TWISTED ORBITAL INTEGRALS AND IRREDUCIBLE COMPONENTS OF AFFINE DELIGNE–LUSZTIG VARIETIES

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Abstract. We analyze the asymptotic behavior of certain twisted orbital integrals arising from the study of affine Deligne–Lusztig varieties. The main tools include the Base Change Fundamental Lemma and q-analogues of the Kostant partition functions. As an application we prove a conjecture of Miaofen Chen and Xinwen Zhu, relating the set of irreducible components of an affine Deligne–Lusztig variety modulo the action of the σ-centralizer group to the Mirković–Vilonen basis of a certain weight space of a representation of the Langlands dual group.

Contents

1. Introduction

1.1. **The main result.** First introduced by Rapoport [Rap05], the affine Deligne–Lusztig varieties play an important role in arithmetic geometry and the Langlands program. One of the main motivations to study affine Deligne–Lusztig varieties comes from the theory of p-adic uniformization, which was studied by various authors including Čerednik [Č76], Drinfeld [Dri76], Rapoport–Zink [RZ96], and more recently Howard–Pappas [HP17] and Kim [Kim18]. In this theory, a p-adic formal scheme known as the Rapoport–Zink space uniformizes a tubular neighborhood in an integral model of a Shimura variety around a Newton stratum. The reduced subscheme of the Rapoport–Zink space is a special example of affine Deligne–Lusztig varieties. In a parallel story over function fields, affine Deligne–Lusztig varieties also arise naturally in the study of local shtukas, see for instance Hartl–Viehmann [HV11].

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Understanding of basic geometric properties of affine Deligne–Lusztig varieties has been fruitful for arithmetic applications. For instance an understanding of the connected components [CKV15] was applied to the proof of a version of Langlands–Rapoport conjecture by Kisin [Kis17]. The geometry of the supersingular locus of Hilbert modular varieties, which is also a question closely related to affine Deligne–Lusztig varieties via p-adic uniformizatoin, was applied to arithmetic level raising in the recent work of Liu–Tian [LT17].

In this paper, we concern the problem of parameterizing the irreducible components of affine Deligne–Lusztig varieties. This problem was initiated in the work of Xiao–Zhu [XZ17]. These authors studied this problem in some special cases, as an essential ingredient in their proof of a version of Tate's conjecture for special fibers of Shimura varieties. After that, Miaofen Chen and Xinwen Zhu formulated a general conjecture, relating the set of top dimensional irreducible components of general affine Deligne–Lusztig varieties to the Mirković–Vilonen cycles in the affine Grassmannian, and thus to the representation theory of the Langlands dual group via the geometric Satake. Partial results towards this conjecture have been obtained by Xiao–Zhu [XZ17], Hamacher–Viehmann [HV17], and Nie [Nie18a], based on a common idea of reduction to the superbasic case (which goes back to [GHKR06]).

In this paper we present a new method and prove:

Theorem. *The Chen–Zhu Conjecture (see Conjecture 1.2.1) holds in full generality.*

Our proof is based on an approach completely different from the previous works. The problem is to compute the number of top dimensional irreducible components of an affine Deligne–Lusztig variety (modulo a certain symmetry group). We use the Lang–Weil estimate to relate this number to the asymptotic behavior of the number of points on the affine Deligne–Lusztig variety over a finite field, as the finite field grows. We show that the number of points over a finite field is computed by a twisted orbital integral, and thus we reduce the problem to the asymptotic behavior of twisted orbital integrals. We study the latter using explicit methods from local harmonic analysis and representation theory, including the Base Change Fundamental Lemma and the Kato–Lusztig formula.

An interesting point in our proof is that we apply the Base Change Fundamental Lemma, which is only available in general for mixed characteristic local fields as the current known proofs of it rely on trace formula methods. Thus our method crucially depends on the geometric theory of mixed characteristic affine Grassmannians as in [BS17] and [Zhu17]. To deduce the Chen–Zhu Conjecture also for equal characteristic local fields, we use results of He [He14], which imply that the number of irreducible components (modulo a symmetry group) only depends on the affine Hecke algebra, and thus the truth of the conjecture transfers between different local fields.

In our proof, certain polynomials that are linear combinations of the q-analogue of Kostant partition functions appear, and the key computation is to estimate the sizes of them. These polynomials can be viewed as a non-dominant generalization of the q-analogue of Kostant's weight multiplicity formula. Some properties of them are noted in [Pan16], but beyond this there does not seem to have been a lot of study into these objects. From our proof, it seems reasonable to expect that a more thorough study of the combinatorial and geometric properties of these polynomials would shed new light on the structure of affine Deligne–Lusztig varieties, as well as the structure of twisted orbital integrals.

In Appendix A, we combine our main result with the work of He [He14] to generalize the Chen–Zhu Conjecture to quasi-split groups that are not necessarily unramified.

1.2. **The precise statement.** We now give a precise statement of the Chen–Zhu Conjecture. Let F be a non-archimedean local field with valuation ring $\mathcal{O}_F$ and residue field $k_F = \mathbb{F}_q$. Let L be the completion of the maximal unramified extension of F. Let G be a connected reductive group scheme over $\mathcal{O}_F$ and σ the

Frobenius automorphism of L over F. We fix $T \subset G$ to be the centralizer of a maximal $\mathcal{O}_F$-split torus, and fix a Borel subgroup $B \subset G$ containing T. For $\mu \in X_*(T)^+$ and $b \in G(L)$, the *affine Deligne–Lusztig variety* associated to (G, μ, b) is defined to be

$$X_\mu(b) = \{g \in G(L)/G(\mathcal{O}_L) | g^{-1} b\sigma(g) \in G(\mathcal{O}_L)\mu(\pi_F)G(\mathcal{O}_L)\},$$

where $\pi_F \in F$ is a uniformizer. More precisely, the above set is the set of $\overline{\mathbb{F}}_q$-points of a locally closed, locally of (perfectly) finite type subscheme of the (Witt vector) affine Grassmannian of G, while the (Witt vector) affine Grassmannian of G is an inductive limit of (perfections of) projective varieties over $\overline{\mathbb{F}}_q$ along closed immersions. Here the parentheses apply when F is of mixed characteristic, and the geometric structure stated in this case is due to the recent breakthrough by Bhatt–Scholze [BS17] (cf. [Zhu17]).

Let Σ^{top} be the set of top-dimensional irreducible components of $X_\mu(b)$. Let

$$J := J_b(F) = \{g \in G(L) | g^{-1} b\sigma(g) = b\}.$$

Then the group J naturally acts by left multiplication on $X_\mu(b)$, and hence on Σ^{top}. The set $J\backslash\Sigma^{\mathrm{top}}$ is in fact finite.

Let $\widehat{G}$ denote the Langlands dual group of G over $\mathbb{C}$, equipped with a Borel pair $\widehat{T} \subset \widehat{B}$, where $\widehat{T}$ is a maximal torus dual to T and equipped with an algebraic action by σ. Let $\widehat{\mathcal{S}}$ be the identity component of the σ-fixed points of $\widehat{T}$. In $X^*(\widehat{\mathcal{S}})$, there is a distinguished element λ_b, determined by b. It is the "best integral approximation" of the Newton cocharacter of b, but we omit its precise definition here (see Definition 2.6.5). For $\mu \in X_*(T)^+ = X^*(\widehat{T})^+$, we write V_μ for the highest weight representation of $\widehat{G}$ with highest weight μ. We write $V_\mu(\lambda_b)_{\mathrm{rel}}$ for the λ_b-weight space in V_μ with respect to the action of $\widehat{\mathcal{S}}$.

Conjecture 1.2.1 (Miaofen Chen, Xinwen Zhu)**.** *There exists a natural bijection between $J\backslash\Sigma^{\mathrm{top}}$ and the Mirković–Vilonen basis of $V_\mu(\lambda_b)_{\mathrm{rel}}$. In particular,*

$$\left|J\backslash\Sigma^{\mathrm{top}}\right| = \dim V_\mu(\lambda_b)_{\mathrm{rel}}. \tag{1.2.1}$$

The first time this conjecture was considered in the form stated above was in [XZ17]. In *loc. cit.* Xiao–Zhu proved the conjecture for general G, general μ, and *unramified* b, meaning that J_b and G are assumed to have equal F-rank.

Hamacher–Viehmann [HV17] proved the conjecture under either of the following assumptions:

- The cocharacter μ is minuscule, and G is split over F.
- The cocharacter μ is minuscule, and b is superbasic in M, where M is the largest Levi of G inside which b is basic. (In particular if b is basic then they assume that b is superbasic).

More recently, Nie [Nie18a] proved the conjecture for arbitrary G under the assumption that μ is a sum of dominant minuscule coweights. In particular it holds when the Dynkin diagram of $G_{\overline{F}}$ only involves factors of type A. Moreover, Nie constructed a surjection from the Mirković-Vilonen basis to the set $J\backslash\Sigma^{\mathrm{top}}$ in all cases. Thus in order to prove the conjecture, it suffices to prove the numerical relation (1.2.1) for groups without type A factors.[1]

1.3. **Overview of the proof.** We now explain our proof of the Chen–Zhu Conjecture. A standard reduction allows us to assume that b is basic, and that G is adjoint and F-simple. Throughout we also assume that G is not of type A, which is already sufficient by the work of Nie [Nie18a]. To simplify the exposition, we also assume that G is split and not of type E_6. Then $\widehat{\mathcal{S}} = \widehat{T}$, and we drop the subscript "rel" for the weight spaces in Conjecture 1.2.1.

[1]After we finished this work, Nie uploaded online a new version of the preprint [Nie18a], in which he also proves the Chen–Zhu Conjecture. However our work only uses the weaker result of Nie as stated here. See Remark 1.3.2.

For any $s \in \mathbb{Z}_{>0}$, we let F_s be the unramified extension of F of degree s, with residue field k_s. We denote by $\mathcal{H}_s$ the spherical Hecke algebra $\mathcal{H}(G(F_s)//G(\mathcal{O}_{F_s}))$. We may assume without loss of generality that b is s_0-decent for a fixed $s_0 \in \mathbb{N}$, meaning that $b \in G(F_{s_0})$ and

$$b\sigma(b)\cdots\sigma^{s_0-1}(b) = 1.$$

As mentioned above, our idea is to use the Lang–Weil estimate to relate the number of irreducible components to the asymptotics of twisted orbital integrals. Since $X_\mu(b)$ is only locally of (perfectly) finite type and we are only counting J-orbits of irreducible components, we need a suitable interpretation of the Lang–Weil estimate. The precise output is the following (Proposition 4.2.4):

$$TO_b(f_{\mu,s}) = \sum_{Z \in J\backslash\Sigma^{\mathrm{top}}} \mathrm{vol}(\mathrm{Stab}_Z\, J)^{-1} q^{s\dim X_\mu(b)} + o(q^{s\dim X_\mu(b)}), \quad s \in s_0\mathbb{N},\ s \gg 0. \tag{1.3.1}$$

Here $f_{\mu,s} \in \mathcal{H}_s$ is the characteristic function of $G(\mathcal{O}_{F_s})\mu(\pi_F)G(\mathcal{O}_{F_s})$, and $TO_b(f_{\mu,s})$ denotes the twisted orbital integral of $f_{\mu,s}$ along $b \in G(F_s)$.

To proceed, we need the following variant of (1.3.1):

$$TO_b(\tau_\mu) = \sum_{Z \in J\backslash\Sigma^{\mathrm{top}}} \mathrm{vol}(\mathrm{Stab}_Z\, J)^{-1} q^{s\delta} + o(q^{s\delta}), \quad s \in s_0\mathbb{N},\ s \gg 0. \tag{1.3.2}$$

Here $\tau_\mu \in \mathcal{H}_s$ denotes the function whose Satake transform is the character of the representation V_μ of $\widehat{G}$, and

$$\delta := -\frac{1}{2}(\mathrm{rk}_F G - \mathrm{rk}_F J_b).$$

The proof of (1.3.2) is based on (1.3.1), the dimension formula for $X_\mu(b)$ by Hamacher [Ham15] and Zhu [Zhu17], and asymptotics of the Kato–Lusztig formula [Kat82].

We next apply the Base Change Fundamental Lemma to compute $TO_b(\tau_\mu)$. There are two problems in this step. Firstly, the Base Change Fundamental Lemma can only be applied to stable twisted orbital integrals. This problem is solved because one can check that the twisted orbital integral in (1.3.1) is already stable. Secondly, the general Base Change Fundamental Lemma is only available for $\mathrm{char}F = 0$. In fact, the proofs of this result by Clozel [Clo90] and Labesse [Lab90] rely on methods only available over characteristic zero, for example the trace formula of Deligne–Kazhdan. To circumvent this, we show using the reduction method of [He14] (in §3) that the truth of the Chen–Zhu Conjecture depends only on the affine root system associated to G. Hence it suffices to prove the conjecture just for p-adic fields.

After computing the left hand side of (1.3.2) using the Base Change Fundamental Lemma, we obtain

$$\sum_{\lambda \in X^*(\widehat{T})^+, \lambda \le \mu} \dim V_\mu(\lambda) \cdot \mathfrak{M}^0_{s\lambda}(q^{-1}) = \pm \sum_{Z \in J\backslash\Sigma^{\mathrm{top}}} \mathrm{vol}(\mathrm{Stab}_Z\, J)^{-1} q^{s\delta} + o(q^{s\delta}),\ s \gg 0, \tag{1.3.3}$$

where each $\mathfrak{M}^0_{s\lambda}(\mathbf{q}) \in \mathbb{C}[\mathbf{q}]$ is a polynomial given explicitly in terms of the $\mathbf{q}$-analogues of Kostant's partition functions (see Definition 5.2.5 and §5.3).

The key computation of our paper is summarized in the following

Proposition 1.3.1. *Let $\lambda_b^+ \in X^*(\widehat{T})^+$ be the dominant conjugate of λ_b. For all $\lambda \in X^*(\widehat{T})^+ - \{\lambda_b^+\}$, we have*

$$\mathfrak{M}^0_{s\lambda}(q^{-1}) = o(q^{s\delta}),\ s \gg 0.$$

When G is the split adjoint E_6, we only prove a weaker form of the proposition, which also turns out to be sufficient for our purpose.

Proposition 1.3.1 tells us that on the left hand side of (1.3.3), only the summand indexed by $\lambda = \lambda_b^+$ has the "right size". We thus obtain

$$\dim V_\mu(\lambda_b) \cdot \mathscr{L}_b = \pm \sum_{Z \in J \backslash \Sigma^{\mathrm{top}}} \mathrm{vol}(\mathrm{Stab}_Z\, J)^{-1}, \tag{1.3.4}$$

where

$$\mathscr{L}_b = \lim_{s \to \infty} \mathfrak{M}^0_{s\lambda_b^+}(q^{-1}) q^{s\delta}$$

is a constant that depends only on b and **not** on μ.

In (1.3.4) we already see both the number $\dim V_\mu(\lambda_b)$ and the set $J\backslash\Sigma^{\mathrm{top}}$. In order to deduce the desired (1.2.1), one still needs some information on the volume terms $\mathrm{vol}(\mathrm{Stab}_Z J)$. It turns out that even very weak information will suffice. In §3 we show that the right hand side of (1.3.4) is equal to $R(q)$, where $R(T) \in \mathbb{Q}(T)$ is a rational function which is *independent of* F in the following sense. It turns out that the triple (G, μ, b) can be encoded combinatorially in terms of the affine root system of G. The rational function $R(T)$ only depends on this combinatorial information and not on F. Moreover we show that

$$R(0) = |J\backslash\Sigma^{\mathrm{top}}|.$$

Therefore, the desired (1.2.1) will follow from (1.3.4), if we can show that

$$\mathscr{L}_b = S(q), \quad \text{for some } S(T) \in \mathbb{Q}(T) \text{ with } S(0) = 1. \tag{1.3.5}$$

A remarkable feature of the formulation (1.3.5) is that it is independent of μ. We recall that in the works of Hamacher–Viehmann and Nie, special assumptions on μ are made. Hence we are able to bootstrap from known cases of the Chen–Zhu Conjecture (for example when $\mu = \lambda_b^+$) to establish (1.3.5), and hence to establish the Chen–Zhu Conjecture in general.

We end our discussion with several remarks.

Remark 1.3.2. We refer to the statements of Theorem 6.3.2 and Corollary 6.3.3 for the logical dependence of our work on previous works of other people.

Remark 1.3.3. In Appendix A, we prove a generalization of Conjecture 1.2.1 for possibly ramified quasi-split groups G over F. See Theorem A.3.1.

Remark 1.3.4. At the moment, we are unable to directly compute the rational functions $S(T)$ appearing in (1.3.5) in general. To do this would require a much better understanding of the polynomials $\mathfrak{M}^0_{s\lambda}(\mathbf{q})$. We are however able to compute $S(T)$ in a very special case. When b is a basic unramified element in the sense of [XZ17], we show directly that (1.3.5) is satisfied by $S(T) \equiv 1$, see §6.2. From this we deduce the conjecture for b, as well as the equality $\mathrm{vol}(\mathrm{Stab}_Z\, J) = 1$ for each $Z \in \Sigma^{\mathrm{top}}$. This last equality implies (according to our normalization) that $\mathrm{Stab}_Z\, J$ is a hyperspecial subgroup of $J = J_b(F)$. This gives another proof of a result in [XZ17], avoiding their use of Littelmann paths.

Remark 1.3.5. Our proof of Proposition 1.3.1 in the case G is split of type D_n with n odd turns out to be different from the other cases. In this case we devise a combinatorial method to compute the polynomials $\mathfrak{M}^0_{s\lambda}(\mathbf{q})$, using certain binary trees whose vertices are decorated by pairs of roots in the root system, see §8.1. This method could possibly be generalized to compute more instances of $\mathfrak{M}^0_{s\lambda}(\mathbf{q})$.

1.4. **Outline of paper.** In §2, we introduce notations and state precisely our formulation of the Chen–Zhu conjecture. In §3, we study the $J_b(F)$-action on the top-dimensional irreducible components of $X_\mu(b)$. We prove using the reduction method of [He14] that the set of orbits and the stabilizers only depend on the affine root system of G. In §4, we prove the relation (1.3.1), and then apply the Base Change Fundamental Lemma to compute twisted orbital integrals. In §5 we review the relationship between the coefficients of the Satake transform and the $\mathbf{q}$-analogue of Kostant's partition functions, and draw some consequences. In §6 we state Proposition 6.3.1 as a more technical version of Proposition 1.3.1. We then deduce Conjecture 1.2.1 from Proposition 6.3.1. The proof of Proposition 6.3.1 is given in §7, §8, and §9, by analyzing each root system case by case. In Appendix A, we generalize our main result to quasi-split groups.

Notation. We order $\mathbb{N}$ by divisibility, and write $s \gg 0$ to mean "for all sufficiently divisible $s \in \mathbb{N}$". Limits $\lim_{s\to\infty}$, as well as the big-O and little-o notations, are all with respect to the divisibility on $\mathbb{N}$. For example, we write $f(s) = o(g(s))$ to mean that f, g are $\mathbb{C}$-valued functions defined for all sufficiently divisible $s \in \mathbb{N}$, such that

$$\forall \epsilon > 0 \ \exists s_0 \in \mathbb{N} \ \forall s \in s_0\mathbb{N}, \ |f(s)/g(s)| < \epsilon.$$

Similarly, we write $f(s) = O(g(s))$ to mean that $f(s)/g(s)$ is **eventually** bounded, namely

$$\exists M > 0 \ \exists s_0 \in \mathbb{N} \ \forall s \in s_0\mathbb{N}, \ |f(s)/g(s)| < M,$$

without requiring boundedness on all of $\mathbb{N}$.

For any finitely generated abelian group X, we write X_{free} for the free quotient of X.

We use $\mathbf{q}$ or $\mathbf{q}^{-1}$ and sometimes $\mathbf{q}^{-1/2}$ to denote the formal variable in a polynomial or power series ring.

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2. Notations and preliminaries

2.1. **Basic notations.** Let F be a non-archimedean local field with valuation ring $\mathcal{O}_F$ and residue field $k_F = \mathbb{F}_q$. Let $\pi_F \in F$ be a uniformizer. Let p be the characteristic of k_F. Let L be the completion of the maximal unramified extension of F, with valuation ring $\mathcal{O}_L$ and residue field $k = \overline{k_F}$. Let $\Gamma = \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group. Let σ be the Frobenius of L over F. We have σ-equivariant isomorphisms

$$L \cong W(k) \otimes_{W(k_F)} F, \quad \mathcal{O}_L \cong W(k) \otimes_{W(k_F)} \mathcal{O}_F.$$

Let G be a connected reductive group over $\mathcal{O}_F$. In particular its generic fiber G_F is an unramified reductive group over F, i.e. is quasi-split and splits over an unramified extension of F. Then $G(\mathcal{O}_F)$ is a hyperspecial subgroup of $G(F)$. Fix a maximal $\mathcal{O}_F$-split torus A of G. Let T be the centralizer of A_F in G_F, and fix a Borel subgroup $B \subset G_F$ containing T. Hence T is an unramified maximal torus of G_F. In the following we often abuse notation and simply write G for G_F.

Note that T_L is a maximal L-split torus of G_L. Let V be the apartment of G_L corresponding to T_L. The hyperspecial vertex $\mathfrak{s}$ corresponding to $G(\mathcal{O}_L)$ is then contained in V. We have an identification $V \cong X_*(T) \otimes \mathbb{R}$ sending $\mathfrak{s}$ to 0. Let $\mathfrak{a} \subset V$ be the alcove whose closure contains $\mathfrak{s}$, such that the image of $\mathfrak{a}$ under $V \cong X_*(T) \otimes \mathbb{R}$ is contained in the anti-dominant chamber. The action of σ induces an action on V, and both $\mathfrak{a}$ and $\mathfrak{s}$ are stabilized under this action. We let $\mathcal{I}$ be the Iwahori subgroup of $G(L)$ corresponding to $\mathfrak{a}$.

2.2. **The Iwahori Weyl group.** The relative Weyl group W_0 over L and the Iwahori–Weyl group W are defined by

$$W_0 = N(L)/T(L), \qquad W = N(L)/T(L) \cap \mathcal{I},$$

where N denotes the normalizer of T in G. Note that W_0 is equal to the absolute Weyl group, as T_L is split.

The Iwahori–Weyl group W is a split extension of W_0 by the subgroup $X_*(T)$. The splitting depends on the choice of a hyperspecial vertex, which we fix to be $\mathfrak{s}$ defined above. See [HR08] for more details. When considering an element $\lambda \in X_*(T)$ as an element of W, we write t^λ. For any $w \in W$, we choose a representative $\dot{w} \in N(L)$.

Let W_a be the associated affine Weyl group, and $\mathbb{S}$ be the set of simple reflections associated to $\mathfrak{a}$. Since $\mathfrak{a}$ is σ-stable, there is a natural action of σ on $\mathbb{S}$. We let $\mathbb{S}_0 \subset \mathbb{S}$ be set of simple reflections fixing $\mathfrak{s}$. The Iwahori–Weyl group W contains the affine Weyl group W_a as a normal subgroup and we have a natural splitting

$$W = W_a \rtimes \Omega,$$

where Ω is the normalizer of $\mathfrak{a}$ and is isomorphic to $\pi_1(G)$. The length function ℓ and the Bruhat order $\leq$ on the Coxeter group W_a extend in a natural way to W.

For any subset P of $\mathbb{S}$, we shall write W_P for the subgroup of W generated by P.

For $w, w' \in W$ and $s \in \mathbb{S}$, we write $w \xrightarrow{s}_\sigma w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leq \ell(w)$. We write $w \to_\sigma w'$ if there is a sequence $w = w_0, w_1, \dots, w_n = w'$ of elements in W such that for any i, $w_{i-1} \xrightarrow{s_i}_\sigma w_i$ for some $s_i \in \mathbb{S}$. Note that if moreover, $\ell(w') < \ell(w)$, then there exists i such that $\ell(w) = \ell(w_i)$ and $s_{i+1} w_i \sigma(s_{i+1}) < w_i$.

We write $w \approx_\sigma w'$ if $w \to_\sigma w'$ and $w' \to_\sigma w$. It is easy to see that $w \approx_\sigma w'$ if $w \to_\sigma w'$ and $\ell(w) = \ell(w')$. We write $w \tilde{\approx}_\sigma w'$ if there exists $\tau \in \Omega$ such that $w \approx_\sigma \tau w' \sigma(\tau)^{-1}$.

2.3. **The set** $B(G)$**.** For any $b \in G(L)$, we denote by $[b] = \{g^{-1} b \sigma(g); g \in G(L)\}$ its σ-conjugacy class. Let $B(G)$ be the set of σ-conjugacy classes of $G(L)$. The σ-conjugacy classes have been classified by Kottwitz in [Kot85] and [Kot97], in terms of the *Newton map* $\bar{\nu}$ and the *Kottwitz map* κ. The Newton map is a map

$$\bar{\nu} : B(G) \to (X_*(T)_{\mathbb{Q}}^+)^\sigma, \tag{2.3.1}$$

where $X_*(T)_{\mathbb{Q}}^+$ is the set of dominant elements in $X_*(T)_{\mathbb{Q}} := X_*(T) \otimes \mathbb{Q}$. The Kottwitz map is a map

$$\kappa = \kappa_G : B(G) \to \pi_1(G)_\Gamma.$$

By [Kot97, §4.13], the map

$$(\bar{\nu}, \kappa) : B(G) \to (X_*(T)_{\mathbb{Q}}^+)^\sigma \times \pi_1(G)_\Gamma$$

is injective.

The maps $\bar{\nu}$ and κ can be described in an explicit way via the map $W \to G(L), w \mapsto \dot{w}$. As a result, we obtain an explicit map $(\bar{\nu}, \kappa) : W \to (X_*(T)_{\mathbb{Q}}^+)^\sigma \times \pi_1(G)_\Gamma$. Moreover, this map descends to the set $B(W, \sigma)$ of σ-conjugacy classes of W. See [HZ16, §1.2] for details. The inclusion map $W \to G(L), w \mapsto \dot{w}$ induces a map $\Psi : B(W, \sigma) \to B(G)$, which is independent of the choice of the representatives $\dot{w}$. By [He14], the map Ψ is surjective and we have a commutative diagram

$$\begin{tikzcd} B(W,\sigma) \arrow[rr, two heads, "\Psi"] \arrow[dr, "(\bar{\nu},\kappa)"'] & & B(G) \arrow[dl, hook, "(\bar{\nu},\kappa)"] \\ & (X_*(T)_{\mathbb{Q}}^+)^\sigma \times \pi_1(G)_\Gamma & \end{tikzcd} .$$

The map $B(W,\sigma) \to B(G)$ is not injective. However, there exists a canonical lifting to the set of *straight σ-conjugacy classes*.

By definition, an element $w \in W$ is called *σ-straight* if for any $n \in \mathbb{N}$,

$$\ell(w\sigma(w)\cdots\sigma^{n-1}(w)) = n\ell(w).$$

This is equivalent to the condition that $\ell(w) = \langle \bar{\nu}_w, 2\rho\rangle$, where ρ is the half sum of all positive roots. A σ-conjugacy class of W is called *straight* if it contains a σ-straight element. It is easy to see that the minimal length elements in a given straight σ-conjugacy class are exactly the σ-straight elements.

The following result is proved in [He14, Theorem 3.7].

Theorem 2.3.1. *The restriction of $\Psi : B(W,\sigma) \to B(G)$ gives a bijection from the set of straight σ-conjugacy classes of W to $B(G)$.* □

2.4. **The affine Deligne–Lusztig variety $X_{P,w}(b)$.** Let $\mathcal{P}$ be a standard σ-invariant parahoric subgroup of $G(L)$, i.e. a σ-invariant parahoric subgroup that contains $\mathcal{I}$. In the following, we will generally abuse of notation to use the same symbol to denote a parahoric subgroup and the underlying parahoric group scheme. We denote by $P \subset \mathbb{S}$ the set of simple reflections corresponding to $\mathcal{P}$. Then $\sigma(P) = P$. We have

$$G(L) = \bigsqcup_{w \in W_P\backslash W/W_P} \mathcal{P}(\mathcal{O}_L)\dot{w}\mathcal{P}(\mathcal{O}_L).$$

For any $w \in W_P\backslash W/W_P$ and $b \in G(L)$, we set

$$X_{P,w}(b)(k) := \{g\mathcal{P}(\mathcal{O}_L) \in G(L)/\mathcal{P}(\mathcal{O}_L) | g^{-1}b\sigma(g) \in \mathcal{P}(\mathcal{O}_L)\dot{w}\mathcal{P}(\mathcal{O}_L)\}.$$

If $\mathcal{P} = \mathcal{I}$ (corresponding to $P = \emptyset$), we simply write $X_w(b)(k)$ for $X_{\emptyset,w}(b)(k)$.

We freely use the standard notations concerning loop groups and partial affine flag varieties (i.e. affine Grassmannians associated to parahoric group schemes over $\mathcal{O}_L$.) See [BS17, §9] or [Zhu17, §1.4]. When $\mathrm{char}F > 0$, it is known that $X_{P,w}(b)(k)$ could be naturally identified with the set of k-points of a locally closed sub-ind scheme $X_{P,w}(b)$ of the partial affine flag variety $\mathrm{Gr}_{\mathcal{P}}$. When $\mathrm{char}F = 0$, thanks to the recent breakthrough by Bhatt–Scholze [BS17, Corollary 9.6] (cf. also [Zhu17]), we can again identify $X_{P,w}(b)(k)$ with the k-points of a locally closed perfect sub-ind scheme $X_{P,w}(b)$ of the Witt vector partial affine flag variety $\mathrm{Gr}_{\mathcal{P}}$. In both cases, the (perfect) ind-scheme $X_{P,w}(b)$ is called an *affine Deligne–Lusztig variety*, and all topological notions related to the Zariski topology on $X_{P,w}(b)$ are well-defined. In particular, we have notions of Krull dimension and irreducible components for $X_{P,w}(b)$.

We are mainly interested in the case when $\mathcal{P} = G_{\mathcal{O}_L}$. In this case the corresponding set of simple reflections is $K = \mathbb{S}_0$. We have an identification

$$X_*(T)^+ \cong X_*(T)/W_0 \cong W_K\backslash W/W_K.$$

For $\mu \in X_*(T)^+$, we write $X_\mu(b)$ for $X_{K,t^\mu}(b)$.

We simply write Gr_G for $\mathrm{Gr}_{G_{\mathcal{O}_L}}$. The relationship between the hyperspecial affine Deligne–Lusztig variety $X_\mu(b) \subset \mathrm{Gr}_G$ and the Iwahori affine Deligne–Lusztig varieties $X_w(b) \subset \mathrm{Gr}_{\mathcal{I}}$ is as follows. We have a projection

$$\pi : \mathcal{FL} \to \mathrm{Gr}_G \tag{2.4.1}$$

which exhibits $\mathcal{FL} := \mathrm{Gr}_{\mathcal{I}}$ as an étale fibration over Gr_G. Indeed the fiber of this map is isomorphic to the fpqc quotient $L^+G/L^+\mathcal{I}$ where $L^+G, L^+\mathcal{I}$ are the positive loop groups attached to G and $\mathcal{I}$. More concretely, $L^+G/L^+\mathcal{I}$ is a finite type flag variety over k when $\mathrm{char}F > 0$, and is the perfection of a finite type flag variety over k when $\mathrm{char}F = 0$. We have

$$\pi^{-1}(X_\mu(b)) = X(\mu, b)^K := \bigcup_{w \in W_0 t^\mu W_0} X_w(b).$$

2.5. **Basic information about $X_\mu(b)$.** Let $\mu \in X_*(T)^+$. We first recall the definition of the neutral acceptable set $B(G, \mu)$ in [RV14]. We have the dominance order $\leq$ on $X_*(T)^+_{\mathbb{Q}}$ defined as follows. For $\lambda, \lambda' \in X_*(T)_{\mathbb{Q}}$, we write $\lambda \leq \lambda'$ if $\lambda' - \lambda$ is a non-negative rational linear combination of positive coroots. Set

$$B(G, \mu) = \{[b] \in B(G); \kappa([b]) = \mu^\natural, \bar{\nu}_b \leq \mu^\diamond\}.$$

Here $\mu^\natural$ denotes the image of μ in $\pi_1(G)_\Gamma$, and $\mu^\diamond \in X_*(T)_{\mathbb{Q}}$ denotes the Galois average of μ.

The following result, conjectured by Kottwitz and Rapoport in [KR03], is proved in [Gas10].

Theorem 2.5.1. *For $b \in G(L)$, the affine Deligne–Lusztig variety $X_\mu(b)$ is non-empty if and only if $[b] \in B(G, \mu)$.* □

We now let $\mu \in X_*(T)^+$ and let $b \in G(L)$ such that $[b] \in B(G, \mu)$.

Theorem 2.5.2. *If $\mathrm{char}F > 0$, then $X_\mu(b)$ is a scheme locally of finite type over k. If $\mathrm{char}F = 0$, then $X_\mu(b)$ is a perfect scheme locally of perfectly finite type over k. In both cases the Krull dimension of $X_\mu(b)$ is equal to*

$$\langle \mu - \bar{\nu}_b, \rho \rangle - \frac{1}{2}\mathrm{def}_G(b),$$

where

$$\mathrm{def}_G(b) := \mathrm{rk}_F G - \mathrm{rk}_F J_b.$$

Proof. The local (perfectly) finiteness is shown in [HV17, Lemma 1.1], using an earlier argument in [HV11]. The dimension formula is proved in [Ham15] and [Zhu17]. □

Corollary 2.5.3. *Let $w \in W_0 t^\mu W_0$. If $\mathrm{char}F > 0$, then $X_w(b)$ is a scheme locally of finite type over k. If $\mathrm{char}F = 0$, then $X_w(b)$ is a perfect scheme locally of perfectly finite type over k.*

Proof. This follows from Theorem 2.5.2 and the fact that $X_w(b)$ is an ind-(perfect) scheme locally closed inside $\pi^{-1}(X_\mu(b))$, where π is the fibration (2.4.1). □

Definition 2.5.4. For any (perfect) ind-scheme X, we write $\Sigma(X)$ for the set of irreducible components of X. When X is of finite Krull dimension, we write $\Sigma^{\mathrm{top}}(X)$ for the set of top dimensional irreducible components of X.

