

CHIRAL CATEGORIES

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1. INTRODUCTION

1.1. The purpose of this text is to introduce a framework of *chiral (DG) categories* (alias: *factorization categories*). This material is of a technical nature and is designed to supplement the works [Ras1], [Ras2], and [Ras3].

1.2. We divide the introduction into two pieces. In §1, we describe what factorization is and why it is important. In §3, we describe what heuristics the reader should keep in mind, and we give a detailed overview of the body of this text.

The intervening §2 sets up our categorical conventions and notations.

1.3. **Factorization.** First, we recall the meaning of the almost synonymous words *chiral* and *factorization*.

The subject begins with the Beilinson-Drinfeld theory of chiral algebras from [BD2], whose features we recall below.

Remark 1.3.1. We will give a somewhat leisurely introduction to the theory of chiral algebras below. The subject carries a reputation of being very technical and for lacking applications, or at least, lacking applications in which the role played by the chiral structure is straightforward and easy to isolate from the arguments. However, there is a rich folklore around this subject, only partially written down, which explains what these things are good for. We hope that in presenting the general aspects of this material, the strategy of the present series of papers will be made more transparent to the reader.

1.4. The Beilinson-Drinfeld theory of chiral algebras on a smooth curve X has the following salient features:

- (1) Chiral algebras are of local origin on the curve X . Many of their invariants (e.g., modules at a point) are closely related to the geometry of the formal punctured disc, especially algebraic loop spaces and de Rham local systems on the formal punctured disc.

Moreover, chiral algebras tend to “decrease the complexity” in the following sense. A chiral algebra whose fibers involve only the disc will have invariants associated with the whole of the formal punctured disc. For instances, the chiral geometry of an arc space tends to encode the usual geometry of the associated algebraic loop space. As another example, the chiral geometry of the Beilinson-Drinfeld affine Grassmannian, recalled below, tends to encode information about the whole of the algebraic loop group, and in particular its group structure.
- (2) For X a proper curve, chiral algebras on X give rise to interesting global invariants (e.g., through chiral homology).
- (3) Chiral algebras appear naturally in much of the geometric representation theory involving the curve X . For example, see [KL], [BFS], [Gai2], [BD1] and [BG]. Note that chiral algebras naturally arise through both algebraic and geometric constructions.

The combination of the above techniques makes the theory of chiral algebras especially relevant to geometric Langlands. Recall that the local geometric Langlands program seeks to decompose representations of the algebraic loop group of a reductive group G , with spectral parameters de Rham local systems on the formal punctured disc with structure group \check{G} the dual reductive group to G .

The geometric and spectral sides each appear in (1) as arising from chiral algebras, and it is therefore natural to expect that local geometric Langlands admits a chiral avatar (c.f. the introduction to [Bei]). Moreover, this should make the subject *easier*: in certain nice settings, we can move from the simple geometry of the disc to the much more complicated geometry of the formal punctured disc.

Then the local-to-global techniques can be brought to bear to give global applications as well.

Example 1.4.1. A primordial example of the above procedure is implicit in [BD1], where the Feigin-Frenkel identification of the critical and infinite level chiral \mathcal{W} -algebras for Langlands dual Lie algebras is used to construct Hecke eigensheaves for regular opers.

1.5. A wonderful discovery of Beilinson-Drinfeld, explained in [BD2], is the two guises of chiral algebras: as chiral Lie algebras and as factorization algebras.

Chiral Lie algebras, a coordinate-free variant of the more classical notion of vertex algebra (see [Bor] and [BF]) are technically convenient in providing an algebraic perspective on chiral algebras. For example, the construction of a chiral Lie algebra from a Lie- $*$ algebra (in the vertex language: vertex Lie algebras) is realized more naturally as an induction functor analogous to the usual enveloping algebra of a Lie algebra.

Factorization algebras, invented by Beilinson-Drinfeld, provide a much more geometric perspective. This is the perspective on which we will presently focus.

1.6. The factorizable Grassmannian. To motivate the definition of factorization algebra, it is convenient to recall the definition and features of the Beilinson-Drinfeld affine Grassmannian.

Let X be a smooth curve over k and let $x \in X$ be a closed point.

Let K_x denote the field of Laurent series at x and let $O_x \subseteq K_x$ denote its subring of integral elements. Let Γ be an affine algebraic group over k .

By Beauville-Laszlo, the affine Grassmannian $\mathrm{Gr}_{\Gamma,x} := \Gamma(K_x)/\Gamma(O_x)$ at x is the moduli space of Γ -bundles on X with a trivialization on the open $X \setminus x \subseteq X$.

For a positive integer n , the Beilinson-Drinfeld affine Grassmannian Gr_{Γ,X^n} is the moduli space of an n -tuple of points x_1, \dots, x_n of X , a Γ -bundle on X and a trivialization of the Γ -bundle away from x_1, \dots, x_n .

The spaces Gr_{Γ,X^n} satisfy the following *factorization* property, say for $n = 2$:

$$\begin{aligned} \mathrm{Gr}_{\Gamma,X^2} |_{(X \times X) \setminus X} &\simeq \mathrm{Gr}_{\Gamma,X} \times \mathrm{Gr}_{\Gamma,X} |_{(X \times X) \setminus X} \\ \mathrm{Gr}_{\Gamma,X^2} |_X &\simeq \mathrm{Gr}_{\Gamma,X} \end{aligned} \tag{1.6.1}$$

where $X \rightarrow X \times X$ is the diagonal embedding.

1.7. Factorization algebras. Let X be a k -scheme of finite type.

A *factorization algebra* \mathcal{A} on X is a rule that assigns to each positive integer n a D -module¹ \mathcal{A}_{X^n} on X^n equivariant for the symmetric group S_n and satisfying a linearized version of (1.6.1) that says e.g. for $n = 2$ that we have S_2 -equivariant equivalences:

$$\begin{aligned} \mathcal{A}_{X^2} |_{(X \times X) \setminus X} &\simeq \mathcal{A}_X \boxtimes \mathcal{A}_X |_{(X \times X) \setminus X} \\ \mathcal{A}_{X^2} |_X &\simeq \mathcal{A}_X. \end{aligned} \tag{1.7.1}$$

In our setting of D -modules, the latter restriction should be understood in !-sense.

For example, we have the trivial example ω defined by the dualizing D -modules $n \mapsto \omega_{X^n}$.

