A GENERALIZATION OF THE b-FUNCTION LEMMA

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Abstract. We establish some cohomological bounds in D-module theory that are known in the holonomic case and folklore in general. The method rests on a generalization of the b-function lemma for non-holonomic D-modules.

Contents

1. Introduction

1.1. This note studies how D-module operations interact with singular support. The main technical result, Theorem 2.5.1, shows that D-module operations preserve a certain numerical obstruction to holonomicity. This result generalizes the usual preservation of holonomic D-modules under such operations, which is essentially equivalent to the b-function lemma: see [Kas] or [Ber1].

1.2. **Affine morphisms.** As an application, we show in Theorem 3.3.1 that $f_!$ is left t-exact for an affine morphism $f : X \to Y$.

This is certainly an old folklore result. Of course it is standard for holonomic D-modules, where it is a consequence of the usual b-function lemma. It is also easy to show for $Y = \mathrm{Spec}(k)$, or for a map of curves (e.g., an open embedding). Otherwise, it does not seem to follow from existing foundational results in the literature, which is quite surprising for something so basic.

We remark that the formulation of this result does not quite make sense, since $f_!(\mathcal{F})$ does not typically make sense as a D-module (although it always does if $\mathcal{F}$ is holonomic). One can rectify this in one of two ways: one can ask to show that if $\mathcal{F}$ is in cohomological degrees $\geqslant 0$ and $f_!(\mathcal{F})$ is defined, $f_!(\mathcal{F})$ is in degrees $\geqslant 0$; or one can work with pro-complexes. We use the latter technique, since it is somewhat more general. For technical reasons, we only work with coherent D-modules $\mathcal{F}$.

Applying this result for non-holonomic D-modules is actually useful in geometric representation theory. The point is that in many settings typical of the subject, $f_!$ is defined on some non-holonomic D-modules of interest even when f is affine. For example, this occurs for the Fourier-Deligne transform, and the results here can be used to show its t-exactness in a conceptual way.[1] For an application of such results to infinite-dimensional Lie theory, see [Ras] or Appendix **??** below.

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[1] C.f. [Gai2] §1.8. Note that *loc. cit.* implicitly assumes the left t-exactness of $f_!$ for affine f.

1.3. **Notation.** We let k denote a field of characteristic zero. We use the phrase "category" to mean ∞-category wherever appropriate. (This language is used only very mildly.)

By a *variety*, we mean a reduced, separated, finite type k-scheme.

For X a variety over k, we let $D(X)$ denote the DG category of D-modules on X. We let $D(X)^{\geqslant i}$ and $D(X)^{\leqslant i}$ respectively denote the subcategories of complexes $\mathcal{F} \in D(X)$ with $H^j(\mathcal{F}) = 0$ for $j < i$ and $j > i$ respectively. We let $D(X)^{\heartsuit} = D(X)^{\geqslant 0} \cap D(X)^{\leqslant 0}$ denote the heart of the t-structure, i.e., the abelian category of D-modules. We let $\tau^{\geqslant i}$ and $\tau^{\leqslant i}$ denote the corresponding truncation functors.

For $f : X \to Y$, we let $f^! : D(Y) \to D(X)$ and $f_{*,dR} : D(X) \to D(Y)$ denote the D-module pullback and pushforward operations. We let $f_!$ and $f^{*,dR}$ denote their left adjoints where appropriate.

We let $D(X)^c \subseteq D(X)$ denote the DG subcategory of coherent complexes, i.e., bounded complexes with coherent (i.e., locally finitely generated) cohomology groups. Recall that $D(X)$ is compactly generated, i.e., $D(X) = \mathsf{Ind}(D(X)^c)$. We let $\mathbb{D} : D(X)^c \xrightarrow{\simeq} D(X)^{c,op}$ denote the Verdier duality functor.

1.4. **Acknowledgements.** We are grateful to Dennis Gaitsgory, Victor Ginzburg, and Masaki Kashiwara for useful correspondence on the subject of this note. The methods owe a great deal to [Ber1], [Gin], and [Kas].

2. Holonomic defect

2.1. In this section, we introduce a generalization of the holonomic condition on a D-module and show that it is preserved under D-module operations.

The method is standard. The main point is Lemma 2.7.1, which is a generalization of the fact that pushforward along an open embedding preserves holonomic objects, which is essentially equivalent to the usual b-function lemma. The main difference is that we cannot use finite length methods. See also Remark 2.7.2 for indications on a different approach.

The presentation is based on [Kas] and [Gin].

2.2. **Gabber-Kashiwara-Sato (GKS) filtration.** We begin by reviewing some material from [Gin] §1.

Let X be a variety and let $\mathcal{F} \in D(X)^{\heartsuit}$ be a given D-module.

