_0^n\tilde{u}\circ g=\underline{\mathrm{Hom}}(D,u)g$

If $\mathcal{C}_1,\mathcal{C}_2$ are simplicial categories then a simplicial functor $F:\mathcal{C}_1\to\mathcal{C}_2$ is a functor from $\mathcal{C}_1$ to $\mathcal{C}_2$ together with maps of simplicial sets $\underline{\mathrm{Hom}}(A,B)\to\underline{\mathrm{Hom}}(FA,FB)$ also denoted $F$, which agrees with $F$ on $\underline{\mathrm{Hom}}(A,B)_n$

**Example 7.3.** For the case of simplicial sets we define

$$\underline{\mathrm{Hom}}(A,B)_n=\underline{\mathrm{Hom}}(A\times\Delta(n),B)$$

with face and degeneracy maps given by

$$d_i(f)=f\circ 1\times d_i \text{ and } s_i(f)=f\circ 1\times s_i$$

The composition map $\underline{\mathrm{Hom}}(A,B)_n\times\underline{\mathrm{Hom}}(B,C)_n\to\underline{\mathrm{Hom}}(A,C)_n$ is given by $g\circ f(x,u)=g(f(x,u),u)$ and this gives $s\mathbf{Sets}$ the structure of a simplicial category.

**Example 7.4.** Consider the category **Top** of topological spaces and let $A$ and $B$ be spaces, we define $\underline{\mathrm{Hom}}(A,B)$ by

$$\underline{\mathrm{Hom}}(A,B)_n=\mathrm{Hom}(A\times|\Delta(n)|,B)$$

and face and degeneracy maps are induced by those $|\Delta(n)|$. Composition is given by $g\circ f(x,u)=g(f(x,u),u)$, in this way $\mathcal{C}$ becomes a simplicial category, in fact this simplicial structure is compatible with the model category structure i.e. it endows $\mathcal{C}$ with the structure of a closed simplicial model category.

*Remark* 7.5. There are evident similarities in the above two constructions which hint at some deeper underlying connection between the two categories. Indeed if we restrict ourselves to the subcategory of compactly generated topological spaces, there is a Quillen adjunction $Sing : \mathcal{T} \to \mathrm{s}\mathbf{Sets}$ and $|.| : \mathrm{s}\mathbf{Sets} \to \mathcal{T}$, where $Sing$ is the singular complex functor. These satisfy the assumptions of Theorem 6.6. and hence induce equivalences on the homotopy categories. One consequence of this is that it gives us a completely combinatorial definition of homotopy groups see [5] for details.

In s**Sets** there is a canonical evaluation map

$$ev : A \times \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(A, B) \to B$$

If we take

$$\begin{aligned}\sigma_n \in \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(A, B)_n &= \mathrm{Hom}_{\mathrm{s}\mathbf{Sets}}(\Delta(n), \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(A, B)) \\ &= \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(A \times \Delta(n), B)\end{aligned}$$

then this map is the unique map which makes the following diagram commute:

$$\begin{tikzcd} A \times \Delta(n) \arrow[d, "id_A \times \sigma_n"'] \arrow[dr, "\sigma_n"] & \\ A \times \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(A, B) \arrow[r, "ev"] & B \end{tikzcd}$$

Concretely, for $a \in A, \sigma_n \in \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(A \times \Delta(n), B)$ and where $i_n \in \Delta(n)_n$ the unique non-degenerate $n$ simplex, the map is given by

$$(x \times \sigma_n) \mapsto \sigma_n(x, i_n)$$

The map $ev$ determines an isomorphism

$$\mathrm{Hom}_{\mathrm{s}\mathbf{Sets}}(K, \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(A, B)) \xrightarrow{\Theta} \mathrm{Hom}_{\mathrm{s}\mathbf{Sets}}(A \times K, B)$$

for any simplicial set $K$, where $\Theta(u) = ev \circ (id \times u)$

We now fix a simplicial category $\mathcal{C}$, it will be convenient to identify $\mathrm{Hom}_{\mathcal{C}}(A, B)$ with $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, B)_0$ and drop the $\sim$ notation. The simplicial structure on a category allows us to define a notion of simplicial homotopy on a simplicial category analogous to the definition for topological spaces.

**Definition 7.6.** A generalized unit interval is a simplicial set $J$ which is a union of copies of $\Delta(1)$'s joined end on end. More concretely it has non-degenerate 1 simplices $a_1, ..., a_n$ and 0 simplices $v_0, ..., v_n$ with $d_0(a_i) = v_{i-1}$ and $d_1(a_i) = v_i$ or $d_0(a_i) = v_i$ and $d_1(a_i) = v_{i-1}$. We let 0 and 1 denote the endpoints $v_0$ and $v_n$ of $J$.

**Definition 7.7.** Let $K$ be a simplicial set. The two elements $x, y$ of $K_0$ are strictly homotopic if there exists an element $h \in K_1$ such that $d_0 h = x$ and $d_1 h = y$ and homotopic if there exists a map $h : J \to K_1$ such that $h(0) = x$ and $h(1) = y$.

It is clear the the condition of being homotopic to is an equivalence relation on $K_0$ and we write $\pi_0(K)$ for the set of equivalence classes.

Applying the above definition to the simplicial function complex in an arbitrary simplicial category we obtain the following.

**Definition 7.8.** Let $A, B \in ob\mathcal{C}$ and $f, g$ two morphisms from $A$ to $B$. We say $f$ and $g$ are strictly homotopic (resp. homotopic) if they are so considered as elements of $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)$. A strict homotopy (resp. homotopy) between $f$ and $g$ will then be an element $h \in \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)_1$ (resp. $h : J \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)$ with $d_0 h = f, d_1 h = g$ (resp. $h(0) = f, h(1) = g$).

In order to relate this notion of homotopy, which we will henceforth refer to as simplicial homotopy, to that given in the context of model categories, we need to have simplicial versions of cylinder/ path objects

**Definition 7.9.** Let $A \in ob\mathcal{C}$, and $K \in ob\mathrm{s}\mathbf{Sets}$. $A \otimes K$ will denote an object of $\mathcal{C}$ together with a morphism of simplicial sets

$$\alpha : K \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, A \otimes K) \tag{7.1}$$

which satisfy the property that for any other object $B$, we have an isomorphism

$$\varphi : \underline{\mathrm{Hom}}_{\mathcal{C}}(A \otimes K, B) \cong \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(K, \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)) \tag{7.2}$$

Where $\Theta(\varphi) : K \times \underline{\mathrm{Hom}}_{\mathcal{C}}(A \otimes K, B) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)$ is the map given by

$$K \times \underline{\mathrm{Hom}}_{\mathcal{C}}(A \otimes K, B) \xrightarrow{\alpha \times id} \underline{\mathrm{Hom}}_{\mathcal{C}}(A, A \otimes K) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(A \otimes K, B) \xrightarrow{\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)$$

Dually $B^K$ will denote an object of $\mathcal{C}$ together with a morphism $B^K \to B$ satisfying the property for any other object $A$ we have an isomorphism

$$\psi : \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B^K) \cong \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(K, \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B))$$

*Remark* 7.10. Any object $A$ determines a simplicial functor $h_A := \underline{\mathrm{Hom}}_{\mathcal{C}}(-, A)$ from $\mathcal{C}^{op}$ to s**Sets**. A functor $F : \mathcal{C}^{op} \to \mathrm{s}\mathbf{Sets}$ is representable if it is naturally isomorphic to some $h^A$. There is an analogue of Yoneda's lemma for this type of representability, and in this language, we have $A \otimes K$ represents the functor $B \mapsto \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(K, \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B))$

To see why these should be considered cylinder/ path objects, consider the category s**Sets**, where the product simplicial sets $A \times K$ gives such an object $A \otimes K$. When $K$ is $\Delta(1)$, the geometric realization of $|X \times \Delta|$ is homeomorphic to $A \times [0,1]$. The reader is encouraged to keep these geometric interpretations in the back of their mind for intuition in later proofs.

These path and cylinder objects satisfy the following properties.

