Let \( K \) be a finite extension of \( \mathbb{Q}_p \). Let \( \mathcal{O}_K \) be its ring of integers with maximal ideal \( p \), and let \( k = \mathcal{O}_K/p \) be the residue field. Let \( v_p \) denote the valuation on \( K \), normalized so that the valuation of a uniformizer is 1.

(1) (a) Show that the subgroup \( (K^\times)^2 \subseteq K^\times \) of squares contains an open neighborhood of the identity, i.e., every element of \( 1 + p^N \) is a square for \( N \) large enough. Give an upper bound on \( N \).

(b) Show that \( (K^\times)^2 \subseteq K^\times \) is a subgroup of index \( 4|k|^{v_p(2)} \).

(c) Show that \( x \in \mathbb{Q}_2^\times \) is a square if and only if \( v_2(x) \in 2\mathbb{Z} \subseteq \mathbb{Z} \), and \( 2^{-v_2(x)} \cdot x \in \mathbb{Z}_2 \) is equal to 1 modulo \( 8\mathbb{Z}_2 \).

(2) (a) For \( a, b \in K^\times \), show that \( a^{v_p(b)} \cdot b^{v_p(a)} \in \mathcal{O}_K^\times \).

(b) Define the tame symbol as the pairing:

\[
K^\times \times K^\times \rightarrow k^\times
\]

\[
(a, b) \mapsto \text{Tame}(a, b) := (\frac{-1}{a^{v_p(a)} b^{v_p(b)}}) \mod \mathfrak{p}.
\]

For \( p \neq 2 \), show that the Hilbert symbol is computed by composing the tame symbol with the unique non-trivial character \( k^\times \rightarrow \{1, -1\} \).

(c) If \( a \in \mathbb{Z}_2^\times \), define \( \varepsilon(a) \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \) as the reduction of \( a \) mod \( 4\mathbb{Z}_2 \) under the isomorphism \( (\mathbb{Z}/4\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \) (i.e., \( \varepsilon(a) = 0 \) if \( a \in 1 + 4\mathbb{Z}_2 \) and \( \varepsilon(a) = 1 \) if \( a \in 3 + 4\mathbb{Z}_2 \)).

If \( a, b \in \mathbb{Z}_2^\times \), show that their Hilbert symbol is computed as:

\[
(a, b) = (-1)^{\varepsilon(a) \varepsilon(b)}.
\]

(d) Further show that \( (2, 2) = 1 \), and that for \( a \in \mathbb{Z}_2^\times \):

\[
(a, 2) = (-1)^{\theta(a)}.
\]

Here \( \theta(a) \in \mathbb{Z}/2\mathbb{Z} \) is the reduction of \( a^2 \) mod \( 16\mathbb{Z}_2 \) under the isomorphism between the squares in \( (\mathbb{Z}/16\mathbb{Z})^\times \) (which are 1 and 9) and \( \mathbb{Z}/2\mathbb{Z} \).

Using bimultiplicativity, deduce an explicit formula for the 2-adic Hilbert symbol (which could be deduced using similarly elementary methods and some more work).

(3) Let \( \ldots \subseteq F_2A \subseteq F_1A \subseteq F_0A = A \) and \( \ldots \subseteq F_2B \subseteq F_1B \subseteq F_0B = B \) be abelian groups with complete filtrations.

Let \( f : A \rightarrow B \) be a map that is not necessarily a homomorphism, but preserves the filtration in the sense that for every \( x \in A \), \( f \) maps \( x + F_nA \) to \( f(x) + F_nB \).
(a) For every $x \in A$, show that the symbol map:

$$F_n A / F_{n+1} A \xrightarrow{y \mapsto f(y) - f(x)} F_n B / F_{n+1} B$$

is well-defined.

(b) Suppose that for all $x \in A$, the associated symbol map is surjective. Show that $f$ is surjective.

(c) Deduce Hensel’s lemma: for $f(t) \in O_K[t]$ a polynomial with $f(p) \subseteq p$ and $f'(p) \subseteq O_K^\times$, $f$ has a zero. (Then look up Hensel’s lemma on Wikipedia and make sure you understand why this statement is equivalent to that one. E.g., use it to show that $-1$ is a square in $\mathbb{Q}_5$.)

Now let $K$ be any local field of characteristic $\neq 2$\footnote{This means we do not allow $F_2^n((t))$, but e.g. $\mathbb{Q}_2$ is still allowed.} You may assume the bimultiplicativity of the Hilbert symbol in the next problems.

(4) For $a, b, \lambda \in K^\times$, so that $ax^2 + by^2 = \lambda$ has a solution if and only if we have the Hilbert symbol equality $(-ab, \lambda) = (a, b)$.

(5) For $a, b \in K^\times$, define the quaternion algebra $H_{a,b}$ to be the (unital, associative) $K$-algebra generated by elements $i, j$ and with relations

$$i^2 = a, j^2 = b, ij = -ji.$$

(a) Show that $H_{a,b}$ is 4-dimensional as a $K$-vector space, with basis $\{1, i, j, ij\}$. 

(b) Show that the Hilbert symbol $(a, b)$ equals 1 if and only $H_{a,b}$ is isomorphic to $M_2(K)$, the algebra of $2 \times 2$-matrices over $K$. 