Definition 2.2.1. For an integer i, we let:

$$F_i^{GKS}\mathcal{F} := \text{Image}(H^0(\mathbb{D}\tau^{\geqslant -i}\mathbb{D}\mathcal{F}) \to \mathcal{F}).$$

Remark 2.2.2. By definition, $\mathbb{D}\tau^{\geqslant -i}\mathbb{D}\mathcal{F} \in D(X)$ means the obvious thing if $\mathcal{F}$ is coherent, and in general, we understand this expression to commute with filtered colimits. (It is equivalent interpret this more literally and consider $\mathbb{D}$ as an equivalence between $D(X)$ and the DG category of pro-coherent D-modules, equipped with the t-structure of §3.2.)

Note that $F_\bullet^{GKS}$ is an increasing filtration on $\mathcal{F}$. Because $\mathbb{D}\mathcal{F}$ is in cohomological degrees $[-\dim X, 0]$, we have $F_i^{GKS}\mathcal{F} = 0$ for $i < 0$, and $F_i^{GKS}\mathcal{F} = \mathcal{F}$ for $i \geqslant \dim X$. Formation of the GKS filtration is functorial for D-module morphisms, i.e., a map $\mathcal{F}_1 \to \mathcal{F}_2 \in D(X)^{\heartsuit}$ sends $F_i^{GKS}\mathcal{F}_1$ to $F_i^{GKS}\mathcal{F}_2$.

Note that if $\mathcal{F} = \text{colim}_j\, \mathcal{F}_j$ is a filtered colimit in $D(X)^{\heartsuit}$, then $F_i^{GKS}\mathcal{F} = \text{colim}_j\, F_i^{GKS}\mathcal{F}_j$.

Lemma 2.2.3. *Formation of $F_\bullet^{GKS}$ commutes with open restriction and pushforwards along closed embeddings.*

Proof. Each of these functors is t-exact and commutes with Verdier duality. □

Therefore, many results about this filtration reduce to the case of smooth X by taking Zariski local closed embeddings into affine space. The key property in the smooth case is:

Theorem 2.2.4. *If X is smooth, then a local section s of $\mathcal{F}$ lies in $F_i^{GKS}\mathcal{F}$ if and only if the D-module generated by it has singular support with dimension $\leqslant \dim X + i$.*[2]

See [Gin] Proposition V.14. Note that it is equivalent to say that $F_i^{GKS}\mathcal{F}$ is the maximal submodule of $\mathcal{F}$ with singular support of dimension $\leqslant \dim X + i$.

2.3. **Holonomic defect.** For $\delta \in \mathbb{Z}^{\geqslant 0}$, we say $\mathcal{F} \in D(X)^{\heartsuit}$ has *holonomic defect* δ if $F_\delta^{GKS}\mathcal{F} = \mathcal{F}$.

Remark 2.3.1. If $\mathcal{F}$ has holonomic defect δ, then it also has holonomic defect $1 + \delta$.

Example 2.3.2. A coherent D-module $\mathcal{F}$ has holonomic defect 0 if and only if $\mathcal{F}$ is holonomic. Indeed, this follows by reduction to the smooth case and Theorem 2.2.4.

Example 2.3.3. Every $\mathcal{F}$ has holonomic defect $\dim X$.

Example 2.3.4. If X is smooth and $\mathcal{F}$ is coherent, then by Theorem 2.2.4, $\mathcal{F}$ has holonomic defect δ if and only if $\mathcal{F}$ has singular support with dimension $\leqslant \dim X + \delta$.

Lemma 2.3.5. *The subcategory of $D(X)^{\heartsuit}$ consisting of objects with holonomic defect δ is closed under submodules, quotient modules, and extensions.*

Proof. The argument reduces to the case of X smooth, and then follows from Theorem 2.2.4 and standard facts about singular support. □

Lemma 2.3.6. *Holonomic defect is preserved under filtered colimits, and $\mathcal{F} \in D(X)^{\heartsuit}$ has holonomic defect δ if and only if $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ with $\mathcal{F}_i$ coherent of holonomic defect δ.*

Proof. The first part is clear since formation of $F_\bullet^{GKS}$ commutes with filtered colimits. For the second part, write $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i'$ with $\mathcal{F}_i'$ coherent, and then set $\mathcal{F}_i = F_\delta^{GKS}\mathcal{F}_i'$. □

2.4. More generally, for $\mathcal{F} \in D(X)$ a complex of D-modules, we say that $\mathcal{F}$ has *holonomic defect* δ if all of its cohomology groups do. By Lemmas 2.3.5 and 2.3.6, this defines a DG subcategory of $D(X)$ closed under colimits.