**Proposition 7.11.** *For $A \in ob\mathcal{C}$ and $L, K \in ob\mathrm{s}\mathbf{Sets}$ we have canonical isomorphisms:*

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L \qquad A^{K \times L} \cong (A^K)^L$$

*Proof.* Let $B \in ob\mathcal{C}$, then we have bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(A \otimes (K \times L), B) \cong & \quad \mathrm{Hom}_{\mathrm{s}\mathbf{Sets}}(K \times L, \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)) \\ \cong & \mathrm{Hom}_{\mathrm{s}\mathbf{Sets}}(L, \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(K, \underline{\mathrm{Hom}}_{\mathcal{C}}(A,B))) \\ \cong & \quad \mathrm{Hom}_{\mathrm{s}\mathbf{Sets}}(L, \underline{\mathrm{Hom}}_{\mathrm{s}\mathbf{Sets}}(A \otimes K, B)) \\ \cong & \quad \mathrm{Hom}_{\mathcal{C}}((A \otimes K) \otimes L, B) \end{aligned}$$

∎

Suppose that cylinder and path objets exists in $\mathcal{C}$, then for $J$ a generalized unit interval, the images of 0 and 1 under the map $\alpha : J \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, A \otimes J)$ correspond to morphisms $j_0, j_1 : A \to A \otimes K$. It is then easy to see that a homotopy from $f$ to $g$ is the same as a map $h : A \otimes J \to B$ such that $hj_0 = f$ and $hj_1 = g$. Similarly a homotopy can be identified with a map $k : A \to B^J$ such $i_0 k = f$ and $j_1 k = g$ where $i_0$ and $i_1$ are the maps $B^J \to B$ determined by $0, 1 \in J_0$.

We now come to the definition of a closed simplicial model. As mentioned before, this is a category with both a closed model structure and simplicial structure which are compatible, and so allows us to relate the two notions of homotopy. We first define the model structure on s**Sets** as we will need this to define a closed simplicial model category. The definitions make s**Sets** into a closed simplicial model category, however the proof of this is rather long so we instead refer to [6] for the proof.

Let $\partial\Delta(n)$ denote the simplicial subset of $\Delta(n)$ generated by the faces $d_i : [n-1] \to [n]$ of $\Delta(n)$, and $V(n,k)$,the $k$ horn of $\Delta(n)$, the simplicial subset of $\Delta(n)$ generated by the face $d_i : [n-1] \to [n]$ for $i \neq k$. Geometrically, $|\partial\Delta(n)|$ is the boundary of the standard $n$ simplex and $|V(n,k)|$ is the boundary minus the $k^{kWh}$ face, i.e. the closed star of the $kth$ vertex, so that the notation is consistent with Chapter 4.

Let $f$ be a morphism in s**Sets**; we will need the following two lemmas:

**Lemma 7.12.** The following are equivalent:

i) $f$ has the right lifting property with respect to $\partial\Delta(n) \hookrightarrow \Delta(n)$.

ii) $f$ has the right lifting property with respect to any injective map.

*Proof.* [3] Prop 3.22 □

**Lemma 7.13.** The following are equivalent:

i) $f$ has the right lifting property with respect to $V(n,k) \hookrightarrow \Delta(n)$.

ii)$f$ has the right lifting property with respect to $\partial\Delta(n) \times \Delta(1) \cup \Delta(n) \times 0 \hookrightarrow \Delta(n) \times \Delta(1)$

ii) $f$ has the right lifting property with respect $K^{\times}\Delta(1) \cup L \times 0 \hookrightarrow L \times \Delta(1)$ where $K \to L$ is any injective map.

*Proof.* [6] □

**Definition 7.14.** A map $f$ in s**Sets** is a trivial fibration (resp. fibration) if it satisfies the equivalent conditions of Lemma 7.12 (resp. Lemma 7.13). Then a cofibration (resp. trial cofibration) in s**Sets** is a map which satisfies the left lifting property with respect to the trivial fibrations (resp. fibrations), in particular a map in s**Sets** is a cofibration if and only if it is injective.

A simplicial set will be called finite if it has only finitely many non-degenerate simplices. Note that a finite simplicial set is always a simplicial object over the category of finite sets, however the converse is not true.

The definition of a closed simplicial model category can now be stated.

**Definition 7.15.** A closed simplicial model category is a closed model category which is also a closed simplicial model category which and who satisfies the following conditions:

SM1) For all $A \in ob\mathcal{C}$ and $K$ a finite simplicial set, the objects $A \otimes K$ and $A^K$ exist.

SM2) If $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration. Then the map

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(B,X) \xrightarrow{(i^*,p_*)} \underline{\mathrm{Hom}}_{\mathcal{C}}(A,X) \times_{\underline{\mathrm{Hom}}_{\mathcal{C}}(A,Y)} \underline{\mathrm{Hom}}_{\mathcal{C}}(B,Y)$$

is a fibration of simplicial sets which is trivial if either $i$ or $p$ is.

It will be convenient to denote the target of the above map by $\underline{\mathrm{Hom}}_{\mathcal{C}}(i,p)$
A more useful version of the axiom SM2) is stated in the next proposition

**Proposition 7.16.** *Suppose $\mathcal{C}$ is a category with 4 classes of morphisms, fibrations, trivial fibrations, cofibrations and trivial cofibrations such that the first and fourth, and second and third classes determine each other via the lifting property as in M6a) (In particular these conditions are satisfied when $\mathcal{C}$ is a closed simplicial model category. Then SM2 is equivalent separately to each of the following two condition:*

*SM2a) If $X \to Y$ is a fibration (resp. trivial fibration), the map $X^{\Delta(n)} \to X^{\Delta.(n)} \times_{Y^{\partial\Delta(n)}} Y^{\partial\Delta(n)}$ is a fibration (resp. trivial fibration) and the map $X^{\Delta(1)} \to X^e \times_{Y^e} Y^{\Delta(1)}$ is a trivial fibration for $e = 0$ or $1$.*

*SM2b) The dual of the SM1a): If $A \to B$ is a cofibration (resp. trivial cofibration), the map $A \otimes \Delta(n) \vee_{A\otimes\partial\Delta(n)} B \otimes \partial\Delta(n) \to B \otimes \Delta(n)$ is a cofibration (resp. trivial cofibration) and the map $A \otimes \Delta(1) \vee_{A\otimes e} B \times e \to B \otimes \Delta(1)$ is a trivial cofibration for $e = 0$ or $1$.*

*Proof.* We prove only the equivalence of SM2) and SM2a) as the other is dual. For $L \to K$ a map of finite simplicial sets, it follows from the definition of path objects that to give a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X^K \arrow[d] \\ B \arrow[r] & X^K \times_{X^L} Y^L \end{tikzcd}$$

is equivalent to giving a square:

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{C}}(B,X) \arrow[d] \\ K \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{C}}(A,X) \times_{\underline{\mathrm{Hom}}_{\mathcal{C}}(A,Y)} \underline{\mathrm{Hom}}_{\mathcal{C}}(B,Y) \end{tikzcd}$$

it can then be shown that the left/ right lifting property for both diagrams are also equivalent.

When $X \to K$ is a fibration, then $X^K \to X^K \times_{X^L} Y^L$ is a fibration if and only if for $A \to B$ a trivial cofibration, there is a lift $B \to X^K$. This is equivalent to the existence of a lifting $K \to \underline{\mathrm{Hom}}_{\mathcal{C}}(B,X)$. From this one sees the equivalence of SM2) and SM2a). □

The above proposition immediately implies the following corollary, which gives a simplicial analogue of the homotopy lifting theorem which was essential in the proof of Theorem 3.9. From now on we fix a closed simplicial model category $\mathcal{C}$. We would like to relate the two notions of homotopy that we now have, we write $f \sim^{s.s} g$ if $f$ and $g$ are strictly homotopic and $f \sim^{s} g$ if $f$ and $g$ are homotopic, where homotopic is meant in the simplicial sense.

