408L CLASS PROBLEMS (!)

MARCH 30TH, 2020

Problem 1. Which of the following sequences converge? For sequences that do converge, find the limit.

(1) $a_n = \frac{1}{n!}$.
(2) $a_n = \frac{2^n+3^n}{3^n}$.
(3) $a_n = e^{\frac{n^2}{n+1}}$.
(4) $a_n = \frac{\log(n)}{\log(2n)}$.

Solution. Clearly (1) converges with limit 0.

For (2), we can rewrite the expression as $(\frac{2}{3})^n + 1$. As $(\frac{2}{3})^n$ converges to 0, this sequence converges to 1.

For (3), the sequence $\frac{n^2}{n+1}$ diverges to ∞, so $e^{\frac{n^2}{n+1}}$ does as well.

Finally, in (4), we write $\log(2n) = \log(2) + \log(n)$ to see:

$$\frac{\log(n)}{\log(2n)} = \frac{\log(n)}{\log(2) + \log(n)} = \frac{1}{\frac{\log(2)}{\log(n)} + 1}.$$

As $\frac{\log(2)}{\log(n)}$ converges to 0, this sequence converges to 1.

Problem 2. Define a sequence by taking $a_1 = 1$ and setting $a_n = \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-1}})$ for $n \geqslant 2$.

Find the limit of the sequence a_n.

Solution. Take $L = \lim_{n\to\infty} a_n$. Then we have:

$$L = \frac{1}{2}(L + \frac{2}{L}) \Rightarrow$$
$$L^2 = 2.$$

As $a_n > 0$ for all n, we must have $L = \boxed{\sqrt{2}}$ (as opposed to $-\sqrt{2}$).

Just for fun: we can view this sequence operating as follows.

Suppose $a > 0$ is some "guess" for $\sqrt{2}$. For the sake of definiteness, let's suppose $a < \sqrt{2}$; otherwise, replace a with $\frac{2}{a}$. Then $\frac{2}{a}$ is bigger than $\sqrt{2}$, and the average $b = \frac{1}{2}(a + \frac{2}{a})$ is a better guess for $\sqrt{2}$ than either a or $\frac{2}{a}$. Indeed, you should check:

$$a < b = \frac{1}{2}(a + \frac{2}{a}) < \frac{2}{a}.$$

So our sequence keeps iterating these better and better guesses to $\sqrt{2}$. (This kind of iterative method is called *Newton's method*.)

Problem 3. Let $1, 1, 2, 3, 5, 8, \ldots$ be the Fibonacci sequence, and let F_n be the nth Fibonacci number. So $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$.

Define $a_n = \frac{F_{n+1}}{F_n}$. Using a calculator, determine the limit $\lim_{n\to\infty} a_n$ to several decimal places. Then find the exact value of the limit.

Solution. We can find the first few terms of the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

We have:

$$a_8 = 34/21 = 1.6190\ldots, a_9 = 55/34 = 1.6176\ldots,$$
$$a_{10} = 89/55 = 1.6182\ldots, a_{11} = 144/89 = 1.6180\ldots,$$

and using a computer can find $a_{10^5} = F_{10^5+1}/F_{10^5} = 1.61803398874989\ldots$.

This number looks an awful lot like the *golden ratio* $\varphi = \frac{1+\sqrt{5}}{2}$, which was beloved in antiquity by the ancient Greeks and is beloved today by Wikipedia.

To see that the limit is the golden ratio, take $L = \lim_{n\to\infty} a_n$. We then have:

$$L = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_n + F_{n-1}}{F_n} = \lim_{n\to\infty} 1 + \frac{F_{n-1}}{F_n} = 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} = 1 + \frac{1}{L}.$$

Multiplying by L and rearranging terms, we obtain the quadratic equation:

$$L^2 - L - 1 = 0 \Rightarrow L = \frac{1 \pm \sqrt{5}}{2}.$$

As $L > 1$, we must have $L = \boxed{\frac{1+\sqrt{5}}{2}}$.

Problem 4. For any x, it is known that the sequence $a_n = (1 + \frac{x}{n})^n$ converges. Define a function $f(x) = \lim_{n\to\infty}(1 + \frac{x}{n})^n$.

(1) Using a calculator, compute $f(1)$ to several decimal places.
(2) Show that $f(0) = 1$.
(3) Show that $\frac{d}{dx} f(x) = f(x)$.
(4) Show that $\frac{d}{dx} \log(f(x)) = 1$.
(5) Deduce that $f(x) = e^x$.

Solution. We have $f(1) \approx (1+\frac{1}{10000})^{10000} = 2.718\ldots$, which looks a lot like e. We will see that $f(1) = e$ later.

That $f(0) = 1$ is clear.

We have:

$$\frac{d}{dx}f(x) = \lim_{n\to\infty}(1+\frac{x}{n})^n = \lim_{n\to\infty}\frac{d}{dx}(1+\frac{x}{n})^n =$$
$$\lim_{n\to\infty}\frac{1}{n}\cdot n\cdot(1+\frac{x}{n})^{n-1} = \lim_{n\to\infty}(1+\frac{x}{n})^{n-1} = \lim_{n\to\infty}\frac{(1+\frac{x}{n})^n}{1+\frac{x}{n}} =$$
$$\lim_{n\to\infty}\frac{(1+\frac{x}{n})^n}{1+\frac{x}{n}} = \frac{\lim_{n\to\infty}(1+\frac{x}{n})^n}{\lim_{n\to\infty}1+\frac{x}{n}} = \frac{f(x)}{1} = f(x).$$

We now have $\frac{d}{dx}\log(f(x)) = \frac{f'(x)}{f(x)} = \frac{f(x)}{f(x)} = 1$.

We deduce that $\log(f(x)) = x + C$ for some constant C; any function with derivative 1 has this form. Then plugging in $x = 0$, we see $\log(f(0)) = 0 + C = C$; by before, $f(0) = 1$, so $C = 0$. This means $\log(f(x)) = x$. We finally obtain:

$$f(x) = e^{\log(f(x))} = e^x.$$