408L THIRD MIDTERM REVIEW

MAY 4TH, 2020

Subjects we have covered so far:

- Sequences
  - Pattern recognition
  - Convergence
    - Q: Does the sequence $a_n = \sin(\frac{1}{n})$ converge? To what?
    - A: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin(\frac{1}{n}) = \sin(0) = 0$.
    - L'Hôpital's rule often useful.
- Series (everything else we did since the last midterm)
  - Σ-notation.
  - Partial sums.
- Telescoping series.
  - Works for series like $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.
  - Find the partial fractions decomposition $\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$.
  - Series telescopes: when you find partial sums, you find cancellations occurring.
- Geometric series
  - $1 + r + r^2 + \ldots = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.
  - Converges for $|r| < 1 \Leftrightarrow -1 < r < 1$.
  - Useful whenever we're summing nth powers.
  - Useful for finding Taylor series.
    - Gives the Taylor series of $f(x) = \frac{1}{1-x}$.
    - Use substitution to find Taylor series of $f(x) = \frac{1}{1+x}$ or $\frac{1}{1-x^2}$, etc.
    - Take the derivative to find the Taylor series of $f(x) = \frac{1}{(1-x)^2}$.
    - Integrate to find the Taylor series of $f(x) = \log(1-x)$.
- Convergence, and tests for it. Q: When does $\sum_{n=0}^{\infty} a_n$ converge?
  - Divergence test: if $\lim_{n\to\infty} a_n \neq 0$, then sum diverges.
    - Converse does not hold. Just because $\lim_{n\to\infty} a_n = 0$, this is not enough to conclude series converges. (E.g., the harmonic series diverges even though $\lim \frac{1}{n} = 0$.)
  - Integral test: if $f(x)$ is positive and decreasing and $a_n = f(n)$, then sum converges $\Leftrightarrow \int_1^{\infty} f(x)dx$ converges.
    - Know the picture for why the integral test works.
    - Use integral test to estimate sums.
  - p-series test: $\sum \frac{1}{n^p}$ converges $\Leftrightarrow p > 1$. Important special case: $p = 1$, get that $\sum \frac{1}{n}$ diverges.

- Root test: $\lim |a_n|^{\frac{1}{n}} < 1$, then series converges absolutely. (Indeterminate when $\lim |a_n|^{\frac{1}{n}} = 1$.)
  Often used when you have $a_n = b_n^n$. Something like $a_n = (1 + \frac{1}{n})^n$.
- Ratio test: $\lim \frac{|a_{n+1}|}{|a_n|} < 1$, then series converges absolutely. (Indeterminate when $\lim \frac{|a_{n+1}|}{|a_n|} = 1$.)
  Often used when you see $n!$ or 2^n.
- Alternating series test. If a_n is alternating (positive - negative - positive - negative) and $\lim_{n\to\infty} a_n = 0$, then the series converges (maybe not absolutely).
- Limit comparison test.

• Estimating values of functions is often useful for applying the comparison test and the divergence test.
  - General principle: n^3 is much larger than n^2 for n very large. So $a_n = \frac{1}{n^3+n^2}$, compare to $b_n = \frac{1}{n^3}$ (and deduce convergence by p-series test).
  - If you have a function $a_n = f(\frac{1}{n})$, use Taylor series of f around 0 for comparison test purposes.

• Power series
  - Radius and interval of convergence.
    * If you have a power series $\sum a_n(x-1)^n$ around 1, then the series converges on an interval that "looks like" $(0,2)$, $[0,2)$ $(0,2]$, $[0,2]$ if the radius of convergence is 1.
    * Always: for radius of convergence R, a power series $\sum a_n(x-a)^n$ about a converges for sure on $(a-R, a+R)$, check endpoints $a-R$ and $a+R$ separately.
    * To find radius of convergence R, usually use the root test.
  - Taylor series: write $f(x)$ as $\sum_{n=0}^{\infty} a_n(x-a)^n$ for Taylor series based at a.
    * $a_0 = f(a)$, $a_1 = f'(a)$, and $a_n = \frac{1}{n!} f^{(n)}(a)$.
    * Taylor series to know by heart:
      · Geometric series: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.
        (See above for how to derive other Taylor series from this one.)
      · $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
      · $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$
      · $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \ldots$
  - Taylor polynomials.
    * These are just the sums of the first few terms in the Taylor series.
    * Often use $a_n = \frac{1}{n!} f^{(n)}(a)$ formula to find the coefficients.
    * Use these for estimating values of our functions, or integrals, etc. To do that, just replace a function by its degree n Taylor polynomial.