**Corollary 7.17.** *i) Suppose $i : A \to B$ is a cofibration and $p : X \to Y$ a fibration. Let $h : B \otimes J \to Y$ be a homotopy and $k : A \otimes J \to X$ be a lift of this homotopy in the sense that the following diagram commutes.*

$$\begin{tikzcd} A \otimes J \arrow[r, "k"] \arrow[d, "i \otimes id_J"'] & X \arrow[d, "p"'] \\ B \otimes J \arrow[r, "h"] & Y \end{tikzcd}$$

*Suppose $\theta : B \to X$ is map such that $p\theta = hj_0$, then there exists a homotopy $H : B \otimes J \to X$ such that $pH = h$, $H \circ \otimes id_J = k$ and $Hj_0 = \theta$.*

*ii) If in addition $i$ or $p$ is trivial and $\theta_0, \theta_1 : B \to X$ are maps such that $p\theta_0 = hj_0$ and $p\theta_1 = hj_1$, then one can find $H : B \otimes J \to X$ as above which satisfies $Hj_0 = \theta_0$ and $Hj_1 = \theta_1$*

*Proof.* i) By induction we may assume $J = \Delta(1)$. Then this follows immediately from SM2b); indeed the map $A \otimes \Delta(1) \vee_{A \otimes e} B \times e \to B \otimes \Delta(1)$ is a trivial fibration, hence we may pick a lifting in the following diagram:

$$\begin{tikzcd} A \otimes \Delta(1) \vee_{A \otimes e} B \times e \arrow[r, "k+\theta"] \arrow[d] & X \arrow[d, "p"'] \\ B \otimes \Delta(1) \arrow[r, "h"] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

which gives our desired homotopy.

ii) The proof is similar to the above, noting that the fact that $i$ or $p$ being trivial implies we can lift in the following diagram:

$$\begin{tikzcd} A \otimes \Delta(1) \vee_{A \otimes \partial\Delta(1)} B \times \partial\Delta(1) \arrow[r, "k+(\theta_0,\theta_1)"] \arrow[d] & X \arrow[d, "p"'] \\ B \otimes \Delta(1) \arrow[r, "h"] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

□

The main result of this section is then the following

**Proposition 7.18.** *i) The conclusion of Theorem 3.7 holds with $\pi\mathcal{C}_{cf}$ replaced with $\pi\mathcal{C}_{cf}$.*

*ii) Let $\mathcal{C}$ be a closed simplicial model category and $f, g : A \to B$ two morphisms in $\mathcal{C}$. The $f \sim^s g$ implies $f \sim^l g$ and $f \sim^r g$. When $A$ is cofibrant and $B$ is fibrant all three relations coincide.*

*Proof.* i) Suppose $A$ is cofibrant, then $0 \hookrightarrow J$ has the left lifting property with respect to fibrations, indeed $0 \hookrightarrow \Delta(1)$ is the inclusion $V(1,0) \hookrightarrow \Delta(1)$ and the general case follows by induction on the length of $J$. Then it follows by the proof of SM2b) that $j_0 : A \to A \otimes J$ has the left lifting property with respect to fibrations hence is a trivial cofibration. Here we use the fact that for the initial object 0 we have $0 \otimes K \cong 0$ for any simplicial set $K$, which follows from equation (7.2), so that the applying SM7b) with $0 \hookrightarrow B$ we obtain the claim. Let $\sigma : A \times J \to A$ denote the constant homotopy, then it follows from M6) that $\sigma$ is a weak equivalence. And since $\partial\Delta(1) \hookrightarrow \Delta(1)$ has the left lifting property with respect to trivial cofibrations, it follows by SM2b) again that $j_0 + j_1; A \vee A \to A \otimes J$ is a cofibration so that $A \times J$ is a cylinder object for $A$.

Examining the Proposition 3.6 part ii), we see that the same proof applies (cf. remark 3.7), so that any map which inverts weak equivalences identifies (simplicially) homotopic maps. Thus $\gamma_c$ induces a functor $\pi_0\mathcal{C}_c \to \mathrm{Ho}\mathcal{C}_c$, and similarly for $\gamma$ and $\gamma_f$.

To construct the quasi-inverse functor, we use the cofibrant and fibrant replacement functors $Q$ and $R$. Again since Proposition 3.5 holds for the simplicial case (cf. remark 3.7) we obtain a well defined functor $\overline{RQ} : \mathcal{C} \to \pi_0\mathcal{C}_{cf}$. The same proof then goes through as before.

ii) [6] Chapter II, Section 2 Proposition 5. □

*Remark* 7.19. In fact we have that Theorem 3.9 holds with $\pi\mathcal{C}_c$, $\pi\mathcal{C}_f$ and $\pi\mathcal{C}_{cf}$ replaced by $\pi_0\mathcal{C}_c$, $\pi_0\mathcal{C}_f$ and $\pi_0\mathcal{C}_{cf}$, the proof is adapted in the same way as above using the previous Corollary.

## 8. More Examples

We now state some examples of closed simplicial categories. The proofs of the axioms for these categories are generally quite difficult and can be found in [6] Chapter 2, Sections 3 and 4. Alternatively there is a more modern treatment of this material in [3]

**Example 8.1.** We have already seen in Section 4 that the category of topological spaces **Top** is a model category and in Example 7.4 we also defined a simplicial structure on it. It turns out that these endow **Top** with he structure of a closed simplicial model category.

**Example 8.2.** Let $\mathcal{A}$ be a category which is closed under finite limits and all colimits, and $s\mathcal{A}$ be the category of simplicial objets over $\mathcal{A}$, i.e. functors $\Delta^{op} \to \mathcal{A}$ and natural transformations between them. For $A$ an object of $s\mathcal{A}$ we let $A_n$ denote the image of $[n]$. For $A, B \in ob\mathcal{C}$, and $K$ a simplicial set, we define a map $f : A \times \Delta(n) \to B$ to be a collection of maps $f(\sigma) : A_n \to B_n$ for each $n$-simplex $\sigma$ such that if $\varphi : [k] \to [n]$ is a map in $\Delta$, the following diagram commutes:

$$\begin{tikzcd} A_n \arrow[r, "f(\sigma)"] \arrow[d, "A(\varphi)"'] & B_n \arrow[d, "B(\varphi)"] \\ A_k \arrow[r, "K(\varphi)\sigma"] & B_k \end{tikzcd}$$

Let $\mathrm{Maps}(A \times K, B)$ be the set of maps $f : A \times K \to B$ (it is an easy exercise to show that for the case of simplicial sets we have $\mathrm{Maps}(A \times K, B) = \underline{\mathrm{Hom}}_{s\mathbf{Sets}}(A \times K, B))$. This allows us to define a simplicial structure on $s\mathcal{A}$ by setting

$$\underline{\mathrm{Hom}}_{s\mathcal{A}}(A, B)_n = \mathrm{Maps}(A \times \Delta(n), B)$$

where boundary and face maps are induced by those on $\Delta(n)$.

There is a composition map

$$\mathrm{Maps}(A \times K, B) \times \mathrm{Maps}(B \times K, C) \to \mathrm{Maps}(A \times K, C)$$

given by $(g \circ f)(\sigma) \mapsto g(\sigma)f(\sigma)$ and this induces a composition:

$$\underline{\mathrm{Hom}}_{s\mathcal{A}}(A, B) \times \underline{\mathrm{Hom}}_{s\mathcal{A}}(B, C) \to \underline{\mathrm{Hom}}_{s\mathcal{A}}(A, C)$$

To define the model category structure, recall that a morphism $f : A \to B$ in $\mathcal{A}$ is called an effective epimorphism if for any other object $C$, the map of sets.

$$\mathrm{Hom}(B, C) \xrightarrow{f^*} \mathrm{Hom}(A, C) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} \mathrm{Hom}(A \times_B A, C)$$

is exact.

An object $P$ of $\mathcal{A}$ is called projective if the map $\mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$ is surjective for all effective epimorphisms $A \to B$. We say the category $\mathcal{A}$ has enough projectives if for all objects $A$ there exists an effective epimorphism $P \to A$.

We call an object $A$ small if $\mathrm{Hom}(A, .)$ commutes with all colimits and a class of objects $I$ are generators of $\mathcal{A}$ if for all objects $A$, there exists an effective epimorphism $Q \to A$ where $Q$ is a direct sum of elements of $I$.

Now let $A$ and $B$ be objects of $s\mathcal{A}$, define a map $f : A \to B$ to be an fibration (resp. a weak equivalence) if $\underline{\mathrm{Hom}}_{s\mathcal{A}}(P, f)$ is a fibration (resp. a weak equivalence) in s**Sets** for all projectives $P \in obs\mathcal{A}$. Cofibrations are then defined to be maps which have the left lifting property with respect to trivial fibrations.

With these definitions the category $s\mathcal{A}$ is a closed simplicial model category if one of the following conditions is satisfied:

- All objects of $s\mathcal{A}$ are fibrant
- A has a set of small projective generators.

