408L CLASS PROBLEMS

JANUARY 31, 2020

Problem 1. Find $\int \frac{dx}{x\sqrt{\log(x)}}$.

Solution. Take $u = \log(x)$, so $du = \frac{dx}{x}$. We obtain:

$$\int \frac{dx}{x\sqrt{\log(x)}} = \int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}} du = \frac{1}{2}u^{\frac{1}{2}} + C = \boxed{\frac{1}{2}\sqrt{\log(x)} + C}.$$

Problem 2. Find $\int_1^2 \frac{e^{\frac{1}{x}}dx}{x^2}$.

Solution. First, we find an anti-derivative to the function (so we can omit the constant C).

Taking $u = \frac{1}{x}$, we have $du = -\frac{dx}{x^2}$, so $-du = \frac{dx}{x^2}$. We then find:

$$\int \frac{e^{\frac{1}{x}}dx}{x^2} = -\int e^u du = -e^u = -e^{\frac{1}{x}}.$$

To calculate the definite integral, we then evaluate:

$$-e^{\frac{1}{x}}\Big|_1^2 = -e^{\frac{1}{2}} - (-e^{\frac{1}{1}}) = \boxed{e - \sqrt{e}}.$$

Alternative solution. We use u-substitution as above, but calculate the definite integral directly:

$$\int_1^2 \frac{e^{\frac{1}{x}}dx}{x^2} = -\int_{u(1)}^{u(2)} e^u du = -\int_{\frac{1}{1}}^{\frac{1}{2}} e^u du = -e^u\Big|_1^{\frac{1}{2}} = \boxed{e - \sqrt{e}}.$$

Problem 3. Find $\int_{-5}^{-1} \frac{dx}{x}$.

Solution. The formula $\frac{d}{dx}\log(x) = \frac{1}{x}$ is valid *only* for $x > 0$, as log is only defined on positive numbers.

For $x < 0$, we can use the formula $\frac{d}{dx}\log(-x) = \frac{1}{x}$; here $-x$ is positive because x is negative, so $\log(-x)$ is defined.

We can summarize by saying $\frac{d}{dx}\log(|x|) = \frac{1}{x}$ for all $x \neq 0$. Often in calculus, we use the formula:

$$\int \frac{dx}{x} = \log(|x|) + C.$$

Note that this is valid when x is positive or negative, but issues may arise when x is allowed to be zero; for example, $\int_{-1}^{1} \frac{dx}{x}$ is not defined.

For our problem, we now have:

$$\int_{-5}^{-1} \frac{dx}{x} = \log(|x|)\Big|_{-5}^{-1} = \log(1) - \log(5) = \boxed{-\log(5)}.$$

(For practice, justify this another way by drawing a picture to show:

$$\int_{-5}^{-1} \frac{dx}{x} = -\int_{1}^{5} \frac{dx}{x}.$$

Justify this identity another way using u-substitution with $u = -x$.)

Problem 4. Find $\int \frac{x^2+x+1}{x^2+1} dx$.

Solution. We break up the integrand as:

$$\frac{x^2+x+1}{x^2+1} = 1 + \frac{x}{x^2+1}.$$

We therefore have:

$$\int \frac{x^2+x+1}{x^2+1} dx = \int dx + \int \frac{x}{x^2+1} dx = x + \int \frac{x}{x^2+1} dx.$$

We can solve the latter integral by substitution: take $u = x^2+1$, so $du = 2xdx$, i.e., $\frac{1}{2}du = xdx$. We then have:

$$\int \frac{x}{x^2+1} dx = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\log(u) + C = \frac{1}{2}\log(x^2+1) + C$$

(Because $u = x^2 + 1 > 0$ for all x, we are justified in saying $\int \frac{1}{u} = \log(u)$, not $\log(|u|)$.)

Combined with the above, we obtain the final answer:

$$\boxed{x + \frac{1}{2}\log(x^2+1) + C}.$$

Problem 5. Find $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x} dx$.

Solution. We use u-substitution with $u = 1 + \tan x$, so $du = \sec^2 x dx$. We then have:

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x} dx = \int_{1+\tan(0)}^{1+\tan(\frac{\pi}{4})} \frac{du}{u} = \int_1^2 \frac{du}{u} = \log(u)\Big|_1^2 = \log(2) - \log(1) = \boxed{\log(2)}.$$

Here we note that $u = 1 + \tan x > 0$ for $0 \leqslant x \leqslant \frac{\pi}{4}$, justifying our use of $\log(u)$: it equals $\log(|u|)$ for such x.