408L CLASS PROBLEMS

APRIL 8TH, 2020

Problem 1. Does $\sum_{n=1}^{\infty} \frac{1}{e^n - n}$ converge or diverge?

Solution. We have:

$$\lim_{n\to\infty} \frac{e^n - n}{e^n} = \lim_{n\to\infty} \frac{e^n}{e^n} - \lim_{n\to\infty} \frac{n}{e^n} = 1 - 0 = 1.$$

Therefore, by the limit comparison test, convergence of our series is equivalent to convergence of $\sum_{n=1}^{\infty} \frac{1}{e^n}$. The latter series is a geometric series for $r = \frac{1}{e} < 1$, so $\boxed{\text{converges}}$.

Problem 2. Does $\sum_{n=1}^{\infty} \frac{e^n}{3^n + n\cos(n)}$ converge or diverge?

Solution. We have:

$$\lim_{n\to\infty} \frac{3^n + n\cos(n)}{3^n} = \lim_{n\to\infty} \frac{3^n}{3^n} + \lim_{n\to\infty} \frac{n\cos(n)}{3^n} = 1 + 0 = 1.$$

Therefore, we also have:

$$\lim_{n\to\infty} \frac{\frac{e^n}{3^n}}{\frac{e^n}{3^n + n\cos(n)}} = 1.$$

As the series $\sum_{n=1}^{\infty} \frac{e^n}{3^n}$ is a geometric series with $r = \frac{e}{3} < 1$, the limit comparison test implies our series $\boxed{\text{converges}}$.

Problem 3. Does $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n^2-1}}$ converge or diverge?

Solution. We have:

$$\lim_{n\to\infty} \frac{1+\sqrt{n^2-1}}{n} = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{\sqrt{n^2-1}}{n} = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \sqrt{1 - \frac{1}{n^2}} = 0 + 1 = 1.$$

Therefore, by the limit comparison test and the divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$, we find that our series $\boxed{\text{diverges}}$.

Problem 4. Does $\sum \frac{e^n+1}{ne^n+1}$ converge or diverge?

Solution. We have:

$$\lim_{n\to\infty} \frac{e^n+1}{e^n} = \lim_{n\to\infty} 1 + \frac{1}{e^n} = 1.$$

Similarly, $\lim_{n\to\infty} \frac{ne^n+1}{ne^n} = 1$. Therefore:

$$\lim_{n\to\infty} \frac{\frac{e^n+1}{ne^n+1}}{\frac{e^n}{ne^n}} = 1.$$

By the limit comparison test, convergence of our series is equivalent to that of $\sum_{n=1}^{\infty} \frac{e^n}{ne^n} = \sum_{n=1}^{\infty} \frac{1}{n}$. The latter series diverges, so our series $\boxed{\text{diverges}}$ as well.