The following is a special case of the above and is an extremely important example.

Fix a commutative ring $R$ and let $\mathbf{Mod}_R$ be the category of modules over $R$, then $s\mathbf{Mod}_R$ is that category of simplcial modules over $R$. One can check that $\mathbf{Mod}_R$ satisfies all the conditions in the above example and that in $s\mathbf{Mod}_R$ all objects are fibrant, so that the above construction endows $\mathbf{Mod}_R$ with the structure of a closed simplicial model category. In this particular case there are more tractable interpretations of the model structure which we will now describe.

Given a simplicial module $M$ (as before we let $M_n$ denote $M([n])$) we define the normalized chain complex $\mathbf{N}(M)$ of $M$ to be the complex of $R$ modules with

$$\mathbf{N}(M)_n = \bigcap_{i=1}^{n} \ker d_i$$

with differential given by $d_0$. The simplicial identities show that $\mathbf{N}(M)$ is indeed a chain complex and indeed $\mathbf{N}$ can be extended to a functor $\mathbf{Mod}_R \to Ch_{\geq 0}(R)$. Define the homotopy groups to be the $R$ modules

$$\pi_n(M) = H_n(\mathbf{N}(M))$$

If $N$ is an $R$-module, define $s(N)$ to be the simplicial $R$ module which is $N$ in every degree and all face and degeneracy maps are the identity. Then

$$\pi_n(s(N)) = \begin{cases} N & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

One checks that $s$ extends to a functor $\mathbf{Mod}_R \to s\mathbf{Mod}_R$ which is right adjoint to $\pi_0$.

**Theorem 8.3.** *(Dold-Kan correspondence) The functor $N : \mathbf{Mod}_R \to Ch_{\geq}(R)$ is an equivalence of model categories.*

Here an equivalence of model categories means an equivalence of categories which preserves the structures associated to a model category, eg. fibrations, cofibrations weak equivalences. Thus comparing with the model category structure on $Ch_{\geq 0}(R)$ we obtain the following:

**Proposition 8.4.** *i) A morphism $f : M \to N$ of simplicial modules is a weak equivalence if and only if the induced maps*

$$\pi_*(f) : \pi_*(M) \to \pi_*(N)$$

*is a weak equivalence.*

*ii) A morphism $f : M \to N$ if simplicial modules is a fibration if and only if*

$$\mathbf{N}(f) : \mathbf{N}(M) \to \mathbf{N}(M)$$

*is surjective.*

*Remark* 8.5. When $R = \mathbb{Z}$, $\mathbf{Mod}_Z$ is just the category of abelian groups, hence the above constructions all go through for the case of simplicial abelian groups.

## 9. Homology and Cohomology

In the context of model categories, there are good notions of homology and cohomology. If one thinks of homotopoical algebra as a non-abelian homological algebra, then a natural question would to ask what the abelianization functor looks like. It turns out that (assuming the existence of a nice abelianization functor) taking derived functors will give us a good notion of homology and which agrees with that defined by suitable a Grothendieck topology.

Although cohomology is defined for general model categories, if we are in the case of a closed simplicial model category, the simplicial structure can be a powerful tool for computing cohomology. We will see some examples of this in Chapter 9, the point is that one can use simplicial resolutions to compute cohomology (in the language of Chapter 3, a simplicial resolution is just a cofibrant replacement). In the special case of simplicial modules over a ring, the Dold-Kan correspondence tells us that this is exactly the same as taking a projective resolution, giving us a beautiful generalization the case of abelian categories.

In the next Chpater we will see how one can interpret taking Kahler differentials of a morphism of rings as an abelianization functor on a suitable simplicial model, so the rest of this section will prove the correct context for talking about the cotangent complex.

Let $\mathcal{C}$ be a fixed model category and $\mathcal{C}_{ab}$ the subcategory of abelian objects of $\mathcal{C}$. Recall that an abelian group object in the category $\mathcal{C}$ is an object $A$ such that the $\mathrm{Hom}(C, A)$ is an abelian group functorially in the object $C$. By Yoneda's lemma this is equivalent to the existence of maps

$$m : A \times A \to A, i : A \to A \text{ and } e : * \to A$$

where $*$ is the terminal object in $A$, such that these satisfy appropriate analogues of the axioms of an abelian group.

We will assume that there exists an abelianization functor $ab : \mathcal{C} \to \mathcal{C}_{ab}$ which is left adjoint to the inclusion $\mathcal{C}_{ab}$. Moreover we assume that $\mathcal{C}_{ab}$ is a model category in such a way that these functors form a Quillen adjunction, so that the total left and right derived $\underline{Lab}$ and $\underline{Ri}$ are canonically adjoint.

**Example 9.1.** For s**Sets** the category of simplicial sets, s$\mathbf{Sets}_{ab}$ is the subcategory of simplicial abelian groups. $i$ : s**Sets** $\to$ s$\mathbf{Sets}_{ab}$ is the forgetful functor and $ab$ : s**Sets** $\to$ s$\mathbf{Sets}_{ab}$ is induced by the the functor taking a set $X$ to the free abelian group with generator set $X$.

We also assume that in the homotopy category the adjunction morhpism

$$\Theta : A \to \Omega\Sigma A$$

is an isomorphism and that if

$$A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{\delta} \Sigma A$$

is a cofibration sequence, then

$$\Omega\Sigma A \xrightarrow{i\Theta^{-1}} A \xrightarrow{j} C \xrightarrow{\delta} \Sigma A$$

is a fibration sequence.

These conditions hold in the situation of example 8.2. More importantly for us they hold for the category of simplicial modules over a simplicial ring.

**Definition 9.2.** The Quillen homology of an object $X$ is defined to be the object $\underline{Lab}X$ in the category $\mathrm{Ho}\mathcal{C}_{ab}$.

The $i^{th}$ cohomology group of $X$ with coefficients in an object $A \in ob\mathrm{Ho}\mathcal{C}_{ab}$ is given by

$$H^i(X, A) = [\underline{Lab}X, \Omega^{i+N}\Sigma^N A]$$

where $N \geq 0$ is sufficiently large that $i + N \geq 0$.

Let's see what this definition gives us in a couple of examples

**Example 9.3.** If $X$ is a simplicial set, then $X$ is cofibrant and we have that $\underline{Lab}X = X_{ab}$. Let $A$ be the simplicial abelian group $s(\mathbb{Z})$, then by the Dold-Kan correspondence, we have that

$$H^i(X, A) = [X_{ab}, \Omega^{i+N}\Sigma^N s(\mathbb{Z})] = \pi(\mathbf{N}(X_{ab}), \mathbf{N}(\Omega^{i+N}\Sigma^N s(\mathbb{Z})))$$

Here $\mathbf{N}(\Omega^{i+N}\Sigma^N s(\mathbb{Z}))$ is the complex which $\mathbb{Z}$ in degree $i$ and zero everywhere else. If $Y$ is a topological space, $\mathbf{N}(SingY)$ is the singular complex associated to $Y$, hence $H^i(SingY, s(\mathbb{Z})) = H^i(Y, \mathbb{Z})$ the usual cohomology of $Y$.

This examples reinforces the notion that homotopy theory gives a non-abelian generalization of homology. Indeed one would would the abelianization of any such theory would recover ordinary homology.

**Definition 9.4.** Let $\mathcal{C}$ be a closed simplicial model category. By a simplicial resolution $P$ of an object $A \in ob\mathcal{C}$ we mean a cofibrant replacement for $A$, in other words a cofibrant object together with a trivial fibration $P \to A$.

Suppose $\mathcal{A}$ satisfies the conditions in example 8.3 and let $X \in ob\mathcal{A}$. Consider $X$ as a constant simplicial object in $s\mathcal{A}$, and let $P \to X$ be simplicial resolution of $X$. Then for an abelian group object $A \in ob\mathcal{A}_{ab}$, define $R^i h_A(P)$ to be $\pi_i(\underline{\mathrm{Hom}}_{s\mathcal{A}}(P_{ab}, A))$, the $i^t h$ homotopy group of the simplicial function complex of maps from $P_{ab}$ to $A$.

The following proposition makes it clear how simplicial resolutions generalize projective resolutions and can be use to compute homology.

