408L CLASS PROBLEMS

FEBRUARY 12TH, 2020

Problem 1. Find $\int x^2 \cos(2x) dx$.

Solution. We use integration by parts with $u = x^2$, $v = \frac{1}{2}\sin(2x)$. We then have $du = 2xdx$ and $dv = \cos(2x)$, so:

$$\int x^2 \cos(2x) dx = \int u dv = uv - \int v du =$$
$$\frac{1}{2}x^2 \sin(2x) - \int x \sin(2x) dx.$$

We evaluate the integral in the last expression using integration by parts again: we take $u = x$, $v = -\frac{1}{2}\cos(2x)$, so $du = dx$ and $dv = \sin(2x)$. We then obtain:

$$\int x \sin(2x) dx = -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x) dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C.$$

Substituting into our earlier expression, we obtain:

$$\boxed{\frac{1}{2}x^2 \sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C}.$$

(Formally, we obtain $-C$ when we substitute. Why are we justified in changing the sign?)

Problem 2. Find $\int \frac{e^{\frac{1}{x}}}{x^3} dx$.

Solution. First, we use substitution. I will reserve the letter u for later in the problem, when we do integration by parts. Instead, I will use t for my new variable.

Set $t = \frac{1}{x}$, so $dt = -\frac{1}{x^2}dx$. We then obtain:

$$\int \frac{e^{\frac{1}{x}}}{x^3} dx = -\int te^t dt.$$

We can evaluation this integral using integration by parts. We set $u = t$ and $v = e^t$ to obtain:

$$\int te^t dt = te^t - \int e^t dt = (t-1)e^t + C.$$

Substituting back in, we obtain:

$$\int \frac{e^{\frac{1}{x}}}{x^3}dx = -\int te^t dt = (1-t)e^t = (1-\frac{1}{x})e^{\frac{1}{x}} = \boxed{\frac{x-1}{x}\cdot e^{\frac{1}{x}} + C}.$$

Alternative solution. First, we do integration by parts. We set $u = \frac{1}{x}$ and $v = -e^{\frac{1}{x}}$, so $du = -\frac{1}{x^2}$ and $dv = \frac{1}{x^2}e^{\frac{1}{x}}$. We obtain:

$$\int \frac{e^{\frac{1}{x}}}{x^3}dx = \int udv = uv - \int vdu =$$
$$-\frac{1}{x}e^{\frac{1}{x}} - \int \frac{e^{\frac{1}{x}}}{x^2}dx.$$

The latter integral may be solved by substitution, taking $t = \frac{1}{x}$ as before.

Problem 3. Find $\int \log(x)dx$.

Solution. Suppose $f(x)$ is a function where we know how to integrate $x \cdot f'(x)$. Then a good way to find $\int f(x)dx$ is with integration by parts, taking $u = f(x)$ and $v = x$.

This is the case for $f(x) = \log(x)$, since $x \cdot f'(x) = 1$ in this case.

Therefore, we take $u = \log(x)$ and $v = x$ and apply integration by parts:

$$\int \log(x)dx = x\log(x) - \int x \cdot \frac{1}{x}dx = x\log(x) - \int dx = \boxed{x\log(x) - x + C}.$$

Problem 4. Find $\int e^x \sin(x)dx$.

Solution. Here is the idea: if we do integration by parts, we can relate this integral to $\int e^x \cos(x)dx$. We can then do integration by parts again, to obtain a further relation to $\int e^x \sin(x)dx$. We will then do some algebra to rearrange the resulting equation and be able to solve for $\int e^x \sin(x)dx$.

To implement this strategy, we take $u = e^x$ and $v = \sin(x)$. We apply integration by parts to obtain:

$$\int e^x \cos(x)dx = e^x \sin(x) - \int e^x \sin(x)dx.$$

We then apply integration by parts again with $u = e^x$ and $v = -\cos(x)$. This yields:

$$\int e^x \sin(x)dx = -e^x \cos(x) + \int e^x \cos(x)dx.$$

Substituting back into our earlier equation, we obtain:

$$\int e^x \cos(x)dx = e^x \sin(x) - \int e^x \sin(x)dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)dx.$$

We deduce:

$$2 \cdot \int e^x \cos(x)dx = e^x \cdot (\sin(x) + \cos(x)) \Rightarrow$$
$$\int e^x \cos(x)dx = \boxed{\frac{1}{2} \cdot e^x \cdot (\sin(x) + \cos(x)) + C}.$$