18.786 PROBLEM SET 5

Due March 17th, 2016

(1) Given an example of an acyclic[1] complex that is not homotopy equivalent to the zero complex.

(2) Let $X = X^\bullet$ and $Y = Y^\bullet$ be complexes of abelian groups. We define their *tensor product* as the complex $X \otimes Y = (X \otimes Y)^\bullet$ where:

$$(X \otimes Y)^i = \oplus_{j+k=i} X^j \otimes Y^k$$

with differential:

$$d(x \otimes y) := dx \otimes y + (-1)^j x \otimes dy$$
$$\text{for } x \in X^j, y \in Y^k,\ j + k = i.$$

(a) Show that this actually defines a complex, i.e., that the above formula really does a map, and that the resulting differentials square to zero.
(b) Briefly, why do we need to incorporate the sign above? Use the sign in the tensor product to explain why we should have a sign in the shift operator, i.e., why we change the sign of the differential when defining $X[1]$.
(c) Show that $X \otimes Y$ is canonically isomorphic (as a complex) to $Y \otimes X$.
(d) Suppose that $x \in X^j$ and $y \in Y^k$ with $dx = 0$, $dy = 0$. Show that $d(x \otimes y) = 0$. Deduce that there is a canonical map:

$$\oplus_{j+k=i} H^j(X) \otimes H^k(Y) \to H^i(X \otimes Y).$$

Show that this map is an isomorphism if X is a chain complex of $\mathbb{Q}$-vector spaces, but provide an example to show that this map is not an isomorphism in general.
(e) Give an example of a pair of complexes X and Y with Y acyclic such that $X \otimes Y$ is not acyclic.

(3) Let X and Y be as above.
(a) Show that there exists a complex $\underline{\mathrm{Hom}}^\bullet(X, Y) = \underline{\mathrm{Hom}}(X, Y)$ characterized by the data of isomorphisms:[2]

$$\mathrm{Hom}(Z, \underline{\mathrm{Hom}}(X, Y)) = \mathrm{Hom}(X \otimes Z, Y)$$

defined for every chain complex Z, with these isomorphisms being functorial in Z.

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[1] Recall that this means that the cohomology of the complex is zero in each degree.

[2] Note that Hom with an underline refers to a certain chain complex, whereas Hom with no underline is the set of morphisms of chain complexes.

(b) Show that a map of chain complexes $X \to Y$ is the same as an element $f \in \underline{\mathrm{Hom}}^0(X,Y)$ with $df = 0$.
(c) For f as above, show that a nullhomotopy of the map $X^\bullet \to Y^\bullet$ is the same as an element $h \in \underline{\mathrm{Hom}}^{-1}(X,Y)$ with $dh = f$.
(d) Deduce that $H^0(\underline{\mathrm{Hom}}^\bullet(X,Y))$ is the set of homotopy classes of maps from $X^\bullet$ to $Y^\bullet$.
(e) Provide a similar interpretation of $H^i(\underline{\mathrm{Hom}}^\bullet(X,Y))$ for all $i \in \mathbb{Z}$.
(f) Suppose that h_1 and h_2 are each nullhomotopies of some map $f : X \to Y$. Define a notion of *homotopy between* h_1 *and* h_2.
(g) Now let A be an associative algebra and suppose that X and Y are complexes of A-modules. Define complexes of abelian groups $X \otimes_A Y$ and $\underline{\mathrm{Hom}}_A(X,Y)$, and show that these are actually complexes of A-modules if A is commutative. For X, Y and Z complexes of A-modules, construct a canonical map:

$$\underline{\mathrm{Hom}}(X,Y) \otimes \underline{\mathrm{Hom}}(Y,Z) \to \underline{\mathrm{Hom}}(X,Z)$$

of complexes, and explain how it encodes the composition of morphisms of complexes.

(4) Let $f : X \to Y$ be a given morphism of complexes. Recall that in this case, the *cone* (aka *homotopy cokernel*) $\mathrm{hCoker}(f)$ is the complex characterized by the fact that giving a map $\mathrm{hCoker}(f) \to Z$ of chain complexes is the same as giving a map $Y \to Z$ plus a nullhomotopy of the resulting map $X \to Z$.

Similarly, recall that $\mathrm{hKer}(f)$ is the complex characterized by the fact that giving a map $Z \to \mathrm{hKer}(f)$ is the same as giving a map $Z \to X$ plus a nullhomotopy of the resulting map $Z \to Y$.

