408L CLASS PROBLEMS

APRIL 15TH, 2020

Problem 1. Use to ratio test to determine whether $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges or diverges.

Solution. Let $a_n = \frac{2^n}{n!}$. Then:

$$a_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2}{n+1} \cdot \frac{2^n}{n!} = \frac{2}{n+1} \cdot a_n.$$

Therefore:

$$\frac{a_{n+1}}{a_n} = \frac{2}{n+1}$$

and:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

By the ratio test, the series $\boxed{\text{converges}}$.

Problem 2. Use to ratio test to determine whether $\sum_{n=0}^{\infty} \frac{n^2+1}{e^n}$ converges or diverges.

Solution. Let $a_n = \frac{n^2+1}{e^n}$. We then have:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \frac{(n+1)^2}{n^2} \cdot \frac{e^n}{e^{n+1}} = \frac{(n+1)^2}{n^2} \cdot \frac{1}{e}.$$

The limit of this sequence as $n \to \infty$ is $1 \cdot \frac{1}{e} = \frac{1}{e} < 1$. By the ratio test, the series $\boxed{\text{converges}}$.

Problem 3. Use the root test to determine whether $\sum_{n=0}^{\infty} \left(\frac{n^2+1}{2n^2+3}\right)^n$ converges or diverges.

Solution. Let $a_n = \left(\frac{n^2+1}{2n^2+3}\right)^n$. We have:

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}.$$

By the ratio test, the series $\boxed{\text{converges}}$.

Problem 4. Use the root test to determine whether the series $\sum_{n=0}^{\infty}(\tan^{-1}(n))^n$ and $\sum_{n=0}^{\infty}(\frac{\tan^{-1}(n)}{2})^n$ converge or diverge.

Solution. Let $a_n = \tan^{-1}(n))^n$ and $b_n = (\frac{\tan^{-1}(n)}{2})^n$. We have:

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2} > 1$$

and:

$$\lim_{n\to\infty} b_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\tan^{-1}(n)}{2} = \frac{\pi}{4} < 1.$$

Therefore, the first series $\boxed{\text{diverges}}$ while the second series $\boxed{\text{converges}}$.

Problem 5. Let $1, 1, 2, 3, 5, 8, \ldots$ be the Fibonacci sequence, and let F_n be the nth Fibonacci number. In other words, $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$.

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$ converges or diverges. (Hint: revisit Problem 3 from Day 22.)

Solution. In the cited problem, we found:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

Therefore, we have:

$$\lim_{n\to\infty} \frac{\frac{1}{F_{n+1}}}{\frac{1}{F_n}} = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{2}{1+\sqrt{5}} < 1.$$

Therefore, the series $\boxed{\text{converges}}$ by the ratio test.