408L CLASS PROBLEMS

MARCH 6TH, 2020

Problem 1. Determine whether the following improper integrals converge or diverge. For the integrals that converge, find the value.

(1) $\int_0^1 x^{-\frac{2}{3}}dx$.
(2) $\int_0^1 x^{-\frac{3}{2}}dx$.
(3) $\int_0^\infty e^{-x}dx$.
(4) $\int_1^\infty \left(x^{-\frac{1}{3}} - x^{-\frac{1}{2}}\right)dx$.
(5) $\int_0^{\frac{\pi}{2}} \tan^2(x)dx$

Solution. For (1), we have:

$$\int_t^1 x^{-\frac{2}{3}}dx = 3x^{\frac{1}{3}}|_t^1 = 3 - 3t^{\frac{1}{3}}.$$

We see the integral converges with value $\lim_{t\to 0} 3 - 3t^{\frac{1}{3}} = \boxed{3}$.

For (2), we have:

$$\int_t^1 x^{-\frac{3}{2}}dx = -2x^{-\frac{1}{2}}|_t^1 = -2 + 2t^{-\frac{1}{2}}.$$

As $\lim_{t\to 0} t^{-\frac{1}{2}} = \infty$, the integral $\boxed{\text{diverges}}$.

For (3), we have:

$$\int_0^t e^{-x}dx = -e^{-x}|_0^t = -e^{-t} + 1.$$

As $\lim_{t\to\infty} e^{-t} = 0$, the integral converges and we have:

$$\int_0^\infty e^{-x}dx = \boxed{1}.$$

For (4), we see:

$$\int_1^t (x^{-\frac{1}{3}} - x^{-\frac{1}{2}})dx = \left(\frac{3}{2}x^{\frac{2}{3}} - 2x^{\frac{1}{2}}\right)|_1^t = \frac{3}{2}t^{\frac{2}{3}} - 2t^{\frac{1}{2}} + \frac{1}{2}.$$

For large t, $t^{\frac{2}{3}}$ is much larger than $t^{\frac{1}{2}}$, so the limit as $t \to \infty$ of the above is ∞ and the integral $\boxed{\text{diverges}}$.

For (5), note that $\tan^2(x) = \sec^2(x) - 1$, so:

$$\int \tan^2(x)dx = \int \sec^2(x)dx - \int dx = \tan(x) - x.$$

As $\lim_{x\to\frac{\pi}{2}} \tan(x) = \infty$, the integral $\boxed{\text{diverges}}$.

Problem 2. Find $\int_1^\infty \frac{\log(x)}{x^2}dx$.

Solution. We use integration by parts with $u = \log(x)$ and $v = \frac{-1}{x}$. Then $du = \frac{dx}{x}$ and $dv = \frac{dx}{x^2}$, so we have:

$$\int \frac{\log(x)}{x^2}dx = -\frac{\log(x)}{x} + \int \frac{dx}{x^2} = -\frac{\log(x)}{x} - \frac{1}{x}.$$

Therefore:

$$\int_1^\infty \frac{\log(x)}{x^2}dx = \big(-\frac{\log(x)}{x} - \frac{1}{x}\big)|_1^\infty = \boxed{1}.$$

Problem 3. Find an anti-derivative of $\frac{1}{x\log(x)}$.

Then, using a calculator, calculate $\int_1^{10^{100}} \frac{dx}{x\log(x)}$ to a few decimal places. Does $\int_1^\infty \frac{dx}{x\log(x)}$ converge or diverge?

Solution. We set $u = \log(x)$, so $du = \frac{dx}{x}$ and find:

$$\int \frac{dx}{x\log(x)} = \int \frac{du}{u} = \log(u) = \log(\log(x)).$$

Therefore:

$$\int_1^{10^{100}} \frac{dx}{x\log(x)} = \log(\log(10^{100})) = \log(100 * \log(10)) = \boxed{5.439\ldots}.$$

Even though the integral is not so large at 10^{100}, we see that $\lim_{x\to\infty} \log(\log(x)) = \infty$, so the integral $\boxed{\text{diverges}}$.

Problem 4. Find $\int_0^\infty \frac{dx}{x^2+3x+2}$.

Solution. We recognize this integral as that of a rational function, so we find it using partial fractions. We factor the denominator as $(x+1)(x+2)$, so setup our equations as:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{(A+B)x + (2A+B)}{(x+1)(x+2)}.$$

Therefore $A + B = 0$ and $2A + B = 1$, so $A = 1$ and $B = -1$. We then obtain:

$$\int \frac{dx}{x^2+3x+2} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \log|x+1| - \log|x+2|.$$

As our problem integrates over $x \geqslant 0$, we can remove the absolute value signs above. We then have:

$$\log(x+1) - \log(x+2) = \log(\frac{x+1}{x+2}).$$

Therefore:

$$\int_0^t \frac{dx}{x^2+3x+2} = \log(\frac{t+1}{t+2}) - \log(\frac{1}{2}).$$

As $\lim_{t\to\infty} \frac{t+1}{t+2} = 1$, we have:

$$\int_0^\infty \frac{dx}{x^2+3x+2} = \log(1) - \log(\frac{1}{2}) = -\log(\frac{1}{2}) = \boxed{\log(2)}.$$