Define the group scheme J_b over F by

$$J_b(R) = \left\{ g \in G(R \otimes_F L) | g^{-1} b \sigma(g) = b \right\} \tag{2.5.1}$$

for any F-algebra R. Then J_b is an inner form of a Levi subgroup of G, see [RZ96, §1.12] or [RR96, §1.11]. The group $J_b(F)$ acts on $X_\mu(b)$ via algebraic automorphisms. In particular $J_b(F)$ acts on $\Sigma(X_\mu(b))$ and on $\Sigma^{\mathrm{top}}(X_\mu(b))$. The following finiteness result is proved in [HV17, Lemma 1.3], building on the result of Rapoport–Zink [RZ99].

Lemma 2.5.5. *The set $J_b(F) \backslash \Sigma(X_\mu(b))$ is finite.* □

Definition 2.5.6. We write $\mathscr{N}(\mu, b)$ for the cardinality of $J_b(F) \backslash \Sigma^{\mathrm{top}}(X_\mu(b))$.

2.6. **The Chen–Zhu conjecture.** In this paper we shall utilize the usual Langlands dual group (as a reductive group over $\mathbb{C}$ equipped with a pinned action by the Galois group), rather than the Deligne–Lusztig dual group which is used in [HV17]. As a result, our formulation of the Chen–Zhu conjecture below differs from [HV17, Conjecture 1.4, §2.1]. However it can be easily checked that the two formulations are equivalent.

The Frobenius σ acts on $X_*(T)$ via a finite-order automorphism, which we denote by θ. Let $\widehat{G}$ be the usual dual group of G over $\mathbb{C}$, which is a reductive group over $\mathbb{C}$ equipped with the following structures:

- a Borel pair $(\widehat{B}, \widehat{T})$.
- isomorphisms $X^*(\widehat{T}) \cong X_*(T), X_*(\widehat{T}) \cong X^*(T)$, which we think of as equalities. These isomorphisms identify the positive roots in $X^*(\widehat{T})$ with the positive coroots in $X_*(T)$, and identify the positive coroots in $X_*(\widehat{T})$ with the positive roots in $X^*(T)$. We denote by $\hat{\theta}$ the automorphism of $X^*(\widehat{T})$ corresponding to the automorphism θ of $X_*(T)$.

For more details on the dual group see §5.1 below.

For any finitely generated abelian group X, we write X_{free} for the free quotient of X. The following lemma is elementary, and we omit its proof.

Lemma 2.6.1. *Let Γ be a finite group. Let X be a $\mathbb{Z}[\Gamma]$-module which is a finite free $\mathbb{Z}$-module. We write X^Γ (resp. X_Γ) for the invariants (resp. coinvariants) of X under Γ. As usual define the norm map:*

$$\mathrm{N} : X \to X, \ x \mapsto \sum_{\gamma \in \Gamma} \gamma(x).$$

Let $Y \subset X$ be a Γ-stable subgroup. Then the following statements hold.

(1) The kernel of the map $Y \to X_{\Gamma,\mathrm{free}}$ is equal to $\{y \in Y | \mathrm{N}(y) = 0\}$. In particular, it is also equal to the kernel of $Y \to Y_{\Gamma,\mathrm{free}}$.

(2) Suppose Y has a finite $\mathbb{Z}$-basis which is stable under Γ as a set. Then the Γ-orbits in this $\mathbb{Z}$-basis define distinct elements of Y_Γ, which form a $\mathbb{Z}$-basis of Y_Γ. In particular Y_Γ is a finite free $\mathbb{Z}$-module.

(3) The map $\mathrm{N} : X \to X$ factors through a map

$$\mathrm{N} : X_\Gamma \to X^\Gamma.$$

Both the compositions

$$X_\Gamma \xrightarrow{\mathrm{N}} X^\Gamma \subset X \to X_\Gamma$$

and

$$X^\Gamma \subset X \to X_\Gamma \xrightarrow{\mathrm{N}} X^\Gamma$$

are given by multiplication by $|\Gamma|$. In particular we have a canonical isomorphism

$$\frac{1}{|\Gamma|}\mathrm{N} : X_\Gamma \otimes \mathbb{Q} \xrightarrow{\sim} X^\Gamma \otimes \mathbb{Q}.$$

□

Definition 2.6.2. Let $\widehat{\mathcal{S}}$ be the identity component of the $\hat{\theta}$-fixed points of $\widehat{T}$. Equivalently, $\widehat{\mathcal{S}}$ is the sub-torus of $\widehat{T}$ such that the map $X^*(\widehat{T}) \to X^*(\widehat{\mathcal{S}})$ is equal to the map $X^*(\widehat{T}) \longrightarrow X^*(\widehat{T})_{\hat{\theta},\mathrm{free}}$.

Definition 2.6.3. For $\mu \in X_*(T)^+ = X^*(\widehat{T})^+$, let V_μ be the highest weight representation of $\widehat{G}$ of highest weight μ. For all $\lambda \in X^*(\widehat{T})$, we write $V_\mu(\lambda)$ for the λ-weight space in V_μ for the action of $\widehat{T}$. For all $\lambda' \in X^*(\widehat{\mathcal{S}})$, we write $V_\mu(\lambda')_{\mathrm{rel}}$ for the λ'-weight space in V_μ for the action of $\widehat{\mathcal{S}}$.

As in §2.4, let $\mu \in X_*(T)^+$, and let $[b] \in B(G,\mu)$. Recall from (2.3.1) that the Newton point of $[b]$ is an element

$$\bar{\nu}_b \in (X_*(T)^+_{\mathbb{Q}})^\sigma \subset X_*(T)^\sigma_{\mathbb{Q}} = X_*(T)^\theta_{\mathbb{Q}}$$

By Lemma 2.6.1 (3) we identify $X_*(T)^\theta_{\mathbb{Q}}$ with

$$X_*(T)_\theta \otimes \mathbb{Q} = X_*(T)_{\theta,\mathrm{free}} \otimes \mathbb{Q} = X^*(\widehat{T})_{\hat{\theta},\mathrm{free}} \otimes \mathbb{Q} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q},$$

and we shall view $\bar{\nu}_b$ as an element of $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$. We also have $\kappa(b) \in \pi_1(G)_\Gamma = \pi_1(G)_\sigma$, which is equal to the image of μ.

Let $\widehat{Q}$ be the root lattice inside $X^*(\widehat{T})$. Applying Lemma 2.6.1 to $X = X^*(\widehat{T})$ and $Y = \widehat{Q}$, we obtain:

- $\widehat{Q}_{\hat{\theta}}$ is a free $\mathbb{Z}$-module. It injects into $X^*(\widehat{T})_{\hat{\theta}}$ and also injects into $X^*(\widehat{T})_{\hat{\theta},\mathrm{free}} = X^*(\widehat{\mathcal{S}})$.
- The image of the simple roots in $\widehat{Q}$ in $\widehat{Q}_{\hat{\theta}}$ (as a set) is a $\mathbb{Z}$-basis of $\widehat{Q}_{\hat{\theta}}$. We call members of this $\mathbb{Z}$-basis the *relative simple roots* in $\widehat{Q}_{\hat{\theta}}$.

Lemma 2.6.4. *There is a unique element $\tilde{\lambda}_b \in X^*(\widehat{T})_{\hat{\theta}}$ satisfying the following conditions:*

(1) The image of $\tilde{\lambda}_b$ in $\pi_1(G)_\sigma$ is equal to $\kappa(b)$.

(2) In $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$, the element $(\tilde{\lambda}_b)|_{\widehat{\mathcal{S}}} - \bar{\nu}_b$ is equal to a linear combination of the relative simple roots in $\widehat{Q}_{\hat{\theta}}$, with coefficients in $\mathbb{Q} \cap (-1, 0]$. Here $(\tilde{\lambda}_b)|_{\widehat{\mathcal{S}}}$ denotes the image of $\tilde{\lambda}_b$ under the map $X^*(\widehat{T})_{\hat{\theta}} \to X^*(\widehat{T})_{\hat{\theta},\mathrm{free}} = X^*(\widehat{\mathcal{S}})$.

Proof. This is just a reformulation of [HV17, Lemma 2.1]. We repeat the proof in our setting for completeness. The uniqueness follows from the fact that the preimage of $\kappa(b) \in \pi_1(G)_\sigma$ in $X^*(\widehat{T})_{\hat{\theta}}$ is a $\widehat{Q}_{\hat{\theta}}$-coset. To show the existence, it suffices to find an element $\lambda \in X^*(\widehat{T})_{\hat{\theta}}$ lifting $\kappa(b)$ such that $\lambda|_{\widehat{\mathcal{S}}} - \bar{\nu}_b$ is equal to a $\mathbb{Q}$-linear combination of the relative simple roots in $\widehat{Q}_{\hat{\theta}}$. In fact, if such λ exists, we can then modify λ by a suitable element in $\widehat{Q}_{\hat{\theta}}$ to obtain the desired $\tilde{\lambda}_b$. Now let λ be any element of $X^*(\widehat{T})_{\hat{\theta}}$ lifting $\kappa(b)$. By the compatibility of the two invariants of b, we know that the image of $\bar{\nu}_b \in X^*(\widehat{T})_{\hat{\theta}} \otimes \mathbb{Q}$ in $\pi_1(G)_\sigma \otimes \mathbb{Q}$ is equal to the natural image of $\kappa(b) \in \pi_1(G)_\sigma$. It follows that the element $\lambda|_{\widehat{\mathcal{S}}} - \bar{\nu}_b \in X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$ should map to zero in $\pi_1(G)_\sigma \otimes \mathbb{Q}$. On the other hand we have a short exact sequence

$$0 \longrightarrow \widehat{Q}_{\hat{\theta}} \longrightarrow X^*(\widehat{T})_{\hat{\theta}} \longrightarrow \pi_1(G)_\sigma \longrightarrow 0,$$

from which we obtain the short exact sequence

$$0 \longrightarrow \widehat{Q}_{\hat{\theta}} \otimes \mathbb{Q} \longrightarrow X^*(\widehat{T})_{\hat{\theta}} \otimes \mathbb{Q} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q} \longrightarrow \pi_1(G)_\sigma \otimes \mathbb{Q} \longrightarrow 0.$$

It follows that the element $\lambda|_{\widehat{\mathcal{S}}} - \bar{\nu}_b \in X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$ is in $\widehat{Q}_{\hat{\theta}} \otimes \mathbb{Q}$, as desired. $\square$

Definition 2.6.5. Let $\tilde{\lambda}_b \in X^*(\widehat{T})|_{\hat{\theta}}$ be as in Lemma 2.6.4. We write λ_b for $(\tilde{\lambda}_b)|_{\widehat{\mathcal{S}}} \in X^*(\widehat{\mathcal{S}})$.

Conjecture 2.6.6 (Miaofen Chen, Xinwen Zhu)**.** *Let $\mu \in X_*(T)^+$ and let $[b] \in B(G,\mu)$. There exists a natural bijection between $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$ and the Mirković–Vilonen basis of $V_\mu(\lambda_b)_{\mathrm{rel}}$.*

In [Nie18b, §1], Nie showed that in order to prove Conjecture 2.6.6, it suffices to prove it when the group G is adjoint and $[b] \in B(G)$ is basic. Moreover he defined a natural surjective map from the Mirković-Vilonen basis of $V_\mu(\lambda_b)_{\mathrm{rel}}$ to the set $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$. Thus in order to prove the conjecture, it suffices to prove the following numerical result.

Conjecture 2.6.7 (Numerical Chen–Zhu)**.** *Let $\mu \in X_*(T)^+$ and let $[b] \in B(G,\mu)$. Let $\mathscr{N}(\mu,b)$ be the cardinality of $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$. We have*

$$\mathscr{N}(\mu,b) = \dim V_\mu(\lambda_b)_{\mathrm{rel}}.$$

A further standard argument, for example [HZ16, §6], shows that one can also reduce to the case when G is F-simple. Therefore we have:

Proposition 2.6.8. *In order to prove Conjecture 2.6.6, it suffices to prove Conjecture 2.6.7 when G is adjoint, F-simple, and $b \in G(L)$ represents a basic σ-conjugacy class.* □

3. The action of $J_b(F)$

3.1. **The stabilizer of a component.** In this section we study the stabilizer in $J_b(F)$ of an irreducible component of $X_\mu(b)$. Here as before we let $\mu \in X_*(T)^+$ and $[b] \in B(G, \mu)$. The first main result is the following.

Theorem 3.1.1. *The stabilizer in $J_b(F)$ of each $Z \in \Sigma(X_\mu(b))$ is a parahoric subgroup of $J_b(F)$.*

We first reduce this statement to a question about the Iwahori affine Deligne–Lusztig varieties $X_w(b), w \in W_0 t^\mu W_0$. Note that $J_b(F)$ acts on each $X_w(b)$ via automorphisms.

Proposition 3.1.2. *The projection π in (2.4.1) induces a bijection between $\Sigma(X_\mu(b))$ and $\Sigma(X(\mu, b)^K)$ compatible with the action of $J_b(F)$. Moreover, this bijection maps $\Sigma^{\mathrm{top}}(X_\mu(b))$ onto $\Sigma^{\mathrm{top}}(X(\mu, b)^K)$.*

Proof. This follows from the fact that the fiber of π is (the perfection of) a flag variety. □

In view of Proposition 3.1.2, the proof of Theorem 3.1.1 reduces to showing that the stabilizer of each irreducible component of $X(\mu, b)^K$ is a parahoric subgroup of $J_b(F)$.

Now let $Y \in \Sigma(X(\mu, b)^K)$. Then since each $X_w(b)$ is locally closed in $\mathcal{FL}$, there exists $w \in W_0 t^\mu W_0$ such that $Y \cap X_w(b)$ is open dense in Y and is an irreducible component of $X_w(b)$. Since the action of $J_b(F)$ on $X(\mu, b)^K$ preserves $X_w(b)$, it follows that $j \in J_b(F)$ stabilizes Y if and only if j stabilizes $Y \cap X_w(b)$. Hence we have reduced to showing that the stabilizer in $J_b(F)$ of any element of $\Sigma(X_w(b))$ is a parahoric subgroup. We will show that this is indeed the case in Proposition 3.1.4 below.

One important tool needed in our proof is the following result, which is [GH10, Corollary 2.5.3].

Proposition 3.1.3. *Let $w \in W$, and let $s \in \mathbb{S}$ be a simple reflection.*

(1) If $\ell(sw\sigma(s)) = \ell(w)$, then there exists a universal homeomorphism $X_w(b) \to X_{sw\sigma(s)}(b)$.

(2) If $\ell(sw\sigma(s)) < \ell(w)$, then there is a decomposition $X_w(b) = X_1 \sqcup X_2$, where X_1 is closed and X_2 is open, and such that there exist morphisms $X_1 \to X_{sw\sigma(s)}(b)$ and $X_2 \to X_{sw}(b)$, each of which is the composition of a Zariski-locally trivial fiber bundle with one-dimensional fibers and a universal homeomorphism.

Moreover the universal homeomorphism in (1) and the morphisms $X_1 \to X_{sw\sigma(s)}(b)$ and $X_2 \to X_{sw}(b)$ in (2) are all equivariant for the action of $J_b(F)$. □

Proposition 3.1.4. *Assume $X_w(b) \neq \emptyset$ and let $Z \in \Sigma(X_w(b))$. The stabilizer in $J_b(F)$ of Z is a parahoric subgroup of $J_b(F)$.*

Proof. We prove this by induction on $\ell(w)$. Assume first that $w \in W$ is of minimal length in its σ-conjugacy class. Then $X_w(b) \neq \emptyset$ implies $\Psi(w) = b$, i.e. w and b represent the same σ-conjugacy class in $B(G)$, by [He14, Theorem 3.5]. In this case, by [He14, Theorem 4.8] and its proof, there is an explicit description of the stabilizer of an irreducible component which we recall.

Let ${}^P W \subset W$ (resp. $W^{\sigma(P)} \subset W$) denote the set of minimal representatives for the cosets $W_P \backslash W$ (resp. $W/W_{\sigma(P)}$). Let ${}^P W^{\sigma(P)}$ be the intersection ${}^P W \cap W^{\sigma(P)}$ (cf. [He14, §1.6]). By [He14, Theorem 2.3], there exists $P \subset \mathbb{S}$, $x \in {}^P W^{\sigma(P)}$, and $u \in W_P$, such that:

- W_P is finite.

- x is σ-straight and $x^{-1}\sigma(P)x = P$.

In this case, there is a $J_b(F)$-equivariant universal homeomorphism between $X_w(b)$ and $X_{ux}(b)$, and we have $\Psi(ux) = \Psi(w)$, see [He14, Corollary 4.4]. Hence we may assume $w = ux$. By [He14, Lemma 3.2] we have $\Psi(x) = \Psi(w)$, and therefore we may assume $b = \dot{x}$. Upon replacing P, we may assume P is minimal with respect to a fixed choice of x and u satisfying the above properties.

Let $\mathcal{P}$ denote the parahoric subgroup of $G(L)$ corresponding to P. The proof of [He14, Theroem 4.8] shows that

$$X_{ux}(\dot{x}) \cong J_{\dot{x}}(F) \times_{J_{\dot{x}}(F)\cap\mathcal{P}} X^{\mathcal{P}}_{ux}(\dot{x}),$$

where $X^{\mathcal{P}}_{ux}(\dot{x})$ is the reduced k-subscheme of the (perfectly) finite type scheme $L^+\mathcal{P}/L^+\mathcal{I}$ whose k-points are

$$X^{\mathcal{P}}_{ux}(\dot{x})(k) = \{g \in \mathcal{P}(\mathcal{O}_L)/\mathcal{I}(\mathcal{O}_L) | g^{-1}\dot{x}\sigma(g) \in \mathcal{I}(\mathcal{O}_L)\dot{u}\dot{x}\mathcal{I}(\mathcal{O}_L)\}.$$

Thus it suffices to show the stabilizer in $J_{\dot{x}}(F)\cap\mathcal{P}(\mathcal{O}_L)$ of an irreducible component of $X^{\mathcal{P}}_{ux}(\dot{x})$ is a parahoric subgroup of $J_{\dot{x}}(F)$.

Let $\overline{\mathcal{P}}$ denote the algebraic group over k, which is the reductive quotient of the special fiber of $\mathcal{P}$. Recall its Weyl group is naturally identified with W_P. Then $\mathcal{I}$ is the preimage of a Borel subgroup $\overline{\mathcal{I}}$ of $\overline{\mathcal{P}}$ under the reduction map $\mathcal{P} \to \overline{\mathcal{P}}$. Let $\sigma_{\dot{x}}$ denote the automorphism of $\overline{\mathcal{P}}$ given by $\overline{p} \mapsto \dot{x}^{-1}\sigma(\overline{p})\dot{x}$. Then the natural map $L^+\mathcal{P}/L^+\mathcal{I} \to \overline{\mathcal{P}}/\overline{\mathcal{I}}$ induces an identification between $X^{\mathcal{P}}_{\dot{u}\dot{x}}(\dot{x})$ and the (perfection of the) finite type Deligne-Lusztig variety

$$X' = \{\overline{p} \in \overline{\mathcal{P}}/\overline{\mathcal{I}} | \overline{p}^{-1}\sigma_{\dot{x}}(\overline{p}) \in \overline{\mathcal{I}}\dot{u}\overline{\mathcal{I}}\}.$$

The natural projection map $\mathcal{P} \to \overline{\mathcal{P}}$ takes $J_{\dot{x}}(F)\cap\mathcal{P}(\mathcal{O}_L)$ to $\overline{\mathcal{P}}^{\sigma_{\dot{x}}}$, and the action of $J_{\dot{x}}(F)\cap\mathcal{P}(\mathcal{O}_L)$ factors through this map. Since P is minimal satisfying $u \in W_P$ and since $x^{-1}\sigma(P)x = P$, it follows that u is not contained in any $\sigma_{\dot{x}}$-stable parabolic subgroup of W_P. Therefore by [Gö9, Corollary 1.2], X' is irreducible. It follows that the stabilizer of the irreducible component $1 \times X^{\mathcal{P}}_{ux}(\dot{x}) \subset X_{ux}(\dot{x})$ is $J_{\dot{x}}(F)\cap\mathcal{P}(\mathcal{O}_L)$, which is a parahoric of $J_{\dot{x}}(F)$. It also follows that the stabilizer of any other irreducible component of $X_{ux}(\dot{x})$ is a conjugate parahoric.

Now we assume w is not of minimal length in its σ-conjugacy class. By [HN14, Corollary 2.10], there exists $w' \approx_\sigma w$ and $s \in \mathbb{S}$ such that $sw'\sigma(s) < w'$. Then by Proposition 3.1.3, there is a $J_b(F)$-equivariant universal homeomorphism between $X_w(b)$ and $X_{w'}(b)$. Thus it suffices to prove the result for $X_{w'}(b)$.

Let $Z' \in \Sigma(X_{w'}(b))$, and let X_1 and X_2 be as in Proposition 3.1.3. We have either $Z' \cap X_1$ or $Z' \cap X_2$ is open dense in Z'. Assume $Z' \cap X_1$ is open dense in Z'; the other case is similar. Since $J_b(F)$ preserves X_1, it suffices to show that the stabilizer of $Z' \cap X_1$ is a parahoric. From the description of X_1, there exists an element $V \in \Sigma(X_{sw'\sigma(s)}(b))$ such that $Z' \cap X_1 \to V$ is a fibration and is $J_b(F)$-equivariant. Therefore by induction, the stabilizer of V is a parahoric of $J_b(F)$, and hence so is the stabilizer of $Z' \cap X_1$. $\square$

3.2. **Volumes of stabilizers and independence of F.** The second main result of this section is that the set of $J_b(F)$-orbits of irreducible components of $X_\mu(b)$ and the volume of the stabilizer of an irreducible component depend only on the affine root system together with the action of the Frobenius. In particular, it is *independent of* F in a manner which we will now make precise. This fact is a key observation that we will need for later applications.

By [He14, §6], the set of $J_b(F)$-orbits of top dimensional irreducible components of $X_w(b)$ depends only on the affine root system of G together with the action of σ. This is proved by using the Deligne–Lusztig reduction method to relate the number of orbits to coefficients of certain *class polynomials*, which can be defined purely in terms of the affine root system for G, see *loc. cit.* for details. In view of the fibration

$$\pi : X(\mu, b)^K \to X_\mu(b)$$

it follows that the same is true for $X_\mu(b)$. In particular, the number $\mathscr{N}(\mu, b)$ depends only on the affine root system and hence does not depend on the local field F.

We will need the following stronger result. To state it, we introduce some notations. Let F' be another local field with residue field $\mathbb{F}_{q'}$. Let G' be a connected reductive group over $\mathcal{O}_{F'}$. Let $T' \subset B' \subset G'_{F'}$ be analogous to $T \subset B \subset G_F$ as in §2.1. Define the hyperspecial vertex $\mathfrak{s}'$, the apartement V', and the anti-dominant chamber $\mathfrak{a}'$ analogously to $\mathfrak{s}, V, \mathfrak{a}$. Assume there is an identification $V \cong V'$ that maps $X_*(T)^+$ into $X_*(T)^+$, maps $\mathfrak{a}$ into $\mathfrak{a}'$, maps $\mathfrak{s}$ to $\mathfrak{s}'$, and induces a σ-σ' equivariant bijection between the affine root systems. Here σ' denotes the q'-Frobenius acting on the affine roots system of G'. We fix such an identification once and for all. To the pair (μ, b), we attach a corresponding pair (μ', b') for G' as follows. The cocharacter $\mu' \in X_*(T')^+$ is defined to be the image of μ under the identification $X_*(T)^+ \cong X_*(T')^+$. To construct b', we note that since b is basic, it is represented by a unique σ-conjugacy class in Ω. The identification fixed above induces an identification of Iwahori–Weyl groups $W \cong W'$, which induces a bijection on length-zero elements. Then b' is represented by the corresponding length-zero element in W'.

By our choice of b', the affine root systems of J_b and $J_{b'}$ together with the actions of Frobenius are identified. We thus obtain a bijection between standard parahorics of J_b and those of $J_{b'}$. Let $\mathcal{J} \subset J_b(F)$ and $\mathcal{J}' \subset J_{b'}(F')$ be parahoric subgroups. We say that $\mathcal{J}$ and $\mathcal{J}'$ are *conjugate*, if the standard parahoric conjugate to $\mathcal{J}$ is sent to the standard parahoric conjugate to $\mathcal{J}'$ under the above-mentioned bijection. In the following, we write $J := J_b(F)$ and $J' := J_{b'}(F')$.

Proposition 3.2.1. *There is a bijection*

$$J\backslash\Sigma^{\mathrm{top}}(X_\mu(b)) \xrightarrow{\sim} J'\backslash\Sigma^{\mathrm{top}}(X_{\mu'}(b'))$$

with the following property. If $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$ and $Z' \in \Sigma^{\mathrm{top}}(X_{\mu'}(b'))$ are such that JZ is sent to $J'Z'$, then the parahoric subgroups $\mathrm{Stab}_Z(J) \subset J$ and $\mathrm{Stab}_{Z'}(J') \subset J'$ are conjugate.

The proposition will essentially follow from the next lemma.

Lemma 3.2.2. *Let $w' \in W'$ correspond to $w \in W$ under the identification $W \cong W'$. Then there is a bijection*

$$\Theta : J\backslash\Sigma^{\mathrm{top}}(X_w(b)) \xrightarrow{\sim} J'\backslash\Sigma^{\mathrm{top}}(X_{w'}(b'))$$

with the following property. If $Z \in \Sigma^{\mathrm{top}}(X_w(b))$ and $Z' \in \Sigma^{\mathrm{top}}(X_{w'}(b'))$ are such that $\Theta(JZ) = J'Z'$, then $\mathrm{Stab}_Z(J)$ and $\mathrm{Stab}_{Z'}(J')$ are conjugate.

Proof. We induct on $\ell(w)$. First assume w is minimal length in its σ-conjugacy class. Then by [He14], $X_w(b) \neq \emptyset$ if and only if $\Psi(w) = b$, which holds if and only if $\Psi(w') = b'$, if and only if $X_{w'}(b') \neq \emptyset$. If this holds, then by [He14] the group J acts transitively on $\Sigma^{\mathrm{top}}(X_w(b))$, and similarly the group J' acts transitively on $\Sigma^{\mathrm{top}}(X_{w'}(b'))$. Hence the two sets $J\backslash\Sigma^{\mathrm{top}}(X_w(b))$ and $J'\backslash\Sigma^{\mathrm{top}}(X_{w'}(b'))$ are both singletons. Let Θ be the unique map between them. The desired conjugacy of the stabilizers follows from the computation of $\mathrm{Stab}_Z(J)$ in Proposition 3.1.4.

Now assume w is not of minimal length in its σ-conjugacy class. Let $Z \in \Sigma^{\mathrm{top}}(X_w(b))$. Then as in the proof of Proposition 3.1.4, there exists $w_1 \tilde{\approx}_\sigma w$ and $s \in \mathbb{S}$ such that $sw_1\sigma(s) < w_1$. Then $X_w(b)$ is universally homeomorphic to $X_{w_1}(b)$. We fix such a universal homeomorphism and we obtain a corresponding element $Z_1 \in \Sigma^{\mathrm{top}}(X_{w_1}(b))$. By Proposition 3.1.3, there exists $U \in \Sigma^{\mathrm{top}}(X_{sw_1\sigma(s)}(b))$ or $U \in \Sigma^{\mathrm{top}}(X_{sw_1}(b))$ such that Z_1 is universally homeomorphic to a fiber bundle over U. We assume $U \in \Sigma^{\mathrm{top}}(X_{sw_1\sigma(s)}(b))$; the other case is similar. Then $\mathrm{Stab}_Z(J) = \mathrm{Stab}_U(J)$. Note that the choice of U depends on the choice of w_1 and a universal homeomorphism $X_w(b) \cong X_{w_1}$. However upon fixing these choices, the J-orbit of U is canonically associated to the J-orbit of Z.

By the induction hypothesis, we have a bijection

$$\Theta_1 : J\backslash\Sigma^{\mathrm{top}}(X_{sw_1\sigma(s)}(b)) \xrightarrow{\sim} J'\backslash\Sigma^{\mathrm{top}}(X_{s'w_1'\sigma'(s')}(b')),$$

where $s', w_1' \in W'$ correspond to s, w_1 respectively. Choose

$$U' \in \Sigma^{\mathrm{top}}(X_{s'w_1'\sigma'(s')}(b'))$$

such that $J'U' = \Theta_1(JU)$. By the induction hypothesis, $\mathrm{Stab}_U(J)$ is conjugate to $\mathrm{Stab}_{U'}(J')$. Reversing the above process we obtain $Z' \in \Sigma^{\mathrm{top}}(X_{w'}(b')))$ such that $\mathrm{Stab}_{U'}(J') = \mathrm{Stab}_{Z'}(J')$. Again the J'-orbit of Z' is canonically associated to U' upon fixing the universal homeomorphism $X_{w'(b')} \cong X_{w_1'}(b')$.

We define the map Θ to send JZ to $J'Z'$. Switching the roles of G and G', we obtain the inverse map of Θ, and so Θ is a bijection as desired. □

Remark 3.2.3. In the situation of Lemma 3.2.2, note that since $\dim X_w(b) = \dim X_{w'}(b')$, we have $\dim Z = \dim Z'$ whenever $\Theta(JZ) = J'Z'$.

Proof of Proposition 3.2.1. For each $w \in W$, fix a bijection

$$\Theta : J\backslash\Sigma^{\mathrm{top}}(X_w(b)) \xrightarrow{\sim} J'\backslash\Sigma^{\mathrm{top}}(X_{w'}(b'))$$

as in Lemma 3.2.2. Let $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$. Then the preimage $\pi^{-1}(Z)$ under the projection $\pi : X(\mu, b)^K \to X_\mu(b)$ is a top dimensional irreducible component of $X(\mu, b)^K$. Hence there exists a unique $w \in W$ such that $X_w(b) \cap \pi^{-1}(Z)$ is open dense in Z. Moreover we have $X_w(b) \cap \pi^{-1}(Z) \in \Sigma^{\mathrm{top}}(X_w(b))$. Write Y for $X_w(b) \cap \pi^{-1}(Z)$, and choose $Y' \in \Sigma^{\mathrm{top}}(X_{w'}(b'))$ such that $\Theta(JY) = J'Y'$. Then since $\dim X(\mu, b)^K = \dim X(\mu', b')^{K'}$, the closure of Y' in $X(\mu', b')^{K'}$ gives an element of $\Sigma^{\mathrm{top}}(X(\mu', b')^{K'})$, whose J'-orbit is independent of the choice of Y'. Taking the image of the last element under the projection $X(\mu', b')^{K'} \to X_{\mu'}(b')$ we obtain an element $Z' \in \Sigma^{\mathrm{top}}(X_{\mu'}(b'))$ by dimension reasons, and the orbit $J'Z'$ is independent of the choice of Y'. Moreover $\mathrm{Stab}_Z(J)$ is conjugate to $\mathrm{Stab}_{Z'}(J')$ since $\mathrm{Stab}_Y(J)$ is conjugate to $\mathrm{Stab}_{Y'}(J')$. The association $JZ \mapsto J'Z'$ gives a well-defined map

$$J\backslash\Sigma^{\mathrm{top}}(X_\mu(b)) \longrightarrow J'\backslash\Sigma^{\mathrm{top}}(X_{\mu'}(b'))$$

which satisfies the condition in the proposition. Switching the roles of G and G' we obtain the inverse map. □

Proposition 3.2.1 implies that the truth of Conjecture 2.6.6 depends only on the affine root system associated to G together with the Frobenius action. In the proof of Conjecture 2.6.6 for unramified elements [XZ17, Theorem 4.4.14], the authors made the assumption that $p \neq 2, 3$. The following corollary is then immediate.

Corollary 3.2.4. *Theorem 4.4.14 in* [XZ17] *also holds for* $p = 2, 3$. □

For later applications we need the following. We now assume that b is basic, so that G and J_b are inner forms. Since b is basic we may choose a representative $\dot\tau$ for b where $\tau \in \Omega \subset W$. Using this one may identify the affine Weyl groups for J_b and G respecting the base alcoves. However the Frobenius action on W (or $\mathbb{S}$), defined by J_b, is given by $\tau\sigma$, where τ acts via left multiplication. See for example [HZ16, §5] for more details.

Since G and J_b are inner forms, the choice of a Haar measure on $G(F)$ determines a Haar measure on $J_b(F)$, and vice versa, see for example [Kot88, §1].

Corollary 3.2.5. *Fix the Haar measure on $J_b(F)$ such that the volume of $G(\mathcal{O}_F)$ is 1. For $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$ there exists a rational function $R(t) \in \mathbb{Q}(t)$ such that*

$$\mathrm{vol}(\mathrm{Stab}_Z(J_b)) = R(q).$$

Moreover this rational function satisfies $R(0) = e(J_b)$ and is independent of the local field F. Here $e(J_b)$ is the Kottwitz sign $(-1)^{\mathrm{rk}_F J_b - \mathrm{rk}_F G}$. More precisely, in the notation of Proposition 3.2.1, if JZ' corresponds to JZ, then $\mathrm{vol}(\mathrm{Stab}_{Z'}(J_{b'})) = R(q')$.

Proof. Since J_b splits over an unramified extension, the volume of a standard parahoric of $J_b(F)$ corresponding to any $\tau\sigma$-stable subset $K_J \subset \mathbb{S}$ can be calculated in terms of the affine root system. More precisely let $\mathcal{K}_J$ be the corresponding parahoric subgroup of $J_b(F)$ and $\mathcal{I}_J$ be the standard Iwahori subgroup of $J_b(F)$. Then we have

$$\begin{aligned}\mathrm{vol}(\mathcal{K}_J(\mathcal{O}_F)) &= \frac{\mathrm{vol}(\mathcal{K}_J(\mathcal{O}_F))}{\mathrm{vol}(\mathcal{I}_J(\mathcal{O}_F))}.\,\mathrm{vol}(\mathcal{I}(\mathcal{O}_F)).\frac{\mathrm{vol}(\mathcal{I}_J(\mathcal{O}_F))}{\mathrm{vol}(\mathcal{I}(\mathcal{O}_F))} \\ &= \frac{[\mathcal{K}_J(\mathcal{O}_F) : \mathcal{I}_J(\mathcal{O}_F)]}{[G(\mathcal{O}_F) : \mathcal{I}(\mathcal{O}_F)]}.\frac{\mathrm{vol}(\mathcal{I}_J(\mathcal{O}_F))}{\mathrm{vol}(\mathcal{I}(\mathcal{O}_F))}\end{aligned}$$

where $\mathcal{I}$ is the standard Iwahori subgroup of $G(F)$ (whereas previously we denoted by $\mathcal{I}$ the standard Iwahori subgroup of $G(L)$). The term $[\mathcal{K}_J(\mathcal{O}_F) : \mathcal{I}_J(\mathcal{O}_F)]$ (resp. $[G(\mathcal{O}_F) : \mathcal{I}(\mathcal{O}_F)]$) is just the number of $\mathbb{F}_q$-points in the finite-type full flag variety corresponding to the reductive quotient of the special fiber of $\mathcal{K}_J$ (resp. G).