2.5. The following is the main result of this section.

Theorem 2.5.1. *Holonomic defect is preserved under D-module operations. That is, if $f : X \to Y$ is a morphism and $\mathcal{F} \in D(X)$ (resp. $\mathcal{G} \in D(Y)$) has holonomic defect δ, then $f_{*,dR}(\mathcal{F})$ (resp. $f^!(\mathcal{G})$) does as well. Moreover, for $\mathcal{F}$ coherent as above, $\mathbb{D}\mathcal{F}$ has holonomic defect δ as well.*

This theorem generalizes the preservation of holonomic objects under D-module operations, so the proof must follow similar lines. It is given below.

2.6. **Verdier duality.** The compatibility with Verdier duality in Theorem 2.5.1 is well-known. Indeed, the result immediately reduces to X being smooth, and then we have:

Proposition 2.6.1. *For $\mathcal{F} \in D(X)^{\heartsuit}$ with singular support of dimension $\leqslant \dim X + i$, we have $H^{-j}\mathbb{D}\mathcal{F} = 0$ unless $0 \leqslant j \leqslant i$. Moreover, $H^{-j}\mathbb{D}\mathcal{F}$ has holonomic defect j.*

See e.g. [Kas] Theorem 2.3.

[2]We regard $\dim X$ as a locally constant function on X if X is not equidimensional.

2.7. **Affine open embeddings.** The main case of Theorem 2.5.1 is pushforward along an open embedding.

Lemma 2.7.1. *Let X be a connected, smooth variety and let $f : X \to \mathbb{A}^1$ be a function. Let $U = \{f \neq 0\}$ be the corresponding basic open and let $j : U \hookrightarrow X$ denote the corresponding affine open embedding.*

Then $j_{*,dR}$ preserves holonomic defect.

Remark 2.7.2. The argument that follows is a version of the standard proof in the holonomic setting via b-functions. Victor Ginzburg communicated to us that the proof of the holonomic version of Lemma 2.7.1 via the Bernstein filtration and Hilbert polynomials has a more straightforward generalization. This argument, which follows early work [Ber2] in the subject, is easily extracted from [HTT] §3.2.2.

Proof of Lemma 2.7.1.

Step 1. We may obviously assume X is connected and affine and that f is non-constant.

We abuse notation slightly in letting D_X and D_U denote the respective rings of differential operators (as opposed to the sheaves of differential operators).

Let $\mathcal{F}$ be a D_U-module. Because we are working with modules rather than sheaves, considering $\mathcal{F}$ as a D_X-module by restriction is the same as considering the sheaf $j_{*,dR}(\mathcal{F}) \in D(X)^{\heartsuit}$.

For $s \in \mathcal{F}$, we write $\mathrm{SS}_U(s) \subseteq T^*U$ for the singular support of $D_U \cdot s$ and $\mathrm{SS}_X(s) \subseteq T^*X$ for the singular support of $D_X \cdot s$. Note that $\mathrm{SS}_X(s)|_{T^*U} = \mathrm{SS}_U(s)$. We always understand singular support as a *reduced* subscheme.

We want to show that if every section $s \in \mathcal{F}$ has $\dim \mathrm{SS}_U(s) \leqslant \dim U + \delta = \dim X + \delta$, then the same is true of $\dim SS_X(s)$.

Step 2. First, we observe that there is a D_X-submodule $\mathcal{G} \subseteq \mathcal{F}$ such that every section of $\mathcal{G}$ has singular support with dimension $\leqslant \dim X + \delta$, and which is a *lattice*, i.e., $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_U \xrightarrow{\simeq} \mathcal{F}$.

Indeed, we can take $\mathcal{G} = F_{\delta}^{GKS}\mathcal{F}$, where the GKS filtration is with $\mathcal{F}$ considered as a D_X-module. Because the GKS filtration commutes with open restriction, we must have $\mathcal{F}_0|_U = \mathcal{F}$.

(Note that by Theorem 2.2.4, we are trying to show that $\mathcal{G} = \mathcal{F}$.)

Step 3. Let λ be an indeterminate. We write $\mathbb{A}^1_\lambda$ for $\mathrm{Spec}(k[\lambda])$. We let k' denote the fraction field $k(\lambda)$ of $k[\lambda]$. We use similar notation for a base-change to k'; e.g., X', or $\mathcal{F}'$, etc. We always consider X' and U' as schemes over k', so e.g. their cotangent bundles are understood relative to k', and $D'_X = D_{X'}$.

Recall that $U := \{f \neq 0\}$. Then we have the D'_U-module "f^λ" $\cdot\, \mathcal{F}'$, the tensor product of the usual D-module "f^λ" with $\mathcal{F}'$.

Step 4. We first show that the result is true for "f^λ" $\cdot\, \mathcal{F}'$, i.e., that every section has $SS_{X'}$ with dimension $\leqslant \dim X + \delta$.

First, note that the singular support in U' of any section has dimension $\leqslant \dim X + \delta$: this follows because "$f^\lambda$" is lisse on U'.

