408L CLASS PROBLEMS

FEBRUARY 3RD, 2020

Problem 1. Work with a partner on this problem.

(1) Find $\int \tan(x)\sec^2(x)dx$ via u-substitution.

Partner A should solve the problem with $u = \tan(x)$; Partner B should use $u = \sec(x)$.

(2) Reconcile.

Solution. Taking $u = \tan(x)$, we have $du = \sec^2(x)dx$, so our integral is:

$$\int \tan(x)\sec^2(x)dx = \int u du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2(x) + C}.$$

Taking $u = \sec(x)$, we have $du = \tan(x)\sec(x)dx$, so our integral is:

$$\int \tan(x)\sec^2(x)dx = \int u du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sec^2(x) + C}.$$

To reconcile these seemingly different answers, note that:

$$\frac{1}{2}\tan^2(x) = \frac{1}{2}\cdot\frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{2}\cdot\frac{1-\cos^2(x)}{\cos^2(x)} = \frac{1}{2}\cdot\sec^2(x) - \frac{1}{2}.$$

Therefore, $\frac{1}{2}\tan^2(x)$ and $\frac{1}{2}\sec^2(x)$ differ by a constant. Our answers reflect that each of these functions is an anti-derivative of $\tan(x)\sec^2(x)$, and there is no contradiction because anti-derivatives are only determined up to constants.

Problem 2. Find $\int_{-3}^{3} \sin(x)\cdot x^4 dx$.

Solution. Because $f(x) = \sin(x)\cdot x^4$ is *odd*, that is, $f(-x) = -f(x)$, we have:

$$\int_{-a}^{a} f(x)dx = \boxed{0}$$

for all a. (Draw a picture to convince yourself of this!)

Problem 3. Suppose f is a function with $\frac{df}{dx} = \frac{\cos x}{2+\sin x}$. Suppose $f(0) = 1$. Find $f(10)$.

Solution. Here is the strategy. We know that f must be some anti-derivative of $\frac{df}{dx} = \frac{\cos x}{2+\sin x}$, which should determine f up to constants. We then use the initial value property $f(0) = 1$ to determine exactly which function f is (not only up to constants).

Setting $u = 2 + \sin x$, so $du = \cos x dx$, and observing that $u > 0$ for all x, we have:

$$\int \cos x \cdot \log(2 + \sin x) dx = \int \frac{du}{u} = \log(u) + C = \log(2 + \sin x) + C.$$

As $f(x)$ is an anti-derivative of $\frac{df}{dx}$, we have:

$$f(x) = \log(2 + \sin x) + C$$

for some value of the constant C. We then have:

$$1 = f(0) = \log(2) + C$$

so $C = 1 - \log(2)$.

Finally, we obtain:

$$f(10) = \boxed{\log(2 + \sin(10)) + 1 - \log(2)}.$$

Problem 4. Find an anti-derivative of the function $\log(\cos(x)^{\tan(x)})$ defined on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solution. Note that for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\cos(x) > 0$ and $\tan(x)$ is defined, so the function $\log(\cos(x)^{\tan(x)})$ is defined and continuous.

We have:

$$\log(\cos(x)^{\tan(x)}) = \tan(x) \cdot \log(\cos(x)) = \frac{\sin(x)\log(\cos(x))}{\cos(x)}.$$

We use $u = \cos(x)$, $du = -\sin(x)dx$ to solve the integral:

$$\int \log(\cos(x)^{\tan(x)}) dx = \int \frac{\sin(x)\log(\cos(x))}{\cos(x)} dx = -\int \frac{\log(u)du}{u}.$$

We can solve latter integral using another substitution: take $v = \log(u)$, so $dv = \frac{du}{u}$. We then have:

$$-\int \frac{\log(u)du}{u} = -\int v dv = -\frac{1}{2}v^2.$$

(We ignore the constant because we are only looking for some anti-derivative.)

We then have:

$$-\frac{1}{2}v^2 = -\frac{1}{2}\log(u)^2 = \boxed{-\frac{1}{2}\log(\cos(x))^2}.$$

Alternative solution. Use u-substitution with $u = \log(\cos(x))$ instead to solve the problem using only one substitution.