Remark 1.7.1. Factorization spaces in geometry such as $n \mapsto \mathrm{Gr}_{G,X^n}$ are a rich source of factorization algebras. For example, taking the (quasi-coherent) global sections of the distributional D -module on the unit $X^n \subseteq \mathrm{Gr}_{G,X^n}$ one obtains a factorization algebra encoding the loop algebra $\mathfrak{g}(K_x) := \mathfrak{g} \otimes_k K_x$ for varying points x . One obtains the so-called chiral algebra of differential operators for the loop group of G by a similar procedure, c.f. [AG].

More generally, correspondences between factorization spaces are very fruitful for producing factorization algebras by means of D -module operations.

1.8. \mathcal{E}_n -algebras. There is a close analogy between factorization algebras on a curve X and algebras over the homotopy theorist's little 2-discs operad. More generally, factorization algebras on a smooth scheme X of dimension n are in analogy with operads over the little $2n$ -discs operad. The reader may safely skip this analogy, as it will play no role in the text below.

Among classical — that is, non-derived — algebras, there are associative algebras and commutative algebras. The \mathcal{E}_n -algebras appear as intermediates in settings of more homotopical complexity, where \mathcal{E}_1 -algebras are associative algebras and \mathcal{E}_0 -algebras are commutative algebras.

In a traditional setting, namely, in a symmetric monoidal $(1,1)$ -category, an \mathcal{E}_n -algebra structure for $n \geq 2$ is equivalent to an \mathcal{E}_0 -algebra structure. However, when there is greater homotopical flexibility, this is no longer the case.

For example, in the 2-category of $(1,1)$ -categories, a \mathcal{E}_2 -algebra is a braided monoidal category, which appeared in the 1980's as an intermediate between monoidal categories and symmetric monoidal categories. Similarly, n -fold loop spaces in topology carry an \mathcal{E}_n -algebra structure that cannot generally be upgraded to an \mathcal{E}_{n+1} -algebra structure.

¹We only take D -modules as a sheaf-theoretic context for concreteness. One can take quasi-coherent sheaves or ℓ -adic sheaves just as well.

Remark 1.8.1. Under this analogy, the factorization structure of the affine Grassmannian appears because the double loop space $\Omega^2(\mathbb{B}G)$ may be realized as the space of continuous maps:

$$D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \rightarrow \mathbb{B}G$$

sending the boundary $\partial D = S^1$ to the base-point. In words, this is the moduli of G -bundles on the disc with trivialization on the boundary S^1 , which functions here as an analogue to the punctured disc.

Perhaps the simplest characterization of \mathcal{E}_2 -algebra in a symmetric monoidal (higher) category \mathcal{C} is the following: the category $\text{Alg}(\mathcal{C}) = \mathcal{E}_1\text{-alg}(\mathcal{C})$ forms a symmetric monoidal category itself, under the usual tensor product of associative algebras. Therefore, we can ask for associative algebras in $\text{Alg}(\mathcal{C})$, i.e., $\text{Alg}(\text{Alg}(\mathcal{C}))$. In other words, we have an algebra $A \in \mathcal{C}$ with defining multiplication $m_1 : A \otimes A \rightarrow A$, and a second multiplication $m_2 : A \otimes A \rightarrow A$ such that m_2 is a morphism of algebras where $A \otimes A$ and A are regarded as algebras via m_1 .

Similarly, one may define an \mathcal{E}_n -algebra by asking for n -compatible multiplications.

We refer to [Lur4] for a greater discussion of this analogy, where it is explained how to relate \mathcal{E}_n -algebras and a topological analogue of factorization algebras.

1.9. Factorization categories. The analogy above suggests that not only the notion of factorization algebra is of relevance to representation theory, but also of factorization *category* as well. Indeed, a factorization category on a smooth curve is analogous to a braided monoidal category, which is well-known to be of great importance in representation theory.

Remark 1.9.1. The mathematical physicist's *fusion* procedure can be implemented mathematically in several different ways to draw a closer connection between braided monoidal categories and factorization categories.

In the case $X = \mathbb{P}^1$, [KL] used analysis to pass from the *algebraically* defined structure of factorization category on Kac-Moody representations, to obtain a braided monoidal category structure defined. Following physicists, Kazhdan and Lusztig referred to the resulting tensor product as *fusion*.

In fact, in some circumstances the fusion product can be constructed algebraically as well, as in [Gai1]. A general theory of fusion by means of nearby cycles remains undeveloped but highly plausible, and would be highly desirable.

A theory of factorization categories has been anticipated for some time now (c.f. [Gai2]), but has not appeared in the literature at this point due to the technical difficulties foundational in the subject.

1.10. A difficulty that one must grapple with in the theory of factorization algebras is the fact that the equivalences (1.7.1) must be understood in the derived category (already in the case of the dualizing sheaf!), and the equivalences must be then be required to be *homotopy compatible* in some appropriate sense.

Beilinson and Drinfeld circumvent this problem in [BD2] by working only with smooth curves and sheaves \mathcal{A}_{X^I} such that $\mathcal{A}_{X^I}[-|I|]$ lies in the heart of the usual (alias: perverse) t -structure on the category of D -modules on X^I (this t -structure is referred to in *loc. cit.* as the t -structure for left D -modules); favorable arithmetic then provides a supply of examples of factorization algebras for which only abelian categories are necessary.

1.11. The recent advances in homotopical algebra, notably in [Lur1] and [Lur4], provide an easy language of higher categories in which the notion of homotopy compatibility may be used in a systematic way, unburdened by the construction of clever resolutions and model category structures.² This language allows for a different approach, working directly with collections of complexes of sheaves with homotopy compatible equivalences (1.7.1).

This approach is pursued in [FG1], where the theory of higher categories is shown to provide adequate foundations to develop the theory of factorization algebras on arbitrary schemes of finite type, allowing for schemes more general than smooth curves and for complexes of sheaves unfettered by any t -structure.

Moreover, many of the factorization algebras constructed in geometric representation theory by means of Remark 1.7.1 are inherently derived: they are constructed by sheaf-theoretic operations that only under limited and special circumstances preserve the heart of any t -structures. That is, they fall under the purview of the theory of [FG1] exclusively.

Remark 1.11.1. Even in the case of a smooth curve, the Francis-Gaitsgory approach provides a conceptually simpler and more unified theory than overlapping material in [BD2].

1.12. **Acknowledgements.** This work is a fleshing out of ideas of Gaitsgory, explained in [Gai2] and in many unpublished sources. This text serves only to flesh out the technical details of his ideas.

We are also grateful to Dario Beraldo, for many helpful conversations regarding this material.

Finally, we note that since the present text was written, [Gai7] has appeared, which has some overlap in its treatment of unital Ran space: we refer the reader there for an alternative treatment of some points.

2. CONVENTIONS

2.1. We fix a k of characteristic 0 throughout this paper (this assumption is only necessary in those parts that reference the theory of D -modules). All schemes, etc, are understood to be defined over k .

