408L CLASS PROBLEMS

JANUARY 27, 2020

Problem 1. Find the derivative of $f(x) = \int_x^0 \sqrt{1+\sec(t)}dt$.

Solution. Let $g(t) = \sqrt{1+\sec(t)}$ and let $G(t)$ be an anti-derivative of $g(t)$. In other words, $G(t)$ is a function with $G'(t) = g(t)$.

By the fundamental theorem of calculus, $f(x) = G(0) - G(x)$. Therefore, we find:

$$f'(x) = -G'(x) = -g(x) = \boxed{-\sqrt{1+\sec(x)}}.$$

Problem 2. Find the derivative of $f(x) = \int_1^{e^x} \log(t)dt$.

Solution. Let $g(t) = \log(t)$ and let $G(t)$ be an anti-derivative of $g(t)$.

By the fundamental theorem of calculus, $f(x) = G(e^x) - G(1)$. Therefore, we find:

$$f'(x) = \frac{d}{dx}G(e^x) - \frac{d}{dx}G(1) = e^x \cdot G'(e^x) = e^x \cdot \log(e^x) = \boxed{e^x \cdot x}$$

where second equality is by the chain rule.

Problem 3. Find the derivative of $f(x) = \int_1^{\sqrt{x}} \frac{t^2}{t^4+1}dt$.

Solution. Let $g(t) = \frac{t^2}{t^4+1}$ and let $G(t)$ be an anti-derivative of $g(t)$.

By the fundamental theorem of calculus, $f(x) = G(\sqrt{x}) - G(1)$. Therefore, we find:

$$f'(x) = \frac{d}{dx}G(\sqrt{x}) - \frac{d}{dx}G(1) = \frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot G'(\sqrt{x}) =$$

$$\frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1} = \boxed{\frac{1}{2} \cdot \frac{\sqrt{x}}{x^2+1}}$$

where second equality is by the chain rule.

Problem 4. Find $\int_0^2 |2t-1|dt$.

Solution. For a function $f(t)$, we can consider $|f(t)|$ as a piecewise function:

$$\begin{cases} |f(t)| = f(t) & \text{if } f(t) \geqslant 0 \\ |f(t)| = -f(t) & \text{if } f(t) < 0. \end{cases}$$

In our example, $f(t) = 2t - 1$. Note that $f(t) \geqslant 0$ exactly when $t \geqslant \frac{1}{2}$. Therefore, we can break up the integral as:

$$\int_0^2 |2t-1|dt = \int_0^{\frac{1}{2}} |2t-1|dt + \int_{\frac{1}{2}}^2 |2t-1|dt = \int_0^{\frac{1}{2}} -(2t-1)dt + \int_{\frac{1}{2}}^2 (2t-1)dt. \qquad (1)$$

As $t^2 - t$ is an anti-derivative of $2t - 1$, we can apply the fundamental theorem of calculus to obtain:

$$\int_0^{\frac{1}{2}} -(2t-1)dt = -t^2 + t\Big|_0^{\frac{1}{2}} = (-\frac{1}{4} + \frac{1}{2}) - 0 = \frac{1}{4}$$
$$\int_{\frac{1}{2}}^2 (2t-1)dt = t^2 - t\Big|_{\frac{1}{2}}^2 = (4-2) - (\frac{1}{4} - \frac{1}{2}) = \frac{9}{4}.$$

Substituting these values into Equation (1), we obtain:

$$\int_0^2 |2t-1|dt = \frac{1}{4} + \frac{9}{4} = \boxed{\frac{5}{2}}.$$

Alternative solution. Try drawing a picture of the graph of $f(x)$. Then use triangle geometry to recover the above answer.

Problem 5. Find the anti-derivative of $f(x) = \cos^2 x$. (Hint: use trig identities to simplify.)

Solution. Recall from trigonometry that:

$$\cos^2 x + \sin^2 x = 1$$
$$\cos^2 x - \sin^2 x = \cos(2x).$$

Summing these two equations, we obtain:

$$2\cos^2 x = 1 + \cos(2x) \Rightarrow$$
$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x).$$

We then obtain:

$$\int \cos^2 x dx = \int \frac{1}{2} + \frac{1}{2}\cos(2x) = \boxed{\frac{x}{2} + \frac{1}{4}\sin(2x) + C}.$$

Problem 6. Find the average value of $\cos^2 x$ for x in $[-\pi/2, \pi/2]$.

Solution. For a continuous function $f(x)$ defined on an interval $[a,b]$, the average value of $f(x)$ on this interval is $\frac{1}{b-a}\int_a^b f(x)dx$. (Why? How does this relate to Riemann sums?)

In our example, this is:

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x dx.$$

Using the previous problem, we can calculate the integral as:

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{\pi}\cdot\left(\frac{x}{2}+\frac{1}{4}\sin(2x)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$\frac{1}{\pi}\cdot\left((\frac{\pi}{4}+\frac{1}{4}\sin(\pi)) - (\frac{-\pi}{4}+\frac{1}{4}\sin(-\pi))\right) = \frac{1}{\pi}\cdot(\frac{\pi}{4}-(\frac{-\pi}{4})) = \boxed{\frac{1}{2}}.$$