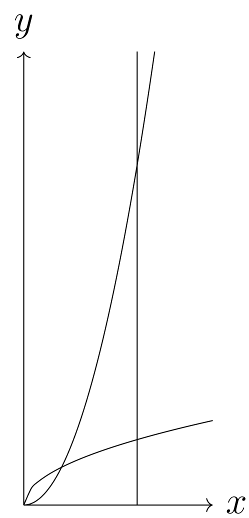
408L CLASS PROBLEMS

FEBRUARY 5TH, 2020

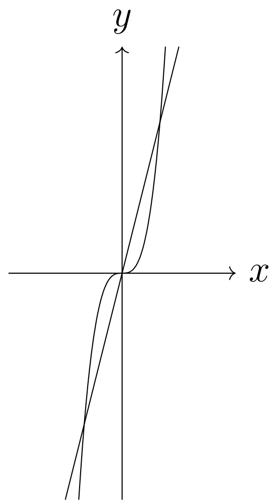
Problem 1. Find the area of the region bounded by the line $x = 3$ and the graphs of the functions $f(x) = x^2$, $g(x) = \sqrt{x}$.



Solution. The graphs of f and g intersect at $x = 1$. Moreover, for $1 \leqslant x$, we have $f(x) \geqslant g(x)$. Therefore, the area is:

$$\int_1^3 (f(x) - g(x))dx = \int_1^3 (x^2 - \sqrt{x})dx = \frac{1}{3}x^3 - \frac{2}{3}x^{\frac{3}{2}}\Big|_1^3 =$$
$$\frac{1}{3}3^3 - \frac{2}{3}3^{\frac{3}{2}} - \frac{1}{3}1^3 + \frac{2}{3}1^{\frac{3}{2}} = \boxed{\frac{28}{3} - 2\sqrt{3}}.$$

Problem 2. Find the area of the region between the graphs of the functions $f(x) = x^3$, $g(x) = 4x$.



Solution. The graphs of f and g intersect when $f(x) = g(x)$, i.e., when $x^3 = 4x$. We can solve this by factoring:

$$x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2).$$

Therefore, the graphs intersect at -2, 0, and 2.

To find the area between the graphs, we calculate:

$$\int_{-2}^{2} |f(x) - g(x)| dx = \int_{-2}^{0} (f(x) - g(x)) dx + \int_{0}^{2} (g(x) - f(x)) dx =$$
$$\int_{-2}^{0} (x^3 - 4x) dx + \int_{0}^{2} (4x - x^3) dx.$$

We can evaluate the second term as:

$$\int_{0}^{2} (4x - x^3) dx = x^2 - \frac{x^3}{3}\Big|_0^2 = 2^2 - \frac{2^3}{3} = \frac{4}{3}.$$

A similar calculation shows that the first term also evaluates to $\frac{4}{3}$. (In fact, the two integrals must be equal because f and g are both odd functions.)

Therefore, the area between the two graphs is:

$$\frac{4}{3} + \frac{4}{3} = \boxed{\frac{8}{3}}.$$

Problem 3. Find the area bound by the line $x = \frac{\pi}{4}$ and the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \tan(x)$.

Solution. For $0 < x < \frac{\pi}{2}$, we have:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} > \sin(x)$$

because $0 < \cos(x) < 1$ in this range.

Therefore, the area in question is:

$$\int_0^{\frac{\pi}{4}} (\tan(x) - \sin(x))dx.$$

To evaluate this integral, we first find the anti-derivative of $\tan(x)$. As $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we can integrate using u-substitution with $u = \cos(x)$, $du = -\sin(x)dx$:

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{du}{u} = -\log(|\cos(x)|) = -\log(\cos(x))$$

where the last equality is valid for $0 < x < \frac{\pi}{2}$.

We now have:

$$\int_0^{\frac{\pi}{4}} (\tan(x) - \sin(x))dx = -\log(\cos(x)) + \cos(x)\Big|_0^{\frac{\pi}{4}} =$$

$$-\log(\cos(\frac{\pi}{4})) + \cos(\frac{\pi}{4}) + \log(\cos(0)) - \cos(0) =$$

$$-\log(\frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}} + \log(1) - 1 = \boxed{\frac{1}{2}\log(2) + \frac{1-\sqrt{2}}{\sqrt{2}}}.$$