Problem 1. Find \( \int \frac{dx}{x^2 - 6x + 5} \).

Solution. First, we factor the denominator \( x^2 - 6x + 5 = (x - 5)(x - 1) \). We then find the partial fractions decomposition by setting up an equation:

\[
\frac{1}{(x - 5)(x - 1)} = \frac{A}{x - 5} + \frac{B}{x - 1} = \frac{A(x - 1) + B(x - 5)}{(x - 5)(x - 1)} = \frac{(A + B)x + (-A - 5B)}{(x - 5)(x - 1)}.
\]

Therefore, we have:

\[
A + B = 0
\]

\[-A - 5B = 1.
\]

Solving this system of equations gives \( A = \frac{1}{4} \), \( B = -\frac{1}{4} \). Summarizing, we obtain:

\[
\frac{1}{x^2 - 5x + 6} = \frac{1}{4(x - 5)} + \frac{-1}{4(x - 1)}.
\]

We now have:

\[
\int \frac{dx}{x^2 - 6x + 5} = \frac{1}{4} \int \frac{dx}{x - 5} - \frac{1}{4} \int \frac{dx}{x - 1} = \frac{1}{4} \log |x - 5| - \frac{1}{4} \log |x - 1| + C.
\]

Problem 2. Find \( \int \frac{4x + 1}{x^2 - x - 2} \, dx \).

Solution. The method is the same as the previous problem. We have \( x^2 - x - 2 = (x - 2)(x + 1) \). To find the partial fraction decomposition, we setup the equation:

\[
\frac{4x + 1}{(x - 2)(x + 1)} = \frac{A}{x - 2} + \frac{B}{x + 1} = \frac{(A + B)x + (A - 2B)}{(x - 2)(x + 1)}.
\]

We now have the system of equations:

\[
A + B = 4
\]

\]

We find the solutions are \( A = 3 \), \( B = 1 \).

Therefore, we obtain:
\[
\int \frac{4x + 1}{x^2 - x - 2} \, dx = \int \frac{3dx}{x - 2} + \int \frac{dx}{x + 1} = 3\log |x - 2| + \log |x + 1| + C.
\]

**Problem 3.** Find \( \int \frac{xdx}{x^2 - 4x + 4} \).

**Solution.** Note that \( x^2 - 4x + 4 = (x - 2)^2 \). Therefore, we find the partial fraction decomposition in the form:

\[
\frac{x}{(x - 2)^2} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2} = \frac{Ax + (-2A + B)}{(x - 2)^2}.
\]

We obtain the system of equations:

\[
A = 1
\]
\[-2A + B = 0.
\]

Clearly \( A = 1 \) and \( B = 2 \).

Therefore, we have:

\[
\int \frac{xdx}{x^2 - 4x + 4} = \int \frac{dx}{x - 2} + \int \frac{2dx}{(x - 2)^2} = \log |x - 2| - \frac{2}{x - 2} + C.
\]

**Problem 4.** Find \( \int \frac{x^3 + 3x + 1}{x^2 - 3x + 2} \, dx \).

**Solution.** We begin with long division. Observe that:

\[
x^3 + 3x + 1 = \left( x(x^2 - 3x + 2) + (3x^2 - 2x) \right) + 3x + 1 = x(x^2 - 3x + 2) + 3(x^2 - 3x + 2) + 10x - 5 = (x + 3)(x^2 - 3x + 2) + 10x - 5.
\]

Therefore, we have:

\[
\frac{x^3 + 3x + 1}{x^2 - 3x + 2} = x + 3 + \frac{10x - 5}{x^2 - 3x + 2}.
\]

We know how to integrate the first two terms. To integrate the third, we use partial fractions. We notice \( x^2 - 3x + 2 = (x - 2)(x - 1) \), so we setup the equation:

\[
\frac{10x - 5}{(x - 2)(x - 1)} = \frac{A}{x - 2} + \frac{B}{x - 1} = \frac{(A + B)x - A - 2B}{(x - 2)(x - 1)}.
\]

We have the system of equations:

\[
A + B = 10
\]
\[-A - 2B = -5.
\]
with solutions $A = 15$, $B = -5$.

Finally, we find:

$$
\int \frac{x^3 + 3x + 1}{x^2 - 3x + 2} \, dx = \int \left( x + 3 + \frac{15}{x-2} - \frac{5}{x-1} \right) \, dx =
$$

$$
\frac{1}{2} x^2 + 3x + 15 \log |x - 2| - 5 \log |x - 1| + C.
$$