408L CLASS PROBLEMS

MARCH 4TH, 2020

Problem 1. Find $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution. We have $\int \frac{dx}{1+x^2} = \tan^{-1}(x)$. We have $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$, so:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \boxed{\pi}.$$

Problem 2. Find $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$.

Solution. We set $u^2 = x$, so $2udu = dx$. We obtain:

$$\int \frac{dx}{\sqrt{x}(x+1)} = 2\int \frac{udu}{u\cdot(u^2+1)} = 2\int \frac{du}{u^2+1} = 2\tan^{-1}(u) = 2\tan^{-1}(\sqrt{x}).$$

Therefore:

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} = \lim_{x\to\infty} 2\tan^{-1}(\sqrt{x}) - 2\tan^{-1}(0) = \boxed{\pi}.$$

Problem 3. Find $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sqrt{\sin(x)}}dx$.

Solution. We set $u = \sin(x)$ so $du = \cos(x)dx$. We obtain:

$$\int \frac{\cos(x)}{\sqrt{\sin(x)}}dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = \frac{1}{2}\sqrt{\sin(x)}.$$

We obtain:

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sqrt{\sin(x)}}dx = 2\sqrt{\sin(\frac{\pi}{2})} - 2\sqrt{\sin(0)} = \boxed{2}.$$

Problem 4. Determine whether the following integrals converge or diverge.

(1) $\int_1^{\infty} \frac{dx}{x^3}$.
(2) $\int_0^{\infty} \frac{dx}{x^2}$.

(3) $\int_{-1}^{1} \frac{dx}{x^2}$.
(4) $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$.
(5) $\int_{1}^{\infty} \frac{dx}{\log(x)}$.
(6) $\int_{1}^{\infty} \frac{dx}{x\log(x)}$.

Solution. For (1), $\int \frac{dx}{x^3} = -\frac{1}{2x^2}$, so $\int_1^t \frac{dx}{x^3} = -\frac{1}{2t^2} + \frac{1}{2}$. We see that the limit as $t \to \infty$ exists: the value is $\frac{1}{2}$. So the integral converges.

For (2), we have $\int \frac{dx}{x^2} = -\frac{1}{x}$, which goes to $-\infty$ as $x \to 0$. Therefore, the integral diverges.

Because $\frac{1}{x^2}$ is only defined for $x \neq 0$, we must interpret the integral in (3) as $\int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$. By the previous problem, the integral diverges. (Even though $\int \frac{dx}{x^2} = -\frac{1}{x}$ is defined at $x = 1$ and $x = -1$!)

For (4), we have $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)$, which is defined at -1 and 1, so the integral converges.

For (5), we have $\log(x) < x$ for all $x > 0$. Therefore, $\frac{1}{x} < \frac{1}{\log(x)}$. So $\int_1^t \frac{dx}{x} = \log(t) < \int_1^t \frac{dx}{\log(x)}$. As the first term goes to ∞ as $t \to \infty$, the integral $\int_1^\infty \frac{dx}{\log(x)}$ diverges.

Finally, for (6), we evaluate the integral by setting $u = \log(x)$, so $du = \frac{dx}{x}$. Then:

$$\int \frac{dx}{x\log(x)} dx = \int \frac{1}{u} du = \log(u) = \log(\log(x)).$$

We have $\lim_{x\to\infty} \log(\log(x)) = \infty$, so the integral diverges.