**Proposition 9.5.** *Suppose $\mathcal{A}$ satisfies the condition in example 8.3 with $X \in ob\mathcal{A}$ and $A \in ob\mathcal{A}_{ab}$, so that the simplicial category $s(\mathcal{A}/X)$ satisfies the conditions at the start of this section. Then if we consider $X$ and $A$ as constant simplicial objects in $s(\mathcal{A}/X)$, there exists an isomorphism*

$$H^i(X, A) \cong R^i h_A(X)$$

*Proof.* Let $\mathcal{B}$ be the category $(\mathcal{A}/X)_{ab}$, then $\mathcal{B}$ has enough projective objects given by $ab(P)$ where $P$ is a projective object in $\mathcal{A}/X$. Then $s\mathcal{B}$ and $Ch_{\geq 0}(\mathcal{B})$ exist, and $\mathbf{N} : s\mathcal{B}_{ab} \to Ch_{\geq 0}(B)$ is an equivalence of model categories by the Dold-Kan theorem. Let $P$ be a simplicial resolution of $X$, and we will consider $A$ as a constant simplicial object then we have the following bijections:

$$\begin{aligned} H^i(X, A) = &\quad \pi(\underline{Lab}, \Omega^i A) \\ &= \pi(P, \Omega^{i+N} i \Sigma^N A) \\ &= \pi_0(P, \Omega^{i+N} \Sigma^N A) \\ &= \pi_0(\mathbf{N}(P), \Omega^i \mathbf{N}(A)) \end{aligned}$$

Now $\Omega^i(\mathbf{N}(A))$ is just the chain complex with $A$ in degree $i$ and zero elsewhere, hence the homotopy classes of maps from $\mathbf{N}(P)$ to $\mathbf{N}(A)$ is just the $i^{th}$ homology of the complex $\mathrm{Hom}(\mathbf{N}(P), A)$, which is just $R^i h_A X$.

□

*Remark* 9.6. Under some further assumptions, the cohomology defined here can be shown to be same as that defined by the sheaf cohomology with respect to a certain Grothendieck topology on the category or by a sets of cotriples, see [6] Chapter 2 section 5 for more details.

If $\mathcal{A}$ itself was an abelian category, for example $\mathbf{Mod}_R$, then the abelianzation functor is just the identity and the proof of the above theorem shows that the $i^{th}$ quillen cohomology is just the $i^{th}$ homology of the complex $\mathrm{Hom}_{\mathcal{A}}(P_\bullet, A)$ where $P_\bullet$ is a projective resolution of $X$ and hence is just the usual $Ext^i(X, A)$.

## 10. The Cotangent Complex

We have now arrived at our final port of call, the definition of the cotangent complex. We will see how the techniques developed in the paper so far allow a very clean definition of this object as the left derived functor taking Kahler differentials. We will explain how this is equivalent to taking the derived functor of abelianization on a suitable category, placing the construction in the general context of Chapter 9, and use the theory developed derive some basic properties. The next two Chpaters follow the material covered in [4] and [1] and we refer the reader to these articles for more detailed explanations. We begin by recalling some results about Kahler differentials; we will not go into any proofs as these can be found in most textbooks on commutative algebra.

Let $R$ be a ring and $S$ and $R$ an algebra. For an $S$-module $M$, and $R$-linear derivation with values in $M$ is a homomorphism of $R$ modules $\delta : S \to M$ satisfying the Leibniz rule:

$$\delta(xy) = x\delta(y) + y\delta(x)$$

The set of such derivations is denoted $\mathrm{Der}_R(S, M)$ and it has the natural structure of an $S$ module. The functor $M \mapsto Der_R(S, M)$ is representable and the representing object is denoted by $\Omega_{S/R}$, the module of Kahler differentials. It is equipped with a universal derivations $\delta_{S/R} : S \to \Omega_{S/R}$ and the bijection

$$\mathrm{Hom}(\Omega_{S/R}, M) \to \mathrm{Der}_R(S, M)$$

is given by $f \mapsto f\delta_{S/R}$.

The module $\Omega_{S/R}$ can be constructed as a quotient of the free module on generators $ds, s \in S$ by a set of universal relations. Alternatively letting $I$ denote the kernel of the map $S \otimes_R S \to S$ given by $s \otimes s' \to ss'$, one can show that $I/I^2$ together with the map $S \to I/I^2$ given by $s \mapsto S \otimes 1 - 1 \otimes s$ satisfies the universal property of $\Omega_{S/R}$.

**Example 10.1.** When $S$ is the polynomial ring $R[X]$ in a set of indeterminates $X = \{x_i : i \in I\}$, one can show

$$\Omega_{S/R} = \bigoplus_{i \in I} dx_i$$

the free module on generators $dx_i$ and $\delta_{S/R}$ is given by $\delta_{S/R}(x_i) = dx_i$. Indeed it is easy to check that this module has the desired universal property.

Let $f : S \to T$ be a homormorphism of $R$ algebras, then composing the universal derivation $\delta_{T/R}$ with $S \to T$, we obtain an $R$-linear derivation of $S$ with coefficients in $\Omega_{T/R}$ and hence a map of $S$ modules $\Omega_{f/R} : \Omega_{S/R} \to \Omega_{T/R}$, in fact it is the unique map which makes the following diagram commute:

$$\begin{tikzcd} \Omega_{S/R} \arrow[r, "\Omega_{f/R}"] & \Omega_{T/R} \\ S \arrow[u, "\delta_{S/R}"] \arrow[r, "f"] & T \arrow[u, "\delta_{T/S}"'] \end{tikzcd} \tag{10.1}$$

Suppose we have homomorphisms of commutative rings

$$R \xrightarrow{f} S \xrightarrow{g} T$$

Then there exists an exact sequence

$$\Omega_{S/R} \otimes_S T \xrightarrow{\alpha} \Omega_{T/R} \xrightarrow{\beta} \Omega_{T/S} \longrightarrow 0 \tag{10.2}$$

The map $\alpha$ is given by the map in (10.1) upon extension of scalars. Similarly restricting the universal derivation $\delta_{T/S}$ to $R$, we obtain an $R$ linear derivation with coefficients in $\Omega_{T/S}$ hence a unique map $\beta : \Omega_{T/R} \to \Omega_{T/S}$.

The exact sequence (10.2) is really the origin of the cotangent complex, it hints that $\Omega_{-/R}$ is behaving somewhat like a right exact functor on the category on $R$ algebras. Obviously this notion does not make any sense since the category of $R$ algebras is not abelian, however it is still a natural question to ask whether one can extend this sequence to the left in the same way that left derived functors do in the abelian case. Moreover when $S \to T$ is a smooth map it can be shown that

$\alpha$ is injective, so whatever this extension, the first term should vanish when $S \to T$ is smooth.

The sequence (10.2) can be extended naturally in the special case when $g : S \to T$ is surjective. In this case $\Omega_{T/S}$ is trivial, and if we let $J$ be the kernel of $g$, (10.2) can be extended to

$$J/J^2 \xrightarrow{\zeta} \Omega_{S/R} \otimes_S S/J \xrightarrow{\alpha} \Omega_{T/R} \longrightarrow 0 \tag{10.3}$$

The map $\zeta$ is defined as follows. Restricting the universal derivation $\delta_{S/R}$ to $J$ gives a map $J \to \Omega_{S/R} \otimes_R R/J$. Using the Leibniz rule, one shows that this map is trivial on $J^2$ and the induced map $J/J^2 \to \Omega_{S/R} \otimes_R R/J$ is precisely the map $\zeta$. Again we would hope any extension of this sequence would give $J/J^2$ for this special case.

So what we are looking for then is a complex $\mathcal{L}_\bullet$ of $S$ modules for any map $R \to S$ of commutative rings, well defined in the homotopy category of complexes over $S$, and such that $H^0(\mathcal{L}_\bullet) \cong \Omega_{S/R}$ and $H^1(\mathcal{L}) = 0$ (resp. $J/J^2$) when $R \to S$ is smooth (resp. when $S = R/J$).

We have seen through our example of $Ch_{\geq 0}(R)$ that model categories give a good generalization of derived functors, so let's what the constructions give in this case.

We are trying to derive the the functor of Kahler differentials on the category of $R$ algebras which is a highly non-abelain category. To compute derived functors in the abelian one uses projective resolutions, we have seen that simplicial resolutions give a good generalization of this. Translating the constructions above into the simplicial setting will turn out to be very fruitful.

