Problem 1. Does $\sum_{n=1}^{\infty} \frac{1}{e^n - n}$ converge or diverge?

Solution. We have:

$$\lim_{n \to \infty} \frac{e^n - n}{e^n} = \lim_{n \to \infty} \frac{e^n}{e^n} - \lim_{n \to \infty} \frac{n}{e^n} = 1 - 0 = 1.$$ 

Therefore, by the limit comparison test, convergence of our series is equivalent to convergence of $\sum_{n=1}^{\infty} \frac{1}{e^n}$. The latter series is a geometric series for $r = \frac{1}{e} < 1$, so \textbf{converges.}

Problem 2. Does $\sum_{n=1}^{\infty} \frac{e^n}{3^n + n \cos(n)}$ converge or diverge?

Solution. We have:

$$\lim_{n \to \infty} \frac{3^n + n \cos(n)}{3^n} = \lim_{n \to \infty} \frac{3^n}{3^n} + \lim_{n \to \infty} \frac{n \cos(n)}{3^n} = 1 + 0 = 1.$$ 

Therefore, we also have:

$$\lim_{n \to \infty} \frac{\frac{e^n}{3^n}}{\frac{e^n}{e^n}} = 1.$$ 

As the series $\sum_{n=1}^{\infty} \frac{e^n}{3^n}$ is a geometric series with $r = \frac{e}{3} < 1$, the limit comparison test implies our series [converges].

Problem 3. Does $\sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{n^2} - 1}$ converge or diverge?

Solution. We have:

$$\lim_{n \to \infty} \frac{1 + \sqrt{n^2} - 1}{n} = \lim_{n \to \infty} \frac{1}{n} + \lim_{n \to \infty} \frac{\sqrt{n^2} - 1}{n} = \lim_{n \to \infty} \frac{1}{n} + \lim_{n \to \infty} \sqrt{1 - \frac{1}{n^2}} = 0 + 1 = 1.$$ 

Therefore, by the limit comparison test and the divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$, we find that our series \textbf{diverges.}

Problem 4. Does $\sum_{n=1}^{\infty} \frac{e^n + 1}{n e^n + 1}$ converge or diverge?

Solution. We have:
\[
\lim_{n \to \infty} \frac{e^n + 1}{e^n} = \lim_{n \to \infty} 1 + \frac{1}{e^n} = 1.
\]

Similarly, \(\lim_{n \to \infty} \frac{ne^n + 1}{ne^n} = 1\). Therefore:

\[
\lim_{n \to \infty} \frac{e^n + 1}{ne^n} = 1.
\]

By the limit comparison test, convergence of our series is equivalent to that of \(\sum_{n=1}^{\infty} \frac{e^n}{ne^n} = \sum_{n=1}^{\infty} \frac{1}{n}\). The latter series diverges, so our series \(\text{diverges}\) as well.