408L CLASS PROBLEMS

APRIL 3RD, 2020

Problem 1.

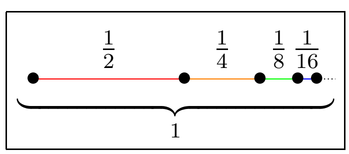
(1) Find $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

(2) Draw a segment of length $\frac{1}{2}$, then one of length $\frac{1}{2} + \frac{1}{4}$, then one of length $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$. How do these pictures relate to your answer to (1)?

Solution. Using the geometric series formula, we have:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots =$$
$$\frac{1}{2} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) =$$
$$\frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2} \cdot 2 = \boxed{1}.$$

The picture for the second part is as follows.



(1)

Problem 2. Find $6/5 + 18/25 + 54/125 + 162/625 + \dots$.

Solution. If we write the terms of the series as $a_0, a_1, \dots$, we recognize the pattern as $a_n = 2 \cdot (\frac{3}{5})^{n+1} = \frac{6}{5} \cdot (\frac{3}{5})^n$. Therefore, using the geometric series formula, we have:

$$6/5 + 18/25 + 54/125 + 162/625 + \dots = \frac{6}{5} \cdot \sum_{n=0}^{\infty} (\frac{3}{5})^n = \frac{6}{5} \cdot \frac{1}{1 - \frac{3}{5}} = \frac{6}{5} \cdot \frac{5}{2} = \boxed{3}.$$

Problem 3. Find $.99999\dots = \sum_{n=1}^{\infty} 9 \cdot (\frac{1}{10})^n$ by evaluating the geometric series.

Solution. We use the geometric series formula:

$$\sum_{n=1}^{\infty} 9 \cdot (\frac{1}{10})^n = \frac{9}{10} \cdot \sum_{n=0}^{\infty} (\frac{1}{10})^n =$$
$$\frac{9}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{9}{10} \cdot \frac{10}{9} = \boxed{1}.$$

If it seems odd to you that .99999... = 1, convince yourself that 1 − .99999... is less than .1, and less than .01, and less than .001, etc., and therefore must be zero.

Problem 4. Find[1] $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

Solution. This is a telescoping series problem.

We begin by finding the partial fractions decomposition:

$$\frac{1}{n^2-1} = \frac{1}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1} = \frac{(A+B)n + (A-B)}{(n-1)(n+1)}$$

giving $A = \frac{1}{2}$, $B = -\frac{1}{2}$ and:

$$\frac{1}{n^2-1} = \frac{1}{2}(\frac{1}{n-1} - \frac{1}{n+1}).$$

Therefore, our sum is:

$$\frac{1}{2} \cdot \Big((\frac{1}{2-1} - \frac{1}{2+1}) + (\frac{1}{3-1} - \frac{1}{3+1}) + (\frac{1}{4-1} - \frac{1}{4+1}) + (\frac{1}{5-1} - \frac{1}{5+1}) \Big) + \ldots =$$
$$\frac{1}{2} \cdot \Big((\frac{1}{1} - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{4} - \frac{1}{6}) + \ldots \Big) =$$
$$\frac{1}{2} \cdot \Big(\frac{1}{1} + \frac{1}{2} \Big)$$

as all of the other terms cancel. (For example, the $-\frac{1}{3}$ in the first term on the second line cancels with the $\frac{1}{3}$ in the third term, etc.)

Evaluating this, we obtain a final answer $\boxed{\frac{3}{4}}$.

[1] The sum begins at 2 not to be tricky, but to avoid dividing by zero.