Let $s\mathbf{Alg}_R$ category of simplcial algebras over $R$. An object $A$ in $s\mathbf{Alg}_R$ is then a collection of $R$ algebras $A_n$, together with face maps which are morphisms of $R$ algebras. If $X$ is a simplicial $R$ algebra, by simplicial $X$-module $M$ we mean collection of $X_n$ modules $M_n$ together with face and degeneracy maps which respect the fact and degeneracy maps on $X$. We denote the category of simplicial $X$ modules my $s\mathbf{Mod}_X$ (note this is not the category of simplcial objects over some category, the $s$ just denotes the fact that the category is simplicial).

$s\mathbf{Alg}_R$ has the structure of a closed simplicial model category, and considering an $R$ algebra as an $R$ algebra via the structure map we obtain, as in the case of simplcial modules, a normalization functor $S \to \mathbf{N}(S)$ from $s\mathbf{Alg}_R$ to the category of chain complexes of $R$-modules and hence the homotopy groups $\pi_n$, which are no longer $R$ algebras, merely $R$-modules. We have the following characterization of weak equivalences and fibrations in $s\mathbf{Alg}_R$

**Proposition 10.2.** *A morphism $f : A \to B$ in $s\mathbf{Alg}_R$ is:*
*i) a weak equivalence if and if $\pi_* A \to \pi_* B$ is an isomorphism.*
*ii) a fibration if and only if the map $X \to \pi_0 X \times_{\pi_0 Y} Y$ is surjective.*

The cofibrations can then be characterized formally by the lifting property in the usual way, however we will need a more concrete characterization.

**Definition 10.3.** Let $A$ be a simplicial algebra over $R$. A free extension of $A$ is a simplicial algebra $B$ over $R$ satisfying the following conditions:
i) $B_n = A_n[x_n]$ where $X_n$ is a set of indeterminates.
ii) $s_j(X_n) \subset X_{n+1}$ for each degeneracy map $s_j$.

ii) The natural map $A \to B$ is a morphism of simplicial algebras.

We usually denote free extensions by $A[X]$ where $X = (X_n)_{n\geq 0}$ is the set of indeterminates.

It is easy to check that free extensions have the left lifting property with respect to the fibrations in $s\mathbf{Alg}_R$, the following also holds.

**Proposition 10.4.** *A morphism $f : A \to B$ in $s\mathbf{Alg}_R$ is a cofibration if and only if it is the retract of a free extension.*

Let us now consider the commutative rings $S$ and $R$ as constant objects in $s\mathbf{Alg}_R$, and let

$$R \xrightarrow{i} P \xrightarrow{p} S$$

be a factorization of the structure map into a cofibration followed by a trivial fibration, in the terminology of the previous chapter, $P$ is a simplicial resolution of $P$. In fact one can show that we can take $P$ to be a free resolution, in the sense that $R \to P$ is a free extensions, this can be proved by a version of the small object argument. Applying the functor $\Omega_{-/R}$ degree wise to $P$ we obtain a simplicial module $\Omega_{P/R}$ over $P$ which is $\Omega_{P_n,/R}$ in the $n^{th}$ degree and face and degeneracy maps are induced by those on $P$ by the diagram (10.1).

**Definition 10.5.** The cotangent complex $\mathcal{L}_{S/R}$ of $S/A$ is the chain complex of $S$ given by the normalization

$$\mathcal{L}_{S/R} = \mathbf{N}(\Omega_{P/R}) \otimes_P S)$$

where the tensor product is taken component wise over $P$.

Note that as in the case of Kahler differentials, the cotangent complex depends on the morphism of rings $R \to S$. The notations $\mathcal{L}_{S/R}$ however will not cause any confusion as we will never consider two different morphisms between rings.

*Remark* 10.6. Some authors define the cotangent complex as the simplicial module $\Omega_{P/R} \otimes_P S$, by the Dold-Kan correspondence these two points of view are equivalent.

The cotangent complex is not a functor from $R$ algebras to another category since the target category depends on the $R$ algebra $S$. Thus this definition does not make it clear that it is a left derived functor, we will see in the next paragraph how we can remedy this situation by restricting ourselves to augmented algebras.

The cotangent complex is only well defined in the homotopy category of chain complexes over $S$, so that its homology is well defined. This is easily seen by putting the construction in the context of section 9, which we will explain below. To prove any basic properties we will use free resolutions to compute $\mathcal{L}_{S/R}$ in which case one can give a direct proof of the well-definedness, we will omit this and instead refer the reader to [1]

Begin going to discuss properties of the cotangent complex, we show how one can think of $\Omega_{S/R}$ can be thought of as a left derived functor of abelianization, so that it can be thought of as the cohomology theory of some category. This cohomology theory is known as Andre-Quillen cohomology.

Remember before we were working in the category $\mathbf{Alg}_R$. The terminal object in this category is the $R$ algebra 0 and as such does not admit maps into any non zero object in $\mathbf{Alg}_R$ thus there are non non-trivial abelian group objects in this category and its cohomology theory is not particularly interesting. The way to remedy this is to work in the category of $S$-augmented $R$ algebras, denoted $\mathbf{Alg}_R^S$ whose objects consists of $R$ algebras $A$ together with an augmentation map $A \to S$.

An abelian group object in this category is then an $S$-augmented $R$ algebra $A$ together with maps

$$m : A \times_S A \to A \qquad e : S \to A \qquad i : A \to A$$

satisfying the axioms for an abelian group.

Suppose $M$ is an $S$-module, then consider the $R$-algebra $S \ltimes M$ whose additive group is $S \oplus M$ and whose multiplication is given by

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_2 m + a_1 m)$$

It is easy to check to if we take $e : S \to S \ltimes M$ to be the inclusion of the first factor, $m : (a, m_1, m_2) \mapsto (a, m_1 + m_2)$ and $i : (a, m_1) \mapsto (a, -m_2)$ then this turns $S \ltimes M$ into an abelian group object of $\mathbf{Alg}_R^S$.

Suppose $X$ is another object in $\mathbf{Alg}_R^S$, then what is the group $\mathrm{Hom}_{\mathbf{Alg}_R^S}(X, S \ltimes M)$? An element of this group can be written as $f \oplus d$, where $f$ is the augmentation map $B \to A$. The algebra structure on $S \ltimes M$ implies $d$ is a an $R$ linear derivation of $X$ with coefficients in $M$. Conversely an element $d \in \mathrm{Der}_R(X, M)$ determines an element of $\mathrm{Hom}_{\mathbf{Alg}_R^S}(X, S \ltimes M)$ by $f \oplus d$ and hence we obtain a bijection

$$\mathrm{Der}_R(X, M) \to \mathrm{Hom}_{\mathbf{Alg}_R^S}(X, S \ltimes M) \tag{10.4}$$

which can be seen to be compatible with the group structure. The following is then an easy exercise.

**Proposition 10.7.** *The functor $M \mapsto S \ltimes M$ is an equivalence of categories between the category of $S$ modules and the category of abelian group objects in $\mathbf{Alg}_R^S$ and it has a left adjoint given by $X \mapsto \Omega_{X/R} \otimes_X S$*

The next corollary is immediate

**Corollary 10.8.** *The abelianization functor in the category $\mathbf{Alg}_R^S$ is given by*

$$X \mapsto \Omega_{X/R} \otimes_X S$$

*under the above identification of categories*

It follows that one can think of the cotangent complex as the Quillen homology of the object $B$.

We finish this section by discussing the basic properties of the the cotangent complex, showing that it satisfies the properties expected of it. First note that by right exactness of (10.2) we have that $H^0(\mathcal{L}_{S/R}) = \Omega_{S/R}$. The following is proved in [1]

**Proposition 10.9.** *(Flat base change) Given a diagram of rings*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "g"'] & S \arrow[d] \\ A \arrow[r] & S \otimes_R A \end{tikzcd}$$

*where either $f$ or $g$ is flat, then there is an isomorphism $\mathcal{L}_{S \otimes_R A/S} \cong \mathcal{L}_{A/R}$*

We now show how it allows us to extend the Jacobi-Zariski exact sequence (9.2).