For any connected reductive group $\overline{H}$ over $\mathbb{F}_q$ and $\overline{B}$ a Borel subgroup, let $W_{\overline{H}}$ denote the absolute Weyl group. Then we have the Bruhat decomposition

$$\overline{H}/\overline{B}(\overline{\mathbb{F}}_q) = \bigsqcup_{w \in W_{\overline{H}}} S_w.$$

We have $S_w(\mathbb{F}_q) \neq \emptyset$ if and only if $\sigma(w) = w$, in which case S_w is an affine space of dimension $\ell(w)$ defined over $\mathbb{F}_q$. In particular

$$\overline{H}/\overline{B}(\mathbb{F}_q) = \sum_{w \in W_{\overline{H}}^{\sigma}} q^{\ell(w)}.$$

It follows that $[\mathcal{K}_J(\mathcal{O}_F) : \mathcal{I}_J(\mathcal{O}_F)]$ and $[G(\mathcal{O}_F) : \mathcal{I}(\mathcal{O}_F)]$ are both polynomials in q with coefficients in $\mathbb{Z}$ and constant coefficient 1, and the polynomials depend only on the root systems of the corresponding reductive quotients of the special fiber.

Similarly the ratio $\frac{\mathrm{vol}(\mathcal{I}_J(\mathcal{O}_F))}{\mathrm{vol}(\mathcal{I}(\mathcal{O}_F))}$ can be computed as the ratio

$$\frac{\det(1 - q^{-1}\varsigma_J|V)}{\det(1 - q^{-1}\varsigma|V)} = \frac{\det(q - \varsigma_J|V)}{\det(q - \varsigma|V)}$$

where ς denotes the linear action of the Frobenius on $V = X_*(T)_{\mathbb{R}}$, and similarly for ς_J, see [Kot88, §1]. This is also a ratio of polynomials in q with coefficients in $\mathbb{Z}$, and the ratio at $q = 0$ is equal to

$$\det(\varsigma_J)/\det(\varsigma) = (-1)^{\mathrm{rk}_F J_b - \mathrm{rk}_F G} = e(J_b).$$

Moreover the polynomials depend only on the affine root system of G and the element b. The result follows. ☐

Finally in this section we record the following immediate consequence of Proposition 3.2.1.

Corollary 3.2.6. *If Conjecture 2.6.7 is true for all p-adic fields F, then it is true for all local fields F.* ☐

From now on we will assume F is a p-adic field. The upshot of this, as we will see in the next section, is that we are able to apply the Base Change Fundamental Lemma to count points on the affine Deligne–Lusztig variety.

4. Counting points

4.1. **The decent case.** For each $s \in \mathbb{N}$, let F_s be the degree s unramified extension of F in L. Let $\mathcal{O}_s$ be the valuation ring of F_s, and let k_s be residue field. The number $\mathscr{N}(\mu, b)$ depends on b only via its σ-conjugacy class $[b] \in B(G)$. Recall that given $b \in G(L)$, one can associate a slope cocharacter $\nu_b \in \mathrm{Hom}_L(\mathbb{D}, G)$, where $\mathbb{D}$ is the pro-torus with character group $\mathbb{Q}$.

Definition 4.1.1. Let $s \in \mathbb{N}$. We say that an element $b \in G(L)$ is *s-decent*, if $s\nu_b$ is an integral cocharacter $\mathbb{G}_m \to G$ (as opposed to a fractional cocharacter), and

$$b\sigma(b)\cdots\sigma^{s-1}(b) = (s\nu_b)(\pi_F). \tag{4.1.1}$$

Lemma 4.1.2. *Assume $b \in G(L)$ is s-decent. Then $s\nu_b$ is defined over F_s, and b belongs to $G(F_s)$.*

Proof. The proof is identical to the proof of [RZ96, Corollary 1.9]. □

Lemma 4.1.3. *Any class in $B(G)$ contains an element b which is s-decent for some $s \in \mathbb{N}$.*

Proof. This follows from [Kot85, §4.3]. □

In the following, we assume that b is s_0-decent, for some fixed $s_0 \in \mathbb{N}$. By the above lemma there is no loss of generality in making this assumption. We may and shall also assume that s_0 is divisible enough so that T is split over F_{s_0}.

Definition 4.1.4. Let $s \in s_0\mathbb{N}$. Let $G_s := \mathrm{Res}_{F_s/F}\, G$, so that $b \in G_s(F)$. Let Θ be the F-automorphism of G_s corresponding to the Frobenius $\sigma \in \mathrm{Gal}(F_s/F)$. Let $G_{s,b\Theta}$ be the centralizer of $b\Theta$ in G_s, which is a subgroup of G_s defined over F. Define

$$G(F_s)_{b\sigma} := \left\{ g \in G(F_s) | g^{-1}b\sigma(g) = b \right\}.$$

Thus $G(F_s)_{b\sigma}$ is naturally identified with $G_{s,b\Theta}(F)$.

Lemma 4.1.5. *For $s \in s_0\mathbb{N}$, there is a natural isomorphism of F-groups:*

$$J_b \cong G_{s,b\Theta}.$$

Moreover, $J_b(F) = G(F_s)_{b\sigma}$ as subgroups of $G(L)$.

Proof. Let R be an F-algebra. Recall from (2.5.1) that

$$J_b(R) = \left\{ g \in G(R \otimes_F L) | g^{-1}b\sigma(g) = b \right\}.$$

It suffices to prove that for any $g \in J_b(R)$ we have $g \in G(R \otimes_F F_{s_0})$. Now such a g commutes with $b \rtimes \sigma$, and so it commutes with $(b \rtimes \sigma)^{s_0}$. By (4.1.1), we have $(b \rtimes \sigma)^{s_0} = (s_0\nu_b)(\pi_f) \rtimes \sigma^{s_0}$. On the other hand, by the functoriality of the association $b \mapsto \nu_b$, we know that g commutes with ν_b. It follows that g commutes with σ^{s_0}, and so $g \in G(R \otimes_F F_{s_0})$ as desired. □

4.1.6. We keep assuming that F is p-adic. In §2.4, we discussed the geometric structure on $X_\mu(b)$, as a locally closed subscheme of the Witt vector Grassmannian over $k = \overline{k_F}$. In the current setting, $X_\mu(b)$ is naturally "defined over k_{s_0}". More precisely, we can work with the version of the Witt vector affine Grassmannian as an ind-scheme over k_{s_0} rather than over k. See [BS17, Corollary 9.6], cf. [Zhu17, §1.4]. Then the affine Deligne–Lusztig variety can be defined as a locally closed k_{s_0}-subscheme of the Witt vector affine Grassmannian, as in [Zhu17, §3.1.1]. The key point here is that since T is split over F_{s_0}, all the Schubert cells in the Witt vector affine Grassmannian are already defined over k_{s_0}, see [Zhu17, §1.4.3]. We denote by $\mathbb{G}\mathrm{r}_G$ and $\mathbb{X}_\mu(b)$ the Witt vector affine Grassmannian and the affine Deligne–Lusztig variety over k_{s_0}, and continue to use Gr_G and $X_\mu(b)$ to denote the corresponding objects over k. Thus

$$\mathrm{Gr}_G = \mathbb{G}\mathrm{r}_G \otimes_{k_{s_0}} k$$

$$X_\mu(b) = \mathbb{X}_\mu(b) \otimes_{k_{s_0}} k.$$

Let us recall the moduli interpretations of $\mathbb{G}\mathrm{r}_G$ and $\mathbb{X}_\mu(b)$. For any perfect k_{s_0}-algebra R, denote

$$W_{s_0}(R) := W(R) \otimes_{W(k_{s_0})} \mathcal{O}_{s_0}.$$

We have (see [Zhu17, Lemma 1.3])

$$\mathbb{G}\mathrm{r}_G(R) = \left\{(\mathcal{E}, \beta) | \mathcal{E} \text{ is a } G_{W_{s_0}(R)}\text{-torsor on } W_{s_0}(R),\ \beta \text{ is a trivialization of } \mathcal{E} \text{ on } W_{s_0}(R)[1/p]\right\}.$$

We also have (see [Zhu17, (3.1.2)])

$$\mathbb{X}_\mu(b)(R) = \left\{(\mathcal{E}, \beta) \in \mathbb{G}\mathrm{r}_G(R) | \mathrm{Inv}_x(\beta^{-1} b\sigma(\beta)) = \mu,\ \forall x \in \mathrm{spec} R\right\}.$$

Lemma 4.1.7. *For any $s \in s_0\mathbb{N}$, we have*

$$\mathbb{X}_\mu(b)(k_s) = \left\{g \in G(F_s)/G(\mathcal{O}_s) | g^{-1} b\sigma(g) \in G(\mathcal{O}_s)\mu(\pi_F)G(\mathcal{O}_s)\right\}.$$

Proof. We only need to show that $\mathbb{G}\mathrm{r}_G(k_s) = G(F_s)/G(\mathcal{O}_s)$. For this it suffices to show that any $G_{\mathcal{O}_s}$-torsor over $\mathcal{O}_s$ is trivial (cf. the proof of [Zhu17, Lemma 1.3]). By smoothness this reduces to the Lang–Steinberg theorem, namely that any G_{k_s}-torsor over the finite field k_s is trivial. □

Lemma 4.1.8. *The action of $J_b(F)$ on $X_\mu(b)$ descends to a natural action on $\mathbb{X}_\mu(b)$ via k_{s_0}-automorphisms.*

Proof. By Lemma 4.1.5, $J_b(F) = G(F_{s_0})_{b\sigma}$. The group $G(F_{s_0})_{b\sigma}$ naturally acts on $\mathbb{X}_\mu(b)(R)$ by acting on the trivializations β, for each perfect k_{s_0}-algebra R. □

Lemma 4.1.9. *Up to enlarging s_0, all the irreducible components of $X_\mu(b)$ are defined over k_{s_0}, i.e. come from base change of irreducible components of $\mathbb{X}_\mu(b)$.*

Proof. This follows from Lemma 2.5.5 and Lemma 4.1.8. □

4.2. **Twisted orbital integrals and point counting.** We fix $s_0 \in \mathbb{N}$ to be divisible enough so as to satisfy all the conclusions in §4.1. In particular G is split over F_{s_0} and the conclusion of Lemma 4.1.9 holds. Let $s \in s_0\mathbb{N}$.

For any $\mathbb{C}$-valued function $f \in C_c^\infty(G(F_s))$, define the twisted orbital integral

$$TO_b(f) := \int_{G(F_s)_{b\sigma}\backslash G(F_s)} f(g^{-1} b\sigma(g)) dg, \tag{4.2.1}$$

where $G(F_s)_{b\sigma}$ is equipped with an arbitrary Haar measure, and $G(F_s)$ is equipped with the Haar measure giving volume 1 to $G(\mathcal{O}_s)$. The general convergence of $TO_b(f)$ follows from the result of Ranga Rao [RR72]. However, in our case, by the decency equation (4.1.1) we know that $b\Theta$ is a semi-simple element of $G_s \rtimes \langle\Theta\rangle$, from which it follows that the twisted orbit is closed in $G(F_s)$. The convergence of $TO_b(f)$ then follows easily, cf. [Clo90, p. 266].

Definition 4.2.1. Let $f_{\mu,s} \in C_c^\infty(G(F_s))$ be the characteristic function of $G(\mathcal{O}_s)\mu(\pi_F)G(\mathcal{O}_s)$.

In the following we study the relationship between the twisted orbital integral $TO_b(f_{\mu,s})$ and point counting on $\mathbb{X}_\mu(b)$.

Lemma 4.2.2. *Each irreducible component Z of $X_\mu(b)$ is quasi-compact, and is isomorphic to the perfection of a quasi-projective variety over k. Moreover, Z has non-empty intersection only with finitely many other irreducible components of $X_\mu(b)$.*

Proof. Since $X_\mu(b)$ is a perfect scheme by Theorem 2.5.2, the generic point η of Z and its residue field $k(\eta)$ make sense. Moreover $k(\eta)$ is a perfect field containing k. Let $(\mathcal{E}, \beta) \in X_\mu(b)(k(\eta)) \subset \mathrm{Gr}_G(k(\eta))$ correspond to η, and define $\lambda := \mathrm{Inv}(\beta) \in X_*(T)^+$. Since $\{\eta\}$ is dense in Z, it follows from [Zhu17, Lemma 1.22] that Z is contained in $\mathrm{Gr}_{G,\leq\lambda}$, the Schubert variety inside Gr_G associated to λ. On the other hand, it follows from [Zhu17, §1.4.1, Lemma 1.22] and [BS17, Theorem 8.3] that $\mathrm{Gr}_{G,\leq\lambda}$ is the perfection of a projective variety over k. Since Z is closed in $X_\mu(b)$ and $X_\mu(b)$ is locally closed in Gr_G, we conclude that Z is locally closed in $\mathrm{Gr}_{G,\leq\lambda}$, and hence Z is quasi-compact and isomorphic to the perfection of a quasi-projective variety over k.

Since $X_\mu(b)$ is locally of perfectly finite type by Theorem 2.5.2, each point in $X_\mu(b)$ has an open neighborhood that intersects with only finitely many irreducible components of $X_\mu(b)$. Since Z is quasi-compact, it also intersects with only finitely many irreducible components of $X_\mu(b)$. □

Lemma 4.2.3. *The set $J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$ is finite. For all $\tilde{x} \in \mathbb{X}_\mu(b)(k_s)$, the stabilizer $\mathrm{Stab}_{\tilde{x}}\, J_b(F)$ in $J_b(F)$ is a compact open subgroup of $J_b(F)$. We have*

$$TO_b(f_{\mu,s}) = \sum_{x\in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)} \mathrm{vol}(\mathrm{Stab}_{\tilde{x}}\, J_b(F))^{-1},$$

where for each $x \in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$ we pick a representative $\tilde{x} \in \mathbb{X}_\mu(b)(k_s)$. Here $\mathrm{vol}(\mathrm{Stab}_{\tilde{x}}\, J_b(F))$ is computed with respect to the chosen Haar measure on $J_b(F) = G(F_s)_{b\sigma}$ (cf. Lemma 4.1.5).

Proof. Write $\mathbb{G} = G(F_s), \mathbb{J} = G(F_s)_{b\sigma}, \mathbb{K} = G(\mathcal{O}_s)$. Let $C = \left\{g \in \mathbb{J}\backslash\mathbb{G} | g^{-1}b\sigma(g) \in \mathbb{K}\mu(\pi_F)\mathbb{K}\right\}$. By the discussion below (4.2.1), we know that C is a compact subset of $\mathbb{J}\backslash\mathbb{G}$ (as C is the intersection of the compact set $\mathbb{K}\mu(\pi_F)\mathbb{K}$ with the closed twisted orbit of $b\sigma$ which is homeomorphic to $\mathbb{J}\backslash\mathbb{G}$), and $TO_b(f_{\mu,s})$ is nothing but the volume of C. Consider the action of $\mathbb{K}$ on C by right multiplication. The orbits are open subsets of C, and form a finite partition of C. On the other hand, by Lemma 4.1.7 and Lemma 4.1.8, these orbits are in one-to-one correspondence with $J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$. In particular $J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$ is finite.[2] Hence we write this finite partition as

$$C = \coprod_{x\in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)} C_x.$$

From the above discussion we obtain

$$TO_b(f_{\mu,s}) = \sum_{x\in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)} \mathrm{vol}(C_x). \tag{4.2.2}$$

Next we compute $\mathrm{vol}(C_x)$ for a fixed $x \in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$. Let $\tilde{x} \in \mathbb{X}_\mu(b)(k_s)$ be a lift of x. Fix $r \in \mathbb{G}$ representing $\tilde{x}$, see Lemma 4.1.7. Then by the definition of the quotient measure we have

$$1 = \int_{\mathbb{G}} 1_{r\mathbb{K}}(g)dg = \int_{\mathbb{J}\backslash\mathbb{G}}\int_{\mathbb{J}} 1_{r\mathbb{K}}(hg)dhdg = \int_{\mathbb{J}\backslash\mathbb{G}} \mathrm{vol}(\mathbb{J}\cap r\mathbb{K}g^{-1})dg. \tag{4.2.3}$$

Here the function

$$\mathbb{G} \ni g \mapsto \mathrm{vol}(\mathbb{J}\cap r\mathbb{K}g^{-1}) \text{ (volume computed with respect to the Haar measure on } \mathbb{J})$$

[2]Alternatively, one could also show the finiteness using the result of Rapoport–Zink [RZ99, Theorem 1.4] as interpreted in [HV17, Lemma 1.3].

descends to $\mathbb{J}\backslash\mathbb{G}$. We have

$$\mathrm{vol}(\mathbb{J}\cap r\mathbb{K}g^{-1}) = \begin{cases} \mathrm{vol}(\mathbb{J}\cap r\mathbb{K}r^{-1}), & g\in C_x \\ 0, & g\in \mathbb{J}\backslash\mathbb{G} - C_x. \end{cases}$$

Hence (4.2.3) reads

$$1 = \mathrm{vol}(C_x)\,\mathrm{vol}(\mathbb{J}\cap r\mathbb{K}r^{-1}).$$

Note that $\mathbb{J} = J_b(F)$ (Lemma 4.1.5), and $\mathbb{J}\cap r\mathbb{K}r^{-1}$ is obviously a compact open subgroup of $\mathbb{J}$. Moreover $\mathbb{J}\cap r\mathbb{K}r^{-1} = \mathrm{Stab}_{\tilde{x}}\, J_b(F)$. Hence we have

$$\mathrm{vol}(C_x) = \mathrm{vol}(\mathrm{Stab}_{\tilde{x}}\, J_b(F))^{-1}. \tag{4.2.4}$$

At this point we have already seen that $J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$ is finite, and that $\mathrm{Stab}_{\tilde{x}}\, J_b(F)$ is a compact open subgroup of $J_b(F)$. The rest of the lemma follows from (4.2.2) and (4.2.4). □

Proposition 4.2.4. *For $s\in\mathbb{N}$ divisible by s_0, we have*

$$TO_b(f_{\mu,s}) = \sum_{Z\in J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))^{-1}\,|k_s|^{\dim X_\mu(b)} + o(|k_s|^{\dim X_\mu(b)}), \quad s\gg 0.$$

Here Z runs over a set of representatives for the $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$, and for each such Z we denote by $\mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))$ *the volume of the compact open subgroup* $\mathrm{Stab}_Z(J_b(F))$ *of $J_b(F)$ (see Theorem 3.1.1) under the Haar measure of $J_b(F)$.*

Proof. Write $d = \dim X_\mu(b)$. By Lemma 2.5.5, we let $Z_1,\cdots,Z_M$ be a complete set of representatives of the $J_b(F)$-orbits in $\Sigma(X_\mu(b))$. Up to reordering, assume that $Z_1,\cdots,Z_N$ are of dimension d, and all of $Z_{N+1},\cdots,Z_M$ (if any) are of smaller dimensions. Write $d_i := \dim Z_i$.

Our starting point is Lemma 4.2.3. Note that if $x\in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$ and if $\tilde{x}\in\mathbb{X}_\mu(b)(k_s)$ is any representative of x, then the term $\mathrm{vol}(\mathrm{Stab}_{\tilde{x}}\, J_b(F))$ only depends on x. We henceforth denote it by vol_x.

For each $1\le i\le M$, define

$$U_i := Z_i - \bigcup_{1\le j<i,\gamma\in J_b(F)} \gamma Z_j - \bigcup_{\gamma\in J_b(F),\gamma Z_i\neq Z_i} \gamma Z_i$$

$$V_i := Z_i \cap \bigcup_{Z\in\Sigma(X_\mu(b)),Z\neq Z_i} Z$$

$$\mathbb{J}_i := \mathrm{Stab}_{U_i}(J_b(F)).$$

By Lemma 4.2.2, we know that U_i is open dense in Z_i, and V_i is a proper closed subset of Z_i. Since the action of $\mathrm{Stab}_{Z_i}(J_b(F))$ on Z_i obviously stabilizes U_i, it immediately follows that

$$\mathbb{J}_i = \mathrm{Stab}_{Z_i}(J_b(F)),$$

and in particular $\mathbb{J}_i$ is open compact in $J_b(F)$ by Theorem 3.1.1. By Lemma 4.1.9, we know that all Z_i, U_i, V_i come from base change of locally closed k_{s_0}-subschemes $\mathbb{Z}_i, \mathbb{U}_i, \mathbb{V}_i$ of $\mathbb{X}_\mu(b)$ respectively, where $\mathbb{Z}_i, \mathbb{U}_i, \mathbb{V}_i$ are perfections of quasi-projective varieties over k_{s_0}. Moreover, $\mathbb{Z}_i, \mathbb{U}_i$ are irreducible.

For each $1\le i\le M$ and each $y\in\mathbb{U}_i(k_s)$, define

$$\epsilon(y) := [\mathrm{Stab}_y(J_b(F)) : \mathrm{Stab}_y(J_b(F))\cap\mathbb{J}_i] = [\mathrm{Stab}_y(J_b(F)) : \mathrm{Stab}_y(\mathbb{J}_i)].$$

Thus $\epsilon(y)$ is not larger than the number of irreducible components of $X_\mu(b)$ that intersect Z_i at y. By Lemma 4.2.2 we know that $\epsilon(y)$ is finite.

By construction, the natural map

$$\pi : \bigsqcup_{1\leq i\leq M} \mathbb{U}_i(k_s) \longrightarrow J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$$

is a surjection (between finite sets). By the construction of the $\mathbb{U}_i$'s, for any $x \in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)$ we know that:

- The fiber $\pi^{-1}(x)$ is contained in $\mathbb{U}_{i(x)}(k_s)$, for a unique $1 \leq i(x) \leq M$.
- The group $\mathbb{J}_{i(x)}$ acts transitively on $\pi^{-1}(x)$.

Note that by the second property above, the function $\epsilon(\cdot)$ descends along π. We have

$$\mathrm{vol}_x \cdot \left|\pi^{-1}(x)\right| = \sum_{y\in\pi^{-1}(x)} \mathrm{vol}(\mathrm{Stab}_y(J_b(F))) = \sum_{y\in\pi^{-1}(x)} \epsilon(y)\,\mathrm{vol}(\mathrm{Stab}_y(\mathbb{J}_{i(x)})) = \epsilon(x)\,\mathrm{vol}(\mathbb{J}_{i(x)}), \tag{4.2.5}$$

where the last equality follows from the orbit-stabilizer relation applied to the $\mathbb{J}_{i(x)}$-orbit $\pi^{-1}(x)$. Hence we compute:

$$TO_b(f_{\mu,s}) \xlongequal{\text{Lemma 4.2.3}} \sum_{x\in J_b(F)\backslash\mathbb{X}_\mu(b)(k_s)} \mathrm{vol}_x^{-1} = \sum_{i=1}^{M}\sum_{y\in\mathbb{U}_i(k_s)} \mathrm{vol}_{\pi(y)}^{-1}\cdot\left|\pi^{-1}(\pi(y))\right|^{-1}$$

$$\xlongequal{(4.2.5)} \sum_{i=1}^{M} \mathrm{vol}(\mathbb{J}_i)^{-1} \sum_{y\in\mathbb{U}_i(k_s)} \epsilon(y)^{-1}. \tag{4.2.6}$$

Now let $\mathcal{U}_i$ be a quasi-projective variety whose perfection is $\mathbb{U}_i$. Then $\mathcal{U}_i$ is irreducible, and $\mathcal{U}_i(k_s) = \mathbb{U}_i(k_s)$. By the Lang–Weil bound (see [LW54]) applied to $\mathcal{U}_i$, we know that

$$|\mathbb{U}_i(k_s)| = |k_s|^{d_i} + o(|k_s|^{d_i}), \quad s \gg 0. \tag{4.2.7}$$

Similarly we have

$$|\mathbb{V}_i(k_s)| = |\mathbb{Z}_i(k_s)| - |(\mathbb{Z}_i - \mathbb{V}_i)(k_s)| = o(|k_s|^{d_i}), \quad s \gg 0. \tag{4.2.8}$$

We observe that for any $y \in \mathbb{U}_i(k_s)$, we have $\epsilon(y) \geq 2$ only if $y \in \mathbb{V}_i(k_s) \cap \mathbb{U}_i(k_s)$. Hence by (4.2.6) (4.2.7) (4.2.8) we have

$$TO_b(f_{\mu,s}) = \sum_{i=1}^{M} \mathrm{vol}(\mathbb{J}_i)^{-1}\left[|k_s|^{d_i} + o(|k_s|^{d_i})\right] = \sum_{i=1}^{N} \mathrm{vol}(\mathbb{J}_i)^{-1}\,|k_s|^{d} + o(|k_s|^{d}), \quad s \gg 0.$$

This is what we want to prove. □

4.3. **Applying the Base Change Fundamental Lemma in the basic case.** Recall that we assumed that $[b] \in B(G,\mu)$ and b is s_0-decent. We now assume in addition that b is basic.

For $s \in \mathbb{N}$, recall from [Kot82, §5] that the *s-th norm map* is a map

$$\mathfrak{N}_s : \{\sigma\text{-conjugacy classes in } G(F_s)\} \longrightarrow \{\text{stable conjugacy classes in } G(F)\}\,.$$

By [Kot82, Proposition 5.7], two σ-conjugacy classes in $G(F_s)$ are in the same fiber of $\mathfrak{N}_s$ precisely when they are *stably σ-conjugate*, a notion that is defined in [Kot82, §5].

Lemma 4.3.1. *Let $s \in s_0\mathbb{N}$. Then $\mathfrak{N}_s(b)$, as a stable conjugacy class in $G(F)$, consists of the single element $(s\nu_b)(\pi_F)$. Moreover, the cocharacter $s\nu_b : \mathbb{G}_m \to G$ is defined over F.*

Proof. By [Kot82, Corollary 5.3], any element in $\mathfrak{N}_s(b)$ is $G(\overline{F})$-conjugate to $b\sigma(b)\cdots\sigma^{s-1}(b) \in G(F_s)$, which is equal to $(s\nu_b)(\pi_F)$ since b is s-decent. Now since $(s\nu_b)(\pi_F)$ is central, we know that $\mathfrak{N}_s(b) = \{(s\nu_b)(\pi_F)\}$ and that $(s\nu_b)(\pi_F) \in G(F)$. It follows from the last statement that $s\nu_b$ is defined over F. □

Lemma 4.3.2. *Let $s \in s_0\mathbb{N}$. Let $b' \in G(F_s)$ be an element in the stable σ-conjugacy class of b. Then $\nu_{b'} = \nu_b$, and b' is s-decent. In particular b' is basic. Moreover, if $[b'] \in B(G, \mu)$, then b' is σ-conjugate to b in $G(F_s)$.*

Proof. By hypothesis we have $\mathfrak{N}_s(b) = \mathfrak{N}_s(b')$. By Lemma 4.3.1 applied to b, we know that the $\mathfrak{N}_s(b)$ consists of the single central element $(s\nu_b)(\pi_F) \in G(F)$. On the other hand any element of $\mathfrak{N}_s(b')$ should be $G(\overline{F})$-conjugate to $b'\sigma(b')\cdots\ \sigma^{s-1}(b')$ (by [Kot82, Corollary 5.3]). Therefore

$$b'\sigma(b')\cdots\ \sigma^{s-1}(b') = (s\nu_b)(\pi_F).$$

By the characterization of $\nu_{b'}$ (see [Kot85, §4.3]), the above equality implies that $\nu_{b'} = \nu_b$ and that b' is s-decent. The first part of the lemma is proved.

Now we assume $[b'] \in B(G, \mu)$. Since $B(G, \mu)$ contains a unique basic class, we have $[b'] = [b]$. Finally, by [RZ96, Corollary 1.10], we know that b and b' must be σ-conjugate in $G(F_s)$, since they are both s-decent and represent the same class in $B(G)$. □

Let $s \in s_0\mathbb{N}$. We now consider stable twisted orbital integrals along b. By our assumption that b is s-decent and basic, we know that

$$b\sigma(b)\cdots\sigma^{s-1}(b) = (s\nu_b)(\pi_F)$$

is a central element of $G(F_s)$, and is in fact an element of $G(F)$ by Lemma 4.3.1. In particular, this element is semi-simple, and the centralizer of this element (namely G) is connected. Therefore, with the terminology of [Kot82], an element $b' \in G(F_s)$ is stably σ-conjugate to b if and only if it is $\overline{F}$-σ-conjugate to b. This observation justifies our definition of the stable twisted orbital integral in the following, cf. [Hai09, §5.1].

For any $\mathbb{C}$-valued function $f \in C_c^\infty(G(F_s))$, we let $STO_b(f)$ be the stable twisted orbital integral

$$STO_b(f) := \sum_{b'} e(G_{s,b'\Theta})TO_{b'}(f),$$

where the summation is over the set of σ-conjugacy classes b' in $G(F_s)$ that are stably σ-conjugate to b, and $e(\cdot)$ denotes the Kottwitz sign. Here each $TO_{b'}$ is defined using the Haar measure on $G(F_s)$ giving volume 1 to $G(\mathcal{O}_s)$, and the Haar measure on $G(F_s)_{b'\sigma} = G_{s,b'\Theta}(F)$ that is transferred from the fixed Haar measure on $G(F_s)_{b\sigma} = G_{s,b\Theta}(F)$.

Definition 4.3.3. Denote by $\mathcal{H}_s$ the unramified Hecke algebra $\mathcal{H}(G(F_s)//G(\mathcal{O}_s))$. Denote by BC_s the base change map $\mathcal{H}_s \to \mathcal{H}_1$.

Definition 4.3.4. For $s \in s_0\mathbb{N}$, we write γ_s for $(s\nu_b)(\pi_F)$, and write γ_0 for γ_{s_0}. Thus γ_0 belongs to $G(F)$ (see Lemma 4.3.1) and $\gamma_s = \gamma_0^{s/s_0}$.

Proposition 4.3.5. *Assume $s \in s_0\mathbb{N}$. For any $f \in \mathcal{H}_s$, we have*

$$STO_b(f) = \mathrm{vol}(G(\mathcal{O}_F))^{-1}(\mathrm{BC}_s\, f)(\gamma_s),$$

where $\mathrm{vol}(G(\mathcal{O}_F))$ is defined in terms of the Haar measure on $G(F)$ transferred from the fixed Haar measure on $G_{s,b\Theta}(F)$, for the inner form $G_{s,b\Theta}$ of G.

Proof. By Lemma 4.3.1, $\mathfrak{N}_s(b)$ consists of the single central semi-simple element $\gamma_s \in G(F)$. By the Base Change Fundamental Lemma proved by Clozel [Clo90, Theorem 7.1] and Labesse [Lab90], we know that $STO_b(f)$ is equal to the stable orbital integral of $\mathrm{BC}_s\, f$ at $\mathfrak{N}_s(b)$. The latter degenerates to

$$e(G_{\gamma_s}) \cdot \frac{\mu_1}{\mu_2} \cdot (\mathrm{BC}_s\, f)(\gamma_s g)$$

since γ_s is central. Here μ_1 denotes the Haar measure on $G(F)$ giving volume 1 to $G(\mathcal{O}_F)$, and μ_2 denotes the Haar measure on $G_{\gamma_s}(F) = G(F)$ transferred from $G_{s,b\Theta}(F)$. The notation $\frac{\mu_1}{\mu_2}$ denotes the ratio between these two Haar measures on the same group $G(F)$. Obviously this ratio is equal to $\mathrm{vol}(G(\mathcal{O}_F))^{-1}$ as in the proposition. Finally, since $G_{\gamma_s} = G$ is quasi-split, $e(G_{\gamma_s}) = 1$. □

Lemma 4.3.6. *For s divisible by s_0, we have*

$$STO_b(f_{\mu,s}) = e(J_b) TO_b(f_{\mu,s}).$$

Proof. Firstly, by Lemma 4.1.5 we have $G_{s,b\Theta} = J_b$. We need to check that $TO_{b'}(f_{\mu,s}) = 0$, for any $b' \in G(F_s)$ that is stably σ-conjugate to b but not σ-conjugate to b in $G(F_s)$. Assume the contrary. Then there exists $g \in G(F_s)$ such that $f_{\mu,s}(g^{-1}b'\sigma(g)) \neq 0$, from which $g^{-1}b'\sigma(g) \in G(\mathcal{O}_s)\mu(\pi_F)G(\mathcal{O}_s)$. Hence $\kappa([b']) = \mu^\natural$ by Theorem 2.5.1. But this contradicts Lemma 4.3.2. □

Corollary 4.3.7. *Keep the notation in Proposition 4.2.4. For $s \gg 0$, we have*

$$e(J_b)\,\mathrm{vol}(G(\mathcal{O}_F))^{-1}(\mathrm{BC}_s\, f_{\mu,s})(\gamma_s) = \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))^{-1} \left|k_s\right|^{\dim X_\mu(b)} + o(\left|k_s\right|^{\dim X_\mu(b)}).$$

Proof. This follows from Proposition 4.2.4, Proposition 4.3.5, and Lemma 4.3.6. □

5. Matrix coefficients for the Satake transform

5.1. **General definitions and facts.** In this subsection we expose general facts concerning the Satake isomorphism, for unramified reductive groups over F. The aim is to give an interpretation of the coefficients for the matrix of the inverse Satake isomorphism in terms of a $\mathbf{q}$-analogue of Kostant's partition function. This is well-known by work of Kato [Kat82] in the case when G is split; we will need the case of non-split G. Our main reference is [CCH16, §1], which we follow closely. We also refer to [KS99, §1] for some of the facts.

Let G be an unramified reductive group over F. At this moment it is not necessary to fix a reductive model over $\mathcal{O}_F$ of G. Inside G we fix a Borel pair, namely a Borel subgroup B and a maximal torus $T \subset B$, both defined over F. In particular, T is a minimal Levi, and is split over F^{un}.

We denote by

$$\mathrm{BRD}(B,T) = (X^*(T), \Phi \supset \Delta, X_*(T), \Phi^\vee \supset \Delta^\vee)$$

the based root datum associated to (B,T). This based root datum has an automorphism θ induced by the Frobenius $\sigma \in \mathrm{Gal}(F^{\mathrm{un}}/F)$. Let $d = d_\theta < \infty$ be the order of θ.

