408L CLASS PROBLEMS

FEBRUARY 5TH, 2020

Problem 1. Find the area of the region bounded by the line \( x = 3 \) and the graphs of the functions \( f(x) = x^2, g(x) = \sqrt{x} \).

\[
\int_{1}^{3} (f(x) - g(x))
\]

Solution. The graphs of \( f \) and \( g \) intersect at \( x = 1 \). Moreover, for \( 1 \leq x \), we have \( f(x) \geq g(x) \). Therefore, the area is:

\[
\int_{1}^{3} (x^2 - \sqrt{x})dx = \int_{1}^{3} x^2 - \frac{2}{3}x^{\frac{3}{2}}\bigg|_{1}^{3} = \\
\frac{1}{3} 3^3 - \frac{2}{3} 3^{\frac{3}{2}} - \frac{1}{3} 1^3 + \frac{2}{3} 1^{\frac{3}{2}} = \frac{28}{3} - 2\sqrt{3}
\]

Problem 2. Find the area of the region between the graphs of the functions \( f(x) = x^3, g(x) = 4x \).
Solution. The graphs of $f$ and $g$ intersect when $f(x) = g(x)$, i.e., when $x^3 = 4x$. We can solve this by factoring:

$$x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2).$$

Therefore, the graphs intersect at $-2$, $0$, and $2$.

To find the area between the graphs, we calculate:

$$\int_{-2}^{2} |f(x) - g(x)|dx = \int_{-2}^{0} (f(x) - g(x))dx + \int_{0}^{2} (g(x) - f(x))dx = \int_{-2}^{0} (x^3 - 4x)dx + \int_{0}^{2} (4x - x^3)dx.$$

We can evaluate the second term as:

$$\int_{0}^{2} (4x - x^3)dx = x^2 - \frac{x^4}{4} \bigg|_{0}^{2} = 2^2 - \frac{2^4}{4} = \frac{4}{3}.$$

A similar calculation shows that the first term also evaluates to $\frac{4}{3}$. (In fact, the two integrals must be equal because $f$ and $g$ are both odd functions.)

Therefore, the area between the two graphs is:

$$\frac{4}{3} + \frac{4}{3} = \frac{8}{3}.$$

Problem 3. Find the area bound by the line $x = \frac{\pi}{4}$ and the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \tan(x)$.

Solution. For $0 < x < \frac{\pi}{2}$, we have:
\[ \tan(x) = \frac{\sin(x)}{\cos(x)} > \sin(x) \]

because \( 0 < \cos(x) < 1 \) in this range.

Therefore, the area in question is:

\[ \int_0^\frac{\pi}{4} (\tan(x) - \sin(x))\,dx. \]

To evaluate this integral, we first find the anti-derivative of \( \tan(x) \). As \( \tan(x) = \frac{\sin(x)}{\cos(x)} \), we can integrate using \( u \)-substitution with \( u = \cos(x) \), \( du = -\sin(x)\,dx \):

\[
\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{du}{u} = -\log(|\cos(x)|) = -\log(\cos(x))
\]

where the last equality is valid for \( 0 < x < \frac{\pi}{2} \).

We now have:

\[
\int_0^\frac{\pi}{4} (\tan(x) - \sin(x))\,dx = -\log(\cos(x)) + \cos(x) \bigg|_0^{\frac{\pi}{4}} =
\]

\[
-\log(\cos(\frac{\pi}{4})) + \cos(\frac{\pi}{4}) + \log(\cos(0)) - \cos(0) =
\]

\[
-\log\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} + \log(1) - 1 = \frac{1}{2} \log(2) + \frac{1 - \sqrt{2}}{\sqrt{2}}.
\]