2.2. **Higher categories.** We rely heavily on the theory of higher categories, whose existence is due to the work of many mathematicians. This theory was systematically developed in Lurie's [Lur1] and [Lur4], and we use these as our preferred reference where appropriate.

We assume that the reader is comfortable with higher category theory and derived algebraic geometry. However, we will carefully establish notation and conventions below, highlighting the points where our terminology differs from [Lur1] and [Lur4].

Unlike [Lur1], our use of the theory is *model independent*: there are different³ models of $(\infty, 1)$ -categories⁴ (quasicategories, Segal sets, etc.), each with its own intrinsic notion of, say, homotopy colimit. We use the theory only in as much as it can be implemented in each of these different models, that is, we allow ourselves to use the language of homotopy colimits, but not to use the language of, say, quasicategories.⁵

²Note that model categories appear inadequate to the problem at hand: compare to [BD2] §0.12.

³However, these theories are mutually Quillen equivalent; see [Toë].

⁴We recall for the reader's convenience that (n, m) -category ($0 \leq m \leq n \leq \infty$) refers to a higher category with possibly non-trivial k -morphisms for $k \leq n$, and in which k -morphisms are assumed invertible for $k \geq m$. E.g., a $(1, 1)$ -category is a usual category, a $(2, 2)$ -category is a usual 2-category, etc.

⁵The reader uncomfortable with this approach may happily understand everything to be implemented in quasicategories as in [Lur1], though our language will differ from *loc. cit.* at some places; the translation should always be clear.

As such, we use terms such as *isomorphism* and *equivalence* interchangeably.

2.3. We find it convenient to assume higher category theory as the basic assumption in our language. That is, we will understand “category” and “1-category” to mean “ $(\infty, 1)$ -category,” “colimit” to (necessarily) mean “homotopy colimit,” “groupoid” to mean “ ∞ -groupoid” (aliases: homotopy type, space, etc.), “2-category” to mean $(\infty, 2)$ -category, etc. “Morphism” means 1-morphism. We use the phrase “set” interchangeably with “discrete groupoid,” i.e., a groupoid whose homotopy groups at any basepoint are singletons.

When we need to refer to the more traditional notion of category, we use the term $(1, 1)$ -category.

In particular, we refer to the notion of “stable ∞ -category” from [Lur4] as a stable category.

When we say that \mathcal{D} is a full subcategory of \mathcal{C} , we mean that there is a given functor $\mathcal{D} \rightarrow \mathcal{C}$ inducing equivalences on all groupoids of morphisms.

2.4. **Conventions regarding 2-categories.** The theory of (unital) chiral categories is most naturally developed using the theory of 2-categories. Recall that Segal categories provide an adequate model for 2-categories, granted a theory of 1-categories (this approach is developed in detail e.g. in [GR]).

Every 2-category has an underlying 1-category in which we forget all non-invertible 2-morphisms. For many purposes (such as computing limits and colimits), this underlying category is perfectly adequate, and where it is irrelevant, we do not pay particular attention to the distinction, hoping that this makes it easier for the reader.

For \mathcal{C} a 2-category, we use the notation $\mathbf{Hom}_{\mathcal{C}}(X, Y)$ (as opposed to $\mathrm{Hom}_{\mathcal{C}}(X, Y)$) to indicate that we take the category of maps $X \rightarrow Y$, not the groupoid of maps.

We say that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of 2-categories is *1-fully faithful* if the induced maps:

$$\mathbf{Hom}_{\mathcal{C}}(X, Y) \rightarrow \mathbf{Hom}_{\mathcal{D}}(F(X), F(Y))$$

are fully-faithful functors. A *1-full subcategory* means the essential image of such a functor. If in practice “full subcategory” means that we impose some conditions on a class of objects, then “1-full” means that we impose conditions on both objects and morphisms.

2.5. **Accessibility.** We will typically ignore cardinality issues that arise in category theory. The standard way to do this is through the use of accessible categories (we recall that this condition is satisfied for essentially small categories and for compactly generated categories). The author’s opinion is that focusing too much on accessibility issues distracts the reader who is not familiar with the ideas, while omitting these points will not create confusion for the reader who is.

But we will enforce the following conventions:

- Categories are assumed to be locally small, i.e., Hom groupoids are essentially small.
- We use the term “indexing category” synonymously with “essentially small category.” A category seen indexing a colimit or limit is assumed to be essentially small. If we use e.g. the term “all colimits” (as in: “such and such functor commutes with all colimits”), this certainly means “all small colimits.”
- All functors between accessible categories are assumed to be accessible.
- DG categories are always assumed to be accessible.
- The term “groupoid” nearly always refers to an essentially small groupoid.

2.6. **Notation.** Let \mathbf{Cat} denote the (2-)category of essentially small categories and let \mathbf{Gpd} denote the category of essentially small groupoids.

Let \mathbf{Cat}_{pres} denote the category of presentable (i.e., cocomplete and accessible) categories under functors that commute with arbitrary colimits. We consider \mathbf{Cat}_{pres} as a symmetric monoidal category equipped with the tensor product \otimes of [Lur4] §4.8.1.14.

For $\mathcal{F} \in \mathcal{C}$ and $\mathcal{G} \in \mathcal{D}$, we let $\mathcal{F} \boxtimes \mathcal{G}$ denote the induced object of $\mathcal{C} \otimes \mathcal{D}$, since this notation is compatible with the usual exterior product operation in algebraic geometry.

For \mathcal{C} and \mathcal{D} categories, we let $\mathbf{Hom}(\mathcal{C}, \mathcal{D})$ denote the category of functors between \mathcal{C} and \mathcal{D} .

For \mathcal{C} an essentially small category, we let $\mathbf{Ind}(\mathcal{C})$ denote the category of its ind-objects, as in [Lur1].

2.7. Grothendieck construction. For $F : \mathcal{J} \rightarrow \mathbf{Cat}$ a functor, we let $\mathbf{Groth}(F) \rightarrow \mathcal{J}$ denote the corresponding coCartesian fibration attached by the (higher-categorical) Grothendieck construction, and we let $\mathbf{coGroth}(F) \rightarrow \mathcal{J}^{op}$ denote the corresponding Cartesian fibration.

For $\alpha : i \rightarrow j$ a morphism in \mathcal{J} and $Y \in F(i) = \mathbf{Groth}(F) \times_{\mathcal{J}} \{i\}$, we will often use the notation $\alpha(Y)$ for the induced object of $F(j) = \mathbf{Groth}(F) \times_{\mathcal{J}} \{j\}$.