Fix an F-pinning $(B, T, \mathbb{X}_+)$ of G. Since the Galois action on $\mathrm{BRD}(B,T)$ factors through the cyclic group generated by θ, we know that θ is a Galois-equivariant automorphism of $\mathrm{BRD}(B,T)$, and so it lifts uniquely to an F-automorphism of G preserving $(B, T, \mathbb{X}_+)$. We denote this F-automorphism of G still by θ. We are thus in a special case of the situation considered in [KS99, §1.3].

Let A be the maximal split sub-torus of T. We have[3]

$$X_*(A) = X_*(T)^\theta$$

$$X^*(A) = [X^*(T)/(1-\theta)X^*(T)]_{\mathrm{free}}.$$

Let ${}_F\Phi \subset X^*(A)$ be the image of $\Phi \subset X^*(T)$. It is well known (see for instance [Spr09, Theorem 15.3.8]) that the triple $(X^*(A), {}_F\Phi, X_*(A))$ naturally extends to a (possibly non-reduced) root datum

$$(X^*(A), {}_F\Phi, X_*(A), {}_F\Phi^\vee).$$

[3]In [CCH16, §1.1], it is stated that $X^*(A) = X^*(T)/(1-\theta)X^*(T)$, which is not true in general.

Elements of ${}_F\Phi$ are by definition θ-orbits in Φ. For $\alpha \in \Phi$, we write $[\alpha]$ for its θ-orbit. The θ-orbits in Δ give rise to a set of simple roots in ${}_F\Phi$, which we denote by ${}_F\Delta$. As usual, we denote the structural bijection ${}_F\Phi \xrightarrow{\sim} {}_F\Phi^\vee$ by $[\alpha] \mapsto [\alpha]^\vee$.

We let $\Phi^1 \subset {}_F\Phi$ be the subset of indivisible elements, namely, those $[\alpha] \in {}_F\Phi$ such that $\frac{1}{2}[\alpha] \notin {}_F\Phi$. The image of Φ^1 under the bijection ${}_F\Phi \xrightarrow{\sim} {}_F\Phi^\vee$ is denoted by $\Phi^{1,\vee}$. The tuple

$$(X^*(A), \Phi^1, X_*(A), \Phi^{1,\vee}) \tag{5.1.1}$$

has the structure of a reduced root datum. We note that ${}_F\Delta$ is also a set of simple roots in Φ^1. We henceforth denote ${}_F\Delta$ also by Δ^1. For the sets $\Phi, {}_F\Phi, \Phi^1, \Phi^\vee, {}_F\Phi^\vee, \Phi^{1,\vee}$, we put a superscript $+$ to denote their respective subsets of positive elements.

Recall that we denoted by W_0 the absolute Weyl group of G. We let $W^1 \subset W_0$ be the subgroup of elements that commute with θ. Then W^1 is a Coxeter group (see [CCH16, §1.1]), and we denote by ℓ_1 the length function on it.

Remark 5.1.1. Following [KS99, §1.1], consider the identity component G^1 of the θ-fixed points of G. Denote $B^1 := B \cap G^1$. Then G^1 is a connected split reductive group over F, and (B^1, A) is a Borel pair in G^1. The root datum of (G^1, A) can be identified with (5.1.1), and the Borel B^1 corresponds to the simple roots $\Delta^1 \subset \Phi^1$. Moreover, we can naturally identify W^1 with the Weyl group of (G^1, A). See *loc. cit.* for these statements.[4]

We next form the complex dual group of G. Let $\widehat{G}$ be the dual group of G over $\mathbb{C}$. Thus $\widehat{G}$ is a connected reductive group over $\mathbb{C}$, equipped with a Borel pair $(\widehat{B}, \widehat{T})$ and an isomorphism

$$\mathrm{BRD}(\widehat{B}, \widehat{T}) \xrightarrow{\sim} \mathrm{BRD}(B, T)^\vee,$$

where $\mathrm{BRD}(\widehat{B}, \widehat{T})$ denotes the based root datum associated to $(\widehat{B}, \widehat{T})$, and $\mathrm{BRD}(B,T)^\vee$ denotes the dual based root datum to $\mathrm{BRD}(B,T)$. In particular, we have canonical identifications $X^*(\widehat{T}) \cong X_*(T)$, $X_*(\widehat{T}) \cong X^*(T)$, which we think of as equalities. We fix a pinning $(\widehat{B}, \widehat{T}, \widehat{\mathbb{X}}_+)$. The action of θ on $\mathrm{BRD}(B,T)$ translates to an action on $\mathrm{BRD}(\widehat{B}, \widehat{T})$, and the latter lifts to a unique automorphism $\hat{\theta}$ of $\widehat{G}$ that preserves $(\widehat{B}, \widehat{T}, \widehat{\mathbb{X}}_+)$. The L-group ${}^LG$ is by definition the semi-direct product $\widehat{G} \rtimes \langle \hat{\theta} \rangle$, where $\langle \hat{\theta} \rangle$ denotes the cyclic group of order d generated by $\hat{\theta}$.

Define the torus

$$\widehat{A} := \widehat{T} / \left\{ t \cdot \hat{\theta}(t)^{-1} | t \in \widehat{T} \right\}.$$

We write Y^* for $X^*(\widehat{A})$. Thus we have

$$Y^* = X^*(\widehat{T})^{\hat{\theta}} = X_*(T)^\theta = X_*(A).$$

(Hence $\widehat{A}$ is indeed the dual torus of A.) Define

$$P^+ := \{\lambda \in Y^* | \langle \lambda, \alpha \rangle \geq 0, \ \forall \alpha \in \Delta\} = \left\{ \lambda \in Y^* | \langle \lambda, [\alpha] \rangle \geq 0, \ \forall [\alpha] \in \Delta^1 \right\}.$$

$$R^+ := \text{ the } \mathbb{Z}_{\geq 0}\text{-span of } {}_F\Phi^{\vee,+} \subset Y^*.$$

In the following we adopt the exponential notation for group algebras. The $\mathbb{C}$-vector space $\mathbb{C}[Y^*]^{W^1}$ has a basis given by

$$m_\mu := \sum_{\lambda \in W^1 \mu} e^\lambda,$$

for $\mu \in P^+$. Here $W^1 \mu$ denotes the orbit of μ under W^1.

[4] It is obvious that the torus $T^1 := G^1 \cap T$ in [KS99, §1.1] is equal to A in our case.

Definition 5.1.2. Let V be a finite dimensional representation of ${}^LG$ over $\mathbb{C}$. Let

$$V = \bigoplus_{\mu \in X^*(\widehat{T})} V(\mu)$$

be the weight space decomposition of V as a representation of $\widehat{T}$. Let γ be an element of $(N_{\widehat{G}}\widehat{T}) \rtimes \langle\hat{\theta}\rangle$ of finite order. Let $w \in W^1$ and $\epsilon \in \{\pm 1\}$. Define a $\mathbb{C}[X^*(\widehat{T})]$-linear operator $E^{\epsilon w}$ on $V \otimes_{\mathbb{C}} \mathbb{C}[X^*(\widehat{T})]$ by letting $E^{\epsilon w}$ act on each $V(\mu)$ via the scalar $e^{\epsilon w \mu} \in \mathbb{C}[X^*(\widehat{T})]$. If $\epsilon = 1$ and $w = 1$, we simply write E for $E^{\epsilon w}$.[5] For a formal variable $\mathbf{q}$, define

$$D(V, \gamma, E^{\epsilon w}, \mathbf{q}) := \det(1 - \mathbf{q}\gamma E^{\epsilon w}, V) \in \mathbb{C}[X^*(\widehat{T})][\mathbf{q}]$$

$$P(V, \gamma, E^{\epsilon w}, \mathbf{q}) := D(V, \gamma, E^{\epsilon w}, \mathbf{q})^{-1} \in \mathrm{Frac}\Big(\mathbb{C}[X^*(\widehat{T})][\mathbf{q}]\Big).$$

Definition 5.1.3. Let $\widehat{\mathfrak{n}}$ be the Lie algebra of the unipotent radical of $\widehat{B}$, equipped with the adjoint action of ${}^LG$ (see [CCH16, §1.3.1]). Let $w \in W^1$ and $\epsilon \in \{\pm 1\}$. We define

$$D(E^{\epsilon w}, \mathbf{q}) := D(\widehat{\mathfrak{n}}, \hat{\theta}, E^{\epsilon w}, \mathbf{q}).$$

$$P(E^{\epsilon w}, \mathbf{q}) := P(\widehat{\mathfrak{n}}, \hat{\theta}, E^{\epsilon w}, \mathbf{q}) = D(E^{\epsilon w}, \mathbf{q})^{-1}.$$

Definition 5.1.4. Let $[\alpha] \in \Phi^1 \subset {}_F\Phi$. We say that $[\alpha]$ is of type I, if $2[\alpha] \notin {}_F\Phi$. Otherwise we say that $[\alpha]$ is of type II. For $[\alpha] \in \Phi^1$, we define

$$\mathtt{b}([\alpha]) := \begin{cases} \#[\alpha], & \text{if } [\alpha] \text{ is of type I} \\ \frac{1}{2}\#[\alpha] & \text{if } [\alpha] \text{ is of type II} \end{cases}$$

where $\#[\alpha]$ denotes the size of $[\alpha]$ viewed as a θ-orbit in Φ. Then $\mathtt{b}([\alpha]) \in \mathbb{Z}_{\geq 1}$, see [CCH16, §1.1], cf. Remark 5.1.7 below.

Definition 5.1.5. For any element $\beta = [\alpha]^\vee \in \Phi^{1,\vee}$ (with $[\alpha] \in \Phi^1$), we define $\mathtt{b}(\beta)$ to be $\mathtt{b}([\alpha])$, and we say that β has type I or II if $[\alpha]$ has type I or II. For any $\beta' \in {}_F\Phi^\vee$ that is homothetic to $\beta \in \Phi^{1,\vee} \subset {}_F\Phi^\vee$, we define $\mathtt{b}(\beta')$ to be $\mathtt{b}(\beta)$. Thus we obtain a function

$$\mathtt{b} : {}_F\Phi^\vee \longrightarrow \mathbb{Z}_{\geq 1}.$$

Definition 5.1.6. For $\beta \in \Phi^{1,\vee}$, we define $d_\beta(\mathbf{q}) \in \mathbb{C}[Y^*][\mathbf{q}]$ as follows:

$$d_\beta(\mathbf{q}) := \begin{cases} 1 - \mathbf{q}^{\mathtt{b}(\beta)} e^{\beta}, & \text{if } \beta \text{ is of type I} \\ (1 - \mathbf{q}^{2\mathtt{b}(\beta)} e^{\beta/2})(1 + \mathbf{q}^{\mathtt{b}(\beta)} e^{\beta/2}), & \text{if } \beta \text{ is of type II.} \end{cases}$$

Here, when β is of type II, $\beta/2$ is always an element of ${}_F\Phi^\vee$ and in particular an element of Y^*, see [CCH16, §1.1] or [KS99, §1.3], cf. Remark 5.1.7 below.

Remark 5.1.7. To compare Definitions 5.1.4, 5.1.5, 5.1.6 with [CCH16, §1.3.2], we inform the reader that the symbols α, $[\beta^\vee]$, β used in [CCH16, §1.3.2] correspond to our β, $[\alpha]$, $\alpha^\vee$, respectively. Our choice of symbols is however compatible with [CCH16, Lemma 1.1.1]. Our type I or type II respectively correspond to diagram A_1 or A_2 in [CCH16, (1.3.6)]. Let $b(\cdot)$ be the function in [CCH16, (1.3.6)]. It is a function on both Φ and $\Phi^\vee$, and satisfies $b(\omega) = b(\omega^\vee)$ for all $\omega \in \Phi$. For any $\omega^\vee \in \Phi^\vee$ with image $[\omega^\vee] \in {}_F\Phi^\vee$, we have $b(\omega^\vee) = \mathtt{b}([\omega^\vee])$.

[5] Note that for $\epsilon = -1$ and $w = 1$, $E^{\epsilon w} = E^{-1}$ is indeed the inverse of E, so the notation is compatible.

Definition 5.1.8. Let R be an integral domain, and let $\mathbf{q}$ be a formal variable. We write $R[[\mathbf{q}]]'$ for the intersection of $\mathrm{Frac}(R[\mathbf{q}])$ and $R[[\mathbf{q}]]$, inside $(\mathrm{Frac}R)((\mathbf{q}))$.

Lemma 5.1.9. *For $\epsilon \in \{\pm 1\}$, we have*

$$D(E^\epsilon, \mathbf{q}) = \prod_{\epsilon\beta \in \Phi^{1,\vee,+}} d_\beta(\mathbf{q})$$

$$P(E^\epsilon, \mathbf{q}) = \prod_{\epsilon\beta \in \Phi^{1,\vee,+}} d_\beta(\mathbf{q})^{-1}$$

In particular, $D(E^\epsilon, \mathbf{q}) \in \mathbb{C}[Y^*][\mathbf{q}]$ and $P(E^\epsilon, \mathbf{q}) \in \mathbb{C}[Y^*][[\mathbf{q}]]'$.

Proof. The case $\epsilon = 1$ is [CCH16, Lemma 1.3.7], cf. Remark 5.1.7. The case $\epsilon = -1$ is proved in the same way, by switching the roles of positive elements and negative elements in $\Phi^{1,\vee}$. □

Definition 5.1.10. For each $\lambda \in Y^*$, we define $\mathcal{P}(\lambda, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$ as follows. In view of Definition 5.1.6 and Lemma 5.1.9, we have an expansion

$$P(E^{-1}, \mathbf{q}) = \sum_{\lambda \in R^+} \mathcal{P}(\lambda, \mathbf{q}) e^{-\lambda}, \tag{5.1.2}$$

where each $\mathcal{P}(\lambda, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$ is a polynomial in $\mathbf{q}$. We set $\mathcal{P}(\lambda, \mathbf{q}) := 0$ for all $\lambda \in Y^* - R^+$.

Corollary 5.1.11. *Let $\lambda \in R^+ - \{0\}$. Then the polynomial $\mathcal{P}(\lambda, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$ has constant term 0.*

Proof. This immediately follows from Lemma 5.1.9 and Definition 5.1.10. □

Definition 5.1.12. Let $\rho^\vee \in X_*(T) \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$ be the half sum of elements in $\Phi^{\vee,+}$, and let $\rho \in X^*(T) \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$ be the half sum of elements in Φ^+. Then $\rho^\vee$ in fact lies in $Y^* \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$, and is equal to the half sum of elements in $\Phi^{1,\vee,+}$, see [CCH16, §1.2]. For $w \in W^1$ and $\mu \in Y^*$, we denote

$$w \bullet \mu := w(\mu + \rho^\vee) - \rho^\vee \in Y^*.$$

We also write $w \bullet (\cdot)$ for the induced action of w on $\mathbb{C}[Y^*]$. Define the operator

$$J : \mathbb{C}[Y^*] \longrightarrow \mathbb{C}[Y^*]$$

$$f \mapsto J(f) := \sum_{w \in W^1} (-1)^{\ell_1(w)} w \bullet f. \tag{5.1.3}$$

Definition 5.1.13. For $\lambda \in P^+$ and for a formal variable $\mathbf{q}$, define

$$\tau_\lambda(\mathbf{q}) := J(e^\lambda) P(E^{-1}, \mathbf{q}) = \sum_{\mu \in R^+} J(e^\lambda)\, \mathcal{P}(\mu, \mathbf{q}) e^{-\mu} \in \mathbb{C}[Y^*][[\mathbf{q}]]'. \tag{5.1.4}$$

In particular, $\tau_\lambda(\mathbf{q})$ defines a $\mathbb{C}[[\mathbf{q}]]'$-valued function on $\widehat{A}$, by evaluating Y^* on $\widehat{A}$. Moreover, for any particular value $q \in \mathbb{C}$ of $\mathbf{q}$, it is clear from Lemma 5.1.9 that $\tau_\lambda(q)$ defines a rational function on $\widehat{A}$.

Definition 5.1.14. Let $\lambda \in P^+ \subset Y^* = X^*(\widehat{T})^{\hat{\theta}}$. Let V_λ be the irreducible representation of $\widehat{G}$ of highest weight λ.

Theorem 5.1.15 ([CCH16, Theorem 1.4.1])**.** *Let $\lambda \in P^+$. The character of V_λ, as a function on $\widehat{T}$, descends to a function on $\widehat{A}$. Moreover the following statements hold.*

(1) (Weyl character formula.) We have $\tau_\lambda(1) \in \mathbb{C}[Y^*]^{W^1}$. Moreover, $\tau_\lambda(1)$ is equal to the character of V_λ, when viewed as a function on $\widehat{A}$.

(2) (Weyl denominator formula.) $\tau_0(1) = 1$.

□

5.2. **Matrix coefficients.** We now fix a reductive model of G over $\mathcal{O}_F$ as in §2.1. As before we denote by $\mathcal{H}_1$ the spherical Hecke algebra $\mathcal{H}(G(F)//G(\mathcal{O}_F))$. For each $\mu \in X_*(A)$, we let $f_\mu \in \mathcal{H}_1$ be the characteristic function of $G(\mathcal{O}_F)\mu(\pi_F)G(\mathcal{O}_F)$. Then the $\mathbb{C}$-vector space $\mathcal{H}_1$ has a basis given by f_μ, for $\mu \in P^+ \subset X_*(A)$.

Recall that the Satake isomorphism is a $\mathbb{C}$-algebra isomorphism

$$\mathrm{Sat} : \mathcal{H}_1 \xrightarrow{\sim} \mathbb{C}[Y^*]^{W^1}.$$

In the following, we simply write f_μ for $\mathrm{Sat}(f_\mu)$, if there is no confusion. At this point we have introduced three bases of the $\mathbb{C}$-vector space $\mathbb{C}[Y^*]^{W^1}$, namely $\{m_\mu\}, \{\tau_\mu\}, \{f_\mu\}$, all indexed by $\mu \in P^+$. We denote some of the transition matrices as follows:

$$m_\mu = \sum_\lambda n_\mu^\lambda \tau_\lambda$$

$$\tau_\mu = \sum_\lambda t_\mu^\lambda f_\lambda$$

$$m_\mu = \sum_\lambda \mathfrak{M}_\mu^\lambda f_\lambda.$$

We are mainly interested in $\mathfrak{M}_\mu^\lambda$. There are known formulas for n_μ^λ and t_μ^λ which we recall below, see Theorem 5.2.2 and Theorem 5.2.3. Then $\mathfrak{M}_\mu^\lambda$ is just given by the multiplication of the other two transition matrices.

Definition 5.2.1. As in [CCH16, §1.7], we have a partition

$$Y^* = Y_0^* \sqcup \bigsqcup_{w \in W^1} Y_w^*,$$

where

$$Y_0^* := \left\{\lambda \in Y^* | \exists w \in W^1, w \text{ is a reflection, } w \bullet \lambda = \lambda\right\}$$
$$Y_w^* := \left\{\lambda \in Y^* | w \bullet \lambda \in P^+\right\}, \ w \in W^1.$$

For each $x \in W^1 \sqcup \{0\}$ we let $e_x : Y^* \to \{0,1\}$ be the characteristic function of Y_x^*.

Theorem 5.2.2 (van Leeuwen's formula, [CCH16, Lemma 1.7.4])**.** *For $\mu, \lambda \in P^+$, we have*

$$n_\mu^\lambda = \sum_{w' \in W^1/W_\mu^1} \sum_{w \in W^1} (-1)^{\ell_1(w)} e_w(w'\mu)\delta(w \bullet (w'\mu), \lambda). \tag{5.2.1}$$

Here $\delta(\cdot,\cdot)$ is the Kronecker delta, and W_μ^1 is the subgroup of W^1 generated by the reflections attached to those $[\alpha] \in \Delta^1$ such that $\langle \mu, [\alpha]\rangle = 0$. □

Combining (5.1.3) (5.1.4), for $\lambda' \in P^+$ we have

$$\tau_{\lambda'}(\mathbf{q}) = \sum_{\mu \in R^+} J(e^{\lambda'})\,\mathcal{P}(\mu,\mathbf{q})e^{-\mu} = \sum_{w \in W^1} (-1)^{\ell_1(w)} \sum_{\mu \in R^+} \mathcal{P}(\mu,\mathbf{q})e^{w\bullet\lambda' - \mu}.$$

For each $\lambda \in P^+$, we denote by $K_{\lambda',\lambda}(\mathbf{q})$ the contribution of e^λ in the above formula, or more precisely

$$K_{\lambda',\lambda}(\mathbf{q}) := \sum_{w \in W^1} (-1)^{\ell_1(w)}\,\mathcal{P}(w \bullet \lambda' - \lambda, \mathbf{q}). \tag{5.2.2}$$

This notation is compatible with [Kat82] when G is split.

Theorem 5.2.3 (Kato–Lusztig formula, [CCH16, Theorem 1.9.1])**.** *For $\mu, \lambda \in P^+$, we have*

$$t_\mu^\lambda = K_{\mu,\lambda}(\mathbf{q}^{-1})\mathbf{q}^{-\langle\lambda,\rho\rangle}\Big|_{\mathbf{q}=|k_F|}. \qquad \square \tag{5.2.3}$$

Corollary 5.2.4. *For $\mu, \lambda \in P^+$, we have*

$$\mathfrak{M}_\mu^\lambda = \mathbf{q}^{-\langle \lambda, \rho \rangle} \sum_{w' \in W^1/W^1_\mu} \sum_{w \in W^1} (-1)^{\ell_1(w)} \left(1 - e_0(w'\mu)\right) \mathcal{P}\left(w \bullet (w'\mu) - \lambda, \mathbf{q}^{-1}\right)\Big|_{\mathbf{q} = |k_F|}.$$

Proof. We write q for $|k_F|$. We have

$$\mathfrak{M}_\mu^\lambda = \sum_{\lambda' \in P^+} n_\mu^{\lambda'} t_{\lambda'}^\lambda$$

$$\overset{(5.2.1),(5.2.3)}{=\!=\!=} \sum_{\lambda' \in P^+} \sum_{w' \in W^1/W^1_\mu} \sum_{w'' \in W^1} (-1)^{\ell_1(w'')} e_{w''}(w'\mu) \delta(w'' \bullet (w'\mu), \lambda') K_{\lambda', \lambda}(q^{-1}) q^{-\langle \lambda, \rho \rangle}$$

$$\overset{5.2.2}{=\!=} \sum_{\lambda' \in P^+} \sum_{w' \in W^1/W^1_\mu} \sum_{w'' \in W^1} (-1)^{\ell_1(w'')} e_{w''}(w'\mu) \delta(w'' \bullet (w'\mu), \lambda') q^{-\langle \lambda, \rho \rangle} \sum_{w \in W^1} (-1)^{\ell_1(w)} \mathcal{P}(w \bullet \lambda' - \lambda, q^{-1})$$

$$\overset{\lambda' = w'' \bullet (w'\mu)}{=\!=\!=} \sum_{w' \in W^1/W^1_\mu} \sum_{w'' \in W^1} (-1)^{\ell_1(w'')} e_{w''}(w'\mu) q^{-\langle \lambda, \rho \rangle} \sum_{w \in W^1} (-1)^{\ell_1(w)} \mathcal{P}\left((ww'') \bullet (w'\mu) - \lambda, q^{-1}\right)$$

$$\overset{ww'' \mapsto w}{=\!=\!=} q^{-\langle \lambda, \rho \rangle} \sum_{w' \in W^1/W^1_\mu} \sum_{w'' \in W^1} \sum_{w \in W^1} (-1)^{\ell_1(w)} e_{w''}(w'\mu) \mathcal{P}\left(w \bullet (w'\mu) - \lambda, q^{-1}\right)$$

$$\overset{\sum_{w'' \in W^1} e_{w''}(\cdot) = 1 - e_0(\cdot)}{=\!=\!=\!=\!=} q^{-\langle \lambda, \rho \rangle} \sum_{w' \in W^1/W^1_\mu} \sum_{w \in W^1} (-1)^{\ell_1(w)} \left(1 - e_0(w'\mu)\right) \mathcal{P}\left(w \bullet (w'\mu) - \lambda, q^{-1}\right).$$

□

Motivated from Corollary 5.2.4, we make the following definition.

Definition 5.2.5. For $\mu, \lambda \in P^+$ and a formal variable $\mathbf{q}^{-1/2}$, we set

$$\mathfrak{M}_\mu^\lambda(\mathbf{q}^{-1}) := \mathbf{q}^{-\langle \lambda, \rho \rangle} \sum_{w' \in W^1/W^1_\mu} \sum_{w \in W^1} (-1)^{\ell_1(w)} \left(1 - e_0(w'\mu)\right) \mathcal{P}\left(w \bullet (w'\mu) - \lambda, \mathbf{q}^{-1}\right) \in \mathbb{C}[Y^*][\mathbf{q}^{-1/2}].$$

As a special case, we define

$$\mathfrak{M}_\mu^0(\mathbf{q}^{-1}) := \sum_{w' \in W^1/W^1_\mu} \sum_{w \in W^1} (-1)^{\ell_1(w)} \left(1 - e_0(w'\mu)\right) \mathcal{P}\left(w \bullet (w'\mu), \mathbf{q}^{-1}\right) \in \mathbb{C}[Y^*][\mathbf{q}^{-1}]. \tag{5.2.4}$$

Lemma 5.2.6. *Let $\mu, \lambda \in P^+$. Let $\nu \in X_*(A \cap Z_G)$. Then $\mathfrak{M}_\mu^\lambda(\mathbf{q}^{-1}) = \mathfrak{M}_{\mu-\nu}^{\lambda-\nu}(\mathbf{q}^{-1})$.*

Proof. In fact, we have

$$\langle \lambda, \rho \rangle = \langle \lambda - \nu, \rho \rangle, W^1_\mu = W^1_{\mu-\nu}, \quad e_0(w'\mu) = e_0(w'(\mu - \nu)),$$

$$w \bullet (w'\mu) - \lambda = w \bullet (w'(\mu - \nu)) - (\lambda - \nu),$$

for all $w' \in W^1/W^1_\mu$ and $w \in W^1$. □

5.3. **Interpretation in terms of Kostant partitions.** In certain cases the polynomials $\mathcal{P}(\lambda, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$ in Definition 5.1.10 have a concrete description as a $\mathbf{q}$-analogue of Kostant's partition function, which we now explain. Let $\mathbb{P}$ be the set of all functions

$$_F\Phi^{\vee,+} \longrightarrow \mathbb{Z}_{\geq 0}.$$

We shall typically denote an element of $\mathbb{P}$ by $\underline{m}$, and denote its value at any $\beta \in {}_F\Phi^{\vee,+}$ by $m(\beta)$. For $\underline{m} \in \mathbb{P}$, we define

$$\Sigma(\underline{m}) := \sum_{\beta \in {}_F\Phi^{\vee,+}} m(\beta)\beta \in R^+ \subset Y^*$$

$$|\underline{m}| := \sum_{\beta \in {}_F\Phi^{\vee,+}} m(\beta)\mathtt{b}(\beta) \in \mathbb{Z}_{\geq 0}.$$

Here $\mathtt{b}(\beta)$ is as in Definition 5.1.5.

For all $\lambda \in Y^*$, we define

$$\mathbb{P}(\lambda) := \{\underline{m} \in \mathbb{P} : \Sigma(\underline{m}) = \lambda\},$$

which is of course empty unless $\lambda \in R^+$. Elements of $\mathbb{P}(\lambda)$ are called *Kostant partitions* of λ. For any $L \in \mathbb{Z}_{\geq 0}$, we define

$$\mathbb{P}(\lambda)_L := \{\underline{m} \in \mathbb{P}(\lambda) : |\underline{m}| = L\}.$$

For $\lambda \in Y^*$, we define

$$\mathcal{P}_{\mathrm{Kos}}(\lambda, \mathbf{q}) := \sum_{\underline{m} \in \mathbb{P}(\lambda)} \mathbf{q}^{|\underline{m}|} \in \mathbb{C}[\mathbf{q}].$$

This is known in the literature as the $\mathbf{q}$-*analogue of Kostant's partition function*, at least when G is split.

Remark 5.3.1. In the sequel the function $\mathcal{P}_{\mathrm{Kos}}(\lambda, \mathbf{q})$ will only be used when ${}_F\Phi = \Phi^1$.

Proposition 5.3.2. *The following statements hold.*

(1) Assume ${}_F\Phi = \Phi^1$. For all $\lambda \in Y^*$ we have $\mathcal{P}(\lambda, \mathbf{q}) = \mathcal{P}_{\mathrm{Kos}}(\lambda, \mathbf{q})$.

(2) In general, to each $\underline{m} \in \mathbb{P}$, we can attach a polynomial $\mathcal{Q}(\underline{m}, \mathbf{q}) \in \mathbb{C}[\mathbf{q}]$, with the following properties:

   (a) For all $0 < x < 1$, we have $|\mathcal{Q}(\underline{m}, x)| \leq 1$.

   (b) For any $\lambda \in Y^*$ we have

$$\mathcal{P}(\lambda, \mathbf{q}) = \sum_{\underline{m} \in \mathbb{P}(\lambda)} \mathcal{Q}(\underline{m}, \mathbf{q})\mathbf{q}^{|\underline{m}|}.$$

Proof. Part (1) immediately follows from Definitions 5.1.6, 5.1.10. For part (2), we note that if $\beta \in \Phi^{1,\vee}$ is of type II, then $\beta' := \beta/2$ is an element of ${}_F\Phi^\vee$, and we have

$$d_\beta(\mathbf{q})^{-1} = \left[\sum_{i=0}^{\infty} (\mathbf{q}^{2\mathtt{b}(\beta)} e^{\beta/2})^i\right] \left[\sum_{i=0}^{\infty} (-\mathbf{q}^{\mathtt{b}(\beta)} e^{\beta/2})^i\right] = \left[\sum_{i=0}^{\infty} (\mathbf{q}^{2\mathtt{b}(\beta')} e^{\beta'})^i\right] \left[\sum_{i=0}^{\infty} (-\mathbf{q}^{\mathtt{b}(\beta')} e^{\beta'})^i\right],$$

which is a formal power series in the variable $e^{\beta'}\mathbf{q}^{\mathtt{b}(\beta')}$, with coefficients in $\mathbb{C}[\mathbf{q}]$. Explicitly we have

$$d_\beta(\mathbf{q})^{-1} = \sum_{n=0}^{\infty} \mathcal{R}_{\beta,n}(\mathbf{q})(\mathbf{q}^{\mathtt{b}(\beta')} e^{\beta'})^n,$$

with

$$\mathcal{R}_{\beta,n}(\mathbf{q}) = \sum_{i=0}^{n} (-1)^{n-i} \mathbf{q}^{i\mathtt{b}(\beta')} \in \mathbb{C}[\mathbf{q}]$$

for each $n \geq 0$. We observe that for all $0 < x < 1$ we have

$$|\mathcal{R}_{\beta,n}(x)| \leq 1. \tag{5.3.1}$$

Now for each $\beta' \in {}_F\Phi^\vee$ and each $n \geq 0$, define

$$\mathcal{Q}_{\beta',n}(\mathbf{q}) := \begin{cases} \mathcal{R}_{2\beta',n}(\mathbf{q}), & \text{if } 2\beta' \in \Phi^{1,\vee} \\ 1, & \text{if } 2\beta' \notin \Phi^{1,\vee}. \end{cases}$$

We take

$$\mathcal{Q}(\underline{m}, \mathbf{q}) := \prod_{\beta' \in {}_F\Phi^\vee} \mathcal{Q}_{\beta',m(\beta')}(\mathbf{q}).$$

Then condition (a) follows from the construction and the observation (5.3.1). Condition (b) follows from Lemma 5.1.9 and Definition 5.1.10. □

5.4. **Computation with the base change.** We keep the setting of §4.3 and §5.1. We assume that s_0 is divisible by the order d of θ, and consider $s \in s_0\mathbb{N}$.

The Satake isomorphism for $\mathcal{H}_s$ is

$$\mathrm{Sat} : \mathcal{H}_s \xrightarrow{\sim} \mathbb{C}[X^*(\widehat{T})]^{W_0}.$$

For each $\mu \in X^*(\widehat{T})^+$, let τ'_μ be the character of the highest weight representation V_μ of $\widehat{G}$ of highest weight μ. Then

$$\{\tau'_\mu\}_{\mu \in X^*(\widehat{T})^+}$$

is a basis of $\mathbb{C}[X^*(\widehat{T})]^{W_0}$. This basis is the absolute analogue of the basis $\{\tau_\mu\}_{\mu \in P^+}$ of $\mathbb{C}[Y^*]^{W^1}$ (i.e., they are the same if $\theta = 1$).

Recall from §2.6 and §5.1 that we have

$$Y^* = X_*(A) = X^*(\widehat{T})^{\hat{\theta}}$$

$$X^*(\widehat{\mathcal{S}}) = X^*(\widehat{T})_{\hat{\theta},\mathrm{free}}.$$

By Lemma 2.6.1 (3), the composition

$$Y^* \otimes \mathbb{Q} \to X^*(\widehat{T}) \otimes \mathbb{Q} \to X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$$

is invertible. We denote its inverse by

$$\lambda \mapsto \lambda^{(1)}.$$

We denote $s\lambda^{(1)}$ by $\lambda^{(s)}$. Then since s is divisible by d, for all $\lambda \in X^*(\widehat{\mathcal{S}})$ we have $\lambda^{(s)} \in Y^*$. Thus we have a map

$$X^*(\widehat{\mathcal{S}}) \longrightarrow Y^*, \quad \lambda \mapsto \lambda^{(s)}, \tag{5.4.1}$$

which is an isomorphism after $\otimes\mathbb{Q}$. In the case $\theta = 1$, this is none other than the multiplication-by-s map from Y^* to itself. In general, we denote by $X^*(\widehat{\mathcal{S}})^+ \subset X^*(\widehat{\mathcal{S}})$ the natural image of $X^*(\widehat{T})^+$. Then (5.4.1) maps $X^*(\widehat{\mathcal{S}})^+$ into $P^+ \subset Y^*$. Moreover, the action of W^1 on $X^*(\widehat{T})$ induces an action of W^1 on $X^*(\widehat{\mathcal{S}})$, and the map (5.4.1) is W^1-equivariant.

