408L CLASS PROBLEMS

FEBRUARY 14TH, 2020

Problem 1. Find $\int \frac{\log(x)}{x^2}dx$.

Solution. Set $u = \log(x)$ and $v = -\frac{1}{x}$. Then $du = \frac{dx}{x}$ and $dv = \frac{dx}{x^2}$, so integration by parts gives:

$$\int \frac{\log(x)}{x^2}dx = \int udv = uv - \int vdu = -\frac{\log(x)}{x} + \int \frac{dx}{x^2} =$$
$$\boxed{-\frac{\log(x)}{x} - \frac{1}{x} + C}.$$

Problem 2. Find $\int_0^{\frac{\pi}{3}} x\sec^2(x)dx$.

Solution. We set $u = x$ and $v = \tan(x)$. Then $du = dx$ and $dv = \sec^2(x)dx$, so integration by parts gives:

$$\int x\sec^2(x)dx = x\tan(x) - \int \tan(x)dx.$$

We find $\int \tan(x)dx$ by substitution. Set $t = \cos(x)$ so $dt = -\sin(x)dx$. We then obtain:

$$\int \tan(x)dx = \int \frac{\sin(x)dx}{\cos(x)} = -\int \frac{dt}{t} = -\log|t| + C = -\log(\cos(x)).$$

(Here we can ignore the absolute value because $\cos(x)$ is positive for $0 \leqslant x \leqslant \frac{\pi}{3}$. Moreover, we omit the constant here because we are finding a definite integral in this problem, so only need to find some anti-derivative, not the general one.)

Substituting back into our earlier expression, we obtain:

$$\int x\sec^2(x)dx = x\tan(x) - \int \tan(x)dx = x\tan(x) + \log(\cos(x)).$$

We therefore have:

$$\int_0^{\frac{\pi}{3}} x\sec^2(x)dx = \big(x\tan(x) + \log(\cos(x))\big)\Big|_0^{\frac{\pi}{3}} =$$
$$\Big(\frac{\pi}{3}\tan(\frac{\pi}{3}) + \log(\cos(\frac{\pi}{3})) - 0 * \tan(0) - \log(\cos(0))\Big) = \boxed{\frac{\pi\sqrt{3}}{3} + \log(\frac{1}{2})}.$$

Problem 3. Find $\int \tan^{-1}(x)dx$.

Solution. Set $u = \tan^{-1}(x)$ so that $du = \frac{dx}{x^2+1}$. Therefore, if we take $v = x$ and apply integration by parts, we obtain:

$$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}dx.$$

We can evaluate the resulting integral using substitution. Set $t = x^2+1$, so $dt = 2xdx$. We then obtain:

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log(t) + C = \frac{1}{2}\log(x^2+1) + C.$$

(We can ignore the absolute value because $t = x^2 + 1 > 0$.)

Substituting back into our earlier expression, we find:

$$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}dx = \boxed{x\tan^{-1}(x) - \frac{1}{2}\log(x^2+1) + C}.$$

Problem 4. Find $\int \cos(x)\sin(2x)dx$.

Solution. Set $u = \sin(2x)$ and $v = \sin(x)$, so $du = 2\cos(2x)dx$ and $dv = \cos(x)dx$. Applying integration by parts, we obtain:

$$\int \cos(x)\sin(2x)dx = \sin(x)\sin(2x) - 2\int \sin(x)\cos(2x)dx.$$

In the latter integral, we apply integration by parts again, this time with $u = \cos(2x)$ and $v = -\cos(x)$ to obtain:

$$\int \sin(x)\cos(2x)dx = -\cos(x)\cos(2x) - 2\int \cos(x)\sin(2x)dx.$$

Substituting into our earlier expression, we obtain:

$$\int \cos(x)\sin(2x)dx = \sin(x)\sin(2x) - 2\int \sin(x)\cos(2x)dx =$$
$$\sin(x)\sin(2x) + 2\cos(x)\cos(2x) + 4\int \cos(x)\sin(2x)dx.$$

Rearranging terms gives:

$$-3\int \cos(x)\sin(2x)dx = \sin(x)\sin(2x) + 2\cos(x)\cos(2x) \Rightarrow$$
$$\int \cos(x)\sin(2x)dx = \boxed{-\frac{1}{3}\big(\sin(x)\sin(2x) + 2\cos(x)\cos(2x)\big) + C}.$$

Alternative solution. This problem may also be solved as follows. Recall that $\sin(2x) = 2\sin(x)\cos(x)$. We then obtain:

$$\int \cos(x)\sin(2x)dx = 2\int \cos^2(x)\sin(x)dx.$$

We now apply substitution with $u = \cos(x)$, so $du = -\sin(x)dx$. We obtain:

$$2\int \cos^2(x)\sin(x)dx = -2\int u^2 du = -\frac{2}{3}u^3 + C = \boxed{-\frac{2}{3}\cos^3(x) + C}.$$

(Using trig identities, one can directly check that this answer actually coincides with our previous one.)

Alternative solution. There is a third way to solve this problem. We haven't taught this method yet – you'll see it on your LM for Monday. I'm including this solution to give you another chance to learn this approach, not because I expected you to try this today.

Recall that:

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Therefore:

$$\sin(3x) = \sin(x + 2x) = \sin(x)\cos(2x) + \cos(x)\sin(2x)$$

$$\sin(x) = \sin(-x + 2x) = \sin(-x)\cos(2x) + \cos(-x)\sin(2x) = -\sin(x)\cos(2x) + \cos(x)\sin(2x).$$

Adding these equations and dividing by 2 yields:

$$\frac{1}{2}\big(\sin(3x) + \sin(x)\big) = \cos(x)\sin(2x).$$

Therefore, we have:

$$\int \cos(x)\sin(2x) = \frac{1}{2}\int \big(\sin(3x) + \sin(x)\big)dx = \boxed{-\frac{1}{6}\cos(3x) - \frac{1}{2}\cos(x) + C}.$$

(Using trig identities, you can again check that this answer coincides with the previous ones, although it looks different from either.)

Problem 5. Find $\int \sin(2x)e^{\sin(x)}dx$.

Solution. This problem uses many of the techniques we have studied so far.

First, using trig identities, we expand $\sin(2x)$ as $2\sin(x)\cos(x)$. Next, we use substitution with $t = \sin(x)$, $dt = \cos(x)dx$ to obtain:

$$\int \sin(2x)e^{\sin(x)}dx = 2\int \sin(x)\cos(x)e^{\sin(x)}dx = 2\int te^t dt.$$

Next, we use integration by parts with $u = t$, $v = e^t$ to obtain:

$$\int te^t dt = te^t - \int e^t dt = (t-1)e^t + C.$$

Substituting back in, we obtain:

$$\int \sin(2x)e^{\sin(x)}dx = 2(t-1)e^t + C = \boxed{2(\sin(x)-1)e^{\sin(x)} + C}.$$

Problem 6. Graph the parametric curve:

$$x = 16\sin^3(t)$$
$$y = 13\cos(t) - 5\cos(2t) - 2\cos(3t) - \cos(4t).$$

Solution. ♡ – Happy Valentine's Day!