408L CLASS PROBLEMS

FEBRUARY 7TH, 2020

Problem 1. Find the volume of the solid by rotating the area under the graph of $\sin(x)$ on $0 \leqslant x \leqslant \pi$ around the x-axis.

Solution. We need to calculate:

$$\int_0^\pi \pi \sin^2(x) dx.$$

We can find the anti-derivative of $\sin^2(x)$ using the identity:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) \Rightarrow$$
$$\sin^2(x) = \frac{1 - \cos(2x)}{2}.$$

We obtain:

$$\int \sin^2(x) dx = \int \frac{1 - \cos(2x)}{2} dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

We can now calculate our definite integral as:

$$\int_0^\pi \pi \sin^2(x) dx = \left(\frac{\pi x}{2} - \frac{\pi \sin(2x)}{4}\right)\Big|_0^\pi = \boxed{\frac{\pi^2}{2}}.$$

Problem 2. Find the volume of the solid obtained by rotating around the x-axis the area under the graph of $f(x) = \frac{x+2}{\sqrt{x^2+1}}$ for $0 \leqslant x \leqslant t$.

Solution. We need to find:

$$\int_0^t \pi\left(\frac{x+2}{\sqrt{x^2+1}}\right)^2 dx = \pi \int_0^t \frac{x^2+4x+4}{x^2+1} dx.$$

To find this integral, we break up the integrand as:

$$\frac{x^2+4x+4}{x^2+1} = 1 + \frac{4x}{x^2+1} + \frac{3}{x^2+1}.$$

We then have standard anti-derivatives:

$$\int 1 \cdot dx = x$$

$$\int \frac{3dx}{x^2+1} = 3\tan^{-1}(x).$$

We can find $\int \frac{4xdx}{x^2+1}$ with u-substitution; for $u = x^2+1$, $du = 2xdx$, so:

$$\int \frac{4xdx}{x^2+1} = \int \frac{2du}{u} = 2\log(|u|) = 2\log(x^2+1).$$

(The last term can also be written as $\log((x^2+1)^2)$.)

We therefore have an anti-derivative:

$$\int \frac{x^2+4x+4}{x^2+1} dx = x + 2\log(x^2+1) + 3\tan^{-1}(x)$$

so that:

$$\pi \int_0^t \frac{x^2+4x+4}{x^2+1} dx = \pi \cdot \Big(x + 2\log(x^2+1) + 3\tan^{-1}(x)\Big)\Big|_0^t =$$

$$\pi \cdot \Big(t + 2\log(t^2+1) + 3\tan^{-1}(t) - 0 - 2\log(0^2+1) - 3\tan^{-1}(0)\Big) =$$

$$\boxed{\pi t + 2\pi\log(t^2+1) + 3\pi\tan^{-1}(t)}.$$

Problem 3. Find the volume of a sphere of radius 1. (Hint: regard a sphere as a surface of revolution.)

Solution. A sphere of radius 1 is obtained as a surface of revolution obtained by rotating the semi-circle where $x^2+y^2=1$ and $y \geqslant 0$ around the x-axis. That semi-circle is the graph of the function $f(x) = \sqrt{1-x^2}$ for $-1 \leqslant x \leqslant 1$.

Therefore, our volume is:

$$\int_{-1}^1 \pi(\sqrt{1-x^2})^2 dx = \pi\int_{-1}^1 (1-x^2)dx = \pi \cdot \big(x - \frac{x^3}{3}\big)\Big|_{-1}^1 = \pi \cdot (1 - \frac{1}{3} - (-1 - \frac{-1}{3})) = \boxed{\pi \cdot \frac{4}{3}}.$$

Problem 4. Find the volume obtained by rotating about the y-axis the area in the first quadrant lying between the graph of $f(x) = x+1$, the graph of $g(x) = \sqrt{x}$, and the line $y = 5$.

Solution. Drawing the picture, we see we need to integrate:

$$\int_0^1 \pi g^{-1}(y)^2 dy + \int_1^5 \pi(g^{-1}(y)^2 - f^{-1}(y)^2)dy = \pi\int_0^5 g^{-1}(y)^2 dy - \pi\int_1^5 f^{-1}(y)^2 dy.$$

We have $g^{-1}(y) = y^2$ and $f^{-1}(y) = y - 1$.

We therefore compute the first integral above as:

$$\int_0^5 g^{-1}(y)^2 dy = \int_0^5 y^4 dy = \frac{1}{5}5^5 = 5^4 = 625.$$

We compute the second integral as:

$$\int_1^5 (y-1)^2 dy = \frac{1}{2}(y-1)^2\Big|_1^5 = \frac{1}{3}(4^2 - 0^2) = \frac{16}{3}.$$

Therefore, our final answer is:

$$(625 - \frac{64}{3})\pi = \boxed{\frac{1811\pi}{3}}.$$