408L CLASS PROBLEMS

FEBRUARY 28TH, 2020

Problem 1. Find $\int \frac{2x+5}{x^2+5x+6} dx$.

Solution. We factor the denominator as $(x+3)(x+2)$ and seek the partial fractions decomposition:

$$\frac{2x+5}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2} = \frac{(A+B)x + 3A + 2B}{(x+3)(x+2)}.$$

We therefore have $A + B = 2$, $3A + 2B = 5$. The solution to this system of equations is $A = 1$, $B = 1$, so we have:

$$\frac{2x+5}{(x+3)(x+2)} = \frac{1}{x+3} + \frac{1}{x+2}$$

We now obtain:

$$\int \frac{2x+5}{(x+3)(x+2)} dx = \int \frac{dx}{x+3} + \int \frac{dx}{x+2} = \boxed{\log|x+3| + \log|x+2| + C}.$$

Problem 2. Find $\int \frac{x+4}{x^2-x-2} dx$.

Solution. We factor the denominator as $(x-2)(x+1)$. To find the partial fractions decomposition, we set:

$$\frac{x+4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{(A+B)x + A - 2B}{(x-2)(x+1)}.$$

Therefore, $A + B = 1$ and $A - 2B = 4$. We find the solutions $A = 2$, $B = -1$, so:

$$\frac{x+4}{(x-2)(x+1)} = \frac{2}{x-2} - \frac{1}{x+1}.$$

Integrating, we obtain the answer:

$$\boxed{2\log|x-2| - \log|x+1| + C}.$$

Problem 3. Find $\int \frac{dx}{x^2+6x+9}$.

Solution. We have:

$$\frac{1}{x^2+6x+9} = \frac{1}{(x+3)^2}.$$

This is already a partial-fractions decomposition, so we do not need to do anything further before integrating!

Namely, if we take $u = x+3$ so $du = dx$, we obtain:

$$\int \frac{dx}{u^2} = -\frac{1}{u} + C = \boxed{-\frac{1}{x+3} + C}.$$

Problem 4. Find $\int \frac{x^2}{x^2-1} dx$.

Solution. As the numerator and denominator have the same degree, we need to do division. We write $x^2 = (x^2-1)+1$, so:

$$\frac{x^2}{x^2-1} = \frac{(x^2-1)+1}{x^2-1} = 1 + \frac{1}{x^2-1}.$$

To integrate the second term, we find its partial fractions decomposition:

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{(A+B)x + (A-B)}{(x-1)(x+1)}$$

giving $A = \frac{1}{2}$, $B = \frac{-1}{2}$. We then get:

$$\int \frac{x^2}{x^2-1} dx = \int dx + \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} = \boxed{x + \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| + C}.$$

Problem 5. Find $\int \frac{x^2+1}{x^2+2x+2} dx$.

Solution. We begin with division:

$$\frac{x^2+1}{x^2+2x+2} = \frac{(x^2+2x+2) - 2x - 1}{x^2+2x+2} = 1 - \frac{2x+1}{x^2+2x+2}.$$

To integrate the second term, we recognize the denominator as $(x+1)^2+1$ (in effect, completing the square). Therefore, setting $u = x+1$ so $du = dx$, we obtain:

$$\int \frac{2x+1}{x^2+2x+2} dx = \int \frac{2(x+1)-1}{(x+1)^2+1} dx = \int \frac{2u-1}{u^2+1} du = \int \frac{2u}{u^2+1} du - \int \frac{1}{u^2+1} du.$$

To evaluate the first term, we use substitution with $v = u^2+1$, $dv = 2u$, giving:

$$\int \frac{2u}{u^2+1} du = \int \frac{dv}{v} = \log|v| + C = \log|u^2+1| + C.$$

The second term is a fundamental integral:

$$\int \frac{1}{u^2+1} du = \tan^{-1}(u) + C.$$

Therefore, we obtain:

$$\int \frac{2x+1}{x^2+2x+2} dx = \int \frac{2u}{u^2+1} du - \int \frac{1}{u^2+1} du =$$

$$\log|u^2+1| - \tan^{-1}(u) + C = \log|(x+1)^2+1| - \tan^{-1}(x+1) + C =$$

$$\log(x^2+2x+2) - \tan^{-1}(x+1) + C.$$

Putting everything together, we conclude:

$$\int \frac{x^2+1}{x^2+2x+2} dx = \int dx - \int \frac{2x+1}{x^2+2x+2} dx =$$

$$\boxed{x - \log(x^2+2x+2) + \tan^{-1}(x+1) + C}.$$