We have a canonical element of the Galois group $\gamma \in \mathrm{Gal}(k'/k)$ sending $\lambda \mapsto \lambda + 1$. Of course, anything obtained by extension of scalars from k to k' also carries such an automorphism γ, in particular, D'_X does (it sends a differential operator $P(\lambda)$ to $P(\lambda+1)$).

Similarly, $\mathcal{F}'$ has such an automorphism: note that this is not an automorphism as a D'_X-module, but rather intertwines the standard action with the one obtained by twisting by the automorphism γ of D'_X. That is, $\gamma(P \cdot s) = \gamma(P) \cdot \gamma(s)$ for $P \in D'_X$ and $s \in \mathcal{F}'$.

Define γ on the D'_X-module "f^λ" by setting:

$$\gamma(``f^{\lambda}" \cdot g) = ``f^{\lambda+1}" \cdot \gamma(g) := ``f^{\lambda}" \cdot f \cdot \gamma(g)$$

for g a function on X'. Again, this morphism intertwines the actions of D'_X up to the automorphism γ of D'_X.

Tensoring, we obtain an automorphism γ of "f^{λ}" $\cdot\, \mathcal{F}'$ with similar semi-linearity.

By the semi-linearity, we have:

$$\mathrm{SS}_{X'}(``f^{\lambda}" \cdot s) = \gamma \cdot \mathrm{SS}_{X'}(\gamma(``f^{\lambda}" \cdot s))$$

where we are using γ to indicate the induced automorphism of T^*X'. In particular, we find that $\dim \mathrm{SS}_{X'}(``f^{\lambda}" \cdot s) = \dim \mathrm{SS}_{X'}(\gamma(``f^{\lambda}" \cdot s))$.

Now let $\mathcal{G} = F_{\delta}^{GKS}(``f^{\lambda}" \cdot \mathcal{F}')$,[3] where the GKS filtration is taken with "$f^{\lambda}$" $\cdot\, \mathcal{F}'$ considered as a $D'_X$-module. By the above and Theorem 2.2.4, "$f^{\lambda}$" $\cdot\, s \in \mathcal{G}$ if and only if $\gamma(``f^{\lambda}" \cdot s) \in \mathcal{G}$.

For any $s \in \mathcal{F}$ (as opposed to $\mathcal{F}'$), we clearly have $\gamma(``f^{\lambda}" \cdot s) = f^{\lambda+1} \cdot s$. Since $\mathcal{G}$ is a lattice (by Step 2), $\gamma^N(s) = f^{\lambda+N} s \in \mathcal{G}$ for $N \gg 0$. But by the above, this means that $s \in \mathcal{G}$. Since "f^{λ}" $\cdot\, \mathcal{F}'$ is k'-spanned by such vectors, this means that $\mathcal{G} = ``f^{\lambda}" \cdot \mathcal{F}'$, as desired.

Step 5. We now show that the result is true for our original $\mathcal{F}$. Let $s \in \mathcal{F}$; we want to show $\dim \mathrm{SS}_X(s) \leqslant \dim X + \delta$.

We now write "f^{λ}" $\cdot\, \mathcal{F}$ for the corresponding $D_X[\lambda]$-module (as opposed to the fiber over the generic point in $\mathbb{A}^1_{\lambda}$, which is what we called by this name previously). Note that "$f^{\lambda} \cdot \mathcal{F}$" $= \mathcal{F} \otimes_k k[\lambda]$ as a $\mathcal{O}_X[\lambda]$-module.

Let $\mathcal{F}_0$ be the $D_X[\lambda]$ submodule generated by "f^{λ}" $\cdot\, s$. Give $\mathcal{F}_0$ the filtration $F_i\mathcal{F}_0 = D_X^{\leqslant i}[\lambda] \cdot$ "f^{λ}" $\cdot\, s$, where $D_X^{\leqslant i}$ are differential operators of order $\leqslant i$.

Then $\mathrm{gr}_{\bullet}(\mathcal{F}_0)$ is the structure sheaf of some closed subscheme $Z \subseteq T^*X \times \mathbb{A}^1_{\lambda}$. We have seen that the base-change of Z to the generic point of $\mathbb{A}^1_{\lambda}$ has dimension $\leqslant \dim X + \delta$, so the same is true for its fibers at closed points with only finitely many possible exceptions.

