408L CLASS PROBLEMS

MARCH 2ND, 2020

Problem 1. Find $\int \frac{x^2-9}{x^2-1}dx$.

Solution. We have:

$$\frac{x^2-9}{x^2-1} = \frac{(x^2-1)-8}{x^2-1} = 1 - \frac{8}{x^2-1}.$$

We find the partial fractions decomposition:

$$\frac{8}{x^2-1} = \frac{4}{x-1} + \frac{4}{x+1}$$

giving:

$$\int \frac{x^2-9}{x^2-1}dx = \int dx - 4\int \frac{dx}{x-1} - 4\int \frac{dx}{x+1} = \boxed{x - 4\log|x-1| - 4\log|x+1| + C}.$$

Problem 2. Find $\int \frac{\tan^{-1}(x)}{x^2}dx$.

Solution. As we see a factor of $\tan^{-1}(x)$, we think to use integration by parts with $u = \tan^{-1}(x)$, so $du = \frac{1}{x^2+1}$. Then $dv = \frac{dx}{x^2}$, so $v = -\frac{1}{x}$. This gives:

$$\int \frac{\tan^{-1}(x)}{x^2}dx = -\frac{\tan^{-1}(x)}{x} - \int \frac{dx}{x(x^2+1)}.$$

We recognize the latter term as the integral of a rational function, so we seek the partial fraction decomposition:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+Bx^2+Cx}{x(x^2+1)} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)}.$$

We see that $A = 1$, $C = 0$, and $B = -1$, so:

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}.$$

To integrate $\frac{x}{x^2+1}$, we use substitution with $u = x^2+1$, $du = 2xdx$ to find:

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C = \frac{1}{2}\log(x^2+1) + C.$$

We now obtain:

$$\int \frac{dx}{x(x^2+1)} = \int \frac{dx}{x} - \frac{x}{x^2+1}dx = \log|x| - \frac{1}{2}\log(x^2+1) + C.$$

Substituting back in, we obtain:

$$\int \frac{\tan^{-1}(x)}{x^2}dx = -\frac{\tan^{-1}(x)}{x} - \int \frac{dx}{x(x^2+1)} =$$
$$\boxed{-\frac{\tan^{-1}(x)}{x} - \log|x| + \frac{1}{2}\log(x^2+1) + C}.$$

Problem 3. Find $\int \frac{dx}{e^x\sqrt{e^{2x}+1}}$.

Solution. We begin with substitution, to try and remove the e^x term. We set $u = e^x$, $du = e^x dx$. We obtain:

$$\int \frac{dx}{e^x\sqrt{e^{2x}+1}} = \int \frac{e^x dx}{e^{2x}\sqrt{e^{2x}+1}} = \int \frac{du}{u^2\sqrt{u^2+1}}.$$

We now set $u = \tan(\theta)$, so $u^2+1 = \tan^2(\theta)+1 = \sec^2(\theta)$ and $du = \sec^2(\theta)d\theta$ and find:

$$\int \frac{du}{u^2\sqrt{u^2+1}} = \int \frac{\sec^2(\theta)d\theta}{\tan^2(\theta)\sec(\theta)} = \int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta = \int \cot(\theta)\csc(\theta)d\theta = -\csc(\theta) + C.$$

Returning to our substitution, we have $u^2+1 = \sec^2(\theta) = \frac{1}{\cos^2(\theta)}$ or $\cos^2(\theta) = \frac{1}{u^2+1}$, giving:

$$\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - \frac{1}{u^2+1} = \frac{u^2}{u^2+1}$$
$$\Rightarrow \sin(\theta) = \frac{u}{\sqrt{u^2+1}}$$
$$\Rightarrow \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\sqrt{u^2+1}}{u}.$$

Therefore, we have:

$$\int \frac{dx}{e^x\sqrt{e^{2x}+1}} = \int \frac{du}{u^2\sqrt{u^2+1}} = -\frac{\sqrt{u^2+1}}{u} + C = \boxed{-\frac{\sqrt{e^{2x}+1}}{e^x} + C}.$$

Problem 4. Find $\int \frac{dx}{\cos(x)+3\tan(x)-3\sec(x)}$.

Solution. Clearly denominators, we note that:

$$\int \frac{dx}{\cos(x) + 3\tan(x) - 3\sec(x)} = \int \frac{dx}{\cos(x) + 3 \cdot \frac{\sin(x)}{\cos(x)} - 3 \cdot \frac{1}{\cos(x)}} =$$
$$\int \frac{\cos(x)dx}{\cos^2(x) + 3\sin(x) - 3}.$$

The numerator $\cos(x)dx$ suggests using a u-substitution with $u = \sin(x)$. Therefore, we replace $\cos^2(x)$ with $1 - \sin^2(x)$ in the denominator to obtain:

$$\int \frac{\cos(x)dx}{\cos^2(x) + 3\sin(x) - 3} = \int \frac{\cos(x)dx}{-\sin^2(x) + 3\sin(x) - 2} =$$
$$-\int \frac{du}{u^2 - 3u + 2}.$$

The latter is a rational function. We can factor the denominator as $(u-1)(u-2)$, so we seek a partial fractions decomposition:

$$\frac{1}{(u-1)(u-2)} = \frac{A}{u-1} + \frac{B}{u-2} = \frac{(A+B)u + (-2A-B)}{(u-1)(u-2)}.$$

We find the solution $A = -1$, $B = 1$, so that:

$$\frac{1}{(u-1)(u-2)} = -\frac{1}{u-1} + \frac{1}{u-2}$$
$$\Rightarrow \int \frac{du}{u^2 - 3u + 2} = -\log|u-1| + \log|u-2| + C.$$

Substituting back in, we conclude:

$$\int \frac{dx}{\cos(x) + 3\tan(x) - 3\sec(x)} = -\int \frac{du}{u^2 - 3u + 2} =$$
$$\log|u-1| - \log|u-2| + C = \log|\sin(x) - 1| - \log|\sin(x) - 2| + C =$$
$$\boxed{\log(1 - \sin(x)) - \log(2 - \sin(x)) + C}$$

where in the last equality we are using e.g. $\sin(x) \leqslant 1 \Rightarrow |\sin(x) - 1| = 1 - \sin(x)$.