2.8. Modules. For an (\mathcal{E}_1) -algebra A (in spectra), we let $A\text{-mod}$ denote the stable category of left A -modules (in spectra). We refer to objects of $A\text{-mod}$ as A -modules.

In the particular case $A = k$, we refer to objects of $k\text{-mod}$ as k -vector spaces, as denote the resulting stable category by \mathbf{Vect}_k or \mathbf{Vect} . Note that objects of this category are complexes of usual k -vector spaces modulo quasi-isomorphism.

2.9. DG categories. By DG category, we mean an (accessible) stable category enriched over k -vector spaces. We denote the category of DG categories under k -linear exact functors by \mathbf{DGCat} and the category of cocomplete DG categories under continuous⁶ k -linear functors by \mathbf{DGCat}_{cont} . As with \mathbf{Cat}_{pres} , we consider \mathbf{DGCat}_{cont} as equipped with the symmetric monoidal structure \otimes from [Lur4] §6.3.

Recall that from the higher categorical perspective, the cone is the cokernel of a map. Therefore, we use the notation \mathbf{Coker} where others might use \mathbf{Cone} .

For \mathcal{C} a DG category equipped with a t -structure, we let $\mathcal{C}^{\geq 0}$ denote the subcategory of *coconnective* objects, and $\mathcal{C}^{\leq 0}$ the subcategory of *connective* objects (i.e., the notation is the standard notation relative to the cohomological grading convention). We let $\mathcal{C}^{\heartsuit} = \mathcal{C}^{\geq 0} \cap \mathcal{C}^{\leq 0}$ denote the heart of the t -structure.

For example, \mathbf{Vect} is a DG category with t -structure whose heart $\mathbf{Vect}^{\heartsuit}$ is the abelian category of k -vector spaces. Similarly, for A a k -algebra (i.e., an algebra in \mathbf{Vect}), $A\text{-mod}$ has a natural structure of DG category, and it has a natural t -structure for A connective.

We use the material of the short note [Gai5] freely, taking for granted the reader's comfort with the ideas of *loc. cit.*

2.10. Monoidal categories. We assume the reader is thoroughly familiar with this theory.

We will use the following conventions.

We use the term *colored operad* in place of the term of ∞ -operad from [Lur4], preferring to use *operad* for a “colored operad with one color.” We assume the presence of units according to standard conventions, so e.g. “commutative operad,” we understand the operad controlling *unital*⁷ commutative algebras. Symmetric monoidal functors between symmetric monoidal categories are

⁶There is some disagreement in the literature of the meaning of this word. By *continuous functor*, we mean a functor commuting with filtered colimits. Similarly, by a *cocomplete* category, we mean one admitting all colimits.

⁷To not be misleading: the phrase “commutative algebra” appearing in isolation indicates a unital commutative algebra.

assumed to be unital, though we allow ourselves to speak of e.g. symmetric monoidal functors between non-unital symmetric monoidal categories, obviously meaning the non-unital version.

Next, we use the term *lax symmetric monoidal functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ between symmetric monoidal categories to refer to a morphism of the underlying colored operads. We recall that such an F is equipped with functorial associative maps:

$$F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$$

for $X, Y \in \mathcal{C}$. We use the term *colax monoidal functor* for the dual notion, in which we have functorial morphisms:

$$F(X \otimes Y) \rightarrow F(X) \otimes F(Y).$$

2.11. Cofinality. There is some disagreement in the literature over the meaning of *cofinal* (typically due to trying to avoid confusion with the word “final,” which ought not to take disparate meanings in category theory). We say that a functor $F : \mathcal{J} \rightarrow \mathcal{I}$ of indexing categories is *cofinal* if for every category \mathcal{C} , a functor $G : \mathcal{I} \rightarrow \mathcal{C}$ admits a colimit if and only its restriction to \mathcal{J} does, and the induced map:

$$\operatorname{colim} G \circ F \rightarrow \operatorname{colim} G$$

is an equivalence. We use the term *op-cofinal* to mean that $F^{op} : \mathcal{J}^{op} \rightarrow \mathcal{I}^{op}$ is cofinal, i.e., that the above conditions are satisfied for limits instead of colimits.

Remark 2.11.1. Our use of *cofinal* is in accordance with [Lur1]. In [Lur4], Lurie uses the terminology *left cofinal* for our cofinal, and *right cofinal* for our op-cofinal.

2.12. Derived algebraic geometry. Following our “permanently derived” convention, algebraic geometry means derived algebraic geometry.

Roughly, the development goes as follows: the category \mathbf{AffSch} is defined to be the opposite category to the category of commutative k -algebras that are connective as vector spaces, i.e., commutative k -algebras in $\mathbf{Vect}^{\leq 0}$. We then define the category \mathbf{PreStk} of *prestacks* as the the category of (accessible) functors $\mathbf{AffSch}^{op} \rightarrow \mathbf{Gpd}$. We have Yoneda embedding $\mathbf{AffSch} \hookrightarrow \mathbf{PreStk}$, and schemes are defined so that this extends to an embedding $\mathbf{AffSch} \hookrightarrow \mathbf{Sch} \hookrightarrow \mathbf{PreStk}$.

We say that an affine scheme is *classical* if it is of the form $\operatorname{Spec}(A)$ with $H^i(A) = 0$ for $i \neq 0$, i.e., if it is a “usual” affine scheme. More generally, we say that a prestack is *classical* if it lies in the subcategory of functors $\mathbf{AffSch}^{op} \rightarrow \mathbf{Gpd}$ that are left Kan extensions of their restriction to the (1,1)-category of classical affine schemes.

For X a prestack, we let $\mathbf{QCoh}(X)$ denote the symmetric monoidal DG category of its quasi-coherent sheaves, defined by right Kan extension from the functor $\operatorname{Spec}(A) \mapsto A\text{-mod}$. A crucial point of derived algebraic geometry (and which is not true in classical algebraic geometry) is that for $X \rightarrow Z \leftarrow Y$ schemes, the map:

$$\mathbf{QCoh}(X) \otimes_{\mathbf{QCoh}(Z)} \mathbf{QCoh}(Y) \rightarrow \mathbf{QCoh}(X \times_Z Y)$$

is an equivalence.

For X a scheme, we let $\Omega_X^1 \in \mathbf{QCoh}(X)^{\leq 0}$ denote the cotangent complex, and let $\Omega_X^{1,cl} := H^0(\Omega_X^1) \in \mathbf{QCoh}(X)^\heartsuit$ denote the classical cotangent sheaf.

To avoid overburdening the terminology, we use “finite type” for a morphism in derived algebraic geometry where others use “almost finite type.” When we say a scheme X is finite type, this certainly means relative to the structure map $X \rightarrow \operatorname{Spec}(k)$.

