# 18.786 PROBLEM SET 2

Due February 18th, 2016

(1) Recall that for $F$ a number field (i.e., a finite extension of $\mathbb{Q}$), we have $\mathbb{A}_F$ the topological ring of *adèles*, and its units $\mathbb{A}_F^\times$ form the topological group of *idèles*.

(a) Show that the canonical map $\widehat{\mathbb{Z}}^\times \times \mathbb{Q}^\times \times \mathbb{R}^{>0} \to \mathbb{A}_\mathbb{Q}^\times$ is an isomorphism.

(b) For $F$ a number field, show that:

$$\Big(\prod_{v \text{ a place of } F} \mathcal{O}_{F_v}^\times\Big)\backslash \mathbb{A}_F^\times / F^\times$$

is canonically isomorphic to the class group of $F$, where if $v$ is an infinite place, $\mathcal{O}_{F_v} := F_v$. Here "canonically" means that you should show that such an isomorphism is *uniquely* characterized by the property that that for each prime ideal $\mathfrak{p}$, the composite map:

$$\mathbb{Z} = \mathcal{O}_{F_\mathfrak{p}}^\times \backslash F_\mathfrak{p}^\times \hookrightarrow \Big(\prod_v \mathcal{O}_{F_v}^\times\Big)\backslash \mathbb{A}_F^\times \twoheadrightarrow \Big(\prod_v \mathcal{O}_{F_v}^\times\Big)\backslash \mathbb{A}_F^\times / F^\times \simeq \mathrm{Cl}(F)$$

maps 1 to the ideal class of $\mathfrak{p}$.

(c) Similarly, show that the profinite completion of:

$$\Big(\prod_{v \text{ a } \textit{finite} \text{ place of } F} \mathcal{O}_{F_v}^\times\Big)\backslash \mathbb{A}_F^\times / F^\times$$

is isomorphic to the *narrow*[1] class group of $F$.

(d) For every number field $F$, show that the canonical map $\big(\widehat{\mathbb{Z}} \times \mathbb{R}\big) \underset{\mathbb{Z}}{\otimes} F \to \mathbb{A}_F$ is an isomorphism.

(2) Recall that $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$ has order 8, so $\mathbb{Q}_2$ has 7 (isomorphism classes of) quadratic extensions, corresponding to $\mathbb{Q}_2[\sqrt{d}]$ for $d$ running over a class of coset representatives for the non-squares in $\mathbb{Q}_2^\times$.

(a) Using the fact that an element of $\mathbb{Z}_2^\times$ is a square if and only if it is congruent to 1 modulo $8\mathbb{Z}_2$, show that these coset representatives can be taken to be $d = 2, 3, 5, 6, 7, 10, 14$.

(b) By the general structure theory of nonarchimedean local fields, $\mathbb{Q}_2$ admits a single *unramified* quadratic extension. Which value of $d$ above does it correspond to? How does this relate to the explicit formula you found last week for the Hilbert symbol for $\mathbb{Q}_2$?

*Date*: February 16, 2016.

[1]This is the group of fractional ideals of $F$ modulo principal ideals defined by *totally positive* elements of $F^\times$, i.e., elements $x$ of $F^\times$ such that for every embedding $F \hookrightarrow \mathbb{R}$, $i(x) > 0$.

(c) For each $d$ as above, find a uniformizer in the field $\mathbb{Q}_2[\sqrt{d}]$, and compute its norm in $\mathbb{Q}_2$.

(3) Recall the definition of the *quaternion algebra* $H_{a,b}$ associated to $a, b \in K$: it is the $K$-algebra with generators $i$ and $j$ with relations $i^2 = a$, $j^2 = b$ and $ij = -ji$.

(a) Let $K = \mathbb{Q}_2$. Show that every $d \in \mathbb{Q}_2$ admits a square root in $H_{-1,-1}$, i.e., for every $d$ there exists $x \in H$ with $x^2 = d$.

(b) Let $K$ be a nonarchimedean local field of odd residue characteristic, and let $a, b \in K^\times$ with Hilbert symbol $(a, b) = -1$. Show that every element of $K$ admits a square root in $H_{a,b}$.

(4) Show that a local field $K \neq \mathbb{C}$ contains only finitely many roots of unity.