Choose a negative integer $-N$ not among this finite number of exceptions. Then the coherent D_X-module $\mathcal{F}_0/(\lambda + N)$ has singular support contained in $Z \times_{\mathbb{A}^1_{\lambda}} \{-N\}$, so has dimension $\leqslant \dim X + \delta$. We have the obvious morphism of D_U-modules (in particular, of D_X-modules):

$$\big(``f^{\lambda}" \cdot \mathcal{F}\big)/(\lambda + N) \to \mathcal{F}$$
$$\sum_{i=0}^{r} ``f^{\lambda}" \cdot \sigma_i \lambda^i \mapsto \sum_{i=0}^{r} f^{-N} \cdot \sigma_i \cdot (-N)^i$$

induces a map $\mathcal{F}_0/(\lambda+N)$ to $\mathcal{F}$ sending the generator to $f^{-N}s$. By functoriality of the GKS filtration (or standard singular support analysis), this means that $f^{-N}s \in F_{\delta}^{GKS}\mathcal{F}$, and since $F_{\delta}^{GKS}\mathcal{F}$ is a D_X-module, this means that $s \in F_{\delta}^{GKS}\mathcal{F}$ as well.

□

2.8. **Preservation of holonomic defect.** We now proceed to prove Theorem 2.5.1. The argument is straightforward at this point, and we proceed in cases.

2.9. First, we treat pushforwards along an open embedding $j : U \to X$.

For X smooth, a Cech argument reduces us to the case of a basic open, which is treated in Lemma 2.7.1. (Recall from §2.4 that D-modules with holonomic defect δ are closed under cones.)

[3] So this is a different $\mathcal{G}$ from Step 2, i.e., we are applying the same construction to a different D-module.

For possibly non-smooth X, note that the problem is Zariski local, so we may assume X is affine. Take a closed embedding $X \subseteq \mathbb{A}^N$. If $U = X \backslash Z$, then we have $U \hookrightarrow \mathbb{A}^N \backslash Z \hookrightarrow \mathbb{A}^N$ with the first map being closed and the second being open. Therefore, this pushforward preserves holonomic defect. Clearly this implies the result for the pushforward along $U \hookrightarrow X$.

2.10. Next, we treat restrictions to closed subschemes.

Let $i : Z \hookrightarrow X$ be closed and let $j : U = X \backslash Z \hookrightarrow X$. Then we have an exact triangle:

$$i_{*,dR} i^!(\mathcal{F}) \to \mathcal{F} \to j_{*,dR} j^!(\mathcal{F}) \xrightarrow{+1} .$$

If $\mathcal{F}$ has holonomic defect δ, we have shown the same for $j_{*,dR} j^!(\mathcal{F})$, so $i_{*,dR} i^!(\mathcal{F})$ has holonomic defect δ, which is equivalent to $i^!(\mathcal{F})$ having holonomic defect δ.

2.11. We can now show the result for restrictions in general.

If $f : X \to Y$ is smooth of relative dimension d, then $f^{*,dR}[d] = f^![-d]$ commutes with Verdier duality and is t-exact. Therefore, it commutes with formation of the GKS filtration, and therefore preserves holonomic defect.

The case of general $f : X \to Y$ is immediately reduced to the case of affine varieties (since holonomic defect is Zariski local). We can find a commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{A}^{N_1} \\ \downarrow f & & \downarrow g \\ Y & \longrightarrow & \mathbb{A}^{N_2} \end{array} \tag{2.11.1}$$

with the horizontal arrows being closed embeddings. This reduces to the case where X and Y are smooth.

Then we can factor f through the graph as $X \to X \times Y \xrightarrow{p_1} Y$. The former map is a closed embedding, and the latter is smooth because X is. We have treated each of these cases, so we obtain the result.

2.12. Next, we treat pushforwards along a proper morphism $f : X \to Y$ between smooth varieties.

This case does not need the work we have done so far. Let $\mathcal{F} \in D(X)^\heartsuit$ with holonomic defect δ be given. By Lemma 2.3.6, we may assume $\mathcal{F}$ is coherent, so the hypothesis is that $\mathcal{F}$ has singular support $\mathrm{SS}_X(\mathcal{F})$ with dimension $\dim X + \delta$.

Recall that $\mathrm{SS}_Y(H^i(f_{*,dR}(\mathcal{F})))$ is bounded in terms of $\mathrm{SS}_X(\mathcal{F})$. More precisely, if we take the diagram:

$$\begin{array}{ccc} T^*Y \underset{Y}{\times} X & \xrightarrow{\alpha} & T^*X \\ \downarrow \beta & & \\ T^*Y & & \end{array}$$

then the singular support of these cohomologies are contained in $\alpha(\beta^{-1} \mathrm{SS}_X(\mathcal{F}))$ (see e.g. [Kas] Theorem 4.2).

Because $\mathrm{SS}(\mathcal{F})$ is coisotropic by [Gab], we have:

$$\dim \alpha(\beta^{-1} \mathrm{SS}_X(\mathcal{F})) \leqslant \dim(\mathrm{SS}_X(\mathcal{F})) + \dim Y - \dim X$$

by usual symplectic geometry. This immediately gives the claim.