**Proposition 10.10.** *(Jacobi-Zariski exact sequence) Let $R \to S \to T$ be morphisms of rings, then the sequence*

$$\mathcal{L}_{S/R} \otimes_S T \to \mathcal{L}_{T/R} \to \mathcal{L}_{T/S}$$

*is a cofibration sequence in the category $Ch_{\geq 0}(T)$*

*Proof.* Let $A$ be a free resolution of $R \to S$ and $B$ a free resolution of $A \to T$, so that we have a diagram:

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, equal] & A \arrow[r, "i'"] \arrow[d, "p"'] & B \arrow[d, "p'"'] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

where $p$ and $p'$ are trivial fibrations, and $i$ and $i'$ are free extensions. Thus $B$ is also a free extension of $R$ and we obtain for each $n$ an exact sequence of $B_n$ modules

$$\Omega_{A_n/R} \otimes_{A_n} B_n \to \Omega_{B_n/R} \to \Omega_{B_n/A_n} \longrightarrow 0$$

In fact this sequence is short exact since $B_n$ is a polynomial algebra in $A_n$. Then tensuring by $T$ we obtain a short exact sequence

$$0 \longrightarrow \Omega_{A_n/R} \otimes_{A_n} T \to \Omega_{B_n/R} \otimes_{B_n} T \to \Omega_{B_n/A_n} \otimes_{B_n} T \longrightarrow 0 \tag{10.5}$$

where the left exactness follows from the fact $\Omega_{B_n/A_n}$ is free over $B_n$ (cf example 10.1).

We then have maps $S \otimes_S B \to T$ which can be shown to be trivial fibration and the map $S \to S \otimes_A B$ is cofibration, being the pushout of $i'$. In other words $S \otimes_A B$ is a simplicial resolution (in fact free) resolution of $T$ in the category of simplicial $S$ algebras. Observing that

$$\Omega_{B_n/A_n} \cong \Omega_{S \otimes_{A_n} B_n}$$

by flat base change and that the normalization of the right hand is just the cotangent complex $\mathcal{L}_{T/S}$, we have that upon taking the normalization of the sequence (9.3), we obtain an exact sequence of complexes,

$$0 \to \mathcal{L}_{S/R} \otimes_S T \to \mathcal{L}_{T/R} \to \mathcal{L}_{T/S} \to 0$$

with the third column consisting of free modules. By definition of the model category structure on $Ch_{\geq 0}(R)$ is precisely a cofibration sequence. □

Taking homology of the above sequence we obtain the desired extension of the exact sequence (10.2).

For a concrete computation of the cotangent complex using free resolutions, we refer to Proposition 5.11 in [1]. We will need this result in the next section so we state it below.

**Proposition 10.11.** *Let $S = R/(r)$ and $R \to S$ the canonical surjection. We have the following homotopy equivalence:*

$$\mathcal{L}_{S/R} \cong \Sigma S$$

*where we consider $S$ as the complex of $S$ modules which is $S$ in degree* $0$ *and* $0$ *everywhere else.*

## 11. Applications: Complete intersections and smooth homomorphisms

In this section we will use the cotangent complex to prove a characterization of complete intersection homomorphisms and smooth homomorphisms under some finiteness assumptions. Just as the $Tor$ functor can be used to detect flatness of homomorphism of rings, in order to study these finer invariants one must use the cotangent complex.

Let us begin with the definitions of complete intersections and smooth homomorphisms. For simplicity we assume throughout that all rings are Noetherian and all morphisms of rings are locally of finite type.

Let $R$ be a ring. A sequence of elements $r_1, ..., r_c$ said to be regular if $r_i$ is a non-zero divisor in $R/(r_1, ..., r_{i-1})$ for $i = 1, ..., c$

**Definition 11.1.** A surjective morphism of rings $\varphi : R \to S$ is a complete intersection if the kernel of $\varphi$ is generated by a regular sequence.

A surjective $\varphi$ is a locally complete intersection if for all prime ideals $\mathfrak{q}$ of $S$, the morphism $R_{\varphi^{-1}\mathfrak{q}} \to S_{\mathfrak{q}}$ is a complete intersection.

In general a $\varphi$ is a locally complete intersection if there is a factorization of $\varphi$ into $R \to R' \to S$ where $R'$ is the localization of a polynomial ring over $R$ at some multiplicativly closed subset, and the second map is a surjective locally complete intersection.

Now let $R$ be a local ring with maximal ideal $\mathfrak{m}$, then it is a regular local ring if the maximal ideal is generated by a regular sequence. In general, a ring $R$ is regular if the localization of $R$ at all prime ideals is a regular local ring.

**Definition 11.2.** A morphism $\varphi : R \to S$ is said to be smooth if it is flat and for all morphisms $S \to l$ where $l$ is a field we have $S \otimes_R l$ is a regular ring.

*Remark* 11.3. Had we not insisted that $\varphi$ was locally of finite type, then the smooth should be replaced by regular.

There is a functorial characterization of smooth morphisms due to Grothendieck that says a (locally of finite type) morphisms of rings is a smooth morphism if and only if it is formally smooth, where $R \to S$ is said to be formally smooth if for all $R$ algebras $A$ with $I \subset A$ a nilpotent ideal, the map

$$\mathrm{Hom}_R(S, A) \to \mathrm{Hom}_R(S, A/I)$$

is surjective, i.e. one can lift any $R$ algebra map $S \to A/I$ to an $R$ algebra map $S \to A$.

Recall that in the characterization of flat morphisms, to show an $R$ module $N$ is flat we need to check that $Tor^1_R(N, M)$ is zero for *every* $R$ module $M$. Similarly in order to characterize complete intersections/ smooth morphisms we will need to consider the cotangent complex with coefficients.

**Definition 11.4.** Let $S$ be an $R$ algebra and $N$ an $S$ module, then the $S$ modules

$$D_n(S/R, N) = H_n(\mathcal{L}_{R/S} \otimes_S N), \quad D^n(S/R, N) = H_n(\mathrm{Hom}_S(\mathcal{L}, N))$$

are defined to be the $n^{th}$ Andre-Quillen homology (resp. cohomology) groups of $R \to S$ with coefficients in $N$.

Note as $\mathcal{L}_{S/R}$ is well defined up to homotopy these modules are well defined. Note also that by the Dold-Kan correspondence the definition of cohomology is consistent with the one given in in section 9.

For a composition $R \to S \to T$, it follows from Theorem 10.10 that

$$\mathcal{L}_{S/R} \otimes_S T \to \mathcal{L}_{T/R} \to \mathcal{L}_{T/S}$$

is a cofibration sequence, and hence induces exact sequences on the Andre-Quileen cohomology (and homology) groups.

The main results we will prove in this section are the following:

**Theorem 11.5.** *For a morphism for rings $\varphi : R \to S$, the following are equivalent:*
*i) $\varphi$ is a locally complete intersection*
*ii) $D_n(S/R, N) = 0$ for all $n \geq 2$ and all $S$ modules $N$.*
*iii) $D_2(S/R, N) = 0$ for all $S$ modules $N$*

**Theorem 11.6.** *For a morphism of rings $\varphi : R \to S$, the following are equivalent:*
*i) $\varphi$ is a smooth morphism*
*ii) $D_n(S/R, N) = 0$ for all $n \geq 1$ and all $S$ modules $N$.*
*iii) $D_1(S/R, N) = 0$ for all $S$ modules $N$*

Let us first prove Theorem. The hard part direction is $iii) \Rightarrow i)$, proof of this is by induction on the minimal number of generators of $\ker \varphi$ and the key input is a relationship between the $Tor$ functor and Andre-Quillen homology.

**Proposition 11.7.** *Let $\varphi : R \to S$ be a surjective morphism of rings with kernel $I$, then there are isomorphisms:*

$$D_1(S/R, N) \cong Tor_R^1 \cong I/I^2 \otimes N$$

For a morphism $R \to S$ let $\mu_r^S : S \otimes_S \to S$ denote the multiplication map.

**Proposition 11.8.** *Suppose $\mu_R^S$ is surjective, then there is an exact sequence of $S$ modules*

$$Tor_R^3(S, N) \longrightarrow D_3(S/R, N) \longrightarrow \wedge_S^2 Tor_R^1(S, S) \otimes_S N \longrightarrow ...$$
$$... \longrightarrow Tor_R^2(S, N) \longrightarrow D_2(S/R, N) \longrightarrow 0$$

*Remark* 11.9. These two results are proved in section 7 of [4]. The exact sequence in Proposition 11.8 is just the terms of low degree in a spectral sequence relating $Tor$ and $D_n$. Note also that Proposition 11.7 verifies a desired property of the cotangent complex resulting from equation 10.3.

