408L CLASS PROBLEMS

FEBRUARY 19TH, 2020

First, go through the problems below and use substitution to reduce the integral to a trigonometric integral. After you have done so for all of the problems, go back and solve the integrals you set up.

Problem 1. Find $\int \frac{x}{\sqrt{1+x^2}}dx$.

Solution. We set $\tan(\theta) = x$, $\sec^2(\theta)d\theta = dx$. As $\sqrt{1+\tan^2(\theta)} = \sec(\theta)$, we obtain:

$$\int \frac{x}{\sqrt{1+x^2}}dx = \int \frac{\tan(\theta)}{\sec(\theta)}\sec^2(\theta)d\theta = \boxed{\int \tan(\theta)\sec(\theta)d\theta}.$$

We then know:

$$\int \tan(\theta)\sec(\theta)d\theta = \sec(\theta) + C = \boxed{\sqrt{1+x^2}+C}$$

as $\sec(\theta) = \sqrt{1+\tan^2(\theta)} = \sqrt{1+x^2}$.

Problem 2. Find $\int \frac{\sqrt{x^2-1}}{x^3}dx$.

Solution. We take $\sec(\theta) = x$, so $\sec(\theta)\tan(\theta)d\theta = dx$. As $\sqrt{\sec^2(\theta)-1} = \tan(\theta)$, we obtain:

$$\int \frac{\sqrt{x^2-1}}{x^3}dx = \int \frac{\tan(\theta)}{\sec^3(\theta)}\sec(\theta)\tan(\theta)d\theta = \boxed{\int \sin^2(\theta)d\theta}.$$

As always, we use the double-angle identity $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$ to find:

$$\int \sin^2(\theta)d\theta = \frac{\theta}{2} - \frac{\sin(2\theta)}{4} + C = \frac{\theta}{2} - \frac{\sin(\theta)\cos(\theta)}{2} + C.$$

As $\sec(\theta) = x$, we have $\cos(\theta) = \frac{1}{x}$ and:

$$\sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{1-\frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{x}$$

so we obtain:

$$\int \frac{\sqrt{x^2-1}}{x^3}dx = \boxed{\frac{\sec^{-1}(x)}{2} - \frac{\sqrt{x^2-1}}{2x^2} + C}.$$

Problem 3. Find $\int \frac{\sqrt{1-x^2}}{x} dx$.

Solution. We use the trig substitution $\sin(\theta) = x$, $\cos(\theta)d\theta = dx$. We obtain the integral:

$$\int \frac{\sqrt{1-x^2}}{x} dx = \int \frac{\sqrt{1-\sin^2(\theta)}}{\sin(\theta)} \cos(\theta)d\theta = \int \frac{\cos(\theta)}{\sin(\theta)} \cos(\theta)d\theta =$$
$$\int \frac{\cos^2(\theta)}{\sin(\theta)} d\theta = \boxed{\int \cos^2(\theta)\csc(\theta)d\theta}.$$

To solve this integral, we expand:

$$\int \cos^2(\theta)\csc(\theta)d\theta = \int (1-\sin^2(\theta))\csc(\theta)d\theta = \int \csc(\theta)d\theta - \int \sin^2(\theta)\csc(\theta)d\theta =$$
$$\int \csc(\theta)d\theta - \int \sin(\theta)d\theta.$$

We recall that we can find $\int \csc(\theta)d\theta = -\log(\cot(\theta)+\csc(\theta)) + C$, so we have:

$$\int \csc(\theta)d\theta - \int \sin(\theta)d\theta = -\log(\cot(\theta)+\csc(\theta)) + \cos(\theta) + C.$$

As $\sin(\theta) = x$, we have $\cos(\theta) = \sqrt{1-x^2}$, so:

$$-\log(\cot(\theta)+\csc(\theta)) + \cos(\theta) + C = -\log(\frac{\sqrt{1-x^2}}{x} + \frac{1}{x}) + \sqrt{1-x^2} + C =$$
$$\boxed{-\log(\frac{1+\sqrt{1-x^2}}{x}) + \sqrt{1-x^2} + C}.$$

Problem 4. Find $\int \sqrt{1+x^2} dx$.

Solution. Set $\tan(\theta) = x$, so $\sec^2(\theta)d\theta = dx$. As $\sqrt{1+\tan^2(\theta)} = \sec(\theta)$, we obtain:

$$\int \sqrt{1+x^2} dx = \boxed{\int \sec^3(\theta)d\theta}.$$

To evaluate this integral, we expand:

$$\int \sec^3(\theta)d\theta = \int \sec(\theta)(1+\tan^2(\theta))d\theta = \int \sec(\theta)d\theta + \int \sec(\theta)\tan^2(\theta)d\theta.$$

We recall that:

$$\int \sec(\theta)d\theta = \log(\tan(\theta)+\sec(\theta)) + C.$$

To evaluate $\int \sec(\theta)\tan^2(\theta)d\theta$, we use integration by parts. We set $u = \tan(\theta)$ and $v = \sec(\theta)$ to obtain:

$$\int \sec(\theta)\tan^2(\theta)d\theta = \int u dv = \tan(\theta)\sec(\theta) - \int \sec^3(\theta)d\theta$$

Substituting into our earlier expression, we obtain:

$$\int \sec^3(\theta)d\theta = \int \sec(\theta)d\theta + \int \sec(\theta)\tan^2(\theta)d\theta =$$

$$\log(\tan(\theta) + \sec(\theta)) + \tan(\theta)\sec(\theta) - \int \sec^3(\theta)d\theta.$$

We obtain:

$$2\int \sec^3(\theta)d\theta = \log(\tan(\theta) + \sec(\theta)) + \tan(\theta)\sec(\theta) + C$$

$$\Rightarrow \int \sec^3(\theta)d\theta = \frac{1}{2}\log(\tan(\theta) + \sec(\theta)) + \frac{1}{2}\tan(\theta)\sec(\theta) + C.$$

We now recall $\tan(\theta) = x$, so $\sec(\theta) = \sqrt{1+x^2}$, and we obtain the final answer:

$$\boxed{\frac{1}{2}\log(x + \sqrt{1+x^2}) + \frac{1}{2}x\sqrt{1+x^2} + C}.$$