Proposition 5.4.1. *Under the Satake isomorphisms, the base change map* $\mathrm{BC}_s : \mathcal{H}_s \to \mathcal{H}_1$ *becomes*

$$\mathrm{BC}_s : \mathbb{C}[X^*(\widehat{T})]^{W_0} \longrightarrow \mathbb{C}[Y^*]^{W^1}$$

$$\forall \mu \in X^*(\widehat{T})^+,\ \tau'_\mu \mapsto \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}} \cdot m_{\lambda^{(s)}}.$$

Proof. To simplify notation we write X^* for $X^*(\widehat{T})$. To compute BC_s as a map $\mathbb{C}[X^*]^{W_0} \to \mathbb{C}[Y^*]^{W^1}$, it suffices to compose the map with the natural inclusion $\mathbb{C}[Y^*]^{W^1} \subset \mathbb{C}[X^*]$. For each $\mu \in X^{*,+}$, let

$$m'_\mu := \sum_{\lambda \in W_0(\mu)} e^\lambda.$$

Then $\{m'_\mu\}_{\mu \in X^{*,+}}$ is a basis of $\mathbb{C}[X^*]^{W_0}$. This basis is just the absolute analogue of the basis $\{m_\mu\}_{\mu \in P^+}$ of $\mathbb{C}[Y^*]^{W^1}$. It easily follows from definitions (see for example [Bor79]) that BC_s as a map

$$\mathbb{C}[X^*]^{W_0} \longrightarrow \mathbb{C}[X^*]$$

sends each m'_μ to

$$\sum_{\lambda \in W_0(\mu)} e^{\lambda + \hat{\theta}\lambda + \cdots + \hat{\theta}^{s-1}\lambda}.$$

It follows that for all $\mu \in X^{*,+}$, we have

$$\mathrm{BC}_s\, \tau'_\mu = \sum_{\lambda \in X^*} \dim V_\mu(\lambda) e^{\lambda + \hat{\theta}\lambda + \cdots + \hat{\theta}^{s-1}\lambda}. \tag{5.4.2}$$

Here the summation is over X^* and not over $X^{*,+}$. For each $\lambda \in X^*$, the element

$$\lambda + \hat{\theta}\lambda + \cdots + \hat{\theta}^{s-1}\lambda \in X^*$$

lies in $Y^* \subset X^*$, and its image under the natural map

$$Y^* = (X^*)^{\hat{\theta}} \longrightarrow X^*(\widehat{\mathcal{S}}) = (X^*)_{\hat{\theta},\mathrm{free}}$$

is equal to the image of $s\lambda \in X^*$ under the natural map

$$X^* \longrightarrow X^*(\widehat{\mathcal{S}}).$$

In other words, we have

$$\lambda + \hat{\theta}\lambda + \cdots + \hat{\theta}^{s-1}\lambda = (\lambda|_{\widehat{\mathcal{S}}})^{(s)},$$

where $\lambda|_{\widehat{\mathcal{S}}} \in X^*(\widehat{\mathcal{S}})$ denotes the image of λ under $X^* \to X^*(\widehat{\mathcal{S}})$. Hence by (5.4.2) we have

$$\mathrm{BC}_s\, \tau'_\mu = \sum_{\lambda \in X^*} \dim V_\mu(\lambda) e^{(\lambda|_{\widehat{\mathcal{S}}})^{(s)}},$$

which is easily seen to be equal to

$$\sum_{\lambda \in X^*(\widehat{\mathcal{S}})} \dim V_\mu(\lambda)_{\mathrm{rel}}\ e^{\lambda^{(s)}} = \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ m_{\lambda^{(s)}}.$$

□

Since b is basic and s_0-decent, and since s is divisible by s_0, by Lemma 4.3.1 the cocharacter $s\nu_b : \mathbb{G}_m \to G$ is a cocharacter of Z_G defined over F. In particular we may view $s\nu_b \in X_*(A) = Y^*$.

Corollary 5.4.2. *For $\mu \in X^*(\widehat{T})^+$, we have*

$$(\mathrm{BC}_s\, \tau'_\mu)(\gamma_s) = \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ \mathfrak{M}^0_{\lambda^{(s)} - s\nu_b}(|k_F|^{-1}).$$

Proof. By Proposition 5.4.1 we have

$$(\mathrm{BC}_s\, \tau'_\mu)(\gamma_s) = \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ m_{\lambda^{(s)}}(\gamma_s).$$

Recall from Lemma 4.3.1 that $s\nu_b$ is a central cocharacter of G defined over F. By Corollary 5.2.4, Definition 5.2.5, and Lemma 5.2.6, each $m_{\lambda^{(s)}}(\gamma_s)$ is equal to

$$\mathfrak{M}^{s\nu_b}_{\lambda^{(s)}}(|k_F|^{-1}) = \mathfrak{M}^{0}_{\lambda^{(s)}-s\nu_b}(|k_F|^{-1}).$$

□

5.5. **Some inductive relations.** We keep the setting and notation of §2.6 and §5.1. We assume in addition that G is adjoint, and that G is F-simple. As in Definition 5.2.5, we have the polynomials

$$\mathfrak{M}^0_\lambda(\mathbf{q}^{-1}) \in \mathbb{C}[Y^*][\mathbf{q}^{-1}], \quad \lambda \in P^+.$$

To emphasize the group G we write $\mathfrak{M}^0_{\lambda,G}(\mathbf{q})$ for $\mathfrak{M}^0_\lambda(\mathbf{q})$. In the following we discuss how to reduce the understanding of these polynomials to the case where G is absolutely simple.

We write Dynk_G for the Dynkin diagram of G (or more precisely that of (G, B, T)). By our assumption that G is adjoint and F-simple, the action of $\langle\theta\rangle$ on Dynk_G is transitive on the connected components. In particular all the connected components of Dynk_G are isomorphic. Let d_0 be the smallest natural number such that θ^{d_0} stabilizes each of the connected components. Thus d_0 is also equal to the total number of connected components of Dynk_G. Fix one connected component Dynk_G^+ of Dynk_G once and for all. The connected Dynkin diagram Dynk_G^+ , together with the automorphism θ^{d_0}, determines an unramified adjoint group G' over F, equipped with an F-pinning $(B', T', \mathbb{X}'_+)$. By construction G' is absolutely simple. We apply the constructions in §5.1 to G', adding an apostrophe in the notation when we denote an object associated to G', e.g., $A', (Y^*)', (P^+)', \mathcal{P}'(\lambda', \mathbf{q})$.

We have natural identifications

$$(X^*(A), {}_F\Phi, X_*(A), {}_F\Phi^\vee) \cong (X^*(A'), ({}_F\Phi)', X_*(A'), ({}_F\Phi^\vee)')$$

$$\Phi^1 \cong (\Phi^1)', \quad \Phi^{1,\vee} \cong (\Phi^{1,\vee})'$$

$$Y^* \cong (Y^*)'$$

$$W^1 \cong (W^1)'.$$

To be more precise, all the above identifications are derived from an identification

$$X_*(T) \cong \bigoplus_{i=0}^{d_0-1} X_*(T'), \tag{5.5.1}$$

under which the automorphism θ on the left hand side translates to the following automorphism on the right hand side:

$$(\chi_0, \chi_1, \cdots, \chi_{d_0-1}) \mapsto (\theta'\chi_{d_0-1}, \chi_0, \chi_1, \cdots, \chi_{d_0-2}).$$

In particular, the identification $(Y^*)' \cong Y^*$, when composed with $Y^* = X_*(A) \subset X_*(T)$ and with (5.5.1), is the diagonal map

$$(Y^*)' \longrightarrow \bigoplus_{i=0}^{d_0-1} X_*(T') \tag{5.5.2}$$

$$\chi' \mapsto (\chi', \cdots, \chi').$$

Proposition 5.5.1. *For $\lambda \in Y^*$ and $\lambda' \in (Y^*)'$ that correspond to each other, we have*

$$\mathfrak{M}^0_{\lambda,G}(\mathbf{q}^{-1}) = \mathfrak{M}^0_{\lambda',G'}(\mathbf{q}^{-d_0}),$$

as an element of $\mathbb{C}[Y^*][\mathbf{q}^{-1}] \cong \mathbb{C}[(Y^*)'][\mathbf{q}^{-1}]$.

Proof. When $\beta \in \Phi^{1,\vee}$ corresponds to $\beta' \in (\Phi^{1,\vee})'$, we know that β is of the same type (I or II) as β', and we have

$$\mathtt{b}(\beta) = d_0 \mathtt{b}'(\beta').$$

It follows from Lemma 5.1.9 and Definition 5.1.10 that

$$\mathcal{P}(\lambda, \mathbf{q}) = \mathcal{P}'(\lambda', \mathbf{q}^{d_0}) \in \mathbb{C}[\mathbf{q}],$$

for all $\lambda \in Y^*$ and $\lambda' \in (Y^*)'$ that correspond to each other. The proposition then follows from Definition 5.2.5. $\square$

Next we deduce a relation between the construction of λ_b in §2.6 for G and for G'. Denote by $\widehat{\mathcal{S}}'$ the counterpart of $\widehat{\mathcal{S}}$ for G'. Since G (resp. G') is adjoint, we know that $X^*(\widehat{T})$ (resp. $X^*(\widehat{T}'))$ has a $\mathbb{Z}$-basis consisting of the fundamental weights. It then easily follows from Lemma 2.6.1 (2) that we have

$$X^*(\widehat{T})_{\hat{\theta}} = X^*(\widehat{T})_{\hat{\theta},\mathrm{free}} = X^*(\widehat{\mathcal{S}})$$

$$X^*(\widehat{T}')_{\hat{\theta}'} = X^*(\widehat{T}')_{\hat{\theta}',\mathrm{free}} = X^*(\widehat{\mathcal{S}}'),$$

and we have natural identifications

$$X^*(\widehat{\mathcal{S}}) \cong X^*(\widehat{\mathcal{S}}')$$

$$\widehat{Q}_{\hat{\theta}} \cong \widehat{Q}'_{\hat{\theta}'}$$

$$\pi_1(G)_\sigma \cong \pi_1(G')_\sigma.$$

Fix an arbitrary $\mu \in X_*(T)$. Choose $\mu' \in X_*(T')$, such that the image of μ' in $X^*(\widehat{\mathcal{S}}')$ corresponds to the image of μ in $X^*(\widehat{\mathcal{S}})$. Such μ' always exists because the map $X_*(T') = X^*(\widehat{T}') \to X^*(\widehat{\mathcal{S}}')$ is surjective. It then follows that the image $\mu^\natural \in \pi_1(G)_\sigma$ of μ and the image $(\mu')^\natural \in \pi_1(G')_\sigma$ of μ' correspond to each other. Let $[b]$ (resp. $[b']$) be the unique basic element of $B(G,\mu)$ (resp. $B(G',\mu')$).

Proposition 5.5.2. *In the above setting, the elements $\lambda_b \in X^*(\widehat{\mathcal{S}})$ and $\lambda_{b'} \in X^*(\widehat{\mathcal{S}}')$ correspond to each other, under the identification $X^*(\widehat{\mathcal{S}}) \cong X^*(\widehat{\mathcal{S}}')$.*

Proof. This immediately follows from the uniqueness in Lemma 2.6.4. $\square$

6. The main result

6.1. **The number of irreducible components in terms of combinatorial data.** We keep the setting of §2.6 and §4.1. Thus we fix a reductive group scheme G over $\mathcal{O}_F$, an element $\mu \in X_*(T)^+$, and a basic class $[b] \in B(G,\mu)$. In this section, we relate the number of irreducible components $\mathscr{N}(\mu, b)$ to some combinatorial data.

As in §4.1, we fix $s_0 \in \mathbb{N}$ such that b is s_0-decent. As in §5.4 we assume s_0 is divisible by the order d of θ, and various natural numbers $s \in \mathbb{N}$ that are divisible by s_0. In particular, G will always be split over the extension F_s of F. We shall write

$$q_s := |k_s| = |k_F|^s.$$

By Corollary 4.3.7, we have

(6.1.1)
$$e(J_b)\operatorname{vol}(G(\mathcal{O}_F))^{-1}(\mathrm{BC}_s f_{\mu,s})(\gamma_s) = \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X_\mu(b))} \operatorname{vol}(\mathrm{Stab}_Z(J_b(F)))^{-1} q_s^{\dim X_\mu(b)} + o(q_s^{\dim X_\mu(b)}).$$

By the dimension formula in Theorem 2.5.2, we have

$$\dim X_\mu(b) = \langle \mu, \rho \rangle - \frac{1}{2}\mathrm{def}_G(b) \tag{6.1.2}$$

(since $\bar{\nu}_b$ is central). In particular, from (6.1.1) we get

$$(\mathrm{BC}_s\, f_{\mu,s})(\gamma_s) = O(q_s^{\langle \mu,\rho\rangle - \frac{1}{2}\mathrm{def}_G(b)}). \tag{6.1.3}$$

Proposition 6.1.1. *With the notation in §5.4, we have*

$$\mathrm{BC}_s(\tau'_\mu)(\gamma_s) = q_s^{-\langle \mu,\rho\rangle}(\mathrm{BC}_s\, f_{\mu,s})(\gamma_s) + o(q_s^{-\frac{1}{2}\mathrm{def}_G(b)}).$$

Proof. For λ running over $X^*(\widehat{T})^+$, the Satake transforms of $f_{\lambda,s}$, which we still denote by $f_{\lambda,s}$, form a basis of $\mathbb{C}[X^*(\widehat{T})]^{W_0}$. By the split case of Theorem 5.2.3, we have

$$\tau'_\mu = \sum_{\lambda \in X^*(\widehat{T})^+} K'_{\mu,\lambda}(q_s^{-1}) q_s^{-\langle \lambda,\rho\rangle} f_{\lambda,s},$$

where $K'_{\mu,\lambda}(\cdot)$ is the absolute analogue of (5.2.2), i.e., it is defined by (5.2.2) with θ replaced by 1. It is clear from Definition 5.1.10, Corollary 5.1.11, and (5.2.2), that $K'_{\mu,\lambda}(q_s^{-1}) = 0$ unless $\lambda \le \mu$, that

$$K'_{\mu,\mu}(q_s^{-1}) = 1 + O(q_s^{-1}),$$

and that

$$K'_{\mu,\lambda}(q_s^{-1}) = O(q_s^{-1})$$

for $\lambda < \mu$. Therefore

$$\tau'_\mu = q_s^{-\langle \mu,\rho\rangle} f_{\mu,s} + \sum_{\lambda \in X^*(\widehat{T})^+,\ \lambda \le \mu} O(q_s^{-1-\langle \lambda,\rho\rangle}) f_{\lambda,s} \tag{6.1.4}$$

Note that (6.1.3) is valid with μ replaced by each $\lambda \in X^*(\widehat{T})^+, \lambda \le \mu$, because we still have $[b] \in B(G,\lambda)$. The proposition then follows from (6.1.4) and the above-mentioned bounds provided by (6.1.3) with μ replaced by each $\lambda \le \mu$. □

Corollary 6.1.2. *We have*

$$\mathrm{BC}_s(\tau'_\mu)(\gamma_s) = e(J_b) \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X)} \mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))^{-1} q_s^{-\frac{1}{2}\mathrm{def}_G(b)} + o(q_s^{-\frac{1}{2}\mathrm{def}_G(b)}). \tag{6.1.5}$$

Proof. This follows by combining (6.1.1), (6.1.2), and Proposition 6.1.1. □

Theorem 6.1.3. *Assume the Haar measures are normalized such that* $\mathrm{vol}(G(\mathcal{O}_F)) = 1$. *There exists a rational function* $S_{\mu,b}(t) \in \mathbb{Q}(t)$ *that is independent of the local field* F *(in the same sense as Corollary 3.2.5), such that*

$$S_{\mu,b}(0) = \mathscr{N}(\mu, b), \tag{6.1.6}$$

$$S_{\mu,b}(q_1) = e(J_b) \sum_{Z \in J_b(F)\backslash \Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))^{-1}, \tag{6.1.7}$$

and such that

$$S_{\mu,b}(q_1) q_s^{-\frac{1}{2}\mathrm{def}_G(b)} = \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ \mathfrak{M}^0_{\lambda^{(s)} - s\nu_b}(q_1^{-1}) + o(q_s^{-\frac{1}{2}\mathrm{def}_G(b)}). \tag{6.1.8}$$

In particular

$$S_{\mu,b}(q_1) = \lim_{s\to\infty} q_s^{\frac{1}{2}\mathrm{def}_G(b)} \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ \mathfrak{M}^0_{\lambda^{(s)} - s\nu_b}(q_1^{-1}). \tag{6.1.9}$$

Proof. Fix a set of representatives $\{Z_i | 1 \leq i \leq \mathscr{N}(\mu, b)\}$ for the $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$. For each Z_i, let $R_{Z_i}(t) \in \mathbb{Q}(t)$ be rational function associated to Z_i as in Corollary 3.2.5. Let

$$S_{\mu,b}(t) := e(J_b) \sum_{i=1}^{\mathscr{N}(\mu,b)} R_{Z_i}(t)^{-1}.$$

Then $S_{\mu,b}(t) \in \mathbb{Q}(t)$ and it satisfies (6.1.6), (6.1.7). It follows from Corollary 3.2.5 and (6.1.5) that

$$\mathrm{BC}_s(\tau'_\mu)(\gamma_s) = S_{\mu,b}(q_1) q_s^{-\frac{1}{2}\mathrm{def}_G(b)} + o(q_s^{-\frac{1}{2}\mathrm{def}_G(b)}).$$

Comparing this with Corollary 5.4.2, we obtain (6.1.8). $\square$

The upshot of this theorem is that the right hand side of (6.1.9) is purely combinatorial and can be computed in certain instances using Kostant's partition function $\mathcal{P}_{\mathrm{Kos}}(\lambda, \mathbf{q})$. Moreover the fact that $S_{\mu,b}(t)$ is a rational function independent of the local field F, means that it is in principle determined by its values $S_{\mu,b}(q_1)$ for infinitely many choices of q_1. Once $S_{\mu,b}(t)$ is determined, the number $\mathscr{N}(\mu, b)$ can be read off from (6.1.6).

6.2. **The case of unramified elements.** In this subsection we apply Theorem 6.1.3 to prove Conjecture 2.6.6 for *unramified* and basic b. This is a new proof of a theorem of Xiao and Zhu [XZ17, Theorem 4.4.14].

We keep the setting of §6.1. Assume in addition that b is *unramified*, in the sense of [XZ17, §4.2]. Then we have $J_b \cong G$, and hence $\mathrm{def}_G(b) = 0, e(J_b) = 1$. By Theorem 6.1.3, we would like to compute

$$\lim_{s\to\infty} \sum_{\lambda \in X^*(\widehat{\mathcal{S}})^+} \dim V_\mu(\lambda)_{\mathrm{rel}}\ \mathfrak{M}^0_{\lambda^{(s)} - s\nu_b}(q_1^{-1}).$$

We have the following result.

Proposition 6.2.1. *Let $\lambda \in X^*(\widehat{\mathcal{S}})^+$. Consider $s \in s_0\mathbb{N}$. We have*

$$\mathfrak{M}^0_{\lambda^{(s)} - s\nu_b}(q_1^{-1}) = \begin{cases} 1, & \text{if } \lambda = \lambda_b \\ O(q_1^{-as}) \text{ for some } a \in \mathbb{R}_{>0}, & \text{otherwise} \end{cases}$$

Proof. Firstly, by [XZ17, Lemma 4.2.3], $\lambda_b \in X^*(\widehat{\mathcal{S}})$ is the unique element such that $\lambda_b^{(s)} = s\nu_b$ for one (and hence all) $s \in s_0\mathbb{N}$. (In particular, $\lambda_b \in X^*(\widehat{\mathcal{S}})^+$ as ν_b is central.) Thus the dichotomy in the proposition is the same as whether $\lambda^{(s)} - s\nu_b = 0$ for one (and hence all) $s \in s_0\mathbb{N}$.

Let $w \in W^1$. Since $w\rho^\vee - \rho^\vee$ is not in R^+ for $w \neq 1$, it follows from Definition 5.1.10 and Definition 5.2.5 that

$$\mathfrak{M}^0_0(q^{-1}) = 1 \in \mathbb{C}[q^{-1}].$$

This proves the case $\lambda^{(s)} = s\nu_b$.

Now assume $\lambda^{(s)} \neq s\nu_b$ (for all s). Fix an arbitrary $\mathbb{Q}$-basis $\{f_1, \cdots, f_r\}$ of

$$\mathrm{span}_{\mathbb{Q}}({}_F\Phi^{\vee,+}) \subset Y^* \otimes \mathbb{Q}.$$

For any $v \in \mathrm{span}_{\mathbb{Q}}({}_F\Phi^{\vee,+})$, we write

$$v = \sum_{i=1}^r c_i(v) f_i,\ c_i(v) \in \mathbb{Q}$$

for the expansion.

Fix $w, w' \in W^1$. We write $\mu_s := \lambda^{(s)} - s\nu_b$ and $\psi_s := w \bullet (w'\mu_s)$. By the formula (5.2.4), it suffices to show that

$$\mathcal{P}(\psi_s, q_1^{-1}) = O(q_1^{-as}) \tag{6.2.1}$$

for some $a > 0$.

If $\psi_s \notin R^+$ for some value of s, then by definition $\mathcal{P}(\psi_s, q^{-1}) = 0 \in \mathbb{C}[q^{-1}]$. Hence we may ignore these values of s.

On the other hand, we claim that if $\psi_s \in R^+$ for some $s \in s_0\mathbb{N}$, then $\psi_{ns} \in R^+$ for all $n \in \mathbb{N}$. In fact, the assumption that $\psi_s \in R^+$ is equivalent to that

$$ww'\mu_s \in R^+ + \rho^\vee - w\rho^\vee. \tag{6.2.2}$$

From (6.2.2) it follows that

$$ww'\mu_{ns} = n(ww'\mu_s) \in nR^+ + n(\rho^\vee - w\rho^\vee) \subset R^+ + \rho^\vee - w\rho^\vee,$$

from which $\psi_{ns} \in R^+$.

We thus assume that $\psi_s \in R^+$ for all sufficiently divisible s. Then (6.2.2) holds, and in particular

$$ww'\mu_s \in \mathrm{span}_{\mathbb{Q}}({}_F\Phi^{\vee,+}) - \{0\}\,.$$

Hence there exists $i_0 \in \{1, \cdots, r\}$ and $c_0 \in \mathbb{Q} - \{0\}$, such that for all sufficiently divisible s, we have

$$c_{i_0}(ww'\mu_s) = s \cdot c_0. \tag{6.2.3}$$

By Proposition 5.3.2 (2), we have

$$\left|\mathcal{P}(\psi_s, q_1^{-1})\right| \le \#\mathbb{P}(\psi_s) \cdot q_1^{-N_s}, \tag{6.2.4}$$

where

$$N_s := \min\left\{|\underline{m}| : \underline{m} \in \mathbb{P}(\psi_s)\right\}.$$

Let

$$A := |c_{i_0}(w\rho^\vee - \rho^\vee)| \in \mathbb{R}_{\ge 0}$$
$$B := \max\left\{|c_{i_0}(\beta)| : \beta \in {}_F\Phi^{\vee,+}\right\} + 1 \in \mathbb{R}_{>0}.$$

Then for all sufficiently divisible s we have

$$|c_{i_0}(\psi_s) - c_{i_0}(ww'\mu_s)| \le A,$$

$$\forall \underline{m} \in \mathbb{P}(\psi_s),\ B\,|\underline{m}| \ge c_{i_0}(\psi_s).$$

It follows that

$$N_s \ge B^{-1} \cdot c_{i_0}(\psi_s) \ge B^{-1}(c_{i_0}(ww'\mu_s) - A) \overset{(6.2.3}{=\!=\!=} B^{-1}(sc_0 - A).$$

Since B^{-1} and c_0 are both non-zero, there is a constant $N_0 > 0$ such that

$$N_s > N_0 \cdot s \tag{6.2.5}$$

for all sufficiently divisible s.

On the other hand, because for each $1 \le i \le r$ the coefficient $c_i(\psi_s) \in \mathbb{Q}$ is an affine function in s, there exists a constant $L > 0$ such that

$$\max\left\{|\underline{m}| : \underline{m} \in \mathbb{P}(\psi_s)\right\} \le Ls$$

for all sufficiently divisible s. It then easily follows that

$$\#\mathbb{P}(\psi_s) = \sum_{l=1}^{Ls} \#\mathbb{P}(\psi_s)_l \le \sum_{l=1}^{Ls} l^{\#({}_F\Phi^{\vee,+})} \le (Ls)^M \tag{6.2.6}$$

for some constant $M > 0$. The desired estimate (6.2.1) then follows from (6.2.4), (6.2.5), and (6.2.6). $\square$

Theorem 6.2.2. *In the current setting, $\mathscr{N}(\mu, b) = \dim V_\mu(\lambda_b)_{\mathrm{rel}}$. Moreover, for any $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, the group $\mathrm{Stab}_Z(J_b(F))$ is a hyperspecial subgroup of $J_b(F) = G(F)$.*

Proof. Let $S_{\mu,b}(t) \in \mathbb{Z}(t)$ be as in Theorem 6.1.3. By (6.1.9) and Proposition 6.2.1, we have

$$S_{\mu,b}(q_1) = \dim V_\mu(\lambda)_{\mathrm{rel}},$$

where λ is the unique element of $X^*(\widehat{\mathcal{S}})^+$ such that $\lambda^{(s_0)} = s_0\nu_b$. By varying the local field F, we see that $S_{\mu,b}(t)$ is the constant $\dim V_\mu(\lambda_b)_{\mathrm{rel}}$. In particular

$$\mathscr{N}(\mu, b) = S_{\mu,b}(0) = \dim V_\mu(\lambda_b)_{\mathrm{rel}}.$$

For the second part, recall that according to our normalization each

$$\mathrm{vol}(\mathrm{Stab}_Z(J_b(F))) \leq 1,$$

where equality holds if and only if $\mathrm{Stab}_Z(J_b(F))$ is hyperspecial. On the other hand, combining (6.1.6) and (6.1.7) and the fact that $S_{\mu,b}(t)$ is constant, we have

$$\mathscr{N}(\mu, b) = \sum_{Z \in J_b(F) \backslash \Sigma^{\mathrm{top}}(X_\mu(b))} \mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))^{-1}.$$

It follows that each $\mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))$ must be 1. □

Remark 6.2.3. Arguably the hardest part of the proof of the corresponding result in [XZ17] is to show that the stabilizer of any irreducible component in $\Sigma^{\mathrm{top}}(X_\mu(b))$ is hyperspecial.

Remark 6.2.4. In Theorem 6.2.2 we assume that b is basic. One can show as in Proposition 2.6.8 that the general unramified case of Conjecture 2.6.6 reduces to the basic unramified case.

6.3. **The general case.** We now prove the general case of Conjecture 2.6.6. By Proposition 2.6.8, there is no loss of generality in assuming that G is adjoint and F-simple, and that $[b] \in B(G)$ is basic. In particular $\bar{\nu}_b = 0$. We keep the setting and notation of §6.1. In the following, we do not fix a prescribed $\mu \in X_*(T)^+$ such that $[b] \in B(G, \mu)$.

As in Definition 2.6.5, we have $\lambda_b \in X^*(\widehat{\mathcal{S}})$. Denote by λ_b^+ the unique element in the W^1-orbit of λ_b that is in $X^*(\widehat{\mathcal{S}})^+$. We define (cf. the discussion before Lemma 2.6.4)

$$\Lambda(b) := \left\{ \lambda \in X^*(\widehat{\mathcal{S}})^+ | \lambda \neq \lambda_b^+, \ \lambda - \lambda_b \in \widehat{Q}_{\hat{\theta}} \right\}.$$

Since G is F-simple, all the simple factors of $G_{\overline{F}}$ have the same Dynkin type. We refer to this type as the type of G. The following proposition is the key result towards the proof of Conjecture 2.6.7.

Proposition 6.3.1 (Key estimate)**.** *Assume G is adjoint, F-simple, and not of type A. Let $[b] \in B(G)$ be a basic class. Assume $[b]$ is not unramified.*

(1) Assume G is not a Weil restriction of the split adjoint E_6. For all $\lambda \in \Lambda(b)$, there exists $a > 0$, such that

(6.3.1) $$\mathfrak{M}^0_{\lambda^{(s)}}(q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)}).$$

Moreover, there exists $\mu_1 \in X^*(\widehat{T})^+$ that is minuscule, such that $[b] \in B(G, \mu_1)$ and $\dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = 1$.

(2) Assume G is a Weil restriction of the split adjoint E_6 (necessarily along an unramified extension of F). Then there is an element $\lambda_{\mathrm{bad}} \in \Lambda(b)$ with the following properties:

- *For all $\lambda \in \Lambda(b) - \{\lambda_{\mathrm{bad}}\}$, there exists $a > 0$, such that*

(6.3.2) $$\mathfrak{M}^0_{\lambda^{(s)}}(q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)}).$$

- *There exist $\mu_1, \mu_2 \in X^*(\widehat{T})^+$, such that μ_1 is minuscule and μ_2 is a sum of dominant minuscule elements, such that $b \in B(G, \mu_1) \cap B(G, \mu_2)$, and such that*

$$\dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = 1, \quad V_{\mu_1}(\lambda_{\mathrm{bad}})_{\mathrm{rel}} = 0, \quad V_{\mu_2}(\lambda_{\mathrm{bad}})_{\mathrm{rel}} \neq 0.$$

The proof of Proposition 6.3.1 will occupy §7, §8, §9 below. We now admit this proposition.

Theorem 6.3.2. *Conjecture 2.6.7 holds for G adjoint, F-simple, not of type A, and for $[b] \in B(G)$ basic. More precisely:*

- *If $[b]$ is unramified, then our proof is logically independent of the approaches in* [XZ17], [HV17], *or* [Nie18a].
- *If $[b]$ is not unramified, and if we are in the situation of Proposition 6.3.1 (1), then our proof depends on results from* [HV17].
- *If $[b]$ is not unramified, and if we are in the situation of Proposition 6.3.1 (2), then our proof depends on results from* [Nie18a].

Proof. If $[b]$ is unramified, then the present theorem is just Theorem 6.2.2 (which is also valid for type A). From now on we assume $[b]$ is not unramified.

Assume we are in the situation of Proposition 6.3.1 (1). By Theorem 6.1.3 and Proposition 6.3.1 (1), for all $\mu \in X_*(T)^+$ such that $[b] \in B(G, \mu)$, we have

$$S_{\mu,b}(q_1) = \dim V_\mu(\lambda_b)_{\mathrm{rel}} \lim_{s \to \infty} q_s^{\frac{1}{2}\mathrm{def}_G(b)} \mathfrak{M}^0_{\lambda_b^{+,(s)}}(q_1^{-1}).$$

In particular, we have

$$S_{\mu,b}(q_1) = \frac{\dim V_\mu(\lambda_b)_{\mathrm{rel}}}{\dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}}} S_{\mu_1,b}(q_1) = \dim V_\mu(\lambda_b)_{\mathrm{rel}} S_{\mu_1,b}(q_1).$$

By varying the local field F (whilst preserving the affine root system of G) we conclude that

$$S_{\mu,b}(t) = \dim V_\mu(\lambda_b)_{\mathrm{rel}} S_{\mu_1,b}(t) \in \mathbb{C}(t).$$

In particular

$$\mathscr{N}(\mu, b) = S_{\mu,b}(0) = \dim V_\mu(\lambda_b)_{\mathrm{rel}} S_{\mu_1,b}(0) = \dim V_\mu(\lambda_b)_{\mathrm{rel}} \mathscr{N}(\mu_1, b).$$

On the other hand, since μ_1 is minuscule, it is shown in [HV17, Theorem 1.5] that

$$\mathscr{N}(\mu_1, b) \leq \dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}}.$$

Since $\mathscr{N}(\mu_1, b)$ is a positive natural number and $\dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = 1$, it follows that $\mathscr{N}(\mu_1, b) = 1$, and that

$$\mathscr{N}(\mu, b) = \dim V_\mu(\lambda_b)_{\mathrm{rel}},$$

as desired.

Now assume we are in the situation of Proposition 6.3.1 (2). Denote

$$d_1 := \dim V_{\mu_2}(\lambda_b)_{\mathrm{rel}}, \quad d_2 := \dim V_{\mu_2}(\lambda_{\mathrm{bad}})_{\mathrm{rel}}.$$

By assumption $d_2 \neq 0$. By Theorem 6.1.3 and Proposition 6.3.1 (2), for all $\mu \in X_*(T)^+$ such that $[b] \in B(G, \mu)$, we have

$$S_{\mu,b}(q_1) = \lim_{s \to \infty} \left[\dim V_\mu(\lambda_b)_{\mathrm{rel}} q_s^{\frac{1}{2}\mathrm{def}_G(b)} \mathfrak{M}^0_{\lambda_b^{+,(s)}}(q_1^{-1}) + \dim V_\mu(\lambda_{\mathrm{bad}})_{\mathrm{rel}} q_s^{\frac{1}{2}\mathrm{def}_G(b)} \mathfrak{M}^0_{\lambda_{\mathrm{bad}}^{(s)}}(q_1^{-1}) \right]. \tag{6.3.3}$$

In particular, taking $\mu = \mu_1$ and μ_2, we obtain

$$S_{\mu_1,b}(q_1) = \lim_{s \to \infty} q_s^{\frac{1}{2}\mathrm{def}_G(b)} \mathfrak{M}^0_{\lambda_b^{+,(s)}}(q_1^{-1}) \tag{6.3.4}$$

$$S_{\mu_2,b}(q_1) = \lim_{s\to\infty}\left[d_1 q_s^{\frac{1}{2}\mathrm{def}_G(b)}\mathfrak{M}^0_{\lambda_b^{+,(s)}}(q_1^{-1}) + d_2 q_s^{\frac{1}{2}\mathrm{def}_G(b)}\mathfrak{M}^0_{\lambda_{\mathrm{bad}}^{(s)}}(q_1^{-1})\right]. \tag{6.3.5}$$

Comparing (6.3.3) (6.3.4) (6.3.5), we obtain

$$S_{\mu,b}(q_1) = \dim V_\mu(\lambda_b)_{\mathrm{rel}} S_{\mu_1,b}(q_1) + \frac{\dim V_\mu(\lambda_{\mathrm{bad}})_{\mathrm{rel}}}{d_2}(S_{\mu_2,b}(q_1) - d_1 S_{\mu_1,b}(q_1)).$$

By varying F, we obtain

$$S_{\mu,b}(t) = \dim V_\mu(\lambda_b)_{\mathrm{rel}} S_{\mu_1,b}(t) + \frac{\dim V_\mu(\lambda_{\mathrm{bad}})_{\mathrm{rel}}}{d_2}(S_{\mu_2,b}(t) - d_1 S_{\mu_1,b}(t)), \tag{6.3.6}$$

as an equality in $\mathbb{C}(t)$. Since μ_1,μ_2 are sums of dominant minuscule elements, the main result of [Nie18a] implies that

$$\mathscr{N}(\mu_1, b) = \dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = 1$$
$$\mathscr{N}(\mu_2, b) = \dim V_{\mu_2}(\lambda_b)_{\mathrm{rel}} = d_1.$$

Consequently we have $S_{\mu_1,b}(0) = 1$ and $S_{\mu_2,b}(0) = d_1$. Evaluating (6.3.6) at $t = 0$, we obtain

$$\mathscr{N}(\mu, b) = S_{\mu,b}(0) = \dim V_\mu(\lambda_b)_{\mathrm{rel}} + \frac{\dim V_\mu(\lambda_{\mathrm{bad}})_{\mathrm{rel}}}{d_2}(d_1 - d_1) = \dim V_\mu(\lambda_b)_{\mathrm{rel}}$$

as desired. □

Corollary 6.3.3. *Conjecture 2.6.7 is true in full generality.*

Proof. By Proposition 2.6.8, we reduce to the case where G is adjoint and F-simple, and $[b]$ is basic. If G is not of type A, the conjecture is proved in Theorem 6.3.2. If G is of type A, the conjecture is proved by Nie [Nie18a]. □

The rest of the paper is devoted to the proof of Proposition 6.3.1.