2.13. D -modules. We use the D -module formalism in the format developed in [GR].

For S a scheme of finite type, we let $D(S)$ denote the DG category of D -modules on S . Recall that the prestack S_{dR} is defined by $S_{dR}(T) := S(T^{cl,red})$ for an affine scheme T , where $T^{cl,red}$ is the reduced classical scheme underlying T ; then we have:

$$D(S) := \mathrm{QCoh}(S_{dR}) \overset{-\otimes_{\omega_{S_{dR}}}}{\simeq} \mathrm{IndCoh}(S_{dR})$$

for ω the dualizing sheaf of the ind-coherent theory.

For $f : S \rightarrow T$ a morphism, we let $f^! : D(T) \rightarrow D(S)$ denote the corresponding map. Recall that this functor is the $*$ -pullback in the QCoh picture and the $!$ -pullback in the IndCoh picture. Let $f_{*,dR} : D(S) \rightarrow D(T)$ denote the de Rham pushforward functor constructed in [GR].

For S a scheme with structure map $p : S \rightarrow \mathrm{Spec}(k)$, we let $\omega_S := p^!(k) \in D(S)$ denote the dualizing D -module.

We use $\overset{!}{\otimes}$ to denote the standard tensor product of D -modules, for which $\mathcal{F} \overset{!}{\otimes} \mathcal{G} = \Delta^!(\mathcal{F} \boxtimes \mathcal{G})$ for Δ the diagonal.

3. A GUIDE FOR THE PERPLEXED

3.1. The goal of the following foundational sections is to develop a theory of chiral categories, chiral algebras in them, and chiral modules for these chiral algebras. This material has been heavily influenced by [BD2], [FG1], [Lur4] §5, [Gai2], and private conversations with Dennis Gaitsgory.

3.2. Our goals in developing the theory of chiral categories are modest, and the material itself is technical. These technicalities largely are due to the use of derived categories: the combinatorial aspects of [BD2] need to be replaced by more abstract formulations to be used in higher category theory.

We find it convenient in presenting this material to describe the goals and motivation in isolation from its technical implementation. The present section is devoted exactly to giving an introduction to these ideas, beyond what was already said in §1.

The hope is to provide some general narrative structure for the technical material that follows, and to offer a user's guide for the rest of the text.

Remark 3.2.1. We note from the onset that most of the technicalities occur only in the unital setting, where the meaning of the word *unital* is indicated below.

Remark 3.2.2. Below, we discuss everything at a very heuristic level. In particular, we ignore *higher compatibilities* (such as associativity) throughout.

3.3. Sheaves of categories. Let X be a scheme of finite type.

To discuss chiral categories in analogy with chiral (or more appropriately: factorization) algebras, we need a “linear algebra” of categories over X , meant to be one categorical level higher than quasi-coherent sheaves or D -modules on X .

This theory is provided by the theory of sheaves of categories from [Gai6] (see also Appendix A). Recall that there is a notion of (DG) category \mathcal{C} over X : for $X = \mathrm{Spec}(A)$ an affine (DG) scheme, this amounts to a cocomplete DG category enriched over the symmetric monoidal DG category $A\text{-mod}$, and for general X the notion is obtain by gluing. Categories over schemes are contravariantly functorial with respect to morphisms of schemes.

Moreover, we have a general notion of category \mathcal{C} over X with a *connection*, also known as a crystal of categories. This amounts to saying that given any two infinitesimally close points of X , we identify the fibers of \mathcal{C} in a functorial way satisfying the (higher) cocycle conditions.

The notion of crystal of categories on X can be summarized more succinctly: we have the prestack X_{dR} , and there is a general notion of sheaf of categories on a prestack. Crystals of categories on X are equivalent to sheaves of categories on X_{dR} , since X_{dR} is the quotient of X by its universal infinitesimal groupoid (c.f. [GR]).

We want to have quasi-coherent and D -module versions of the theory of chiral algebras and chiral categories, and therefore we replace X with a general prestack \mathcal{X} , so that for $\mathcal{X} = X$ we obtain the quasi-coherent version and for $\mathcal{X} = X_{dR}$ we obtain the D -module version.

Note that there is a canonical sheaf of categories $\mathbf{QCoh}_{\mathcal{X}}$ on the prestack \mathcal{X} , whose global sections (in the sense of sheaves of categories) is the category $\mathbf{QCoh}(\mathcal{X})$ of quasi-coherent sheaves on \mathcal{X} . This sheaf of categories plays the role that $\mathcal{O}_{\mathcal{X}}$ plays one categorical level down.

Convention 3.3.1. We use the language of quasi-coherent sheaves in what follows, noting that the D -module language is a special case by the above.

Terminology 3.3.2. Recall that [BD2] defines notions of both *chiral* and *factorization* algebra on X_{dR} , and proves that the two notions are equivalent by means of a non-trivial functor (e.g., it doesn't commute with the forgetful functor to D -modules).

The notion of chiral algebra is much less flexible than that of factorization algebra: e.g., it can only be defined in the de Rham setting, not in the general quasi-coherent setting. In particular, only the factorization perspective generalizes to categories.

Therefore, we use the terms *chiral category* and *factorization category* interchangeably in the categorical setting because there is no risk for ambiguity. However, for sheaves, we will be much more conservative in the use of the word *chiral*.

3.4. Ran space. Next, we recall the Ran space construction from [BD2].

The idea of Ran space Ran_X is to parametrize non-empty finite subsets of a space X .

Remark 3.4.1. Any construction of Ran_X builds it out of the schemes X^I for I a finite set. This translates to saying that specialization in Ran_X allows points to collide.

It has been treated formally in algebraic geometry in a number of ways, and we follow [FG1] and [Gai4] in treating it as a prestack. The construction is defined for any prestack \mathcal{X} , giving rise to a prestack $\mathrm{Ran}_{\mathcal{X}}$.

The key point is that quasi-coherent sheaves \mathcal{F} on $\mathrm{Ran}_{\mathcal{X}}$ are equivalent to systems of quasi-coherent sheaves $\mathcal{F}_{\mathcal{X}^I}$ on each \mathcal{X}^I as I varies under non-empty finite sets, and such that these sheaves are compatible along diagonal restrictions (note that we consider the reordering of coordinates as a diagonal restriction, so these quasi-coherent sheaves are automatically equivariant for the symmetric group). The same holds for sheaves of categories.

