# 408L CLASS PROBLEMS

FEBRUARY 10TH, 2020

*Problem* 1. Find the volume of the solid by rotating the area under the graph of $\sqrt{x} \cdot e^{x^2}$ for $0 \leqslant x \leqslant 1$ around the $x$-axis.

*Solution.* The volume is:

$$\pi \int_0^1 (\sqrt{x} \cdot e^{x^2})^2 dx = \pi \int_0^1 x \cdot e^{2x^2} dx.$$

We evaluate the integral using $u$-substitution with $u = 2x^2$, $du = 4xdx$:

$$\int_0^1 x \cdot e^{2x^2} dx = \frac{1}{4} \int_0^2 e^u du = \frac{1}{4} e^u |_0^2 = \frac{1}{4}(e^2 - 1).$$

Reincorporating the factor of $\pi$, we have a final answer:

$$\boxed{\frac{\pi(e^2 - 1)}{4}}.$$

*Problem* 2. Find the volume of the solid obtained by rotating the area between the $x$-axis and the graph of $\cos(x)$ for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ about the line $y = 1$.

*Solution.* The cross-section for fixed $x$ of this solid is a washer with outer radius 1 and inner radius $1 - \cos(x)$.

The area of this washer is:

$$\pi 1^2 - \pi(1 - cos(x))^2 = \pi(2\cos(x) - \cos^2(x)).$$

Therefore, the volume of our solid is:

$$\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos(x) - \cos^2(x))dx = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx - \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(x)dx.$$

The first integral above is easy to calculate:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx = \sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2}) = 1 - (-1) = 2.$$

To calculate the second integral, we use the identity:

$$\cos^2(x) = \frac{\cos^2(x) - \sin^2(x) + \cos^2(x) + \sin^2(x)}{2} = \frac{\cos(2x)+1}{2}.$$

We obtain:

$$\int \cos^2(x)dx = \int \frac{\cos(2x)+1}{2} = \frac{\sin(2x)}{4} + \frac{x}{2}.$$

For our definite integral, we see:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(x)dx = \left(\frac{\sin(2x)}{4} + \frac{x}{2}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

Finally, we obtain:

$$2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx - \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(x)dx = 2\pi \cdot 2 - \pi \cdot \frac{\pi}{2} = \boxed{4\pi - \frac{\pi^2}{2}}.$$

*Problem* 3. In this question, we find the area of a circle of radius 1. (Of course you know the answer, but the point of the problem is to calculate it yourself.)

(1) Using the cross-section method, show that the area of a circle with radius 1 is:

$$2\int_{-1}^{1} \sqrt{1-x^2}dx.$$

(2) Evaluate this integral. (Hint: use $u$-substitution with $u = \sin^{-1}(x)$.)

*Solution.* The interior of a circle with radius 1 is all $(x, y)$ with $x^2 + y^2 \leqslant 1$. For a fixed value of $x$ between $-1$ and 1, this is all $y$ with $|y| \leqslant \sqrt{1-x^2}$. Clearly that is a line of length $2\sqrt{1-x^2}$.

To evaluate the integral, we instead use the perspective $\sin(u) = x$. Then $\cos(x)dx = du$, so we obtain:

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2(u)}\cos(u)du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du.$$

We evaluated this integral in the previous problem and found it was $\frac{\pi}{2}$.

Reincorporating the factor of 2, we find that the area of a circle is $\boxed{\pi}$, as expected.

*Alternative solution.* The above $u$-substitution was a little tricky, so here is a different perspective.

We have $u = \sin^{-1}(x)$, so $du = \frac{1}{\sqrt{1-x^2}}dx$.

As $\sin^{-1}(x)$ does not appear in the integrand, the problem is a little non-standard. But the idea is the same as always: we want to convert every $x$ appearing in the problem by a $u$. Since we have an inverse function here, there is a great way to do that: just write $x = \sin(u)$ at every stage.

We then have:

$$\int \sqrt{1-x^2}dx = \int \sqrt{1-\sin^2(u)}dx = \int \sqrt{1-\sin^2(u)}\sqrt{1-x^2}du =$$
$$\int \sqrt{1-\sin^2(u)}\sqrt{1-\sin^2(u)}du = \int (1-\sin^2(u))du.$$

Here in the second equality, we used $du = \frac{1}{\sqrt{1-x^2}}dx \Rightarrow \sqrt{1-x^2}du = dx$.

From the perspective of either solution, the key thing that made this argument work is that we we had an inverse to the function $u = u(x) = \sin^{-1}(x)$.

*Problem* 4. Find the area lying above the line $y = \frac{1}{\sqrt{2}}$ and within a circle of radius 1 about the origin.

*Solution.* We prefer to rotate by $90^\circ$ to ask about the area bound by the circle and the line $x = \frac{1}{\sqrt{2}}$.

Then the previous solution allows us to calculate this integral as:

$$\int_{\frac{1}{\sqrt{2}}}^{1} \sqrt{1-x^2}dx = \int_{\sin^{-1}(\frac{1}{\sqrt{2}})}^{\sin^{-1}(1)} \cos^2(u)du = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2(u)du =$$
$$(\frac{\cos(2u)}{4} + \frac{u}{2})\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\cos(\pi)}{4} + \frac{\pi}{4} - \frac{\cos(\frac{\pi}{2})}{4} - \frac{\pi}{8} =$$
$$-\frac{1}{4} + \frac{\pi}{4} - 0 - \frac{\pi}{8} = \boxed{\frac{\pi}{8} - \frac{1}{4}}.$$