6.4. Reduction to the absolutely simple case.

Lemma 6.4.1. *Proposition 6.3.1 holds true if it holds for all G that are absolutely simple and adjoint, not of type A.*

Proof. Let G be as in Proposition 6.3.1, not necessarily absolutely simple. Fix a basic $[b] \in B(G)$ as in Proposition 6.3.1. Let G' be the auxiliary absolutely simple and adjoint group over F, constructed in §5.5. We keep the notation established there. Note that $[b]$ is completely determined by $\kappa_G(b) \in \pi_1(G)_\sigma$. We construct a basic $[b'] \in B(G')$ as in §5.5, such that $\kappa_G(b)$ and $\kappa_{G'}(b')$ correspond to each other under the identification

$$\pi_1(G)_\sigma \cong \pi_1(G')_\sigma.$$

We write EDynk_G for the extended Dynkin diagram of G and write $\mathrm{Aut}(\mathrm{EDynk}_G)$ for its automorphism group. We write $|\mathrm{EDynk}_G|$ for the set of nodes in EDynk_G. Similarly for G'.

We claim that

$$\mathrm{def}_G(b) = \mathrm{def}_{G'}(b'). \tag{6.4.1}$$

In fact, there is a natural embedding $\pi_1(G) \rtimes \langle\theta\rangle \hookrightarrow \mathrm{Aut}(\mathrm{EDynk}_G)$ given by the identification of $\pi_1(G)$ with the stabilizer of the base alcove Ω, and $\mathrm{def}_G(b)$ is computed as the number of θ-orbits minus the number of $[\mu] \rtimes \theta$-orbits in $|\mathrm{EDynk}_G|$. Here $[\mu] \in \pi_1(G)$ is any lift of $\kappa_G(b) \in \pi_1(G)_\sigma$. Similarly, choosing a lift $[\mu'] \in \pi_1(G')$ of $\kappa_{G'}(b')$, we compute $\mathrm{def}_{G'}(b')$ as the number of θ'-orbits minus the number of $[\mu'] \rtimes \theta'$-orbits in $|\mathrm{EDynk}_{G'}|$. Now by construction, $\mathrm{EDynk}_{G'}$ is identified with a particular connected component of

EDynk_G. We may thus embed $\mathrm{Aut}(\mathrm{EDynk}_{G'})$ into $\mathrm{Aut}(\mathrm{EDynk}_G)$ by extending the action trivially to other connected components. Then inside $\mathrm{Aut}(\mathrm{EDynk}_G)$ we have the following relations:

$$\theta' = \theta^{d_0}, \quad \pi_1(G) = \bigoplus_{i=0}^{d_0-1} \theta^i \pi_1(G')\theta^{-i}.$$

In particular, we have an embedding $\pi_1(G') = \theta^0\pi_1(G')\theta^0 \hookrightarrow \pi_1(G)$. We may arrange that $[\mu]$ is the image of $[\mu']$ under this embedding. Then we have

$$\#\left\{\theta\text{-orbits in } |\mathrm{EDynk}_G|\right\} = \#\left\{\theta'\text{-orbits in } |\mathrm{EDynk}_{G'}|\right\}$$

$$\#\left\{[\mu] \rtimes \theta\text{-orbits in } |\mathrm{EDynk}_G|\right\} = \#\left\{[\mu'] \rtimes \theta'\text{-orbits in } |\mathrm{EDynk}_{G'}|\right\}.$$

The claim is proved.

Next, we naturally identify $X^*(\widehat{\mathcal{S}})$ with $X^*(\widehat{\mathcal{S}}')$. Then it is easy to see that λ_b corresponds to $\lambda_{b'}$ under this identification. For clarity, we denote the analogue of the map (5.4.1) for G' as:

$$X^*(\widehat{\mathcal{S}}') \longrightarrow (Y^*)', \quad \lambda \mapsto \lambda^{((s))}.$$

The target of the above map is identified with Y^*. Then since the identification $Y^* \cong (Y^*)'$ amounts to the diagonal map (5.5.2), we see that

$$\lambda^{(d_0 s)} = \lambda^{((s))} \tag{6.4.2}$$

for all $\lambda \in X^*(\widehat{\mathcal{S}})$.

Combining (6.4.1), (6.4.2) with Propositions 5.5.1, 5.5.2, we see that the bounds (6.3.1) and (6.3.2) in Proposition 6.3.1 for $(G, b, s := d_0 s')$ reduce to the corresponding bounds for (G', b', s'). In the situation of Proposition 6.3.1 (2), we define λ_{bad} for (G, b) to be equal to that for (G', b'), under the identification $\Lambda(b) \cong \Lambda(b')$.

Finally, by hypothesis the desired μ_1' or $\{\mu_1', \mu_2'\}$ are already defined for (G', b'), as in Proposition 6.3.1. Under the identification (5.5.1) we define $\mu_i \in X_*(T)^+$ to be $(\mu_i', 0, \cdots, 0)$ for $i = 1, 2$. □

7. Proof of the key estimate, Part I

In this section we provide the first part of the proof of Proposition 6.3.1. In Lemma 6.4.1 we already reduced to the absolutely simple case. **From now on until the end of the paper, we assume that G is an absolutely simple adjoint group over F, not of type A.**

As in the proof of Lemma 6.4.1, we denote by EDynk_G the extended Dynkin diagram of G, denote by $\mathrm{Aut}(\mathrm{EDynk}_G)$ its automorphism group, and denote by $|\mathrm{EDynk}_G|$ the set of nodes. Since b is not unramified, we have $\kappa_G(b) \neq 0$, and in particular the groups $\pi_1(G)$ and $\mathrm{Aut}(\mathrm{EDynk}_G)$ are non-trivial. Since G is absolutely simple, adjoint, and not of type A, we see that the following are the only possibilities for Dynk_G and θ (viewed as an automorphism of Dynk_G):

(1) Type $B_n, n \geq 2, \theta = \mathrm{id}$.
(2) Type $C_n, n \geq 3, \theta = \mathrm{id}$.
(3) Type $D_n, n \geq 4, \theta = \mathrm{id}$.
(4) Type $D_n, n \geq 5, \theta$ has order 2.
(5) Type D_4, θ has order 2.
(6) Type D_4, θ has order 3.
(7) Type $E_6, \theta = \mathrm{id}$.
(8) Type E_6, θ has order 2.
(9) Type $E_7, \theta = \mathrm{id}$.

In fact, the above are the only cases where $\mathrm{Aut}(\mathrm{EDynk}_G)$ is non-trivial.

In this part of the proof, we define an explicit subset $\Lambda(b)_{\mathrm{good}}$ of $\Lambda(b)$, and prove the estimate (6.3.1) or (6.3.2) for all $\lambda \in \Lambda(b)_{\mathrm{good}}$.

7.1. **Types** $B, C, D, \theta = \mathrm{id}$.

7.1.1. *The norm method.* We follow [Bou68, Chapitre VI §4] for the presentation of the root systems of types B_n, C_n, D_n, and for the choice of simple roots. The root systems will be embedded in a vector space $E = \mathbb{R}^n$, with standard basis $e_1, \cdots, e_n$, and standard inner product $\langle e_i, e_j \rangle = \delta_{ij}$ so that we may identify the coroots and coweights with subsets of the same vector space. Following *loc. cit.*, we define the following lattices in E:

$$L_0 := \{(\xi_1, \cdots, \xi_n) \in E | \xi_i \in \mathbb{Z}\}$$

$$L_1 := \left\{(\xi_1, \cdots, \xi_n) \in L_0 | \sum_{i=1}^n \xi_i \in 2\mathbb{Z}\right\}$$

$$L_2 := L_0 + \mathbb{Z}(\frac{1}{2}\sum_{i=1}^n e_i).$$

We assume $\theta = \mathrm{id}$, so that $T = A, \widehat{\mathcal{S}} = \widehat{T} = \widehat{A}$. The cocharacter lattice $X_*(T)$ is identified with the coweight lattice in E. Moreover $\pi_1(G)$ is equal to the quotient of the coweight lattice modulo the the coroot lattice in E.

Since $[b] \in B(G)$ is basic, it is uniquely determined by $\kappa_G(b) \in \pi_1(G)_\sigma = \pi_1(G)$. The defect $\mathrm{def}_G(b)$ of b is computed in the way indicated in the proof of Lemma 6.4.1.

For any $v = (\xi_1, \cdots, \xi_n) \in E$, we write

$$|v| := |\xi_1| + \cdots + |\xi_n|. \tag{7.1.1}$$

It is easy to verify the following three facts.

(1) $|\cdot|$ is a norm on E.
(2) $|wv| = |v|$ for any $w \in W_0$ and $v \in E$.
(3) For any coroot $\alpha^\vee \in \Phi^\vee$, we have $|\alpha^\vee| \le \delta$, where $\delta = 2$.

Now given any subset S of $\Lambda(b)$, we define

$$\mathscr{D}(S) := \min_{\lambda \in S} |\lambda|\,. \tag{7.1.2}$$

(The minimum obviously exists.) In the following, we will specify a subset $\Lambda(b)_{\mathrm{good}}$ of $\Lambda(b)$, satisfying

$$\mathscr{D}(\Lambda(b)_{\mathrm{good}}) > \delta \cdot \mathrm{def}_G(b)/2. \tag{7.1.3}$$

We show how to get the bound (6.3.1) for all $\lambda \in \Lambda(b)_{\mathrm{good}}$, from (7.1.3).

Let $\lambda \in \Lambda(b)_{\mathrm{good}}$. Fix $w, w' \in W^1$. We write $\psi_s := w \bullet (w'\lambda^{(s)})$. By the formula (5.2.4), it suffices to show that

$$\mathcal{P}(\psi_s, q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)}) \tag{7.1.4}$$

for some $a > 0$.

The same as (6.2.4), we have the bound

$$\left|\mathcal{P}(\psi_s, q_1^{-1})\right| \le \#\mathbb{P}(\psi_s) \cdot q_1^{-N_s}, \tag{7.1.5}$$

where

$$N_s := \min\{|\underline{m}| : \underline{m} \in \mathbb{P}(\psi_s)\}\,.$$

Suppose $\underline{m} \in \mathbb{P}(\psi_s)$. Then

$$\delta \sum_{\beta \in {}_F\Phi^{\vee,+}} m(\beta) \geq \left| \sum_{\beta \in {}_F\Phi^{\vee,+}} m(\beta)\beta \right| = |\psi_s| \geq \left| ww'\lambda^{(s)} \right| - |w\rho^\vee - \rho^\vee| = s\,|\lambda| - C \geq s \cdot \mathscr{D}(\Lambda(b)_{\mathrm{good}}) - C,$$

where C is a constant independent of s, and $|\cdot|$ is the norm defined in (7.1.1). Since $\theta = \mathrm{id}$, we have ${}_F\Phi^{\vee,+} = \Phi^{\vee,+}$, and $\mathtt{b}(\beta) = 1$ for all $\beta \in \Phi^{\vee,+}$. Hence the leftmost term in the above inequalities is none other than $\delta\,|\underline{m}|$. It follows that

$$N_s \geq (s\mathscr{D}(\Lambda(b)_{\mathrm{good}}) - C)\delta^{-1}.$$

By the above estimate and (7.1.3), we have

$$q_1^{-N_s} = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b) + a')}) \tag{7.1.6}$$

for some $a' > 0$.

On the other hand, by the same argument as in the proof of Proposition 6.2.1, we have

$$\#\mathbb{P}(\psi_s) \leq (Ls)^M \tag{7.1.7}$$

for some constants $L, M > 0$. The desired estimate (7.1.4) then follows from (7.1.5) (7.1.6) (7.1.7).

In the following we specify the definition of $\Lambda(b)_{\mathrm{good}}$ satisfying (7.1.3), for types B, C, D with $\theta = \mathrm{id}$.

7.1.2. *Type $B_n, n \geq 2, \theta = \mathrm{id}$.* The simple roots are

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n.$$

The simple coroots are

$$\alpha_i^\vee = \alpha_i, 1 \leq i < n,\ \alpha_n^\vee = 2e_n.$$

The fundamental weights are

$$\varpi_i = e_1 + \cdots + e_i, 1 \leq i < n,\ \varpi_n = \frac{1}{2}(e_1 + \cdots + e_n).$$

The coroot lattice is L_1, the coweight lattice is L_0. We have

$$P^+ = \{(\xi_1, \cdots, \xi_n) | \xi_i \in \mathbb{Z}, \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \geq 0\}.$$

We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element is represented by $e_1 \in L_0$. Recall that we assumed that $\kappa_G(b)$ is non-trivial, so there is only one choice of $\kappa_G(b)$ (and hence only one choice of the basic $b \in B(G, \mu)$). We have

$$\lambda_b = -e_n, \quad \lambda_b^+ = e_1.$$

Since $\kappa_G(b)$ acts on EDynk_G via its unique non-trivial automorphism, we easily see (both for $n = 2$ and for $n \geq 3$) that

$$\mathrm{def}_G(b) = 1.$$

We take

$$\Lambda(b)_{\mathrm{good}} := \Lambda(b),$$

and we have $\mathscr{D}(\Lambda(b)) = 2$. The inequality (7.1.3) is satisfied.

7.1.3. *Type* $C_n, n \geq 3, \theta = \mathrm{id}$. The simple roots are

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n.$$

The simple coroots are

$$\alpha_i^\vee = \alpha_i, 1 \leq i < n,\ \alpha_n^\vee = e_n.$$

The fundamental weights are

$$\varpi_i = e_1 + \cdots + e_i,\ 1 \leq i \leq n.$$

The coroot lattice is L_0, the coweight lattice is L_2. We have

$$P^+ = \left\{(\xi_1, \cdots, \xi_n) \in L_2 | \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \geq 0\right\}.$$

We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element is represented by $(\frac{1}{2}, \cdots, \frac{1}{2}) \in L_2$. Since $\kappa_G(b)$ is non-trivial, we have

$$\lambda_b = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \cdots, (-1)^n \frac{1}{2}), \quad \lambda_b^+ = (\frac{1}{2}, \cdots, \frac{1}{2}).$$

Since $\kappa_G(b)$ acts on the EDynk_G via its unique non-trivial automorphism, we easily see that

$$\mathrm{def}_G(b) = \lceil \frac{n}{2} \rceil$$

(i.e. the smallest integer $\geq n/2$.) We take

$$\Lambda(b)_{\mathrm{good}} := \Lambda(b)$$

and we have $\mathscr{D}(\Lambda(b)) = (n+2)/2$. The inequality (7.1.3) is satisfied.

7.1.4. *Type* $D_n, n \geq 4, \theta = \mathrm{id}$. The simple roots are

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n.$$

The simple coroots are

$$\alpha_i^\vee = \alpha_i.$$

The fundamental weights are

$$\varpi_i = e_1 + \cdots + e_i,\ 1 \leq i \leq n-2$$
$$\varpi_{n-1} = \frac{1}{2}(e_1 + e_2 + \cdots + e_{n-1} - e_n)$$
$$\varpi_n = \frac{1}{2}(e_1 + e_2 + \cdots + e_n).$$

The coroot lattice is L_1, the coweight lattice is L_2. We have

$$P^+ = \left\{(\xi_1, \cdots, \xi_n) \in L_2 | \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq |\xi_n|\right\}.$$

Case: n **is odd.** We have $\pi_1(G) \cong \mathbb{Z}/4\mathbb{Z}$, and a generator is represented by $(\frac{1}{2}, \cdots, \frac{1}{2}) \in L_2$. For $i = 1, 2, 3$, we let $b_i \in B(G)$ correspond to the image of $i(\frac{1}{2}, \cdots, \frac{1}{2})$ in $\pi_1(G)$. Then

$$\lambda_{b_1} = \sum_{i=1}^{n-2} \frac{(-1)^i}{2} e_i - \frac{1}{2} e_{n-1} + \frac{(-1)^{(n+1)/2}}{2} e_n, \qquad \lambda_{b_1}^+ = (\frac{1}{2}, \cdots, \frac{1}{2}) \tag{7.1.8}$$

$$\lambda_{b_2} = -e_{n-1}, \qquad \lambda_{b_2}^+ = e_1 \tag{7.1.9}$$

$$\lambda_{b_3} = \sum_{i=1}^{n-2} \frac{(-1)^i}{2} e_i - \frac{1}{2} e_{n-1} + \frac{(-1)^{(n-1)/2}}{2} e_n, \qquad \lambda_{b_3}^+ = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}). \tag{7.1.10}$$

Since up to automorphisms of $\mathbb{Z}/4\mathbb{Z}$, there is only one way that $\mathbb{Z}/4\mathbb{Z}$ could act on EDynk_G, we easily see that

$$\mathrm{def}_G(b_1) = \mathrm{def}_G(b_3) = \frac{n+3}{2},\ \mathrm{def}_G(b_2) = 2.$$

Let

$$\lambda_{1,\mathrm{bad}} := (\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$$
$$\lambda_{3,\mathrm{bad}} := (\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{1}{2}).$$

For $i = 1, 3$, we obviously have $\lambda_{i,\mathrm{bad}} \in \Lambda(b_i)$. We take

$$\Lambda(b_i)_{\mathrm{good}} := \Lambda(b_i) - \{\lambda_{i,\mathrm{bad}}\}. \tag{7.1.11}$$

Then

$$\mathscr{D}(\Lambda(b_i)_{\mathrm{good}}) = \frac{n+4}{2}.$$

The inequality (7.1.3) is satisfied.

For $i = 2$, we take

$$\Lambda(b_2)_{\mathrm{good}} := \Lambda(b_2),$$

and we have $\mathscr{D}(\Lambda(b_2)) = 3$. The inequality (7.1.3) is satisfied.

Case: n is even. We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The three non-trivial elements are represented by $(\frac{1}{2}, \cdots, \frac{1}{2}), e_1, (\frac{1}{2}, \cdots, \frac{1}{2}) + e_1 \in L_2$. Correspondingly we have

$$\lambda_{b_1} = \sum_{i=1}^{n-2} \frac{(-1)^i}{2} e_i - \frac{1}{2} e_{n-1} + \frac{(-1)^{n/2}}{2} e_n, \qquad \lambda_{b_1}^+ = (\frac{1}{2}, \cdots, \frac{1}{2}) \tag{7.1.12}$$

$$\lambda_{b_2} = -e_{n-1}, \qquad \lambda_{b_2}^+ = e_1 \tag{7.1.13}$$

$$\lambda_{b_3} = \sum_{i=1}^{n-2} \frac{(-1)^i}{2} e_i - \frac{1}{2} e_{n-1} + \frac{(-1)^{n/2+1}}{2} e_n, \qquad \lambda_{b_3}^+ = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}). \tag{7.1.14}$$

Since $\kappa_G(b_1)$ and $\kappa_G(b_3)$ are related to each other by the automorphism of the based root system $e_n \mapsto -e_n$, it is clear that they correspond to the two horizontal symmetries of order two of EDynk_G. On the other hand, the action of $\kappa_G(b_2)$ on EDynk_G is of order two, is distinct from the two horizontal symmetries, and commutes with the two horizontal symmetries. Hence this must correspond to the vertical symmetry of EDynk_G that has precisely two orbits of size two and fixes all the other nodes. Thus we have

$$\mathrm{def}_G(b_1) = \mathrm{def}_G(b_3) = \frac{n}{2}, \quad \mathrm{def}_G(b_2) = 2.$$

For $i = 1, 2, 3$ we take

$$\Lambda(b_i)_{\mathrm{good}} := \Lambda(b_i).$$

Then we have

$$\mathscr{D}(\Lambda(b_1)) = \mathscr{D}(\Lambda(b_3)) = \frac{n+2}{2}, \quad , \mathscr{D}(\Lambda(b_2)) = 3.$$

The inequality (7.1.3) is satisfied.

7.2. **Type $D_n, n \geq 5, \theta$ has order 2.** The simple (absolute) roots and coroots are the same as in §7.1.4, embedded in $E = \mathbb{R}^n$. We identify E with $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Then θ acts on E by

$$(\xi_1, \cdots, \xi_n) \mapsto (\xi_1, \cdots, \xi_{n-1}, -\xi_n).$$

The subgroup $X_*(A) \subset X_*(T) = L_2$ is given by $\{(\xi_1, \cdots, \xi_n) \in L_2 | \xi_n = 0\}$. Let $L_2' \subset \mathbb{Q}^{n-1}$ be the analogue of L_2, namely $L_2' = \mathbb{Z}^{n-1} + \mathbb{Z}(\frac{1}{2}, \cdots, \frac{1}{2})$. The quotient $X_*(T) = L_2 \to X^*(\widehat{\mathcal{S}})$ is the same as

$$L_2 \longrightarrow L_2', \ (\xi_1, \cdots, \xi_n) \mapsto (\xi_1, \cdots, \xi_{n-1}).$$

The map

$$(s) : X^*(\widehat{\mathcal{S}}) \longrightarrow X_*(A), \ \lambda \mapsto \lambda^{(s)}$$

(for $s \in 2\mathbb{Z}_{\geq 1}$) is given by

$$L_2' \longrightarrow X_*(A),\ (\xi_1, \cdots, \xi_{n-1}) \mapsto (s\xi_1, \cdots, s\xi_{n-1}, 0). \tag{7.2.1}$$

The set ${}_F\Phi^{\vee,+}$, as a subset of $X_*(A)$, is equal to

$$\{e_i \pm e_j | 1 \leq i < j \leq n-1\} \cup \{2e_i | 1 \leq i \leq n-1\}.$$

We have

$$\mathtt{b}(e_i \pm e_j) = 1,\ 1 \leq i < j \leq n-1$$
$$\mathtt{b}(2e_i) = 2,\ 1 \leq i \leq n-1.$$

Moreover ${}_F\Phi^\vee$ is reduced. We have

$$P^+ = \{(\xi_1, \cdots, \xi_n) \in L_2 | \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq \xi_n = 0\}$$
$$X^*(\widehat{\mathcal{S}})^+ = \left\{(\xi_1, \cdots, \xi_{n-1}) \in X^*(\widehat{\mathcal{S}}) = L_2' | \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq 0\right\}.$$

We again write $e_1, \cdots, e_{n-1}$ for the standard basis of $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q} = \mathbb{Q}^{n-1}$. The relative simple roots in $\widehat{Q}_{\hat{\theta}} \subset X^*(\widehat{\mathcal{S}})$ are:

$$e_1 - e_2, e_2 - e_3, \cdots, e_{n-2} - e_{n-1}, e_{n-1}$$

(i.e., the same as type B_{n-1}.)

Case: n is odd. We have $\pi_1(G) \cong \mathbb{Z}/4\mathbb{Z}$, and σ acts on $\pi_1(G)$ by the unique non-trivial automorphism of $\pi_1(G)$. Hence $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element is represented by

$$-\frac{1}{2}(e_1 - e_2) - \frac{1}{2}(e_3 - e_4) \cdots - \frac{1}{2}(e_{n-2} - e_{n-1}) + \frac{1}{2}e_n \in L_2 = X_*(T).$$

The image of the above element in $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q} = \mathbb{Q}^{n-1}$ is obviously equal to a linear combination of the relative simple roots in $\widehat{Q}_{\hat{\theta}}$ with coefficients in $\mathbb{Q} \cap (-1, 0]$. Hence this image is λ_b, and so

$$\lambda_b = (-\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{2}, \frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$$
$$\lambda_b^+ = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}.$$

If γ is any generator of $\pi_1(G) \cong \mathbb{Z}/4\mathbb{Z}$, then the number of orbits of $\gamma \rtimes \theta$ in $|\mathrm{EDynk}_G|$ is $2 + \frac{n-3}{2}$, while the number of orbits of θ in $|\mathrm{EDynk}_G|$ is n. Hence

$$\mathrm{def}_G(b) = \frac{n-1}{2}. \tag{7.2.2}$$

We have

$$\Lambda(b) = \left\{(\xi_1 + \frac{1}{2}, \cdots, \xi_{n-1} + \frac{1}{2}) \in X^*(\widehat{\mathcal{S}}) = L_2' | \xi_i \in \mathbb{Z},\ \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq 0, \xi_1 > 0\right\}. \tag{7.2.3}$$

We take

$$\Lambda(b)_{\mathrm{good}} := \Lambda(b).$$

In the following we show (6.3.1) for all $\lambda \in \Lambda(b)$. The proof is similar to the argument in §7.1.1.

Fix $w, w' \in W^1$. We write $\psi_s := w \bullet (w'\lambda^{(s)})$. By the formula (5.2.4), it suffices to show that

$$\mathcal{P}(\psi_s, q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b) + a)}) \tag{7.2.4}$$

for some $a > 0$. Again we have the bound

$$\left|\mathcal{P}(\psi_s, q_1^{-1})\right| \leq \#\mathbb{P}(\psi_s) \cdot q_1^{-N_s}, \tag{7.2.5}$$

where

$$N_s := \min\{|\underline{m}| : \underline{m} \in \mathbb{P}(\psi_s)\}.$$

Suppose $\underline{m} \in \mathbb{P}(\psi_s)$. We keep the definition (7.1.1) of the norm $|\cdot|$ on $E = \mathbb{R}^n$. Then

$$2 \sum_{\beta \in_F \Phi^{\vee,+}} m(\beta) \geq \left| \sum_{\beta \in_F \Phi^{\vee,+}} m(\beta)\beta \right| = |\psi_s| \geq \left| ww'\lambda^{(s)} \right| - |w\rho^\vee - \rho^\vee| = \left| \lambda^{(s)} \right| - C,$$

where C is a constant independent of s. By (7.2.1) and (7.2.3), we have

$$\left| \lambda^{(s)} \right| \geq s \cdot \frac{n+1}{2}.$$

On the other hand

$$2\,|\underline{m}| := 2 \sum_{\beta \in_F \Phi^{\vee,+}} m(\beta)\mathsf{b}(\beta) \geq 2 \sum_{\beta \in_F \Phi^{\vee,+}} m(\beta).$$

In conclusion we have

$$2\,|\underline{m}| \geq s \cdot \frac{n+1}{2} + C. \tag{7.2.6}$$

Combining (7.2.2) (7.2.6) and (7.1.7) (which holds in general), we obtain the desired (7.2.4). Note that in the above proof, we only used the fact that $\frac{n+1}{2} > \mathrm{def}_G(b)$.

Case: n is even. We have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The action of σ on $\pi_1(G)$ swaps the classes represented by $(\frac{1}{2}, \cdots, \frac{1}{2})$ and $(\frac{1}{2}, \cdots, \frac{1}{2}) + e_1 \in L_2$, and fixes the class represented by e_1. Hence $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element is represented by

$$-\frac{1}{2}(e_1 - e_2) - \frac{1}{2}(e_3 - e_4) \cdots - \frac{1}{2}(e_{n-1} - e_n) \in L_2 = X_*(T).$$

The image of the above element in $X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q} = \mathbb{Q}^{n-1}$ is obviously equal to a linear combination of the relative simple roots in $\widehat{Q}_{\hat{\theta}}$ with coefficients in $\mathbb{Q} \cap (-1, 0]$. Hence this image is λ_b, and so

$$\lambda_b = (-\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}$$

$$\lambda_b^+ = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}.$$

Let $\gamma \in \pi_1(G)$ be the class of $(\frac{1}{2}, \cdots, \frac{1}{2}) \in L_2$. We have seen in §7.1.4 that γ acts on EDynk_G via one of the two order-two horizontal symmetries of EDynk_G. Hence $\gamma \rtimes \sigma$ acts on EDynk_G via one of the two order-four horizontal symmetries of EDynk_G, and the number of orbits is $1 + \frac{n-2}{2} = \frac{n}{2}$. On the other hand the number of orbits of θ in $|\mathrm{EDynk}_G|$ is n. Hence

$$\mathrm{def}_G(b) = \frac{n}{2}.$$

The set $\Lambda(b)$ is again given by (7.2.3). We take

$$\Lambda(b)_{\mathrm{good}} := \Lambda(b).$$

The proof of (6.3.1) for all $\lambda \in \Lambda(b)$ is exactly the same as in the odd case, using the fact that $\frac{n+1}{2} > \mathrm{def}_G(b)$.

7.3. **Type D_4, θ has order 2.** The difference between this case and §7.2 is that the D_4 Dynkin diagram has three (rather than one) automorphisms of order two. However we explain why the proof of (6.3.1) for all $\lambda \in \Lambda(b)_{\mathrm{good}} := \Lambda(b)$ is the same. In fact, there exists a permutation τ of $\{1, 3, 4\}$, such that the root system can be embedded into $\mathbb{R}^4$ with simple roots:

$$\alpha_{\tau(1)} = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_{\tau(3)} = e_3 - e_4,\ \alpha_{\tau(4)} = e_3 + e_4,$$

and such that θ acts on $\mathbb{R}^4$ by $e_4 \mapsto -e_4$.

If $\tau = 1$, then the extra node in EDynk_G is given by $\alpha_0 = -e_1 - e_2$, and the proof is exactly the same as §7.2. For general τ, we still have $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$ and hence a unique choice of b, and the only place in the proof in §7.2 that could change is the computation of $\mathrm{def}_G(b)$, as the extra node in EDynk_G is no longer given by $-e_1 - e_2$. However, it can still be easily checked that as long as $\kappa_G(b) \in \pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$ is non-trivial, we have

$$\mathrm{def}_G(b) = 2 = \frac{4}{2}.$$

In fact, this follows from the observation that for any order-two element $\gamma \in \pi_1(G) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ that is not fixed by σ, the action of $\gamma \rtimes \theta$ on $|\mathrm{EDynk}_G|$ must be of order four and have two orbits.

7.4. **Type D_4, θ has order 3.** In this case $\pi_1(G) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We know that θ acts on $\pi_1(G)$ by an order-three permutation of the three non-trivial elements. Thus $\pi_1(G)_\sigma = 0$ and any basic b is unramified.

7.5. **Type $E_6, \theta = \mathrm{id}$.** We consider the root system E_6 embedded in $\mathbb{R}^9$, which we will consider as $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. The set of roots is given by the 18 elements consisiting of permutations of

$$(1, -1, 0; 0, 0, 0; 0, 0, 0)$$
$$(0, 0, 0; 1, -1, 0; 0, 0, 0)$$
$$(0, 0, 0; 0, 0, 0; 1, -1, 0)$$

under the group $S_3 \times S_3 \times S_3$, together with the 54 elements given by the permutations of

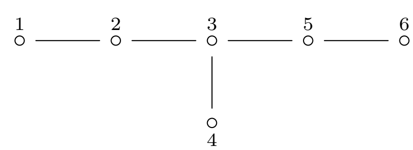
$$(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} : \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$$
$$(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}; -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} : -\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$

under the same group. We will call the first set of roots *type A roots*, and the second set *type B roots*. A type A root is positive if and only if the coordinate 1 appears to the left of the -1. A type B root is positive if and only if the first coordinate is positive.

A choice of simple roots is given by

$$\alpha_1 = (0, 0, 0; 0, 1, -1; 0, 0, 0)$$
$$\alpha_2 = (0, 0, 0; 1, -1, 0; 0, 0, 0)$$
$$\alpha_3 = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}; -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} : -\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$
$$\alpha_4 = (0, 1, -1; 0, 0, 0; 0, 0, 0)$$
$$\alpha_5 = (0, 0, 0; 0, 0, 0; 1, -1, 0)$$
$$\alpha_6 = (0, 0, 0; 0, 0, 0; 0, 1, -1).$$

The corresponding Dynkin diagram is



Under the standard pairing of $\mathbb{R}^9$ with itself, each root is equal to its own corresponding coroots. We therefore identify $\mathbb{R}^9$ with its dual and do not distinguish between roots and coroots. The subspace of $\mathbb{R}^9$ generated by the roots is given by the equations

$$x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = x_7 + x_8 + x_9 = 0 \tag{7.5.1}$$

where x_i are the standard coordinates. The fundamental weights are given by

$$\varpi_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 0, 0, 0)$$

$$\varpi_2 = (\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; 0, 0, 0)$$
$$\varpi_3 = (2, -1, -1; 0, 0, 0; 0, 0, 0)$$
$$\varpi_4 = (1, 0, -1; 0, 0, 0; 0, 0, 0)$$
$$\varpi_5 = (\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 0, 0, 0; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$$
$$\varpi_6 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; 0, 0, 0; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$$

For an element $\lambda = \sum_{i=1}^{6} a_i \varpi_i$ with $a_i \in \mathbb{Z}$, we have λ lies in the root lattice if and only

$$a_5 - a_6 - a_2 + a_1 \equiv 0 \mod 3 \tag{7.5.2}$$

and we have $\pi_1(G) \cong \mathbb{Z}/3\mathbb{Z}$, with the isomorphism being given by

$$\lambda = \sum_{i=1}^{6} a_i \varpi_i \mapsto a_5 - a_6 - a_2 + a_1 \mod 3.$$

Moreover λ is dominant if and only $a_i \geq 0$ for $i = 1, \dots, 6$.