Remark 3.4.2. One may heuristically think that a quasi-coherent sheaf \mathcal{F} on $\mathrm{Ran}_{\mathcal{X}}$ is an assignment of a vector space $\mathcal{F}_{x_1, \dots, x_n}$ for every finite subset $\{x_i\} \subseteq X$, such that these vector spaces behave “continuously” as points move and collide. Similarly, a sheaf of categories on $\mathrm{Ran}_{\mathcal{X}}$ is a continuous assignment of cocomplete DG categories $\mathcal{C}_{x_1, \dots, x_n}$.

3.5. Unital sheaves on $\mathrm{Ran}_{\mathcal{X}}$. There is also a notion of *unital* quasi-coherent sheaf of $\mathrm{Ran}_{\mathcal{X}}$, implicit in [BD2] §3.4.5, and appearing again in [Gai3], [Gai4], and [Bar].

Here we are again given quasi-coherent sheaves $\mathcal{F}_{\mathcal{X}^I}$ for each finite set I , now also allowing the empty set as well. For every morphism $f : I \rightarrow J$ of finite sets, giving rise to the map $\Delta_f : \mathcal{X}^J \rightarrow \mathcal{X}^I$, we should be given:

$$\Delta_f^*(\mathcal{F}_{\mathcal{X}^I}) \rightarrow \mathcal{F}_{\mathcal{X}^J}$$

in a way compatible with compositions of morphisms of finite sets, and such that, if Δ_f is a diagonal embedding (i.e., f is surjective), this map should be an isomorphism. In particular, for every I we have a canonical *unit* map:

$$\mathcal{F}_\emptyset \otimes_k \mathcal{O}_{X^I} \rightarrow \mathcal{F}_{X^I}.$$

Similarly, we have a notion of unital sheaf of categories on $\text{Ran}_\mathcal{X}$.

Obviously, unital quasi-coherent sheaves on $\text{Ran}_\mathcal{X}$ are quasi-coherent sheaves on $\text{Ran}_\mathcal{X}$ with additional structure.

Remark 3.5.1. Unital quasi-coherent sheaves on $\text{Ran}_\mathcal{X}$ do not quite fall under the purview of quasi-coherent sheaves on prestacks. However, in §4, we show that the language of *lax prestacks* — moduli problems valued in categories rather than groupoids — does suffice.

Namely, we define a lax prestack $\text{Ran}_\mathcal{X}^{un}$ whose points are morally the (possibly empty) finite subsets of \mathcal{X} , considered as a category by taking morphisms that are inclusions of finite subsets, and show that this lax prestack gives a good theory of unital quasi-coherent sheaves.

Remark 3.5.2. In the heuristic of Remark 3.4.2, a unital quasi-coherent sheaf \mathcal{F} on $\text{Ran}_\mathcal{X}$ is a continuous assignment:

$$(\{x_1, \dots, x_n\} \subseteq X) \mapsto \mathcal{F}_{x_1, \dots, x_n} \in \text{Vect}$$

as before (now allowing $n = 0$), and such that for every inclusion:

$$\{x_1, \dots, x_n\} \subseteq \{x_1, \dots, x_n, x_{n+1}, \dots, x_m\} \subseteq X \quad (3.5.1)$$

we have a map:

$$\mathcal{F}_{x_1, \dots, x_n} \rightarrow \mathcal{F}_{x_1, \dots, x_m} \quad (3.5.2)$$

satisfying the natural compatibilities.

Remark 3.5.3 (Lax unital functors). The heuristic notion of unital sheaf of categories is identical to the discussion of Remark 3.5.2. However, a difference emerges in the notion of *morphism* of unital sheaves of categories.

Given unital sheaves of categories \mathcal{C} and \mathcal{D} on $\text{Ran}_\mathcal{X}$, we have two notions functor $\mathcal{C} \rightarrow \mathcal{D}$, *strict* and *lax*.

For a *strict* functor, we require that we are given functors:

$$F_{x_1, \dots, x_n} : \mathcal{C}_{x_1, \dots, x_n} \rightarrow \mathcal{D}_{x_1, \dots, x_n}$$

such that, for every inclusion (3.5.1), the diagram:

$$\begin{array}{ccc} \mathcal{C}_{x_1, \dots, x_n} & \xrightarrow{F_{x_1, \dots, x_n}} & \mathcal{D}_{x_1, \dots, x_n} \\ \downarrow & & \downarrow \\ \mathcal{C}_{x_1, \dots, x_m} & \xrightarrow{F_{x_1, \dots, x_m}} & \mathcal{D}_{x_1, \dots, x_m} \end{array}$$

commutes, where the vertical arrows come from the unital structure.

For a *lax* functor, we merely require that the diagram *lax commute*, i.e., we are given a natural transformation:

$$\begin{array}{ccc}
\mathcal{C}_{x_1, \dots, x_n} & \xrightarrow{F_{x_1, \dots, x_n}} & \mathcal{D}_{x_1, \dots, x_n} \\
\downarrow & \swarrow & \downarrow \\
\mathcal{C}_{x_1, \dots, x_m} & \xrightarrow{F_{x_1, \dots, x_m}} & \mathcal{D}_{x_1, \dots, x_n}
\end{array}$$

This difference is a general feature of working with sheaves of categories on lax prestacks that is different from the more restricted theory of sheaves of categories on usual prestacks. It is discussed in detail in §4, where we remove the adjective “lax” from the term “lax functor.”

For the importance of working with lax functors of unital sheaves of categories, see the discussion of Remark 3.6.3 below.

3.6. Factorization algebras. The heuristic idea of a *factorization algebra* in a factorization category \mathcal{C} is that we are given have $\mathcal{A} \in \mathbf{QCoh}(\mathrm{Ran}_X)$, and for $\{x_1, \dots, x_n\} \subseteq X$, we are given isomorphisms:

$$\mathcal{A}_{x_1, \dots, x_n} \simeq \mathcal{A}_{x_1} \otimes \dots \otimes \mathcal{A}_{x_n} \quad (3.6.1)$$

that are continuous as we vary the points x_i . There is a somewhat subtle requirement as points collide: if we choose $1 \leq k < n$, then we require that the induced isomorphisms:

$$\mathcal{A}_{x_1, \dots, x_n} \simeq \mathcal{A}_{x_1, \dots, x_k} \otimes \mathcal{A}_{x_{k+1}, \dots, x_n}$$

extend only when we allow points x_i to collide with points x_j only when $1 \leq i, j \leq k$ or $k < i, j \leq n$. In particular, for a pair $\{x, y\}$ of distinct points of X , we do not at all specify the behavior of the isomorphism:

$$\mathcal{A}_{x, y} \simeq \mathcal{A}_x \otimes \mathcal{A}_y \quad (3.6.2)$$

as x and y collide.

