408L CLASS PROBLEMS

APRIL 10TH, 2020

Problem 1. Does $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ converge or diverge?

Solution. As $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ and $\frac{1}{\sqrt{n}}$ is decreasing, the alternating series test applies and we see that the series $\boxed{\text{converges}}$.

Problem 2. Does $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\log(n)}$ converge or diverge?

Solution. Again, As $\lim_{n\to\infty}\frac{1}{\log n}=0$ and $\frac{1}{\log n}$ is decreasing, so the alternating series test applies and we see that the series $\boxed{\text{converges}}$.

Problem 3. Does $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{4n^2-1}}$ converge or diverge?

Solution. We have:

$$\lim_{n\to\infty}\frac{n}{\sqrt{4n^2-1}}=\lim_{n\to\infty}\frac{1}{\sqrt{4-\frac{1}{n^2}}}=\frac{1}{2}\neq 0$$

so the series $\boxed{\text{diverges}}$.

Problem 4. Does $\sum_{n=0}^{\infty}(-1)^n\cos(n\pi)$ converge or diverge?

Solution. We have $\cos(0\cdot\pi)=1$, $\cos(1\cdot\pi)=-1$, $\cos(2\cdot\pi)=1$, etc., so $\cos(n\pi)=(-1)^n$. Therefore, we can write the series as:

$$\sum_{n=0}^{\infty}(-1)^n\cos(n\pi)=\sum_{n=0}^{\infty}(-1)^n(-1)^n=\sum_{n=0}^{\infty}(-1)^{2n}=$$
$$\sum_{n=0}^{\infty}\left((-1)^2\right)^n=\sum_{n=0}^{\infty}1.$$

Obviously this series $\boxed{\text{diverges}}$.

Problem 5. Does $\sum_{n=1}^{\infty}(-1)^n\cdot n\cdot\sin(\frac{1}{n})$ converge or diverge?

Solution. Recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Therefore, $\lim_{x\to\infty} x \cdot \sin(\frac{1}{x}) = 1$ as well. We therefore see that the series $\boxed{\text{diverges}}$.

Problem 6. Does $\sum_{n=1}^{\infty}(-1)^n\big(\frac{\pi}{2} - \tan^{-1}(n)\big)$ converge or diverge?

Solution. We have $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$, so the terms in the series do limit to 0.

Moreover, we have:

$$\frac{d}{dx}\Big(\frac{\pi}{2} - \tan^{-1}(x)\Big) = \frac{-1}{x^2+1} < 0.$$

Therefore, the function $\frac{\pi}{2} - \tan^{-1}(x)$ is decreasing.

Finally, we note that $\tan^{-1}(x) < \frac{\pi}{2}$ for all x, so the function $\frac{\pi}{2} - \tan^{-1}(x)$ is positive.

Therefore, the alternating series test applies and we deduce that the series $\boxed{\text{converges}}$.