408L CLASS PROBLEMS

FEBRUARY 26TH, 2020

Problem 1. Find $\int \frac{dx}{x^2-6x+5}$.

Solution. First, we factor the denominator $x^2 - 6x + 5 = (x-5)(x-1)$.
We then find the partial fractions decomposition by setting up an equation:

$$\frac{1}{(x-5)(x-1)} = \frac{A}{x-5} + \frac{B}{x-1} = \frac{A(x-1)+B(x-5)}{(x-5)(x-1)} = \frac{(A+B)x+(-A-5B)}{(x-5)(x-1)}.$$

Therefore, we have:

$$A + B = 0$$
$$-A - 5B = 1.$$

Solving this system of equations gives $A = \frac{1}{4}$, $B = -\frac{1}{4}$. Summarizing, we obtain:

$$\frac{1}{x^2-5x+6} = \frac{1}{4(x-5)} + \frac{-1}{4(x-1)}.$$

We now have:

$$\int \frac{dx}{x^2-6x+5} = \frac{1}{4}\int \frac{dx}{x-5} - \frac{1}{4}\frac{dx}{x-1} = \boxed{\frac{1}{4}\log|x-5| - \frac{1}{4}\log|x-1| + C}.$$

Problem 2. Find $\int \frac{4x+1}{x^2-x-2}dx$.

Solution. The method is the same as the previous problem. We have $x^2 - x - 2 = (x-2)(x+1)$. To find the partial fraction decomposition, we setup the equation:

$$\frac{4x+1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{(A+B)x+(A-2B)}{(x-2)(x+1)}.$$

We now have the system of equations:

$$A + B = 4$$
$$A - 2B = 1.$$

We find the solutions are $A = 3$, $B = 1$.
Therefore, we obtain:

$$\int \frac{4x+1}{x^2-x-2}dx = \int \frac{3dx}{x-2} + \int \frac{dx}{x+1} = \boxed{3\log|x-2| + \log|x+1| + C}.$$

Problem 3. Find $\int \frac{xdx}{x^2-4x+4}$.

Solution. Note that $x^2 - 4x + 4 = (x-2)^2$. Therefore, we find the partial fraction decomposition in the form:

$$\frac{x}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{Ax + (-2A+B)}{(x-2)^2}.$$

We obtain the system of equations:

$$A = 1$$
$$-2A + B = 0.$$

Clearly $A = 1$ and $B = 2$.

Therefore, we have:

$$\int \frac{xdx}{x^2-4x+4} = \int \frac{dx}{x-2} + \int \frac{2dx}{(x-2)^2} = \boxed{\log|x-2| - \frac{2}{x-2} + C}.$$

Problem 4. Find $\int \frac{x^3+3x+1}{x^2-3x+2}dx$.

Solution. We begin with long division. Observe that:

$$x^3 + 3x + 1 = \underbrace{\left(x(x^2-3x+2) + (3x^2-2x)\right)}_{x^3} + 3x + 1 =$$
$$x(x^2-3x+2) + 3x^2 + x + 1 = x(x^2-3x+2) + 3(x^2-3x+2) + 10x - 5 =$$
$$(x+3)(x^2-3x+2) + 10x - 5.$$

Therefore, we have:

$$\frac{x^3+3x+1}{x^2-3x+2} = x + 3 + \frac{10x-5}{x^2-3x+2}.$$

We know how to integrate the first two terms. To integrate the third, we use partial fractions. We notice $x^2 - 3x + 2 = (x-2)(x-1)$, so we setup the equation:

$$\frac{10x-5}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{(A+B)x - A - 2B}{(x-2)(x-1)}.$$

We have the system of equations:

$$A + B = 10$$
$$-A - 2B = -5$$

with solutions $A = 15$, $B = -5$.

Finally, we find:

$$\int \frac{x^3+3x+1}{x^2-3x+2}dx = \int (x+3+\frac{15}{x-2}-\frac{5}{x-1})dx =$$
$$\boxed{\frac{1}{2}x^2+3x+15\log|x-2|-5\log|x-1|+C}.$$