408L CLASS PROBLEMS

FEBRUARY 17TH, 2020

Problem 1. Find $\int \sin^2(x)\cos(x)dx$.

Solution. We use u-substitution with $u = \sin(x)$, $du = \cos(x)dx$, so:

$$\int \sin^2(x)\cos(x)dx = \int u^2 du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\sin^3(x) + C}.$$

Problem 2. Find $\int \cos^3(x)dx$.

Solution. We write:

$$\cos^3(x) = \cos^2(x)\cdot\cos(x) = (1-\sin^2(x))\cdot\cos(x).$$

We obtain:

$$\int \cos^3(x)dx = \int \cos(x)dx - \int \sin^2(x)\cos(x)dx = \boxed{\sin(x) - \frac{1}{3}\sin^3(x) + C}$$

where we have used our answer to the previous problem.

Problem 3. Find $\int \cos^2(x)dx$.

Solution. As on previous days, we use the identity:

$$\cos^2(x) = \frac{\cos(2x)+1}{2}$$

to obtain:

$$\int \cos^2(x)dx = \frac{1}{2}\int (\cos(2x)+1)dx = \boxed{\frac{1}{4}\sin(2x) + \frac{x}{2} + C}.$$

Problem 4. Find $\int \sin^2(x)\cos^2(x)dx$.

Solution. First, we write:

$$\sin^2(x)\cos^2(x) = (1-\cos^2(x))\cdot\cos^2(x) = \cos^2(x) - \cos^4(x).$$

We found the integral of $\cos^2(x)$ in the previous problem, so it remains to find the integral of $\cos^4(x)$.

We use a similar method as for $\cos^2(x)$, repeatedly using $\cos^2(x) = \frac{\cos(2x)+1}{2}$. Namely, we have:

$$\cos^4(x) = \big(\cos^2(x)\big)^2 = \big(\frac{\cos(2x)+1}{2}\big) = \frac{\cos^2(2x)}{4} + \frac{cos(2x)}{2} + \frac{1}{4}.$$

We can further expand the first term as $\frac{\cos^2(2x)}{4} = \frac{1+\cos(4x)}{8}$ to obtain:

$$\cos^4(x) = \frac{\cos(4x)+1}{8} + \frac{cos(2x)}{2} + \frac{1}{4} = \frac{\cos(4x)}{8} + \frac{cos(2x)}{2} + \frac{3}{8}.$$

We now obtain:

$$\int \cos^4(x)dx = \frac{\sin(4x)}{32} + \frac{\sin(2x)}{4} + \frac{3x}{8}.$$

We finally obtain:

$$\int \sin^2(x)\cos^2(x)dx = \int \cos^2(x)dx - \int \cos^4(x)dx =$$

$$\frac{1}{4}\sin(2x) + \frac{x}{2} - \frac{\sin(4x)}{32} - \frac{\sin(2x)}{4} - \frac{3x}{8} + C = \boxed{-\frac{\sin(4x)}{32} + \frac{x}{8} + C}.$$

Alternative solution. Write $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$ to obtain:

$$\int \sin^2(x)\cos^2(x)dx = \frac{1}{4}\int \sin^2(2x)dx.$$

The method from the previous problem gives $\int \sin^2(u)du = \frac{u}{2} - \frac{\sin(2u)}{4} + C$, so setting $u = 2x$ we obtain:

$$\frac{1}{2}\int \sin^2(2x)dx = \boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}.$$

Obviously this method is simpler. But it is also special to this particular integral. For a more general problem, you would need to use our favorite method: use $\cos^2(x)+\sin^2(x) = 1$ to reduce to integrals of the form $\int cos^n(x)\sin(x)dx / \int sin^n(x)\cos(x)dx$ (which you solve by substitution) or $\int \cos^{2n}(x)dx / \int \sin^{2n}(x)dx$ (which you reduce using the double-angle identities).

Problem 5. Find $\int \frac{\sin(x)}{\cos^3(x)}dx$.

Solution. We have $\frac{\sin(x)}{\cos^3(x)} = \tan(x)\cdot\sec^2(x)$. Therefore, we apply substitution with $u = \tan(x)$, $du = \sec^2(x)dx$ to obtain:

$$\int \frac{\sin(x)}{\cos^3(x)}dx = \int \tan(x) \cdot \sec^2(x)dx = \int udu = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2(x) + C}.$$

Problem 6. Find $\int \frac{\sin^3(x)}{\cos^3(x)}dx$.

Solution. The integrand is $\tan^3(x)$. We use the identity:

$$\sec^2(x) - \tan^2(x) = 1$$

to obtain:

$$\tan^3(x) = \tan^2(x) \cdot \tan(x) = (\sec^2(x) - 1) \cdot \tan(x).$$

We now integrate:

$$\int \tan^3(x)dx = \int \sec^2(x)\tan(x)dx - \int \tan(x)dx.$$

We have $\int \tan(x)dx = -\log(\cos(x)) + C$ via u-substitution with $u = \cos(x)$. Therefore, we have:

$$\int \sec^2(x)\tan(x)dx - \int \tan(x)dx = \boxed{\frac{1}{2}\tan^2(x) + \log(\cos(x)) + C}$$

using the previous problem.