408L CLASS PROBLEMS

APRIL 17TH, 2020

Problem 1. Determine whether $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ converges or diverges.

Solution. We can apply the $\boxed{\text{comparison test}}$ with the harmonic series $\sum \frac{1}{n}$ to find this series $\boxed{\text{diverges}}$.

Problem 2. Determine whether $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ converges or diverges.

Solution. The exponent suggests the $\boxed{roottest}$. For $a_n = (\frac{n}{n+1})^{n^2}$, we have:

$$a_n^{\frac{1}{n}} = (\frac{n}{n+1})^n = \frac{1}{(1+\frac{1}{n})^n}.$$

We have $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, so $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \frac{1}{e}$. As $\frac{1}{e} < 1$, the root test implies that the series $\boxed{\text{converges}}$.

Problem 3. Determine whether $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot 2n-1}{1\cdot 4\cdot 7\cdot \ldots \cdot 3n-2}$ converges or diverges.

Solution. Let $a_n = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot 2n-1}{1\cdot 4\cdot 7\cdot \ldots \cdot 3n-2}$. We then have the (*recursive*) equation:

$$a_{n+1} = \frac{2n+1}{3n+1}\cdot a_n.$$

This suggests using the $\boxed{\text{ratio test}}$.

By the above, we have:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n+1}{3n+1} = \frac{2}{3}.$$

As this limit is less than 1, the series $\boxed{\text{converges}}$.

Problem 4. Determine whether $\sum_{n=1}^{\infty} \tan(\frac{1}{n})$ converges or diverges.

Solution. We use the $\boxed{\text{comparison test}}$ with the harmonic series. Let $a_n = \tan(\frac{1}{n})$ and $b_n = \frac{1}{n}$. We then have:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \cdot \frac{1}{\cos(\frac{1}{n})} = 1\cdot 1 = 1.$$

Therefore, the comparison test applies, and we deduce that the series $\boxed{\text{diverges}}$ as the harmonic series does.

Problem 5. Determine whether $\sum_{n=1}^{\infty}(-1)^n n^{-\frac{1}{100}}\log(n)$ converges or diverges.

Solution. The series $\boxed{\text{converges}}$ by the $\boxed{\text{alternating series test}}$.

Problem 6. Determine whether $\sum_{n=1}^{\infty}(\sin(\frac{1}{n}) - \frac{1}{n})^n$ converges or diverges.

Solution. The exponent suggests using the $\boxed{\text{root test}}$. For $a_n = (\sin(\frac{1}{n}) - \frac{1}{n})^n$, we have:

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \sin(\frac{1}{n}) - \frac{1}{n} = \lim_{n\to\infty} \frac{n\sin(\frac{1}{n}) - 1}{n}.$$

The numerator has limit 0 while the denominator has limit ∞, so this limit is 0. Therefore, the root test implies the series $\boxed{\text{converges}}$.

Problem 7. Determine whether $\sum_{n=1}^{\infty} 2^{-\log(n)}$ converges or diverges. Determine whether $\sum_{n=1}^{\infty} 3^{-\log(n)}$ converges or diverges.

Solution. We have:

$$2^{-\log(n)} = e^{-\log(2)\cdot\log(n)} = n^{-\log(2)} = \frac{1}{n^{\log(2)}}.$$

Therefore, the series $\boxed{\text{diverges}}$ by the $\boxed{p\text{-series}}$ test as $\log(2) < 1$. However, the second series $\boxed{\text{converges}}$ by the same reasoning, as $\log(3) > 1$.

(Of course, log indicates natural log throughout this problem.)