(a) Give explicit formulae for these two complexes. Deduce that $\mathrm{hCoker}(f) = \mathrm{hKer}(f)[1]$.
(b) Show that $\mathrm{hCoker}(X \xrightarrow{0} 0) = X[1]$.
(c) Show that there is an exact sequence:

$$H^0(X) \to H^0(Y) \to H^0(\mathrm{hCoker}(f)).$$

(d) Let g denote the map $Y \to \mathrm{hCoker}(f)$. Show that the canonical map $X \to \mathrm{hKer}(g)$ is a homotopy equivalence. Remark on how different this is from the situation with abelian groups (as opposed to complexes of abelian groups).
(e) Use the above to deduce that there is a long exact sequence:

$$\ldots \to H^{-1}(\mathrm{hCoker}(f)) \to H^0(X) \to H^0(Y) \to H^0(\mathrm{hCoker}(f)) \to H^1(X) \to \ldots$$

(f) Suppose that X and Y are complexes concentrated in degree 0, i.e., $X^i = Y^i = 0$ for $i \neq 0$. What is the long exact sequence saying in this case?
(g) Suppose that we have a commutative square:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & Y_1 \arrow[d, "g_2"] \\ X_2 \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

of chain complexes. Use universal properties to show that there is a canonical isomorphism of chain complexes between $\mathrm{hCoker}\big(\mathrm{hCoker}(f_1) \to \mathrm{hCoker}(f_2)\big)$ and $\mathrm{hCoker}\big(\mathrm{hCoker}(g_1) \to \mathrm{hCoker}(g_2)\big)$.

(h) Suppose that $f : X \to Y$ is a morphism of chain complexes that is injective in each degree, i.e., each of the given maps $f^i : X^i \to Y^i$ is injective. Show that the canonical map:

$$\mathrm{hCoker}(f) \to Y/X$$

is a quasi-isomorphism.[3] Here the right hand side is the complex that is Y^i/X^i is degree i, with differentials induced by the differentials of Y. Give an example to show that this map need not be a homotopy equivalence.

(i) Use (4g) to prove the snake lemma.

(5) Show that $f : X \to Y$ is a quasi-isomorphism (resp. a homotopy equivalence) if and only if $\mathrm{hCoker}(f)$ is acyclic (resp. $\mathrm{id}_{\mathrm{hCoker}(f)}$ is nullhomotopic).

(6) Suppose that $f : X \to Y$ is given, and we have a map $g : Y \to Z$ with two nullhomotopies h_1 and h_2 of the induced map $g \circ f : X \to Z$. We obtain two maps $\varepsilon_1, \varepsilon_2 : \mathrm{hCoker}(f) \to Z$ corresponding to h_1 and h_2 respectively.

(a) Give an example to show that ε_1 and ε_2 may not be homotopic.

(b) Show that a homotopy between h_1 and h_2 in the sense of (3f) induces a homotopy between ε_1 and ε_2.

(7) Let $G = \mathbb{Z}/n\mathbb{Z}$ with generator σ. For M a G-module, define the *Tate complex* M^{tG} as the chain complex:[4]

$$\ldots \xrightarrow{\sum_{i=0}^{n-1} \sigma^i} M \xrightarrow{1-\sigma} M \xrightarrow{\sum_{i=0}^{n-1} \sigma^i} M \xrightarrow{1-\sigma} \ldots$$

Recall that Tate cohomology is defined as the cohomology of M^{tG}.

Given a short exact sequence:

$$0 \to M \to E \to N \to 0$$

of G-modules, show that there is a canonical quasi-isomorphism:

$$\mathrm{hCoker}(M^{tG} \to E^{tG}) \simeq N^{tG}.$$

Deduce the long exact sequence on Tate cohomology from here.

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[3]Recall that a map of complexes $g : A \to B$ is a *quasi-isomorphism* if $H^i(g) : H^i(A) \to H^i(B)$ is an isomorphism for all i.

[4]We should normalize the parity so that the differential from the 0th term to the 1st is $1 - \sigma$ (rather than $\sum_{i=0}^{n-1} \sigma^i$).

(8) Go for a long walk (this weather is so wonderful!), and try to reconstruct as many of these problems as you can. E.g., recall the basic facts about cones, and reduce the construction of long exact sequences to statements you know you could readily check if only you'd brought pen and paper.