We can reduce $iii) \Rightarrow i)$ to the following special case:

**Proposition 11.10.** *Let $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism of local rings, then $D_2(S/R, N) = 0$ for all $S$-modules $N$ implies $\varphi$ is complete intersection.*

Before we prove the result, let us briefly explain how one can use it to deduce the implication $iii) \Rightarrow i)$. By definition of locally complete intersection, we need to check that for all prime ideals $\mathfrak{q} \subset S$, the map $R_{\varphi^{-1}\mathfrak{q}} \to S_{\mathfrak{q}}$ is a locally complete intersection. The in order to apply the Proposition we need to show that condition $D_1(R/S, N) = 0$ implies $D_n(S_{\mathfrak{q}}, R_{\varphi^{-1}\mathfrak{q}}, l)$.

**Lemma 11.11.** Let $U \subset R$ be a multiplicatively closed subset, and set $S = R[U^{-1}]$, then $\mathcal{L}_{S/R} = 0$

*Proof.* Since the functor $\otimes_R S$ is the identity on the category of $S$ modules, we have the homotopy equivalences:

$$\mathcal{L}_{S/R} \cong \mathcal{L}_{S/R} \otimes_R S \cong \mathcal{L}_{S/S\otimes_R S} \cong \mathcal{L}_{S/S} \cong 0$$

where the map $S \otimes_R S \to S$ is given by $\varphi \otimes_R S$.

Since $R \to S$ is flat, the second isomorphism follows by flat base change and the third isomorphism follows from the fact that $\varphi \otimes_R S$ is non other than the identity map on $S$. □

Now consider the diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_{\varphi^{-1}\mathfrak{q}} \arrow[d] \\ S \arrow[r] & S_{\mathfrak{q}} \end{tikzcd}$$

Applying the Jacobi-Zariski exact sequence to the two compositions $R \to R_{\varphi^{-1}\mathfrak{q}} \to S_{\mathfrak{q}}$ and $R \to S \to S_{\mathfrak{q}}$ and using the above lemma, one shows that

$$\mathcal{L}_{S/R} \otimes_S S_{\mathfrak{q}} \cong \mathcal{L}_{S_{\mathfrak{q}}/R_{\varphi^{-1}\mathfrak{q}}}$$

In particular the vanishing of $D_1(S/R, N)$ implies the vanishing of $D_1(S_{\mathfrak{q}}/R_{\varphi^{-1}\mathfrak{q}}, N)$.

*Proof.* (of Proposition 11.10)We can factor the map $\varphi$ as

$$R \xrightarrow{i} R' \xrightarrow{\varphi'} S$$

where $R' = R[X]_{\mathfrak{q}}$ where $X$ is a finite set of indeterminates, $\mathfrak{q}$ a prime ideal in $R'$ and $\varphi'$ a surjective map. Since $\mathcal{L}_{S/R'}$ vanishes, it follows by the Jacobi-Zariski sequence that $D_n(S/R, N) = D_n(S/R', N)$, and hence we may assume that $\varphi$ is surjective.

Suppose $\ker \varphi$ is minimally generated by $c$ elements $r_1, ..., r_c$, we prove by induction on $c$ that $r_1, ..., r_c$ is a regular sequence in $R$.

For the base case $c = 1$ we have $S = R/(r)$, we would like to show $r$ is a non-zero-divisor. Lemma 11.8 gives us the exact sequence

$$\Lambda^2 Tor_R^1(S, S) \otimes_S k \to Tor_R^2(S, k) \to D_2(S/R, k) \longrightarrow 0$$

Since $Tor^1(S, S) \cong r/r^2 \cong k$ we have that $Tor_R^2(S, k) = 0$. This implies that the projective dimension of $S$ over $R$ is at most 1, hence the complex

$$0 \to R \to R \to 0$$

is exact as it is the start of a resolution of $S$, in particular $r$ is a non-zero-divisor.

Suppose that we know the claim for $c - 1$, then let $T = R/(r_1, .., r_{c-1})$ and $S = T/(r_c)$. The Jacobi-Zariski sequence of $R \to T \to S$ is the following:

$$D_2(T/R, N) \to D_2(S/T, N) \to D_1(T/R, N) \to D_1(S/R, N) \to D_1(S/T) \longrightarrow 0$$

We have that the following isomorphisms

$$D_1(T/R, N) \cong k^{c-1} \quad D_1(S/R, N) \cong k^c \quad D_1(S/R, N) \cong k$$

so that $S_2(S/T, N) = 0$. By the base case of the induction we have that $r_c$ is a non-zero divisor in $T$, hence $r_c$ is a regular sequence. □

*Proof.* (of Theorem 11.5) $iii) \to i)$ was just proved and $ii) \to iii)$ is obvious.

$i) \to ii)$ Using the Jacobi-Zariski exact sequence we may again reduce to the case $\varphi$ is surjective. In this case, suppose $S = R/(r_1, ..., r_c)$ where $r_1, .., r_c$ is a regular sequence. We induct on $c$, the case $c = 1$ being Proposition 10.11.

Now applying the Jacobi-Zariski exact sequence to the morphism of rings $R \to R/(r_1, ..., r_{c-1}) \to S$ we obtain the isomorphism $\mathcal{L}_{S/R} \cong \Sigma S$ and hence the desired vanishing of $D_n$.

□

We now turn to the proof of Theorem 11.6. Arguing in the same way as for Theorem 11.5, we may reduce to the special case:

**Proposition 11.12.** *For a morphism $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ the following are equivalent:*

*i) $\varphi$ is smooth*

*ii) $D_n(S/R, N) = 0$ for all $n \geq 1$ and all $S$ modules $N$, and $\Omega_{S/R}$ is finite free.*

*iii) $D_1(S/R, l) = 0$*

*Proof.* $i) \Rightarrow iii)$ We will first show that $D_n(S/R, l) = 0$ for all $n \geq 1$. Let $S' = S \otimes_R l$, since $R \to S$ is flat, it follows that $D_n(S/R, l) = D_n(S'/l, l)$ for all $n$. Consider the map $S' \to l$ induced by the identity on $l$ and the natural map $S \to l$. Composing with $l \to S'$ is the identity, thus applying the Jacobi Zariski sequence to the composition $l \to S' \to l$ we obtain isomorphisms:

$$D_n(S'/l, l) \cong D_{n+1}(l/S', l)$$

The smoothness of $\varphi$ implies $S'$ is regular, and letting $\mathfrak{n}'$ be the ideal of $S'$ with $S'/\mathfrak{n}' = l$, we have $S'_{\mathfrak{n}'}$ is regular local ring, which is equivalent to the morphism $S'_{\mathfrak{n}'} \to l$ being a complete intersection homomorphism. We thus obtain the isomorphisms

$$D_n(l/S', l) = D_n(l, S'_{\mathfrak{n}'}, l) = 0, \quad n \geq 2$$

where the last isomorphism follows from Theorem 11.5

The vanishing of $D_n(S/R, N)$ for all $S$-modules $N$ and the finite freeness of $\Omega_{S/R}$ follows immediately from the next lemma, stated without proof

**Lemma 11.13.** Let $\varphi : (R, \mathfrak{n}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism of rings. Then the complex of $\mathcal{L}_{S/R}$ is a homotopy equivalent to a complex:

$$... \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow ... \longrightarrow L_1 \longrightarrow L_0 \longrightarrow 0$$

where the $L_i$ are finite free $S$-modules with $d(L_i) \subset \mathfrak{n}L_{i-1}$ In particular if $D_n(S/R, l) = 0$ then $D_n(S.R, N) = 0$ for all $S$-modules $N$.

*Proof.* [4] Lemma 8.7 □

*ii)* $\Rightarrow$ *iii)* is clear

*iii)* $\Rightarrow$ *i)* Since $D_1(S/R, l) = 0$ it follows from Lemma 11.13 that $L_{S/R}$ is homotopy equivalent to a complex with $L_1 = 0$, so that

$$D_1(S/R, l) = H_1(\mathrm{Hom}_S(L, R)) = 0$$

But this last condition is the merely the statement that an extension of $S$ by a square zero ideal is necessarily split. It is then easy to deduce the functorial criterion for smoothness from this condition, hence $\varphi$ is smooth.

$\square$

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