408L CLASS PROBLEMS

APRIL 3RD, 2020

Problem 1.

1) Find $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

2) Draw a segment of length $\frac{1}{2}$, then one of length $\frac{1}{2} + \frac{1}{4}$, then one of length $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$. How do these pictures relate to your answer to (1)?

Solution. Using the geometric series formula, we have:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots =$$

$$\frac{1}{2} \cdot (1 + \frac{1}{2} + \frac{1}{4} + \ldots) =$$

$$\frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2} \cdot 2 = [1].$$

The picture for the second part is as follows.

![Diagram](image)

Problem 2. Find $6/5 + 18/25 + 54/125 + 162/625 + \ldots$

Solution. If we write the terms of the series as $a_0, a_1, \ldots$, we recognize the pattern as $a_n = 2 \cdot (\frac{3}{5})^{n+1}$. Therefore, using the geometric series formula, we have:

$$6/5 + 18/25 + 54/125 + 162/625 + \ldots = \frac{6}{5} \cdot \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{6}{5} \cdot \frac{1}{1 - \frac{3}{5}} = \frac{6}{5} \cdot \frac{5}{2} = [3].$$

Problem 3. Find .99999... = $\sum_{n=1}^{\infty} 9 \cdot (\frac{1}{10})^n$ by evaluating the geometric series.

Solution. We use the geometric series formula:
\[
\sum_{n=1}^{\infty} 9 \cdot \left(\frac{1}{10}\right)^n = \frac{9}{10} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{9}{10} \cdot \frac{10}{9} = 1
\]

If it seems odd to you that \(.99999\ldots = 1\), convince yourself that \(1 - .99999\ldots\) is less than \(.1\), and less than \(.01\), and less than \(.001\), etc., and therefore must be zero.

**Problem 4.** Find \(\sum_{n=2}^{\infty} \frac{1}{n^2-1}\).

**Solution.** This is a telescoping series problem.

We begin by finding the partial fractions decomposition:

\[
\frac{1}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1} = \frac{(A + B)n + (A - B)}{(n-1)(n+1)}
\]

giving \(A = \frac{1}{2}\), \(B = -\frac{1}{2}\) and:

\[
\frac{1}{n^2-1} = \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1}\right).
\]

Therefore, our sum is:

\[
\frac{1}{2} \cdot \left(\left(\frac{1}{2-1} - \frac{1}{2+1}\right) + \left(\frac{1}{3-1} - \frac{1}{3+1}\right) + \left(\frac{1}{4-1} - \frac{1}{4+1}\right) + \left(\frac{1}{5-1} - \frac{1}{5+1}\right) + \ldots\right) = \frac{1}{2} \cdot \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots \right) = \frac{1}{2} \cdot \left(\frac{1}{1} + \frac{1}{2}\right)
\]

as all of the other terms cancel. (For example, the \(-\frac{1}{3}\) in the first term on the second line cancels with the \(\frac{1}{3}\) in the third term, etc.)

Evaluating this, we obtain a final answer \(\frac{3}{4}\).

---

\(^1\)The sum begins at 2 not to be tricky, but to avoid dividing by zero.