2.13. Now observe that preservation of holonomic defect under pushforward along a general morphism $f : X \to Y$ of smooth varieties follows: by Nagata and resolution of singularities,[4] we may find smooth $\overline{X}$ and a factorization:

$$X \xrightarrow{j} \overline{X} \xrightarrow{\overline{f}} Y$$

of f with $\overline{f}$ proper and j an open embedding, so we are reduced to our previous work.

2.14. We can now treat a general pushforward along $f : X \to Y$ a morphism between possibly singular varieties.

Because we know pushforward along open embeddings preserves holonomic defect, Cech reduces us to the case where X and Y are affine. Then we can find a commutative diagram (2.11.1) as before. This reduces to the case with X and Y smooth, which we have already treated.

3. Cohomological bounds

3.1. The main result of this section says that $f_!$ is left t-exact for an affine morphism f. We also show that for $i : X \to Y$ a closed embedding, $i^{*,dR}$ has cohomological amplitude $\geqslant -\dim(Y) + \dim(X)$, i.e., $i^{*,dR}[-\dim(Y) + \dim(X)]$ is left t-exact. Since $f_!$ and $i^{*,dR}$ are not defined on every D-module (e.g., on non-holonomic ones), we use the language of pro-categories to formulate this result.

3.2. **Pro-categories.** For $\mathcal{C}$ a[5] category, we have $\mathsf{Pro}(\mathcal{C})$ the corresponding pro-category. If $\mathcal{C}$ is a DG category, $\mathsf{Pro}(\mathcal{C})$ is as well. If $\mathcal{C}$ admits small colimits, then so does $\mathsf{Pro}(\mathcal{C})$. For $F : \mathcal{C} \to \mathcal{D}$, there is an induced functor $\mathsf{Pro}(\mathcal{C}) \to \mathsf{Pro}(\mathcal{D})$, which we denote again by F where there is no risk for confusion.

For any functor $G : \mathcal{D} \to \mathcal{C}$ commuting with finite colimits (e.g., a DG functor), the induced functor $\mathsf{Pro}(\mathcal{D}) \to \mathsf{Pro}(\mathcal{C})$ admits a left adjoint F. We say that F *is defined* on an object $\mathcal{F} \in \mathcal{C}$ if $F(\mathcal{F}) \in \mathcal{D} \subseteq \mathsf{Pro}(\mathcal{D})$. (This coincides with the usual notion of a left adjoint being defined on some object.)

If $\mathcal{C}$ is a DG category equipped with a t-structure, the $\mathsf{Pro}(\mathcal{C})$ inherits one as well. It is characterized by the equality $\mathsf{Pro}(\mathcal{C})^{\leqslant 0} = \mathsf{Pro}(\mathcal{C}^{\leqslant 0})$. Truncation functors are the pro-extensions of the truncation functors on $\mathcal{C}$. In particular, we find that $\mathcal{C}$ is closed under truncations and inherits its given t-structure. We also find that $\mathsf{Pro}(\mathcal{C})^{\geqslant 0} = \mathsf{Pro}(\mathcal{C}^{\geqslant 0})$: if $\mathcal{F} = \lim_i \mathcal{F}_i \in \mathsf{Pro}(\mathcal{C})^{\geqslant 0}$, then $\mathcal{F} = \tau^{\geqslant 0}\mathcal{F} = \lim_i \tau^{\geqslant 0}\mathcal{F}_i$.

3.3. **Affine morphisms.** For $f : X \to Y$, we have the functor $f_! : \mathsf{Pro}(D(X)) \to \mathsf{Pro}(D(Y))$ left adjoint to $f^!$.

Theorem 3.3.1. *For f affine, the induced functor $f_! : D(X)^c \to \mathsf{Pro}(D(Y))$ is left t-exact.*

Proof. The problem is[6] Zariski local on Y, so we may assume X and Y are affine.

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[4]Of course, one may easily use the more elementary de Jong alterations instead.

[5]Really $\mathcal{C}$ should be *accessible*. Recall that this is a robust set-theoretic condition satisfied by any small category and by any compactly generated category. One should be aware that $\mathsf{Pro}(\mathcal{C})$ is almost never accessible itself.

[6]Indeed, if $Y = U_1 \cup U_2$ with embeddings $j_i : U_i \hookrightarrow Y$ and $j_{12} : U_1 \cap U_2 \hookrightarrow Y$, then for $\mathcal{G} \in \mathsf{Pro}(D(Y))$ with $j_i^!(\mathcal{G}) \in \mathsf{Pro}(D(U_i))^{\geqslant 0}$, we want to see that $\mathcal{G} \in \mathsf{Pro}(D(Y))^{\geqslant 0}$.

