408L CLASS PROBLEMS

APRIL 6TH, 2020

Problem 1. Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge? If it converges, find an upper bound for its value.

Solution. By the integral test, the series $\boxed{\text{converges}}$ because:

$$\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = 1$$

converges.

We can bound the sum using:

$$1 = \int_1^\infty \frac{dx}{x^2} > \sum_{n=2}^{\infty} \frac{1}{n^2}$$
$$\Rightarrow \boxed{2} = \frac{1}{1^2} + 1 > \frac{1}{1^2} + \sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(In fact, in 1735, Euler famously showed that this sum is exactly $\frac{\pi^2}{6} \approx 1.645 < 2$.)

Problem 2. Does $\sum_{n=1}^{\infty} ne^{-n}$ converge or diverge? If it converges, find an upper bound for its value.

Solution. We can apply the integral test with $f(x) = xe^{-x}$.

Using familiar integration by parts, we find:

$$\int xe^{-x}dx = -(x+1)e^{-x}$$

so:

$$\int_1^\infty xe^{-x}dx = -(x+1)e^{-x}\Big|_1^\infty = \big(-\lim_{x\to\infty}(x+1)e^{-x}\big) - (-2e^{-1}).$$

Using L'Hôpital's rule, we find that the limit exists and is zero, so the integral $\boxed{\text{converges}}$ with value $2e^{-2}$.

As in the previous problem, we have:

$$\sum_{n=1}^{\infty} ne^{-n} < 1 * e^{-1} + \int_1^\infty xe^{-x}dx = \boxed{3e^{-1}}.$$

(For a challenge beyond the scope of this class, you can try to show that this sum is exactly $\frac{e}{(e-1)^2}$. There is a tricky way to apply the geometric series formula to find this. As a sanity check, note that $\frac{e}{(e-1)^2} \approx .921 < 1.104 \approx 3e^{-1}$.)

Problem 3. Does the series:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} \dots$$

converge or diverge? If it converges, find an upper bound for its value.

Solution. The integral:

$$\int_1^\infty \frac{dx}{x^{\frac{1}{2}}} = 2x^{\frac{1}{2}}\Big|_1^\infty$$

diverges, so the sum $\boxed{\text{diverges}}$ as well.

Problem 4. Does $\sum_{n=2}^\infty \frac{1}{n\log(n)}$ converge or diverge? If it converges, find an upper bound for its value.

Solution. Using u-substitution with $u = \log(x)$, we find:

$$\int \frac{dx}{x\log(x)} = \int \frac{du}{u} = \log(u) = \log(\log(x)).$$

We see that the integral:

$$\int_2^\infty \frac{dx}{x\log(x)}$$

diverges, because $\log(\log(x)) \to \infty$ as $x \to \infty$. Therefore, the sum $\boxed{\text{diverges}}$ as well.

Problem 5. Does $\sum_{n=2}^\infty \frac{1}{n\log(n)^2}$ converge or diverge? If it converges, find an upper bound for its value.

Using u-substitution with $u = \log(x)$ as in the previous problem, we find:

$$\int \frac{dx}{x\log(x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\log(x)}.$$

Therefore, we can evaluate the integral:

$$\int_2^\infty \frac{dx}{x\log(x)^2} = -\frac{1}{\log(x)}\Big|_2^\infty = -\frac{1}{\log(\infty)} - (-\frac{1}{\log(2)}) = \frac{1}{\log(2)}.$$

As the integral converges, the sum $\boxed{\text{converges}}$ as well.

We find an upper bound as in the first problem:

$$\sum_{n=2}^{\infty} \frac{1}{n\log(n)^2} = \frac{1}{2\log(2)} + \sum_{n=3}^{\infty} \frac{1}{n\log(n)^2} < \frac{1}{2\log(2)} + \int_2^{\infty} \frac{dx}{x\log(x)^2} = \boxed{\frac{3}{2\log(2)}}.$$

(Here the infinite sum does not admit a pithy closed form expression, but you can show that its decimal expansion begins $2.109742801\ldots$, so our sanity check is $\sum_{n=2}^{\infty} \frac{1}{n\log(n)^2} = 2.1097\ldots < 2.1640\ldots = \frac{3}{2\log(2)}$.)