Remark 3.6.1. In practice, it is unreasonable (except for $\mathcal{A} = \mathcal{O}_X$) to require that the isomorphisms (3.6.2) to extend when x and y collide. However, we may require a map to exist in one direction: this gives the theory of *commutative* factorization sheaves, which we develop in §7.

Similarly, we have the notion of *unital* factorization sheaf. Here we require that the isomorphisms (3.6.1) be compatible in the natural sense with the unital maps (3.5.2).

Again, the notion of (resp. unital) chiral category can be described similarly. Note that we can speak about factorization algebras inside of a chiral category \mathcal{C}_x : this is a continuous assignment of objects $\mathcal{A}_{x_1, \dots, x_n} \in \mathcal{C}_{x_1, \dots, x_n}$ with identifications:

$$\mathcal{A}_{x_1, \dots, x_n} \simeq \mathcal{A}_{x_1} \boxtimes \dots \boxtimes \mathcal{A}_{x_n}$$

in:

$$\mathcal{C}_{x_1, \dots, x_n} \simeq \mathcal{C}_{x_1} \otimes \dots \otimes \mathcal{C}_{x_n}.$$

Remark 3.6.2 (Unit objects). The unital factorization conditions force $\mathcal{C}_\emptyset \simeq \mathbf{Vect}$ canonically. Considering $\emptyset \leftrightarrow \{x\}$, we see that \mathcal{C}_x contains a canonical *unit object* $\mathrm{unit}_{\mathcal{C}_x}$ which by definition is the image of $k \in \mathbf{Vect}$ under the induced functor:

$$\mathbf{Vect} = \mathcal{C}_\emptyset \rightarrow \mathcal{C}_x.$$

Remark 3.6.3 (Unital factorization functors). What a factorization functor should be should be clear in the above heuristics: it is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of categories over $\text{Ran}_{\mathcal{X}}$, such that, e.g., for every pair of distinct points $x, y \in \mathcal{X}$, the diagram:

$$\begin{array}{ccc} \mathcal{C}_{x,y} & \xrightarrow{F_{x,y}} & \mathcal{D}_{x,y} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{C}_x \otimes \mathcal{C}_y & \xrightarrow{F_x \otimes F_y} & \mathcal{D}_x \otimes \mathcal{D}_y. \end{array} \quad (3.6.3)$$

As in Remark 3.5.3, there are two notions of unital factorization functor, *lax* and *strict*.

The difference primarily occurs at the level of underlying sheaves of categories, i.e., in the setting of *loc. cit.* That is to say, we still require the diagram (3.6.3)

The key distinction between lax and strict here is that a strictly unital factorization functor preserves unit objects, while for a lax unital factorization functor, we only have a morphism:

$$\text{unit}_{\mathcal{D}} \rightarrow F(\text{unit}_{\mathcal{C}}).$$

Remark 3.6.4. In [Ras2], the factorization functor we are interested in *does not* preserve unit objects; rather, it is merely lax unital. See [Gai2] for a similar construction.

3.7. The idea for implementing §3.6 is to exploit the *chiral multiplication* of $\text{Ran}_{\mathcal{X}}$ and $\text{Ran}_{\mathcal{X}}^{un}$, that we describe below.

Recall that if $\mathcal{S} \in \text{PreStk}$ is equipped with a commutative and associative multiplication, we can speak of *multiplicative* quasi-coherent sheaves on \mathcal{S} ; for m the multiplication operation, these are quasi-coherent sheaves $\mathcal{A} \in \text{QCoh}(\mathcal{S})$ with isomorphisms:

$$m^*(\mathcal{A}) \simeq \mathcal{A} \boxtimes \mathcal{A}$$

satisfying the natural commutativity and associativity requirements.

Note that $\text{Ran}_{\mathcal{X}}$ admits a natural commutative semigroup structure: the multiplication operation is given by union of subsets of \mathcal{X} . Similarly, $\text{Ran}_{\mathcal{X}}^{un}$ has a commutative monoid structure given in the same way.

Remark 3.7.1. We only say “semigroup” here because $\text{Ran}_{\mathcal{X}}$ does not contain the empty subset of \mathcal{X} , which would correspond to the unit: this should only ever be regarded as a minor issue.

The *chiral multiplication* can be thought of as a *partially-defined* multiplication, where we are only allowed to add two subsets of $\text{Ran}_{\mathcal{X}}$ if they are disjoint.

Then we say that e.g. a factorization sheaf on $\text{Ran}_{\mathcal{X}}$ is a multiplicative sheaf with respect to this partially-defined multiplication.

3.8. **Correspondences.** However, there is still a substantive technical issue: what do we mean by “partially-defined multiplication?”

One convenient approach here is to use the formalism of *correspondences* here, developed in the homotopical setting in [GR].

Recall that if \mathcal{C} is a category with fiber products, the category \mathcal{C}_{corr} is defined to have the same objects as \mathcal{C} , with morphisms $X \rightarrow Y$ given by hats:

$$\begin{array}{ccc} & H & \\ & \swarrow & \searrow \\ X & & Y \end{array}$$

in \mathcal{C} . Composition of morphisms is defined by fiber products, i.e., we regard diagrams:

$$\begin{array}{ccccc}
 & & H_3 & & \\
 & \swarrow & & \searrow & \\
 & H_1 & & H_2 & \\
 \swarrow & & & & \searrow \\
 X & & Y & & Z
 \end{array}$$

with inner square Cartesian as realizing the correspondence $(X \leftarrow H_3 \rightarrow Z)$ as the composition of the morphisms $X \rightarrow Y$ and $Y \rightarrow Z$ in \mathcal{C}_{corr} .

If \mathcal{C} is equipped with a symmetric monoidal structure, then \mathcal{C}_{corr} inherits a symmetric monoidal structure in the obvious way.

Remark 3.8.1. We recall the construction from [GR] in more detail in Appendix B.

3.9. Chiral multiplication via correspondences. We can now say that chiral multiplication is a non-unital commutative algebra structure on $\text{Ran}_{\mathcal{X}}$ when regarded as an object of PreStk_{corr} , where the multiplication operation is defined by the correspondence:

$$\begin{array}{ccc}
 & [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} & \\
 \swarrow & & \searrow \\
 \text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}} & & \text{Ran}_{\mathcal{X}}
 \end{array}$$

where the notation *disj* indicates that we take the locus of this product where subsets are disjoint, and where the right map is the map taking the union of two subsets.

In §5, we develop a theory of multiplicative sheaves of categories on lax prestacks with commutative algebra structures defined using correspondences, giving a definition of factorization category. This is specialized to the case of Ran space in §6.