Note that:

$$\mathcal{G} = \mathrm{Ker}\left(j_{1,*,dR}j_1^!(\mathcal{G}) \oplus j_{2,*,dR}j_2^!(\mathcal{G}) \to j_{12,*,dR}j_{12}^!(\mathcal{G})\right).$$

Indeed, this follows by pro-extension from the corresponding fact for usual D-modules. Since t-exact functors induce t-exact functors on pro-categories as well, we obviously obtain the claim.

Note that D-module pushforward along closed embeddings remains fully-faithful on pro-categories: the identity $i^! i_{*,dR} = \mathrm{id}$ induces the same for the pro-functors. Therefore, the same argument as in §2.11 allows us to assume X and Y are smooth.

Recall that we have a Verdier duality equivalence $\mathbb{D} : D(X) \xrightarrow{\simeq} \mathsf{Pro}(D(X)^c)$ induced by the usual Verdier duality equivalence $\mathbb{D} : D(X)^c \xrightarrow{\simeq} D(X)^{c,op}$, and similarly for Y.

We then claim that:

$$f_!(\mathcal{F}) = \mathbb{D} f_{*,dR} \mathbb{D}(\mathcal{F}).$$

This follows formally from the fact that $f_{*,dR}$ and $f^!$ are *dual* functors in the sense of [Gai1], but here is a direct proof anyway. Note that in this formula, $f_{*,dR}\mathbb{D}(\mathcal{F}) \in D(Y)$, and we are then using $\mathbb{D}$ to convert it to a pro-coherent object. Since this object is pro-coherent, it suffices to observe that for $\mathcal{G} \in D(Y)^c$, we have:

$$\mathrm{Hom}_{\mathsf{Pro}(D(Y)^c)}(\mathbb{D} f_{*,dR}\mathbb{D}(\mathcal{F}), \mathcal{G}) = \mathrm{Hom}_{D(Y)}(\mathbb{D}\mathcal{G}, f_{*,dR}\mathbb{D}(\mathcal{F})) = \Gamma_{dR}(Y, f_{*,dR}\mathbb{D}(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}) =$$
$$\Gamma_{dR}(X, \mathbb{D}(\mathcal{F}) \overset{!}{\otimes} f^!(\mathcal{G})) = \mathrm{Hom}_{D(X)}(\mathcal{F}, f^!(\mathcal{G})).$$

Here Γ_{dR} is the complex of de Rham cochains of a D-module, and we are repeatedly using the formula that if $\mathcal{F}_1$ is coherent, then:

$$\mathrm{Hom}_{D(X)}(\mathcal{F}_1, \mathcal{F}_2) = \Gamma_{dR}(X, \mathbb{D}(\mathcal{F}_1) \overset{!}{\otimes} \mathcal{F}_2).$$

Note that $\mathbb{D}\mathcal{F}$ carries the canonical filtration with subquotients $H^{-j}\mathbb{D}\mathcal{F}[-j]$.

By Proposition 2.6.1, $H^{-j}\mathbb{D}\mathcal{F}$ has holonomic defect j. By Theorem 2.5.1, $f_{*,dR}H^{-j}\mathbb{D}\mathcal{F}$ has holonomic defect j as well. Moreover, by affineness of f, this latter complex is in cohomological degrees $\leqslant 0$.

Note that by Proposition 2.6.1, if $\mathcal{G} \in D(Y)^{\heartsuit}$ has holonomic defect δ, then $\mathbb{D}\mathcal{G} \in \mathsf{Pro}(D(Y)^c)$ is in cohomological degrees $[-\delta, 0]$: indeed, this immediately reduces to the coherent case.

Therefore, $\mathbb{D}H^{-k}f_{*,dR}H^{-j}\mathbb{D}\mathcal{F}$ is in cohomological degrees $[-j, 0]$ for every k, which means $\mathbb{D}\big(H^{-k}(f_{*,dR}H^{-j}\mathbb{D}\mathcal{F})[k]\big)$ is in cohomological degrees $[-j+k, k]$. This complex vanishes unless $k \geqslant 0$, so $\mathbb{D}f_{*,dR}H^{-j}\mathbb{D}\mathcal{F}$ is in cohomological degrees $\geqslant -j$. Finally, this means that $\mathbb{D}\big((f_{*,dR}H^{-j}\mathbb{D}\mathcal{F})[j]\big)$ is in cohomological degrees $\geqslant 0$, so the same follows for $\mathbb{D}f_{*,dR}\mathbb{D}(\mathcal{F}) = f_!(\mathcal{F})$.

□

Remark 3.3.2. More generally, this argument shows that if $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ has amplitude $\leqslant n$, then $f_! : D(X)^c \to \mathsf{Pro}(D(Y))$ has amplitude $\geqslant -n$.

