18.786 PROBLEM SET 7

Due April 7th, 2016

(1) Let $G$ be a finite group and let $H$ be a subgroup.
(a) Consider $Z[G]$ as a left $Z[H]$-module. Show that it is a finite rank free module over $Z[H]$.
(b) For a finitely generated projective left $Z[H]$-module $P$, show that its dual $P^\vee := \text{Hom}_H(P, Z[H])$ is naturally a right $Z[H]$-module, and as such is finitely generated and projective. Then show that the dual to $Z[G]$ is canonically isomorphic to $Z[G]$, thought of now as a right $Z[H]$-module (by letting $H$ act on $G$ on the right).
(c) Show that for any complex $X$ of $H$-modules, there is a canonical quasi-isomorphism:

$$(Z[G] \otimes_{Z[H]} X)^{hG} \simeq X^{hH}.$$

(2) Let $G$ and $H$ be as above. Let $X$ be a complex of $G$-modules.
(2.1) Recall from class that we have a restriction map $X^{hG} \to X^{hH}$ and an inflation map $X^{hH} \to X^{hG}$ and that the composition $X^{hG} \to X^{hH} \to X^{hG}$ is homotopic to multiplication by the index $[G : H]$.
(a) Show that there are canonical maps $X^{hH} \to X^{hG}$ and $X^{hG} \to X^{hH}$, such that the composed endomorphism of $X^{hG}$ is again homotopic to multiplication by $[G : H]$.
(b) Do the same for Tate cohomology.
(c) Show that multiplication by $|G|$ is nullhomotopic on $X^{tG}$. Deduce that $\hat{H}^i(G, X) := H^i(X^{tG})$ is a $\mathbb{Z}/[G]$-module.

(3) Let $L/K$ be a Galois extension of fields with Galois group $G$. In this problem, we will show that $H^1(G, L^\times) = 0$, a generalization of Hilbert’s Theorem 90 due to Noether (and often just referred to as Hilbert’s Theorem 90).

Suppose that $\varphi : G \to L^\times$ is a group 1-cocycle, i.e., we have the identity:

$$...$$

\begin{footnotesize}

Date: April 6, 2016.

1 Briefly, the construction went by identifying $X^{hH}$ with $\text{Hom}_H(Z[G/H], X)$ and then using the canonical $G$-equivariant maps $Z[G/H] \to Z$ and $Z \to Z[G/H]$. You should think that restriction is about regarding a $G$-invariant vector as an $H$-invariant one, and inflation is about averaging an $H$-invariant vector to a $G$-invariant vector via $x \mapsto \sum_{g \in G/H} g \cdot x$.

\end{footnotesize}
\[ \varphi(gh) = \varphi(g) \cdot (g \cdot \varphi(h)). \]

(Here the first \( \cdot \) is multiplication in \( L \), and the second \( \cdot \) is the action of \( g \) on \( L \).)

We need to show that \( \varphi \) is coboundary, i.e., that there exists \( x_0 \in L^\times \) with:

\[ \varphi(g) = \frac{x_0}{g \cdot x_0} \]

for all \( g \in G \).

(a) Remind me: why is this enough to deduce that \( H^1(G, L^\times) = 0 \)?

(b) Define \( T_g : L \to L \) by \( T_g(x) = \varphi(g) \cdot (g \cdot x) \). Show that \( T_{gh} = T_g \circ T_h \).

(c) Show that \( \sum_{g \in G} T_g \) is a non-zero \( K \)-linear map \( L \to L \).

(d) Deduce that \( \varphi \) is a coboundary.