We let b_i, $i = 1, 2$ denote the non-trivial elements in $\pi_1(G)$. We have

$$\lambda_{b_1}^+ = \varpi_1, \quad \lambda_{b_2}^+ = \varpi_6$$

We set

$$\Lambda(b_1)_{\text{good}} := \Lambda(b_1) - \{\varpi_5, \varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\}$$
$$\Lambda(b_2)_{\text{good}} := \Lambda(b_2) - \{\varpi_2, \varpi_4 + \varpi_6, \varpi_5 + \varpi_1, 2\varpi_1\}$$

We let $|\cdot|$ be the standard Euclidean norm on $\mathbb{R}^9$. Then $|\cdot|$ is W_0-invariant, and we have $|\alpha^\vee| \leq \delta := \sqrt{2}$, for all $\alpha^\vee \in \Phi^\vee$. Given any subset S of $\Lambda(b_i)$, we define

$$\mathscr{D}(S) := \min_{\lambda \in S} |\lambda|.$$

We claim that

$$\mathscr{D}(\Lambda(b_1)_{\text{good}}) > \sqrt{8}, \quad \mathscr{D}(\Lambda(b_2)_{\text{good}}) > \sqrt{8}.$$

Since $\mathrm{def}_G(b) = 4$ (which we know by counting orbits of the unique non-trivial symmetry of EDynk_G) and $\delta = \sqrt{2}$, the claim will imply the inequality (7.1.3), and by exactly the same argument as in §7.1.1, we conclude that (6.3.1) holds for all $\lambda \in \Lambda(b_i)_{\text{good}}$.

We now prove the claim. By the obvious symmetry of the Dynkin diagram

$$1 \leftrightarrow 6$$
$$2 \leftrightarrow 5$$
$$3 \leftrightarrow 3$$
$$4 \leftrightarrow 4,$$

it suffices to only discuss $\Lambda(b_1)_{\text{good}}$.

Let $\lambda = \sum_{i=1}^{6} a_i \varpi_i \in \Lambda(b_1)$, with $a_i \in \mathbb{Z}_{\geq 0}$, and suppose $|\lambda| \leq \sqrt{8}$. We will show

$$|\lambda| \in \{\varpi_5, \varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\}.$$

Since $\lambda \in \Lambda(b_1)$, we have by (7.5.2) that

$$a_5 - a_6 - a_2 + a_1 \equiv 1 \mod 3. \tag{7.5.3}$$

By looking at the first three coordinates of λ and using the triangle inequality, we easily obtain the inequalities

$$a_1 \leq 3, \ a_2 \leq 1, \ a_3 \leq 1, \ a_4 \leq 3, \ a_5 \leq 1, \ a_6 \leq 3.$$

If $a_3 = 1$, then we have $a_i > 0$ for some $i \neq 3$ since $\lambda \in \Lambda(b_1)$, hence $a_3 = 0$.

If $a_2 = 1$, we have $a_5 = 0$ and $a_1, a_4, a_6 \leq 1$ (by looking at the first 3 coordinates). We check each case and see that only $\lambda = \varpi_2 + \varpi_6$ is possible.

If $a_5 = 1$, we similarly obtain that $\lambda = \varpi_5$ is the only possibility using (7.5.3).

The only cases left are when the only non-zero coefficients are a_1, a_4, a_6. Again by looking at the first three coordinates, we see that $a_1 + a_4 + a_6 \leq 3$. We check each case and see that the only possibilities are $\lambda = \varpi_1 + \varpi_4$ and $\lambda = 2\varpi_6$.

7.6. **Type E_6, θ has order 2.** We keep the notation §7.5. Then θ acts on the root system via the action on $\mathbb{R}^9$ given by

$$(x_1, x_2, x_3; x_4, x_5, x_6; x_7, x_8, x_9) \mapsto (x_1, x_2, x_3; x_7, x_8, x_9; x_4, x_5, x_6).$$

It therefore acts on $\pi_1(G)$ by switching the two non-trivial elements. Hence $\pi_1(G)_\sigma = 0$ and all basic elements are unramified.

7.7. **Type $E_7, \theta = \mathrm{id}$.** We consider the root system E_7 as a subset of $\mathbb{R}^8$. The set of roots is given by the 56 permutations of

$$(1, -1, 0, 0, 0, 0, 0, 0)$$

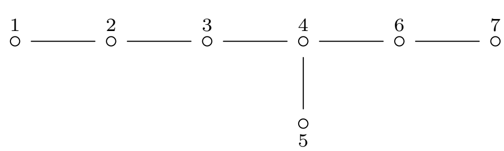
and the $\binom{8}{4}$ permutations of

$$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}).$$

A set of simple roots is given by

$$\alpha_1 = (0, 0, 0, 0, 0, 0, -1, 1)$$
$$\alpha_2 = (0, 0, 0, 0, 0, -1, 1, 0)$$
$$\alpha_3 = (0, 0, 0, 0, -1, 1, 0, 0)$$
$$\alpha_4 = (0, 0, 0, -1, 1, 0, 0, 0)$$
$$\alpha_5 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$$
$$\alpha_6 = (0, 0, -1, 1, 0, 0, 0, 0)$$
$$\alpha_7 = (0, -1, 1, 0, 0, 0, 0, 0)$$

The corresponding Dynkin diagram is



Under the standard pairing of $\mathbb{R}^8$ with itself, roots correspond to coroots and we therefore do not distinguish between them. The subspace of $\mathbb{R}^8$ generated by the roots is the hyperplane given by the equation $\sum_{i=1}^{8} x_i = 0$.

The corresponding fundamental weights are given by

$$\varpi_1 = (\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4})$$
$$\varpi_2 = (\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$$
$$\varpi_3 = (\frac{9}{4}, -\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$
$$\varpi_4 = (3, -1, -1, -1, 0, 0, 0, 0)$$

$$\varpi_5 = (\frac{7}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$$
$$\varpi_6 = (2, -1, -1, 0, 0, 0, 0, 0)$$
$$\varpi_7 = (1, -1, 0, 0, 0, 0, 0, 0).$$

For an element $\lambda = \sum_{i=1}^{7} a_i \varpi_i, a_i \in \mathbb{Z}$, we know λ lies in the root lattice if and only if

$$a_1 + a_3 + a_5 \equiv 0 \mod 2 \tag{7.7.1}$$

and we have $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$. By assumption $\kappa_G(b) \in \pi_1(G)$ is the non-trivial element. Then we have

$$\lambda_b^+ = \varpi_1.$$

We set

$$\Lambda(b)_{\mathrm{good}} := \Lambda(b) - \{\varpi_5\}.$$

We let $|\cdot|$ be the standard Euclidean norm on $\mathbb{R}^8$. Then $|\cdot|$ is W_0-invariant, and we have $|\alpha^\vee| \leq \delta := \sqrt{2}$, for all $\alpha^\vee \in \Phi^\vee$. Given any subset S of $\Lambda(b)$, we define

$$\mathscr{D}(S) := \min_{\lambda \in S} |\lambda|.$$

We claim that

$$\mathscr{D}(\Lambda(b)_{\mathrm{good}}) \geq \frac{\sqrt{22}}{2}.$$

Since $\mathrm{def}_G(b) = 3$ and $\delta = \sqrt{2}$, the claim will imply the inequality (7.1.3), and by exactly the same argument as in §7.1.1, we conclude that (6.3.1) holds for all $\lambda \in \Lambda(b)_{\mathrm{good}}$.

We now prove the claim. Suppose $\lambda = \sum_{i=1}^{7} a_i \lambda_i \in \Lambda(b)$ with $a_i \in \mathbb{Z}_{\geq 0}$, and $|\lambda| \leq \sqrt{22}/2$. We will show that $\lambda = \varpi_5$. By looking at the first four coordinates, we obtain the trivial inequalities:

$$\varpi_1 \leq 2,\ \varpi_2 \leq 1,\ \varpi_3 = 0,\ \varpi_4 = 0,\ \varpi_5 \leq 1,\ \varpi_6 \leq 1,\ \varpi_7 \leq 1.$$

We also obtain $\sum_{i=1}^{7} a_i \leq 2$. It is not hard to see that $\lambda = \varpi_5$ is the only possibility.

8. Proof of the key estimate, Part II

In this part, we finish the proof of the bounds (6.3.1) and (6.3.2) in Proposition 6.3.1.

In §7, we already proved (6.3.1) and (6.3.2) for all $\lambda \in \Lambda(b)_{\mathrm{good}}$. Moreover the $\Lambda(b)_{\mathrm{good}} \subset \Lambda(b)$ is a proper subset only in the following three cases:

Proper-I: Type $D_n, n \geq 5, n$ is odd, $\theta = \mathrm{id}$, $b = b_1$ or b_3. See §7.1.4.
Proper-II: Type $E_6, \theta = \mathrm{id}, b = b_1$ or b_2. See §7.5.
Proper-III: Type $E_7, \theta = \mathrm{id}$. See §7.7.

8.1. Combinatorics for D_n.

In order to treat the case **Proper-I**, we need some combinatorics for the type D_n root system. The material in this subsection is only needed in the proof of Proposition 8.2.1 below.

Let n be an integer ≥ 5. We keep the presentation of the type D_n root system in a vector space $\mathbb{R}^n$, as in §7.1.4. In particular we keep the choice of positive roots. We do not distinguish between roots and coroots. Let Φ_{D_n} be the set of roots and let $\Phi_{D_n}^+$ be the set of positive roots. Thus

$$\Phi_{D_n}^+ = \{e_i \pm e_j | 1 \leq i < j \leq n\}.$$

If $m > n$ is another integer, we embed $\mathbb{R}^n$ into $\mathbb{R}^m$ via the inclusion $\{e_1, \cdots, e_n\} \hookrightarrow \{e_1, \cdots, e_m\}$ of the standard bases. Thus we view Φ_{D_n} (resp. $\Phi_{D_n}^+$) as a natural subset of Φ_{D_m} (resp. $\Phi_{D_m}^+$).

8.1.1. We introduce some terminology for trees. An *out-tree* is a tree where one vertex is specified as the *origin*[6], denoted by O, and all the edges are oriented away from the origin. On an out-tree $\mathscr{T}$, for every vertex v except the origin, there is a unique vertex w that is connected to v by an edge pointing from w to v. This vertex w is called the *parent* of v. Conversely, v is a *child* of its parent. A vertex that does not have children is called an *end vertex*. By a *path* on $\mathscr{T}$, we mean a sequence of vertices $(v_0, v_1, \cdots, v_k)$, such that each v_i is the parent of v_{i+1}. Such a path is called *complete*, if $v_0 = O$ and v_k is an end vertex.

Definition 8.1.2. By an *admissible D_n-decorated tree*, we mean a triple $(\mathscr{T}, \underline{\alpha}, \underline{\beta})$, where $\mathscr{T}$ is a finite out-tree that has at least three vertices, and $\underline{\alpha}, \underline{\beta}$ are maps

$$\underline{\alpha} : \{\text{vertices of } \mathscr{T} \text{ except the origin}\} \longrightarrow \Phi^+_{D_n}$$

$$v \mapsto \alpha(v)$$

$$\underline{\beta} : \{\text{vertices of } \mathscr{T} \text{ except the end vertices}\} \longrightarrow \Phi^+_{D_n}$$

$$v \mapsto \beta(v),$$

satisfying the following conditions:

(1) Each vertex is either an end vertex or has precisely two children. (In other words $\mathscr{T}$ is *binary*.)
(2) For each vertex w that is not an end vertex, let $v, \bar{v}$ be the two children of w. We require that $\alpha(v) - \alpha(\bar{v}) \in \{\pm\beta(w)\}$. Moreover, if $\alpha(v) - \alpha(\bar{v}) = \beta(w)$, then we call v a positive vertex and call $\bar{v}$ a negative vertex.
(3) For every complete path $(v_0 = O, v_1, \cdots, v_k)$, we require that $\beta(v_i)$ are distinct, for $0 \le i \le k-1$.
(4) For every complete path $(v_0 = O, v_1, \cdots, v_k)$, we require that $\alpha(v_i)$ are distinct, for $1 \le i \le k$.

Example 8.1.3. We can visualize an admissible D_n-decorated tree in a diagram as follows. At the location of the origin, we mark $\boxed{O \parallel \beta(O)}$. At the location of each vertex $v \neq O$, we mark $\boxed{\alpha(v) \parallel \beta(v)}$ if v is not an end vertex, and we mark $\boxed{\alpha(v)}$ if v is an end vertex. For example, the diagram

$$\begin{tikzcd}[column sep=small] & & \boxed{O \parallel e_2 - e_3} \arrow[dl, dash] \arrow[dr, dash] & \\ & \boxed{e_1 - e_2 \parallel e_3 - e_4} \arrow[dl, dash] \arrow[d, dash] & & \boxed{e_1 - e_3} \\ \boxed{e_1 - e_3} & \boxed{e_1 - e_4} & & \end{tikzcd}$$

depicts an admissible D_5-decorated tree, with five vertices and four edges.

Definition 8.1.4. Fix an arbitrary sequence of signs $\underline{\nu} = (\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$. We say that an admissible D_n-decorated tree $(\mathscr{T}, \underline{\alpha}, \underline{\beta})$ is *good with respect to $\underline{\nu}$*, if for every complete path $(O, v_1, \cdots, v_k)$ the intersection

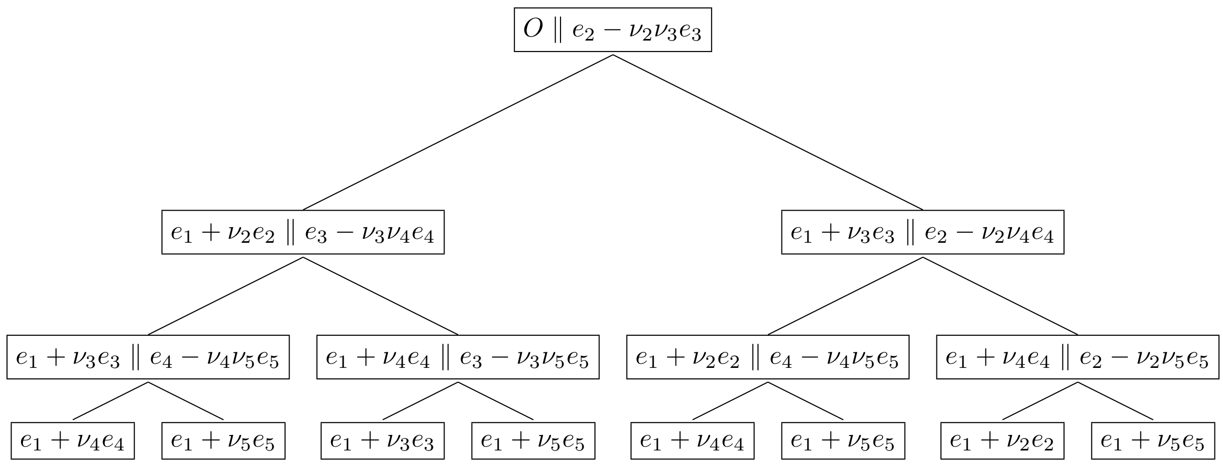
$$\{e_1 + \nu_j e_j | 2 \le j \le n\} \cap \{\alpha(v_i) | 1 \le i \le k\}$$

has exactly $n-2$ elements.

Proposition 8.1.5. *For each odd $n \ge 5$ and each sequence of signs $\underline{\nu} = (\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$, there exists an admissible D_n-decorated tree that is good with respect to $\underline{\nu}$.*

[6]This is usually called the *root* of the tree, but we avoid this terminology to prevent confusion.

Proof. We prove by induction on n. We use the graphic presentation introduced in Example 8.1.3. The base case is $n = 5$. It can be easily checked that the following is an admissible D_5-decorated tree that is good with respect to $\underline{\nu}$.

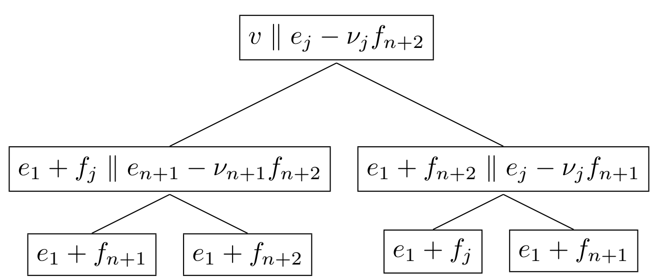


Now assume the proposition is proved for n, and we prove it for $n+2$. Let $\underline{\nu} = (\nu_2, \cdots, \nu_{n+2}) \in \{\pm 1\}^{n+1}$ be arbitrary. Denote by $\underline{\nu}'$ the sequence $(\nu_2, \cdots, \nu_n)$. By induction hypothesis there exists an admissible D_n-decorated tree $(\mathscr{T}', \underline{\alpha}', \underline{\beta}')$ that is good with respect to $\underline{\nu}'$. For each end vertex v of $\mathscr{T}'$, we shall glue a new admissible D_{n+2}-decorated tree to v (i.e., we identify v with the origin of this new out-tree). We then check that after all the gluing we obtain an admissible D_{n+2}-decorated tree that is good with respect to $\underline{\nu}$. In the following we denote for simplicity $f_j := \nu_j e_j$, for $2 \le j \le n+2$.

Let v be an end vertex of $\mathscr{T}'$, and let $(O, v_1, \cdots, v_k = v)$ be the complete path from O to v. By assumption, the intersection

$$I = \{e_1 + f_j | 2 \le j \le n\} \cap \{\alpha(v_i) | 1 \le i \le k\}$$

has exactly $n-2$ elements. Let $2 \le j \le n$ be the unique index such that $e_1 + f_j \notin I$. In this case we glue the following admissible D_{n+2}-decorated tree to v:



We check that after all the gluing we obtain an admissible D_{n+2}-decorated tree $(\mathscr{T}, \underline{\alpha}, \underline{\beta})$ which is good with respect to $\underline{\nu}$. Conditions (1) and (2) in Definition 8.1.2 are obviously satisfied. To see condition (3), note that each complete path on $\mathscr{T}$ is of the form $(v_0 = O, v_1, \cdots, v_k, v_{k+1}, v_{k+2})$, where $(v_0, v_1, \cdots, v_k)$ is a complete path on $\mathscr{T}'$. By induction hypothesis $\beta(v_i)$ are distinct for $0 \le i \le k-1$. By construction $\beta(v_k)$ and $\beta(v_{k+1})$ are always distinct, and they cannot be the same as any of the $\beta(v_i), 0 \le i \le k-1$, because $\beta(v_k), \beta(v_{k+1})$ are in $\Phi^+_{D_{n+2}} - \Phi^+_{D_n}$. Thus condition (3) from Definition 8.1.2 is satisfied. Similarly, condition (4) from Definition 8.1.2, and the statement that $(\mathscr{T}, \underline{\alpha}, \underline{\beta})$ is good with respect to $\underline{\nu}$, follow easily from the construction and the induction hypothesis. $\square$

Remark 8.1.6. In fact, the admissible D_n-decorated tree constructed in the above proof satisfies the following extra property: For any complete path $(O, v_1, \cdots, v_k)$, we have $k = n-2$, and the set $\{\alpha(v_i) | 1 \leq i \leq k\}$ consists of $n-2$ distinct elements of $\{e_1 + \nu_i e_i | 2 \leq i \leq n\}$.

Lemma 8.1.7. *Let $n \geq 5$ be an odd integer. Fix a real number $M > n/2$. Let $\Phi^+_{D_n}$ be the positive (co)roots in the type D_n root system in $\mathbb{R}^n$, and let $|\cdot|$ be the norm on V, as in (7.1.1). Let $\underline{\nu} = (\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$ and $t \in \mathbb{N}$ be arbitrary. Let λ be an element in the (co)root lattice, such that*

$$|\lambda - (6t, 2t\nu_2, 2t\nu_3, \cdots, 2t\nu_n)| < t/M.$$

We keep the notation in §5.3, with respect to ${}_F\Phi^{\vee,+} = \Phi^+_{D_n}$ and $\mathsf{b} \equiv 1$. Let $\underline{m} \in \mathbb{P}(\lambda)$. Assume there is a subset $I \subset \{2, \cdots, n\}$ of cardinality $n-2$, such that $m(e_1 + \nu_i e_i) = 0$ for all $i \in I$. Then

$$|\underline{m}| \geq (n + 4 - \frac{n}{2M})t.$$

Proof. Assume the contrary. For each $2 \leq i \leq n$ we write f_i for $\nu_i e_i$. Let j_0 be the unique element of $\{2, \cdots, n\} - I$. Define $\underline{m}' \in \mathbb{P}$ by:

$$\forall \beta \in \Phi^+_{D_n},\ m'(\beta) := \begin{cases} 0, & \beta \in \{e_1 \pm f_i | 2 \leq i \leq n\} \\ m(\beta), & \text{else} \end{cases}$$

Define

$$\lambda' := \Sigma(\underline{m}').$$

Write

$$\lambda = \lambda_1 e_1 + \sum_{i=2}^{n} \lambda_i f_i,$$

with $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$. In fact it easily follows from our assumption that each $\lambda_i > 0$, as λ_i is close to either $6t$ or $2t$. We have

$$\lambda' = \sum_{i \in I} (\lambda_i + m(e_1 - f_i)) f_i + \lambda'_{j_0} f_{j_0},$$

for some $\lambda'_{j_0} \in \mathbb{R}$. Obviously

$$|\underline{m}'| = |\underline{m}| - \lambda_1,$$

so we have

$$2\,|\underline{m}'| = 2\,|\underline{m}| - 2\lambda_1 \geq |\lambda'| \geq \sum_{i \in I} \lambda_i + m(e_1 - f_i),$$

from which we get

$$0 \leq \sum_{i \in I} m(e_1 - f_i) \leq 2\,|\underline{m}| - 2\lambda_1 - \sum_{i \in I} \lambda_i$$
$$< 2(n + 4 - \frac{n}{2M})t - 2(6t - t/M) - |I|\,(2t - t/M) = 0,$$

i.e., $0 < 0$, a contradiction. $\square$

Proposition 8.1.8. *Let $n \geq 5$ be an odd integer. Let $\underline{\nu} = (\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$ and $t \in \mathbb{N}$. Let*

$$\lambda = \lambda_t := (6t, 2t\nu_2, 2t\nu_3 \cdots, 2t\nu_n).$$

We keep the notation in §5.3, with respect to ${}_F\Phi^{\vee,+} = \Phi^+_{D_n}$ and $\mathsf{b} \equiv 1$. We write Φ^+ for $\Phi^+_{D_n}$. When t is sufficiently large, the following is true: For any integer L in the interval $[0, (n+3.5)t]$, we have

$$\sum_{S \subset \Phi^+} (-1)^{|S|} \# \mathbb{P}(\lambda - \sum_{\beta \in S} \beta)_L = 0.$$

Proof. To simplify notation, in the proof we write $\lambda - S$ for $\lambda - \sum_{\beta\in S}\beta$, for any subset $S \subset \Phi^+$.

We fix an admissible D_n-decorated tree $(\mathscr{T}, \underline{\alpha}, \underline{\beta})$ which is good with respect to $\underline{\nu}$. This exists by Proposition 8.1.5. As usual let O denote the origin of $\mathscr{T}$. Let 2^{Φ^+} denote the power set of Φ^+. For each vertex v of $\mathscr{T}$ not equal to O, we define a subset $C_v \subset 2^{\Phi^+}$ as follows. Let w be the parent of v. If v is a positive vertex, we let

$$C_v := \left\{ S \in 2^{\Phi^+} | \beta(w) \notin S \right\}.$$

If v is a negative vertex, we let

$$C_v := \left\{ S \in 2^{\Phi^+} | \beta(w) \in S \right\}.$$

Now for each vertex $v \neq O$, we let $(O, v_1, \cdots, v_k = v)$ be the unique path from O to v, and define

$$D_v := \bigcap_{i=1}^{k} C_{v_i} \subset 2^{\Phi^+}.$$

Also define

$$D_O := 2^{\Phi^+}.$$

From condition (3) in Definition 8.1.2, it is easy to see that if $v, \bar{v}$ are the two children of any vertex w, then

$$D_w = D_v \sqcup D_{\bar{v}}. \tag{8.1.1}$$

More precisely, if v is the positive vertex among $v, \bar{v}$, then

$$D_v = \{S \in D_w | \beta(w) \notin S\} \tag{8.1.2}$$

$$D_{\bar{v}} = \{S \in D_w | \beta(w) \in S\}, \tag{8.1.3}$$

and there is a bijection

$$\begin{aligned} D_v &\xrightarrow{\sim} D_{\bar{v}} \\ S &\mapsto S \cup \{\beta(w)\}. \end{aligned} \tag{8.1.4}$$

Next, for any subset $\{\gamma_1, \cdots, \gamma_k\} \subset \Phi^+$ and any λ' in the root lattice, we define the following subset of $\mathbb{P}(\lambda')_L$:

$$\mathbb{P}(\lambda')_L^{\gamma_1,\cdots,\gamma_k} := \{\underline{m} \in \mathbb{P}(\lambda')_L | m(\gamma_1) = \cdots = m(\gamma_k) = 0\}.$$

For any vertex $v \neq O$ with $(O, v_1, \cdots, v_k = v)$ the unique path from O to v, we define

$$\mathbb{P}(\lambda')_L^v := \mathbb{P}(\lambda')_L^{\alpha(v_1),\cdots,\alpha(v_k)}.$$

Also define

$$\mathbb{P}(\lambda')_L^O := \mathbb{P}(\lambda')_L.$$

For any two vertices v_1, v_2, we define

$$\mathbb{P}(\lambda')_{L,v_2}^{v_1} := \mathbb{P}(\lambda')_L^{v_1} - \mathbb{P}(\lambda')_L^{v_2}.$$

Claim 1. Assume $v, \bar{v}$ are the two children of a vertex w. Then

$$\sum_{S\in D_w} (-1)^{|S|} \# \mathbb{P}(\lambda - S)_L^w = \sum_{S\in D_v} (-1)^{|S|} \# \mathbb{P}(\lambda - S)_L^v + \sum_{S\in D_{\bar{v}}} (-1)^{|S|} \# \mathbb{P}(\lambda - S)_L^{\bar{v}}.$$

To prove the claim, we may assume v is positive. In view of (8.1.1), it suffices to show that

$$\sum_{S\in D_v} (-1)^{|S|} \# \mathbb{P}(\lambda - S)_{L,v}^w + \sum_{S\in D_{\bar{v}}} (-1)^{|S|} \# \mathbb{P}(\lambda - S)_{L,\bar{v}}^w = 0.$$

In view of (8.1.2), (8.1.3), and the bijection (8.1.4), it suffices to show that for each $S \in D_v$ we have

$$\# \mathbb{P}(\lambda - S)_{L,v}^w = \# \mathbb{P}(\lambda - S - \beta(w))_{L,\bar{v}}^w.$$

To show this we construct a bijection

$$\phi : \mathbb{P}(\lambda - S)^w_{L,v} \xrightarrow{\sim} \mathbb{P}(\lambda - S - \beta(w))^w_{L,\bar{v}}$$

sending $\underline{m}$ to $\phi(\underline{m})$ given by:

$$\forall \beta \in \Phi^+,\ \phi(\underline{m})(\beta) := \begin{cases} m(\beta) - 1, & \beta = \alpha(v) \\ m(\beta) + 1, & \beta = \alpha(\bar{v}) \\ m(\beta), & \text{else} \end{cases}$$

Note that we indeed have $m(\alpha(v)) \geq 1$ because $\underline{m} \in \mathbb{P}(\lambda - S)^w_{L,v}$. Also it is obvious that $\phi(\underline{m})$ lies in the desired set, using the fact that $\alpha(v) - \alpha(\bar{v}) = \beta(v)$ and condition (4) from Definition 8.1.2. Finally, ϕ is a bijection because by the same reasoning the following map is a well-defined inverse map:

$$\psi : \mathbb{P}(\lambda - S - \beta(w))^w_{L,\bar{v}} \to \mathbb{P}(\lambda - S)^w_{L,v}$$

$$\underline{m}' \mapsto \psi(\underline{m}')$$

$$\forall \beta \in \Phi^+,\ \psi(\underline{m}')(\beta) := \begin{cases} m'(\beta) + 1, & \beta = \alpha(v) \\ m'(\beta) - 1, & \beta = \alpha(\bar{v}) \\ m'(\beta), & \text{else} \end{cases}$$

We have proved **Claim 1**.

Using **Claim 1**, we deduce that

$$\sum_{S \subset \Phi^+} (-1)^{|S|} \# \mathbb{P}(\lambda - S)_L = \sum_{S \in D_O} (-1)^{|S|} \# \mathbb{P}(\lambda - S)^O_L = \sum_v \sum_{S \in D_v} (-1)^{|S|} \# \mathbb{P}(\lambda - S)^v_L,$$

where v runs over all the end vertices of $\mathscr{T}$. Hence the proposition is proved once we show the following claim:

Claim 2. When t is sufficiently large, the following is true. For each $L \in \mathbb{Z} \cap [0, 3.5t]$ and for each end vertex v, we have

$$\sum_{S \in D_v} (-1)^{|S|} \# \mathbb{P}(\lambda - S)^v_L = 0.$$

To prove the claim, we fix a real number $M > n$. It is obvious that when t is sufficiently large (in a way that depends on M), the following is true: For all $S \in 2^{\Phi^+}$,

$$\left| \sum_{\beta \in S} \beta \right| < t/M.$$

Thus we can apply Lemma 8.1.7 to each $\lambda - S$. Since our $(\mathscr{T}, \underline{\alpha}, \underline{\beta})$ is good with respect to $\underline{\nu}$, we know that any $\underline{m} \in \mathbb{P}(\lambda - S)^v_L$ satisfies the hypothesis in Lemma 8.1.7 (with respect to $\lambda - S$) about the vanishing of $m(e_1 + \nu_i e_i)$. Hence by that lemma we know that $\mathbb{P}(\lambda - S)^v_L$ is non-empty only if

$$L \geq (n + 4 - \frac{n}{2M})t > (n + 3.5)t.$$

We have proved **Claim 2**. The proof of the proposition is complete. $\square$

8.2. **The case Proper-I.** We treat type D_n with $n \geq 5$ odd, and $\theta = \mathrm{id}$, $b = b_1$ or b_3. See §7.1.4. By symmetry we only need to consider $b = b_1$. Recall in this case

$$\Lambda(b_1) - \Lambda(b_1)_{\mathrm{good}} = \left\{ \lambda_{1,\mathrm{bad}} = (\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}) \right\},$$

and we have $\mathrm{def}_G(b_1) = \frac{n+3}{2}$.

Proposition 8.2.1. *The bound (6.3.1) holds for $\lambda = \lambda_{1,\mathrm{bad}}$.*

Proof. The proof uses the results from §8.1. By the formula (5.2.4), it suffices to show for each $w'' \in W^1 = W_0$, that

$$\sum_{w \in W^1 = W_0} (-1)^{\ell_1(w)} \mathcal{P}(w'' \lambda_{1,\mathrm{bad}}^{(s)} + w\rho^\vee - \rho^\vee, q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b)+a)}) \tag{8.2.1}$$

for some $a > 0$. Here we have made the change of variable $ww' \mapsto w''$ in (5.2.4), and have used the fact that $e_0(w'\lambda_{1,\mathrm{bad}}^{(s)}) \equiv 0$ for all $w' \in W^1$, as long as $s \gg 0$.

Fix w'', and write $\zeta_s := w''\lambda_{1,\mathrm{bad}}^{(s)}$. Let $|\cdot|$ be the norm on $\mathbb{R}^n$ defined in (7.1.1). Since $W_0 \subset \{\pm 1\}^n \rtimes S_n$, there exist $1 \leq j \leq n$, $\varepsilon \in \{\pm 1\}$, and $\underline{\nu} = (\nu_2, \cdots, \nu_n) \in \{\pm 1\}^{n-1}$, such that

$$\zeta_s = (\frac{1}{2}s\nu_2, \frac{1}{2}s\nu_3, \cdots, \frac{1}{2}s\nu_j, \frac{3}{2}s\varepsilon, \frac{1}{2}s\nu_{j+1}, \cdots, \frac{1}{2}s\nu_n),$$

where $\frac{3}{2}s\varepsilon$ is at the j-th place.

Assume either $j \neq 1$ or $\varepsilon = -1$. Then for $s \gg 0$ and all $w \in W_0$, we have $\zeta_s + w\rho^\vee - \rho^\vee \notin R^+$, and so $\mathcal{P}(\zeta_s + w\rho^\vee - \rho^\vee, q_1^{-1}) = 0$. We are done in this case.

Hence we assume $j = 1$ and $\varepsilon = 1$. Assume without loss of generality that $s = 4t$ for $t \in \mathbb{N}$. By the Weyl character formula, the left hand side of (8.2.1) is equal to

$$\sum_{S \subset \Phi^{\vee,+}} (-1)^{|S|} \mathcal{P}(\zeta_s - \sum_{\beta \in S} \beta, q_1^{-1}).$$

By Proposition 5.3.2 (1) and Proposition 8.1.8, the above is equal to

$$\sum_{L \in \mathbb{Z},\ L > (n+3.5)t} q_1^{-L} \sum_{S \subset \Phi^{\vee,+}} (-1)^{|S|} \# \mathbb{P}(\zeta_s - \sum_{\beta \in S} \beta)_L, \tag{8.2.2}$$

By the same argument as in the proof of Proposition 6.2.1, the expression

$$\left| \sum_{L \in \mathbb{Z}_{\geq 0}} \sum_{S \subset \Phi^{\vee,+}} (-1)^{|S|} \# \mathbb{P}(\zeta_s - \sum_{\beta \subset S} \beta)_L \right|$$

is of polynomial growth in s (or in t). Hence (8.2.2) is bounded by $O(q_1^{-(n+3.4)t})$. Since $s \cdot \mathrm{def}_G(b)/2 = (n+3)t$, the desired bound (8.2.1) follows. $\square$

8.3. **The case Proper-II.** We now treat type $E_6, \theta = \mathrm{id}, b = b_1$ or b_2. See §7.5. By symmetry, it suffices to treat the case of b_1. Recall in this case

$$\Lambda(b_1) - \Lambda(b_1)_{\mathrm{good}} = \{\varpi_5, \varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\},$$

and we have $\mathrm{def}_G(b_1) = 4$.

Proposition 8.3.1. *The bound (6.3.2) holds for all $\lambda \in \{\varpi_4 + \varpi_1, \varpi_2 + \varpi_6, 2\varpi_6\}$. In other words, in Proposition 6.3.1 (2) (for $b = b_1$) we may take λ_{bad} to be ϖ_5.*

Proof. We define a function $|\cdot|' : \mathbb{R}^9 \to \mathbb{R}_{\geq 0}$ in the following way. For any $v = \sum_{i=1}^9 x_i e_i \in \mathbb{R}^9$, we define

$$|v|' := \max_{\substack{i,j\in\{0,1,2\}, i\neq j;\\ k,l\in\{1,2,3\}}} |x_{3i+k} - x_{3j+l}|.$$

In other words, we think of $\mathbb{R}^9$ as $(\mathbb{R}^3)^3$, and we take the largest difference between a coordinate in one factor of $\mathbb{R}^3$ and a coordinate in a different factor. Then $|\cdot|'$ is a semi-norm, i.e., it is compatible with scalar multiplication by $\mathbb{R}$ and satisfies the triangle inequality. Note that $|\cdot|'$ is not W_0-invariant. By the explicit description of the roots, we have $|\alpha|' = 1$ for all positive roots α.