3.4. **Closed embeddings.** Similarly, we have:

Theorem 3.4.1. *For $i : X \to Y$ a closed embedding, $i^{*,dR} : D(Y)^c \to \mathsf{Pro}(D(X))$ has cohomological amplitude $\geqslant -\dim(Y) + \dim(X)$.*

Proof. The argument is the same as the above: one writes $i^{*,dR} = \mathbb{D}i^!\mathbb{D}$ and applies Theorem 2.5.1 and Proposition 2.6.1, plus the fact that $i^!$ has amplitude $\leqslant \dim(Y) - \dim(X)$.

□

Appendix A. Exactness properties of Kostant's functor

A.1. In this appendix, we briefly give an application of Theorem 3.3.1 to representation theory. The result below is a toy model of the applications in [Ras].

A.2. Let $\mathfrak{g}$ be a semi-simple Lie algebra, let $\mathfrak{b}, \mathfrak{b}^- \subseteq \mathfrak{g}$ be opposed Borels with radicals $\mathfrak{n}$ and $\mathfrak{n}^-$. Let $\psi : \mathfrak{n}^- \to k$ be a non-degenerate character, i.e., ψ is non-zero on weight spaces corresponding to negative simple roots.

Let $\mathfrak{g}\text{–}\mathsf{mod}$ denote the DG category of $\mathfrak{g}$-modules. Let $\Psi^{fin} : \mathfrak{g}\text{–}\mathsf{mod} \to \mathsf{Vect}$ be the functor computing Lie algebra homology of $\mathfrak{n}^-$ twisted by ψ. That is, $M \in \mathfrak{g}\text{–}\mathsf{mod}$ maps to $C_\bullet(\mathfrak{n}^-, M \otimes \psi)$, where $M \otimes \psi$ indicates the $\mathfrak{n}^-$-module with the same underlying vector space as M but action twisted as $x \overset{new}{\cdot} v := x \overset{old}{\cdot} v + \psi(x)$ for $x \in \mathfrak{n}^-$ and $v \in M$.

Let $\mathfrak{g}\text{–}\mathsf{mod}^N \subseteq \mathfrak{g}\text{–}\mathsf{mod}$ be the full subcategory consisting of complexes such that $\mathfrak{n}$ acts locally nilpotently on cohomology.

Theorem A.2.1. *The functor Ψ^{fin} is exact when restricted to $\mathfrak{g}\text{–}\mathsf{mod}^N$.*

Remark A.2.2. This theorem is well-known (c.f. [Kos]) when we replace $\mathfrak{g}\text{–}\mathsf{mod}^N$ by the BGG category $\mathcal{O}$. Indeed, because the latter category is Artinian, it suffices to verify that $\Psi^{fin}(L_\lambda) \in \mathsf{Vect}^\heartsuit$ for any λ, where L_λ is the simple of highest weight λ. But this is easy to see: if either L_λ is a Verma module and therefore free over $U(\mathfrak{n}^-)$, or L_λ is partially integrable, in which case $\Psi^{fin}(L_\lambda) = 0$.

However, this method does not work in the above setting, since $\mathfrak{g}\text{–}\mathsf{mod}^N$ is not Artinian (the Cartan subalgebra may not act locally finitely). I do not know another reference for it in this generality.

Proof. Beilinson-Bernstein localization gives a commutative diagram:

$$\begin{array}{ccc} \mathfrak{g}\text{–}\mathsf{mod}^N & \longrightarrow & D(G/N) = D(G)^N \\ \downarrow & & \downarrow \\ \mathfrak{g}\text{–}\mathsf{mod}^{N,\psi} & \longrightarrow & D(G)^{N^-,\psi}. \end{array}$$

Here the horizontal arrows are localization, and the vertical arrows are $*$-*averaging* functors in the sense of the theory of group actions on categories. We claim the left vertical arrow is t-exact up to shift by the dimension, i.e., it maps objects in $\mathfrak{g}\text{–}\mathsf{mod}^{N,\heartsuit}$ to objects in cohomological degree $\dim(N)$.

The localization functors are tautologically t-exact. Moreover, the $*$-averaging functor on the right coincides with the !-averaging functor up to cohomological shift $2\dim N$ by [BBM] Theorem 1.5 (1). The $*$-averaging functor has cohomological amplitude $\leqslant \dim N$ because N^- is affine, and the !-averaging functor has amplitude $\geqslant -\dim N$ by Theorem 3.3.1. (Note that the D-modules arising by localization here are not necessarily holonomic!) Identifying the shifts then gives the result.

Finally, we recall that by Skryabin's theorem, the functor of Lie algebra cohomology twisted by $-\psi$ is t-exact on $\mathfrak{g}\text{–}\mathsf{mod}^{N,\psi}$. Noting that Lie algebra homology and cohomology coincide up to shift by $\dim N$, and (switching the role of ψ and $-\psi$) we obtain the desired result.

□

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