408L CLASS PROBLEMS

MARCH 9TH, 2020

Problem 1. For each of the following functions $f(x,y)$, find $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$.

(1) $f(x,y) = x^2 + 2xy + y^2$.
(2) $f(x,y) = \sin(xy)$.
(3) $f(x,y) = ye^{x^2+y^2}$.
(4) $f(x,y) = \frac{x}{y}$.
(5) $f(x,y) = x^y$.

Solution.

$f(x,y)$	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial y}$
$x^2 + 2xy + y^2$	$2x + 2y$	$2x + 2y$
$\sin(xy)$	$y\cos(x)$	$x\cos(y)$
$ye^{x^2+y^2}$	$2xye^{x^2+y^2}$	$(2y^2+1)e^{x^2+y^2}$
$\frac{x}{y}$	$\frac{1}{y}$	$-\frac{x}{y^2}$
x^y	yx^{y-1}	$\log(x)x^y$

Problem 2. The *Laplace equation* $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f(x,y) = 0$ tells when a function $f(x,y)$ returns the temperature of a 2-dimensional room *at thermal equilibrium.*

Which of the following functions satisfy the heat equation?

(1) $f(x,y) = xy$.
(2) $f(x,y) = x^2 - y^2$.
(3) $f(x,y) = x^2 + y^2$.
(4) $f(x,y) = \log(x^2 + y^2)$.
(5) $f(x,y) = \frac{x}{x^2+y^2}$.
(6) $f(x,y) = \sin(y)e^x$.

Solution. All of these equations satisfy the Laplace equation except for the third.
For (1), note that:

$$\frac{\partial^2}{\partial x^2}(xy) = \frac{\partial^2}{\partial y^2}(xy) = 0$$

giving the claim.

For (2) and (3), we have:

$$\frac{\partial^2}{\partial x^2}(x^2) = 2$$
$$\frac{\partial^2}{\partial x^2}(y^2) = 2$$

showing that (2) satisfies the Laplace equation and (3) does not.

For (4), we have:

$$\frac{\partial}{\partial x}\log(x^2+y^2) = \frac{x}{x^2+y^2}$$
$$\frac{\partial^2}{\partial x^2}\log(x^2+y^2) = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

By symmetry, we also have:

$$\frac{\partial^2}{\partial y^2}\log(x^2+y^2) = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

This shows that $\log(x^2+y^2)$ satisfies the Laplace equation.

We deduce (5) from (4). Take $g(x,y) = \frac{1}{2}\log(x^2+y^2)$. As above, we have $\frac{\partial g}{\partial x} = \frac{x}{x^2+y^2} = f(x,y)$. Therefore:

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f(x,y) = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\frac{\partial}{\partial x}g(x,y) = \frac{\partial}{\partial x}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})g(x,y) = 0$$

where the last equality is by our calculation for (4). (Of course, it is possible to explicitly compute the derivatives of f explicitly here, but it is a little unwieldy without this trick.)

Finally, for (6) we have:

$$\frac{\partial^2}{\partial x^2}\sin(y)e^x = \sin(y)e^x$$
$$\frac{\partial^2}{\partial y^2}\sin(y)e^x = -\sin(y)e^x.$$