Claim. $|\mu|' \geq 7/3$ for all $\mu \in W_0\lambda$.

We prove the claim. We first record explicitly the W_0-orbit of λ. To state it we need some notation. Let C_3 be the cyclic group of order 3 with a fixed generator c. We let C_3 act on $S_3 \times S_3 \times S_3$ via

$$c : (\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_2, \sigma_3, \sigma_1).$$

Let H denote the semi-direct product $(S_3 \times S_3 \times S_3) \rtimes C_3$. Then we have an action of H on $\mathbb{R}^9$, where $S_3 \times S_3 \times S_3$ acts naturally on the coordinate indices, and $c \in C_3$ acts via

$$c : (x_1, x_2, x_3; x_4, x_5, x_6; x_7, x_8, x_9) \mapsto (x_4, x_5, x_6; x_7, x_8, x_9; x_1, x_2, x_3)$$

For $\lambda = \varpi_2 + \varpi_6$, its W_0-orbit is given by the union of the H-orbits of the following vectors:

$$\begin{gathered}
(2, -1, -1; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}) \\
(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 3, 3, -6) \\
(\frac{4}{3}, \frac{1}{3}, -\frac{5}{3}; 1, 0, -1; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \\
(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; 1, 0, -1; \frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}) \\
(\frac{4}{3}, \frac{1}{3}, -\frac{5}{3}; 0, 0, 0; \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}) \\
(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 0, 0, 0; \frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}) \\
(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 1, 0, -1; \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}).
\end{gathered}$$

For $\lambda = \varpi_1 + \varpi_4$, its W_0-orbit is the union of the H-orbits of the following:

$$\begin{gathered}
(\frac{4}{3}, -\frac{1}{3}, -\frac{5}{3}; 0, 0, 0; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \\
(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 0, 0, 0; \frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}) \\
(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 1, 0, -1; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \\
(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 1, 0, -1; \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}).
\end{gathered}$$

For $\lambda = 2\varpi_6$, its W_0-orbit is the H-orbit of

$$(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 0, 0, 0; \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}).$$

One sees easily that $|\cdot|'$ of all the above vectors are $\geq 7/3$. Since $|.|'$ is invariant under the action of H, it follows that $|\mu|' \geq 7/3$ holds for all $\mu \in W_0\lambda$. The claim is proved.

Based on the claim, we prove (6.3.2) for $\lambda \in \{\varpi_4+\varpi_1, \varpi_2+\varpi_6, 2\varpi_6\}$, using an argument similar to §7.1.1. By the formula (5.2.4), it suffices to show for each $w, w' \in W_0$ that

$$\mathcal{P}(\psi_s, q_1^{-1}) = O(q_1^{-s(\frac{1}{2}\mathrm{def}_G(b_1)+a)})$$

for some $a>0$, where $\psi_s := w \bullet (w'\lambda^{(s)})$. By the same argument as in §7.1.1, we easily reduce to proving: For some constant $a>0$,

$$\min\{|\underline{m}| : \underline{m} \in \mathbb{P}(\psi_s)\} \geq s(\frac{1}{2}\mathrm{def}_G(b_1)+a) = s(2+a), \tag{8.3.1}$$

for all $s \gg 0$. By the previous claim, for all $\underline{m} \in \mathbb{P}(\psi_s)$ we have

$$1 \cdot \sum_{\beta\in\Phi^{\vee,+}} m(\beta) \geq \left|\sum_{\beta\in\Phi^{\vee,+}} m(\beta)\beta\right|' = |\psi_s|' \geq \left|ww'\lambda^{(s)}\right|' - |w\rho^\vee - \rho^\vee|' = s\,|ww'\lambda|' - C \geq s\cdot\frac{7}{3} - C,$$

where C is a constant independent of s. Here the number 1 appearing in the leftmost term is equal to $\min_{\beta\in\Phi^{\vee,+}}|\beta|'$. Since $\theta = \mathrm{id}$, the leftmost term in the above inequalities is none other than $|\underline{m}|$. The desired (8.3.1) follows. □

8.4. **The case Proper-III.** We now treat type $E_7, \theta = \mathrm{id}$, and $[b] \in B(G)$ being the unique basic class such that $\kappa_G(b)$ is the non-trivial element of $\pi_1(G) = \mathbb{Z}/2\mathbb{Z}$. See §7.7. Recall in this case

$$\Lambda(b) - \Lambda(b)_{\mathrm{good}} = \{\varpi_5\},$$

and $\mathrm{def}_G(b) = 3$.

Proposition 8.4.1. *The bound (6.3.1) holds for $\lambda = \varpi_5$.*

Proof. Firstly, the W_0-orbit of λ is given by all permutations under S_8 of the following vectors:

$$\lambda_1 = (\frac{7}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$$
$$\lambda_2 = (\frac{5}{4}, -\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$
$$\lambda_3 = (-\frac{7}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$
$$\lambda_4 = (-\frac{5}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}).$$

Indeed it is easy to see that all these elements lie in $W_0\lambda$ (using the fact that the W_0 contains the copy of S_8), and one easily computes the size of $W_0\lambda$ to prove that these are all the elements of $W_0\lambda$.

For $1 \leq i \leq 8$ and $\tau \in S_8$, we define functions

$$|\cdot|_i : \mathbb{R}^8 \to \mathbb{R}_{\geq 0}$$
$$|\cdot|_\tau : \mathbb{R}^8 \to \mathbb{R}_{\geq 0}$$

in the following way. For any $v = \sum_{i=1}^8 x_i e_i \in \mathbb{R}^8$, we define

$$|v|_i := |x_i|, \quad |v|_\tau := \left|x_{\tau(1)}\right| + \left|x_{\tau(2)}\right| + \left|x_{\tau(3)}\right| + \left|x_{\tau(4)}\right|.$$

Then $|\cdot|_i, |\cdot|_\tau$ are all semi-norms.

Note that in the proof of Proposition 8.3.1, we reduced to proving (8.3.1) for **each fixed** $w, w' \in W_0$. During the proof (8.3.1) for the fixed w, w', we only needed to apply the semi-norm $|\cdot|'$ to $ww'\lambda$, and **not to** any other element of $W_0\lambda$. Hence for each element in $W_0\lambda$, we could use a semi-norm, which is specifically designed for that element, to finish the proof. In the current case, we reduce to proving that each $\mu \in W_0\lambda$ satisfies at least one of the following inequalities:

$$|\mu|_i > \frac{\mathrm{def}_G(b)}{2} \min_{\beta\in\Phi^{\vee,+}} |\beta|_i = \frac{3}{2} \min_{\beta\in\Phi^{\vee,+}} |\beta|_i \tag{8.4.1}$$

$$|\mu|_\tau > \frac{\mathrm{def}_G(b)}{2} \min_{\beta\in\Phi^{\vee,+}} |\beta|_\tau = \frac{3}{2} \min_{\beta\in\Phi^{\vee,+}} |\beta|_\tau \tag{8.4.2}$$

for some $1 \le i \le 8$ or some $\tau \in S_8$.

When μ is an S_8-permutation of λ_1 or λ_3, assume the i_0-th coordinate of μ is $\pm 7/4$. Then μ satisfies the inequality (8.4.1) indexed by i_0. In fact, any $\beta \in \Phi^{\vee,+}$ satisfies $|\beta|_{i_0} \le 1$, and we have $|\mu|_{i_0} = 7/4$.

When μ is an S_8-permutation of λ_2 or λ_4, there exists $\tau \in S_8$ such that

$$|\mu|_\tau = \frac{5}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{7}{2}.$$

On the other hand $|\beta|_\tau \le 2$ for all $\beta \in \Phi^{\vee,+}$ and all $\tau \in S_8$. Therefore (8.4.2) holds for some τ. □

9. Proof of the key estimate, Part III

We have proved all the statements in Proposition 6.3.1, except the existence of μ_1 in Proposition 6.3.1 (1), and the existence of μ_1, μ_2 in Proposition 6.3.1 (2). In this section we construct these.

First assume that G is not of type E_6, and that $\theta = \mathrm{id}$. We easily examine all such cases in §7 and see that $\lambda_b^+ \in X^*(\widehat{T})^+$ is always minuscule. Hence we may take $\mu_1 := \lambda_b^+$. Since G is adjoint and $\theta = \mathrm{id}$, the condition that $b \in B(G, \mu_1)$ is equivalent to the condition that b and μ_1 have the same image in $\pi_1(G)_\sigma = \pi_1(G)$, which is true by construction. Moreover, we have $\dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = \dim V_{\lambda_b^+}(\lambda_b^+) = 1$. The proof of Proposition 6.3.1 is complete in these cases.

The only remaining cases are the following:

Nonsplit-I: Type $D_n, n \ge 5, \theta$ has order 2. See §7.2.
Nonsplit-II: Type D_4, θ has order 2. See §7.3.
Split-E_6: Type $E_6, \theta = \mathrm{id}$. See §7.5

In fact, in all the other cases listed in the beginning of §7 where θ is non-trivial, namely cases (6) and (8), we have shown in §7.4 and §7.6 that any basic $[b] \in B(G)$ is unramified, so we do not need to consider these cases.

9.1. **The case Nonsplit-I.** As we showed in §7.2, we have $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, and there is a unique choice of basic $[b] \in B(G)$ corresponding to the non-trivial element in $\pi_1(G)_\sigma$. Moreover we have

$$\lambda_b^+ = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{Q}^{n-1} = X^*(\widehat{\mathcal{S}}) \otimes \mathbb{Q}.$$

Recall that $X_*(T) = X^*(\widehat{T}) = L_2 \subset \mathbb{R}^n$. We take

$$\mu_1 := (\frac{1}{2}, \cdots, \frac{1}{2}) \in L_2 = X_*(T) = X^*(\widehat{T}).$$

Then μ_1 is in $X^*(\widehat{T})^+$ and is minuscule. From the description of the action of σ on $\pi_1(G)$ in §7.2, the image of μ_1 in $\pi_1(G)_\sigma$ is the non-trivial element, and hence $[b] \in B(G, \mu_1)$. Finally, the only weights in $X^*(\widehat{T})$ of V_{μ_1} are the elements of $W_0\mu_1$. Among all these weights, there is precisely one that restricts to $\lambda_b^+ \in X^*(\widehat{\mathcal{S}})$, namely μ_1. Hence we have

$$\dim V_{\mu_1}(\lambda_b)_{\mathrm{rel}} = \dim V_{\mu_1}(\lambda_b^+)_{\mathrm{rel}} = \dim V_{\mu_1}(\mu_1) = 1.$$

9.2. **The case Nonsplit-II.** We keep the notation of §7.3. Note that what τ is does not affect $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ as a subset of $\mathbb{R}^4$. Nor does it affect the coroot lattice and the coweight lattice. Moreover, no matter what τ is, the quotient map $X_*(T) \to X^*(\widehat{\mathcal{S}})$ is always the same as

$$L_2 \to L_2', \ (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_1, \xi_2, \xi_3),$$

where $L_2 = \mathbb{Z}^4 + \mathbb{Z}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $L_2' = \mathbb{Z}^4 + \mathbb{Z}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence the situation is precisely the same as the case **Nonsplit-I**, see §9.1. Namely, for the unique basic $[b] \in B(G)$ that maps to the non-trivial element in $\pi_1(G)_\sigma \cong \mathbb{Z}/2\mathbb{Z}$, we have $\lambda_b = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), \lambda_b^+ = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and we take $\mu_1 := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

9.3. **The case Split-E_6.** We keep the notation of §7.5. To finish the proof of Proposition 6.3.1 (2), we need to construct μ_1 and μ_2. By symmetry we only need to consider b_1, among b_1, b_2. Recall from §7.5 that $\lambda_{b_1}^+ = \varpi_1$. Recall from Proposition 8.3.1 that the distinguished element λ_{bad} in $\Lambda(b_1)$ is ϖ_5. Since $\theta = \mathrm{id}$, we have $\widehat{\mathcal{S}} = \widehat{T}$.

Note that $\lambda_{b_1}^+ = \varpi_1$ is minuscule. We take $\mu_1 := \varpi_1$. Then the only weight of V_{μ_1} in $X^*(\widehat{\mathcal{S}})^+ = X^*(\widehat{T})^+$ is $\mu_1 = \lambda_{b_1}^+$, and $\dim V_{\mu_1}(\lambda_{b_1})_{\mathrm{rel}} = \dim V_{\mu_1}(\mu_1) = 1$. We have $b_1 \in B(G, \mu_1)$, because the image of $\mu_1 = \lambda_{b_1}^+$ in $\pi_1(G)_\sigma = \pi_1(G)$ is the same as that of λ_{b_1}, which is the same as $\kappa(b_1)$.

Note that ϖ_6 is also minuscule. We take $\mu_2 := 2\varpi_1 + \varpi_6$. Then μ_2 is a sum of dominant minuscule coweights. By (7.5.2) we know that $\mu_2 - \varpi_1$ is in the coroot lattice. Hence μ_2 represents the same element in $\pi_1(G)_\sigma = \pi_1(G)$ as ϖ_1, and in particular $b_1 \in B(G, \mu_2)$. We are left to check that $V_{\mu_2}(\lambda_{\mathrm{bad}})_{\mathrm{rel}}$, which is $V_{\mu_2}(\varpi_5)$, is non-zero. One computes that $\dim V_{\mu_2}(\varpi_5) = 14$, see for example [LiE]. [7] The proof of Proposition 6.3.1 is complete.

[7] Note that in [LiE], our $\alpha_2, \alpha_3, \alpha_4$ are indexed by $3, 4, 2$ respectively.

Appendix A. Irreducible components for quasi-split groups

We explain in this appendix how we can use our results combined with [He14] to obtain a description of the number of $J_b(F)$-orbits of irreducible components of affine Deligne-Lusztig varieties associated to a group which is quasi-split but not necessarily unramified. The main result Theorem A.3.1 is a generalization of Conjecture 2.6.7.

A.1. **Basic definitions.** We extend the notations introduced in §2. We let $F, L, k_F, k, \sigma, \Gamma$ be as in §2. However now we only assume that G is a quasi-split reductive group over F. Let $T \subset G$ be the centralizer of a fixed maximal F-split torus. Then T is a maximal torus of G since G is quasi-split. We fix B to be a Borel subgroup of G (over F) containing T. Let $\breve{A} \subset T_L$ be the maximal L-split sub-torus of T_L. Note that T_L is a minimal Levi of G_L, so $\breve{A}$ is also a maximal L-split torus of G_L. Let $N \subset G_L$ denote the normalizer of $\breve{A}$. Let V be the apartment of G_L corresponding to $\breve{A}$. Let $\mathfrak{a}$ be a σ-stable alcove in V, and let $\mathfrak{s}$ be a σ-stable special vertex lying in the closure of $\mathfrak{a}$. Denote by $\mathcal{I}$ the Iwahori group scheme over $\mathcal{O}_F$ associated to $\mathfrak{a}$, and denote by $\mathcal{K}$ the special parahoric group scheme over $\mathcal{O}_F$ associated to $\mathfrak{s}$. Let $\Gamma_0 \subset \Gamma$ denote the inertia subgroup, which is also identified with $\mathrm{Gal}(\overline{L}/L)$. The choice of $\mathfrak{s}$ gives an identification

$$V \cong X_*(T)_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R},$$

sending $\mathfrak{s}$ to 0. In the following we freely use the identification in Lemma 2.6.1 (3). We assume that under the identification

$$V \cong X_*(T)_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R} \cong X_*(T)_{\mathbb{R}}^{\Gamma_0}, \tag{A.1.1}$$

the image of $\mathfrak{a}$ is contained in the anti-dominant chamber $-X_*(T)_{\mathbb{R}}^{+}$.

The Iwahori–Weyl group is defined to be

$$W := N(L)/(T(L) \cap \mathcal{I}(\mathcal{O}_L)).$$

For any $w \in W$ we choose a representative $\dot{w} \in N(L)$. We write $W_0 := N(L)/T(L)$ for the relative Weyl group of G over L. Then we have a natural exact sequence:

$$0 \longrightarrow X_*(T)_{\Gamma_0} \longrightarrow W \longrightarrow W_0 \longrightarrow 0.$$

For $\underline{\mu} \in X_*(T)_{\Gamma_0}$ we write $t^{\underline{\mu}}$ for the corresponding element in W. The Frobenius σ induces an action on W which preserves the set of simple roots $\mathbb{S}$. See [HR08] for more details.

Let $B(G)$ (resp. $B(W, \sigma)$) denote the set of σ-conjugacy classes of $G(L)$ (resp. W). Let $X_*(T)^+_{\Gamma_0,\mathbb{Q}}$ denote the intersection of $X_*(T)_{\Gamma_0} \otimes \mathbb{Q} \cong X_*(T)_{\mathbb{Q}}^{\Gamma_0}$ with $X_*(T)_{\mathbb{Q}}^{+}$. Similar to §2, we have an injective map

$$(\bar{\nu}, \kappa) : B(G) \to (X_*(T)^+_{\Gamma_0,\mathbb{Q}})^{\sigma} \times \pi_1(G)_{\Gamma},$$

a surjective map

$$\Psi : B(W, \sigma) \to B(G),$$

and a commutative diagram

$$\begin{tikzcd} B(W,\sigma) \arrow[rr, two heads, "\Psi"] \arrow[dr, "(\bar{\nu},\kappa)"'] & & B(G) \arrow[dl, hook', "(\bar{\nu},\kappa)"] \\ & (X_*(T)^+_{\Gamma_0,\mathbb{Q}})^{\sigma} \times \pi_1(G)_{\Gamma} & \end{tikzcd} , \tag{A.1.2}$$

where the map $(\bar{\nu}, \kappa)$ on $B(W, \sigma)$ can be described explicitly. These are proved in [He14]. See [HZ16, §1.2] for more details.

Let $w \in W$ and $b \in G(L)$. We define the affine Deligne–Lusztig variety $X_w(b)$ as follows:

$$X_w(b) := \{g\mathcal{I}(\mathcal{O}_L) \in G(L)/\mathcal{I}(\mathcal{O}_L) | g^{-1}b\sigma(g) \in \mathcal{I}(\mathcal{O}_L)\dot{w}\mathcal{I}(\mathcal{O}_L)\}.$$

Now let $\underline{\mu} \in X_*(T)_{\Gamma_0}$ be the image of an element $\mu \in X_*(T)^+$. Similarly we define the affine Deligne–Lusztig variety $X_{\underline{\mu},K}(b)$ as follows:

$$X_{\underline{\mu},K}(b) := \{g\mathcal{K}(\mathcal{O}_L) \in G(L)/\mathcal{K}(\mathcal{O}_L) | g^{-1}b\sigma(g) \in \mathcal{K}(\mathcal{O}_L)t^{\underline{\mu}}\mathcal{K}(\mathcal{O}_L)\}.$$

When non-empty, $X_w(b)$ and $X_{\underline{\mu},K}(b)$ are (perfect) schemes locally of (perfectly) finite type over k.

We define the set

$$B(G,\underline{\mu}) = \{[b] \in B(G); \kappa([b]) = \mu^\natural, \bar{\nu}_b \leq \mu^\diamond\}.$$

Here $\mu^\natural$ denotes the image of μ in $\pi_1(G)_\Gamma$, and $\mu^\diamond \in (X_*(T)^+_{\Gamma_0,\mathbb{Q}})^\sigma$ denotes the average over the σ-orbit of $\underline{\mu} \in X_*(T)_{\Gamma_0}$. Note that both $\mu^\natural$ and $\mu^\diamond$ depend only on $\underline{\mu}$.

For $b \in G(L)$, the group $J_b(F)$ acts by left multiplication on $X_{\underline{\mu},K}(b)$ via algebraic automorphisms. Our goal is to understand the cardinality

$$\#\Big(J_b(F)\backslash\Sigma^{\mathrm{top}}(X_{\underline{\mu},K}(b))\Big). \tag{A.1.3}$$

Along the way we shall also show that $X_{\underline{\mu},K}(b) \neq \emptyset$ if and only if $[b] \in B(G,\underline{\mu})$, which should be well known to experts.

For simplicity, from now on we assume that G is adjoint. The general case reduces to this case by a standard argument.

A.2. **Dual group construction.** The desired formula for (A.1.3) will be expressed in terms of a canonical **reductive subgroup** of the dual group $\widehat{G}$. We keep the assumption that G is adjoint.

As in §5.1, we let

$$\mathrm{BRD}(B,T) = (X^*(T), \Phi \supset \Delta, X_*(T), \Phi^\vee \supset \Delta^\vee)$$

be the based root datum associated to (B,T), equipped with an action by Γ. Let $\widehat{G}$ be the dual group of G over $\mathbb{C}$, which is equipped with a Borel pair $(\widehat{B},\widehat{T})$ and an isomorphism

$$\mathrm{BRD}(\widehat{B},\widehat{T}) \xrightarrow{\sim} \mathrm{BRD}(B,T)^\vee.$$

We fix a pinning $(\widehat{B},\widehat{T},\widehat{\mathbb{X}}_+)$. The action of Γ on $\mathrm{BRD}(B,T)$ translates to an action on $\mathrm{BRD}(\widehat{B},\widehat{T})$, and the latter lifts to a unique action of Γ on $\widehat{G}$ via algebraic automorphisms that preserve $(\widehat{B},\widehat{T},\widehat{\mathbb{X}}_+)$.

We define

$$\widehat{H} := \widehat{G}^{\Gamma_0,0},$$

namely the identity component of the Γ_0-fixed points of $\widehat{G}$. This construction was also considered by Zhu [Zhu15] and Haines [Hai18]. By [Hai18, Proposition 5.1], the group $\widehat{H}$ is a reductive subgroup of $\widehat{G}$, and it has a pinning of the form $(\widehat{B}^{\Gamma_0,0},\widehat{T}^{\Gamma_0,0},\widehat{\mathbb{X}}'_+)$. Moreover, the induced action of the Frobenius $\sigma \in \Gamma/\Gamma_0$ on $\widehat{H}$ preserves this pinning. We write $\widehat{B}_H := \widehat{B}^{\Gamma_0,0}$ and $\widehat{T}_H := \widehat{T}^{\Gamma_0,0}$. Let $\hat{\theta}$ denote the automorphism of $\widehat{H}$ given by σ. We define

$$\widehat{\mathcal{S}} := (\widehat{T}_H)^{\hat{\theta},0}.$$

Note that since G is adjoint, the fundamental coweights of G form a Γ-stable $\mathbb{Z}$-basis of $X_*(T)$. It then follows from Lemma 2.6.1 (2) that $X_*(T)_{\Gamma_0}$ and $X_*(T)_\Gamma$ are both free. Hence we in fact have $\widehat{T}_H = \widehat{T}^{\Gamma_0}$ and $\widehat{\mathcal{S}} = \widehat{T}^\Gamma$. This observation will simplify our exposition.

Lemma A.2.1. *Let $b \in G(L)$. There is a unique element $\lambda_b \in X^*(\widehat{\mathcal{S}})$ satisfying the following conditions:*

(1) The image of λ_b under $X^*(\widehat{\mathcal{S}}) = X_*(T)_\Gamma \to \pi_1(G)_\Gamma$ is equal to $\kappa(b)$.

(2) In $X^*(\widehat{\mathcal{S}})_{\mathbb{Q}} = X^*(\widehat{T})_\Gamma \otimes \mathbb{Q} = (X_*(T)_{\Gamma_0})_\sigma \otimes \mathbb{Q} \cong (X_*(T)_{\Gamma_0,\mathbb{Q}})^\sigma$, the element $\lambda_b - \bar{\nu}_b$ is equal to a linear combination of the restrictions to $\widehat{\mathcal{S}}$ of the simple roots in $\Phi^\vee \subset X^*(\widehat{T})$, with coefficients in $\mathbb{Q} \cap (-1, 0]$.

Proof. The proof is the same as Lemma 2.6.4. $\square$

A.3. **The main result.**

Theorem A.3.1. *Assume G is adjoint and quasi-split over F. Let $\underline{\mu} \in X_*(T)_{\Gamma_0}$ be the image of an element $\mu \in X_*(T)^+$. Let $b \in G(L)$.*

(1) We have $X_{\underline{\mu},K}(b) \neq \emptyset$ if and only if $[b] \in B(G, \underline{\mu})$.

(2) Assume that $[b] \in B(G, \underline{\mu})$. Then

$$\#\Big(J_b(F)\backslash\Sigma^{\mathrm{top}}(X_{\underline{\mu},K}(b))\Big) = \dim V_{\underline{\mu}}^{\widehat{H}}(\lambda_b)_{\mathrm{rel}}.$$

Here $V_{\underline{\mu}}^{\widehat{H}}$ denotes the highest weight representation of $\widehat{H}$ of highest weight $\underline{\mu} \in X^*(\widehat{T}_H)^+$. We denote by $V_{\underline{\mu}}^{\widehat{H}}(\lambda_b)_{\mathrm{rel}}$ the λ_b-weight space in $V_{\underline{\mu}}^{\widehat{H}}$, for the action of $\widehat{\mathcal{S}}$. The element $\lambda_b \in X^*(\widehat{\mathcal{S}})$ is defined in Lemma A.2.1.

Remark A.3.2. The appearance of the representation $V_{\underline{\mu}}^{\widehat{H}}$ of the subgroup $\widehat{H}$ of $\widehat{G}$ in Theorem A.3.1 is compatible with the ramified geometric Satake in [Zhu15].

Proof of Theorem A.3.1. The idea of the proof is to reduce to the unramified case. For this we first construct an auxiliary unramified reductive group over F.

From $\widehat{H}$ and its pinned automorphism $\hat{\theta}$, we obtain an unramified reductive group H over F, whose dual group is $\widehat{H}$. By definition H is equipped with a Borel pair (B_H, T_H), and a σ-equivariant isomorphism of based root data $\mathrm{BDR}(B_H, T_H) \xrightarrow{\sim} \mathrm{BDR}(\widehat{B}_H, \widehat{T}_H)^\vee$. We write

$$\mathrm{BDR}(B_H, T_H) = (X^*(T_H), \Phi_H, X_*(T_H), \Phi_H^\vee).$$

Then we have canonical σ-equivariant identifications

$$X^*(T_H) \cong X_*(\widehat{T}_H) \cong X_*(\widehat{T})^{\Gamma_0} \cong X^*(T)^{\Gamma_0}$$

and

$$X_*(T_H) \cong X^*(\widehat{T}_H) \cong X^*(\widehat{T})_{\Gamma_0} \cong X_*(T)_{\Gamma_0},$$

which we shall think of as identities. Here as we noted before $X_*(T)_{\Gamma_0}$ is indeed free.

Note that $T_{H,L}$ is a maximal split torus of H_L. Let V_H be the corresponding apartment, and fix a hyperspecial vertex $\mathfrak{s}_H$ in V_H (coming from the apartment of H corresponding to the maximal F-split sub-torus of T_H). We fix a σ-stable alcove $\mathfrak{a}_H \subset V_H$ whose closure contains $\mathfrak{s}_H$. We identify

$$V_H \cong X_*(T_H) \otimes \mathbb{R}, \tag{A.3.1}$$

sending $\mathfrak{s}_H$ to 0, such that the image of $\mathfrak{a}_H$ is in the anti-dominant chamber.

Since $X_*(T_H) = X_*(T)_{\Gamma_0}$, the two identifications (A.1.1) and (A.3.1) give rise to a σ-equivariant identification

$$V \cong V_H, \tag{A.3.2}$$

which maps $\mathfrak{a}$ onto $\mathfrak{a}_H$, and maps $\mathfrak{s}$ to $\mathfrak{s}_H$.

By [Hai18, Corollary 5.3], the set of coroots $\Phi_H^\vee \subset X_*(T_H) = X_*(T)_{\Gamma_0}$ is given by $\breve{\Sigma}^\vee$, where $\breve{\Sigma}$ is the échelonnage root system of Bruhat–Tits, see [Hai18, §4.3]. In particular, the coroot lattice in $X_*(T_H)$ is isomorphic to the Γ_0-coinvariants of the coroot lattice in $X_*(T)$. Moreover from $\Phi_H^\vee = \breve{\Sigma}^\vee$ we know that the affine Weyl group of G and the affine Weyl group of H are equal, under the identification (A.3.2. See [HR08] for more details. Moreover, since the translation groups $X_*(T_H)$ and $X_*(T)_{\Gamma_0}$ are also identified, we have an identification between the Iwahori–Weyl group W of G and the Iwahori–Weyl group W_H of H. This identification is σ-equivariant.

Note that the bottom group in the diagram (A.1.2) and its analogue for H are identified. Using the identification of W and W_H, and using the surjectivity of the map $\Psi : B(W,\sigma) \to B(G)$ and its analogue $\Psi_H : B(W_H,\sigma) \to B(H)$, we construct $[b_H] \in B(H)$ whose invariants are the same as those of $[b]$. Since the set $B(G,\underline{\mu})$ is defined in terms of the invariants $(\overline{\nu},\kappa)$ and ditto for $B(H,\underline{\mu})$, we see that $[b] \in B(G,\underline{\mu})$ if and only if $[b_H] \in B(H,\underline{\mu})$. Here in writing $B(H,\underline{\mu})$ we view $\underline{\mu}$ as an element of $X_*(T_H)^+$.

To relate the geometry of $X_{\underline{\mu},K}(b)$ with the geometry of $X_{\underline{\mu}}(b_H)$, we use the class polynomials in [He14]. For each $w \in W$ and each σ-conjugacy class $\mathcal{O}$ in W, we let

$$f_{w,\mathcal{O}} \in \mathbb{Z}[v - v^{-1}]$$

denote the class polynomial defined in [He14, §2.3].

Proof of (1)

Using the fibration

$$\bigcup_{w \in W_0 t^{\underline{\mu}} W_0} X_w(b) \to X_{\underline{\mu},K}(b), \tag{A.3.3}$$

we see that $X_{\underline{\mu},K}(b)$ is non-empty if and only $X_w(b)$ is non-empty for some $w \in W_0 t^{\underline{\mu}} W_0$. The proof of [He14, Theorem 6.1] shows that $X_w(b) \neq \emptyset$ if and only if $f_{w,\mathcal{O}} \neq 0$ for some $\mathcal{O}$ such that $(\overline{\nu},\kappa)(\mathcal{O}) = (\overline{\nu},\kappa)(b)$. This latter condition is a condition on the quadruple $Q(b) := (W,\sigma,\underline{\mu},(\overline{\nu},\kappa)(b))$. By construction we have an identification of quadruples $Q(b) \cong Q(b_H)$, where $Q(b_H) := (W_H,\sigma,\underline{\mu},(\overline{\nu},\kappa)(b_H))$. Hence $X_{\underline{\mu},K}(b) \neq \emptyset$ if and only if $X_{\underline{\mu}}(b_H) \neq \emptyset$. On the other hand we have already seen that $[b] \in B(G,\underline{\mu})$ if and only if $[b_H] \in B(H,\underline{\mu})$. Thus the result reduces to the statement that $X_{\underline{\mu}}(b_H) \neq \emptyset$ if and only if $b_H \in B(H,\underline{\mu})$. But this is the main result of [Gas10].

Proof of (2)

We write $\mathscr{N}(\underline{\mu},b)$ for the cardinality of $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_{\underline{\mu},K}(b))$. Using the fibration A.3.3 we have an identification

$$J_b(F)\backslash\Sigma^{\mathrm{top}}\left(\bigcup_{w \in W_0 t^{\underline{\mu}} W_0} X_w(b)\right) \cong J_b(F)\backslash\Sigma^{\mathrm{top}}(X_{\underline{\mu},K}(b)). \tag{A.3.4}$$

By [He14, Theorem 6.1], we have the formula

$$\dim X_w(b) = \max_{\mathcal{O}} \frac{1}{2}(\ell(w) + \ell(\mathcal{O}) + \deg f_{w,\mathcal{O}})) - \langle \overline{\nu}_b, 2\rho \rangle.$$

where $\mathcal{O}$ runs through σ-conjugacy classes in W such that $(\overline{\nu},\kappa)(\mathcal{O}) = (\overline{\nu},\kappa)(b)$ and where $\ell(\mathcal{O})$ denotes the length of a minimal length element in $\mathcal{O}$. Moreover the proof of [He14, Theorem 6.1] also shows that the cardinality of $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_w(b))$ is equal to the leading coefficient of $\sum_{\mathcal{O}} v^{\ell(w)+\ell(\mathcal{O})} f_{w,\mathcal{O}}$. Since each $X_w(b)$ is locally closed in the union $\bigcup_{w \in W_0 t^{\underline{\mu}} W_0} X_w(b)$, any top dimensional irreducible component in the union is the closure of a top dimensional irreducible component in $X_w(b)$ for a unique w. It follows that the cardinality of

$$J_b(F)\backslash\Sigma^{\mathrm{top}}\left(\bigcup_{w \in W_0 t^{\underline{\mu}} W_0} X_w(b)\right)$$

is equal to the leading coefficient of

$$\sum_{w\in W_0 t^{\underline{\mu}} W_0} \sum_{\mathcal{O}} v^{l(w)+l(\mathcal{O})} f_{w,\mathcal{O}}. \tag{A.3.5}$$

By (A.3.4), this number is just $\mathscr{N}(\underline{\mu}, b)$. Since the term (A.3.5) only depends on the quadruple $Q(b)$, the same is true for $\mathscr{N}(\underline{\mu}, b)$.

Applying the same argument to H, we see that $\mathscr{N}(\underline{\mu}, b_H)$ only depends on the quadruple $Q(b_H)$. Again, since the quadruples $Q(b)$ and $Q(b_H)$ are identified, we have $\mathscr{N}(\underline{\mu}, b) = \mathscr{N}(\underline{\mu}, b_H)$.

It thus remains to check

$$\mathscr{N}(\underline{\mu}, b_H) = \dim V_{\underline{\mu}}^{\widehat{H}}(\lambda_b)_{\mathrm{rel}}. \tag{A.3.6}$$

By assumption $[b] \in B(G, \underline{\mu})$, and so $[b_H] \in B(H, \underline{\mu})$. The right hand side of (A.3.6) is easily seen to be the same as the right hand side of Conjecture (2.6.7), with respect to $(H, \underline{\mu}, b_H)$. Hence the desired (A.3.6) follows from the main result of the paper Corollary 6.3.3. □

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