408L CLASS PROBLEMS
FEBRUARY 10TH, 2020

Problem 1. Find the volume of the solid by rotating the area under the graph of $\sqrt{x} \cdot e^{x^2}$ for $0 \leq x \leq 1$ around the $x$-axis.

Solution. The volume is:

$$
\pi \int_0^1 (\sqrt{x} \cdot e^{x^2})^2 dx = \pi \int_0^1 x \cdot e^{2x^2} dx.
$$

We evaluate the integral using $u$-substitution with $u = 2x^2$, $du = 4x dx$:

$$
\int_0^1 x \cdot e^{2x^2} dx = \frac{1}{4} \int_0^2 e^u du = \frac{1}{4} e^u |_0^2 = \frac{1}{4} (e^2 - 1).
$$

Reincorporating the factor of $\pi$, we have a final answer:

$$
\frac{\pi(e^2 - 1)}{4}.
$$

Problem 2. Find the volume of the solid obtained by rotating the area between the $x$-axis and the graph of $\cos(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ about the line $y = 1$.

Solution. The cross-section for fixed $x$ of this solid is a washer with outer radius 1 and inner radius $1 - \cos(x)$.

The area of this washer is:

$$
\pi 1^2 - \pi (1 - \cos(x))^2 = \pi (2 \cos(x) - \cos^2(x)).
$$

Therefore, the volume of our solid is:

$$
\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2 \cos(x) - \cos^2(x)) dx = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx - \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(x) dx.
$$

The first integral above is easy to calculate:

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx = \sin(x) \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin \left( \frac{\pi}{2} \right) - \sin \left( -\frac{\pi}{2} \right) = 1 - (-1) = 2.
$$

To calculate the second integral, we use the identity:
\[
\cos^2(x) = \frac{\cos^2(x) - \sin^2(x) + \cos^2(x) + \sin^2(x)}{2} = \frac{\cos(2x) + 1}{2}.
\]

We obtain:

\[
\int \cos^2(x)dx = \int \frac{\cos(2x) + 1}{2} = \frac{\sin(2x)}{4} + \frac{x}{2}.
\]

For our definite integral, we see:

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(x)dx = \left( \frac{\sin(2x)}{4} + \frac{x}{2} \right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}.
\]

Finally, we obtain:

\[
2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx - \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(x)dx = 2\pi \cdot 2 - \pi \cdot \frac{\pi}{2} = 4\pi - \frac{\pi^2}{2}.
\]

**Problem 3.** In this question, we find the area of a circle of radius 1. (Of course you know the answer, but the point of the problem is to calculate it yourself.)

1. Using the cross-section method, show that the area of a circle with radius 1 is:

\[
2 \int_{-1}^{1} \sqrt{1 - x^2}dx.
\]

2. Evaluate this integral. (Hint: use $u$-substitution with $u = \sin^{-1}(x)$.)

**Solution.** The interior of a circle with radius 1 is all $(x, y)$ with $x^2 + y^2 \leq 1$. For a fixed value of $x$ between $-1$ and 1, this is all $y$ with $|y| \leq \sqrt{1 - x^2}$. Clearly that is a line of length $2\sqrt{1 - x^2}$.

To evaluate the integral, we instead use the perspective $\sin(u) = x$. Then $\cos(x)dx = du$, so we obtain:

\[
\int_{-1}^{1} \sqrt{1 - x^2}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2(u)} \cos(u)du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du.
\]

We evaluated this integral in the previous problem and found it was $\frac{\pi}{2}$.

Reincorporating the factor of 2, we find that the area of a circle is $\frac{4\pi}{2}$ as expected.

**Alternative solution.** The above $u$-substitution was a little tricky, so here is a different perspective.

We have $u = \sin^{-1}(x)$, so $du = \frac{1}{\sqrt{1-x^2}}dx$. 


As \( \sin^{-1}(x) \) does not appear in the integrand, the problem is a little non-standard. But the idea is the same as always: we want to convert every \( x \) appearing in the problem by a \( u \). Since we have an inverse function here, there is a great way to do that: just write \( x = \sin(u) \) at every stage.

We then have:

\[
\int \sqrt{1-x^2} \, dx = \int \sqrt{1-\sin^2(u)} \, dx = \int \sqrt{1-\sin^2(u)} \sqrt{1-x^2} \, du = \\
\int \frac{1-\sin^2(u)}{\sqrt{1-\sin^2(u)}} \sqrt{1-\sin^2(u)} \, du = \int (1-\sin^2(u)) \, du.
\]

Here in the second equality, we used \( du = \frac{1}{\sqrt{1-x^2}} \, dx \Rightarrow \sqrt{1-x^2} \, du = dx \).

From the perspective of either solution, the key thing that made this argument work is that we we had an inverse to the function \( u = u(x) = \sin^{-1}(x) \).

**Problem 4.** Find the area lying above the line \( y = \frac{1}{\sqrt{2}} \) and within a circle of radius 1 about the origin.

*Solution.* We prefer to rotate by 90° to ask about the area bound by the circle and the line \( x = \frac{1}{\sqrt{2}} \).

Then the previous solution allows us to calculate this integral as:

\[
\int_{\sqrt{2}}^{1} \sqrt{1-x^2} \, dx = \int_{\sin^{-1}(\frac{1}{\sqrt{2}})}^{\sin^{-1}(1)} \cos^2(u) \, du = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2(u) \, du = \\
\left( \frac{\cos(2u)}{4} + \frac{u}{2} \right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\cos(\pi)}{4} + \frac{\pi}{4} - \frac{\cos(\frac{\pi}{4})}{4} - \frac{\pi}{8} = \\
-\frac{1}{4} + \frac{\pi}{4} - 0 - \frac{\pi}{8} = \frac{\pi - 1}{8 - 4}.
\]