3.10. Factorization modules. Next, we discuss the idea of factorization modules.

Let \mathcal{A} be a factorization algebra and let x_0 be a point of X . A factorization module structure at x_0 for a vector space M is essentially a rule that associates to every finite set $\{x_0, x_1, \dots, x_n\}$ of points of X a vector space M_{x_0, x_1, \dots, x_n} such that, for every $0 \leq k < n$ we have identifications:

$$M_{x_0, \dots, x_n} \simeq M_{x_0, \dots, x_k} \otimes \mathcal{A}_{x_{k+1}, \dots, x_n}$$

compatible with refinements in the obvious sense.

This notion generalizes in the usual ways: we can allow the x_0 to move, or to take factorization modules at several points at once, or to take unital factorization modules, or to take factorization module categories for a chiral category, etc.

An important point is Theorem 6.13.2, which says that under certain hypotheses, modules for the unit factorization algebra in a unital chiral category are just objects of the underlying category.

A second important point is the construction of *external fusion* from §6.12, that takes chiral modules at two distinct points (or disjoint subsets of points) and produces a module at their union.

Remark 3.10.1. Heuristically, external fusion should make factorization modules for a factorization algebra into a factorization category. However, since the tensor product of DG categories is unwieldy in many respects, we expect that this is only true after appropriate *renormalization* in the sense of [FG2]. In general, the only structure is that of *lax factorization category*, as is discussed in §8.

3.11. Factorization without $\text{Ran}_{\mathcal{X}}$. In §8, we present an alternative approach to chiral categories.

This approach is much more combinatorial than the approach using prestacks and correspondences. Proofs of foundational results, while largely possible in this setting, are much less clean. However, this second approach has the advantage that it only uses finite-dimensional geometry (say if $\mathcal{X} = X$ or X_{dR}), without explicit recourse to the Ran space.

Roughly, in this perspective a factorization sheaf \mathcal{A} on $\text{Ran}_{\mathcal{X}}$ is a compatible system $\mathcal{A}_{\mathcal{X}^I}$ of D -modules on each \mathcal{X}^I , and with identifications:

$$\mathcal{A}_{\mathcal{X}^I} \boxtimes \mathcal{A}_{\mathcal{X}^J} |_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}} \simeq \mathcal{A}_{\mathcal{X}^I \sqcup J} |_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}}.$$

3.12. User's guide. There are two basic results in this text that we will need for [Ras2].

- (1) Proposition-Construction 4.26.1, and its consequence Proposition 6.4.2. These results will be used for constructing unital chiral category structures on various Whittaker categories.

For simplicity, here is what these propositions say we should do to construct a unital structure on $\text{Whit}^{sph} := \text{Whit}(D(\text{Gr}_G))$ (i.e., Whittaker sheaves on Gr_G).

First, we construct a unital structure on $D(\text{Gr}_G)$. For $\{x_1, \dots, x_n\} \subseteq \{x_1, \dots, x_n, x_{n+1}, \dots, x_m\} \subseteq X$ as in Remark 3.5.2, the corresponding unit maps (3.5.2) are given by:

$$\begin{aligned} D(\text{Gr}_{G,x_1}) \otimes \dots \otimes D(\text{Gr}_{G,x_n}) &\simeq D(\text{Gr}_{G,x_1}) \otimes \dots \otimes D(\text{Gr}_{G,x_n}) \otimes \mathbf{Vect} \otimes \dots \otimes \mathbf{Vect} \rightarrow \\ &D(\text{Gr}_{G,x_1}) \otimes \dots \otimes D(\text{Gr}_{G,x_n}) \otimes D(\text{Gr}_{G,x_{n+1}}) \otimes \dots \otimes D(\text{Gr}_{G,x_m}) \end{aligned}$$

where for each $n < i \leq m$, the map $\mathbf{Vect} \rightarrow D(\text{Gr}_{G,x_i})$ sends k to the delta D -module concentrated at the unit point in Gr_{G,x_i} .

For Whittaker sheaves, this construction does not work verbatim because $\mathbf{Vect} \rightarrow D(\text{Gr}_{G,x_i})$ does not factor through the subcategory of Whittaker sheaves. Therefore, we further compose it with the functor of !-averaging against the Whittaker character.

The precise conditions that are needed for this format — which are somewhat more subtle than they appear above because we need to allow points to collide — are discussed in Remark 4.26.2.

- (2) Next, under certain favorable circumstances, we show in Theorem 6.13.2 that for a unital chiral category \mathcal{C} with unit object $\text{unit}_{\mathcal{C}}$, we have $\text{unit}_{\mathcal{C}}\text{-mod}_{un}^{\text{fact}}(\mathcal{C}) \simeq \mathcal{C}$, where these symbols are made sense of in §6. I.e., the result says that the structure of unital module for the unit object is no extra structure at all — certainly a familiar kind of statement!

By functoriality of chiral modules, this result implies that every lax unital functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of unital chiral categories upgrades to a functor:

$$\mathcal{C} \simeq \text{unit}_{\mathcal{C}}\text{-mod}_{un}^{\text{fact}}(\mathcal{C}) \rightarrow F(\text{unit}_{\mathcal{C}})\text{-mod}_{un}^{\text{fact}}(\mathcal{D}).$$

We remark that this generalizes a phenomenon from the theory of monoidal categories: a lax monoidal (in particular, lax unital) functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of monoidal categories induces a functor $\mathcal{C} \rightarrow F(\mathbf{1}_{\mathcal{C}})\text{-mod}(\mathcal{D})$ for $\mathbf{1}_{\mathcal{C}}$ the unit object.

In [Ras2], we will combine the above two techniques and the computations of [Ras1] to give applications to the local geometric Langlands program, as described in the introduction to [Ras1].

4. LAX PRESTACKS AND THE UNITAL RAN SPACE

4.1. In this section, we introduce Ran space as a prestack and its unital counterpart as a lax prestack. We discuss sheaves on lax prestacks in detail.

An important point is Proposition-Construction 4.26.1, which we will use to construct certain important unital sheaves of categories on Ran space.

4.2. Notation for categories of sets. Let \mathbf{Set} denote the $(1,1)$ -category of sets. Let $\mathbf{Set}_{<\infty} \subseteq \mathbf{Set}$ denote the full subcategory of finite sets. Let $\mathbf{fSet}_{\emptyset} \subseteq \mathbf{Set}_{<\infty}$ denote the non-full subcategory with the same objects, but in which we only allow surjective morphisms. Finally, let $\mathbf{fSet} \subseteq \mathbf{fSet}_{\emptyset}$ denote the full subcategory of *non-empty* finite sets.

We consider each of these categories as a non-unital symmetric monoidal category under disjoint unions. Of course, in all cases except \mathbf{fSet} , this symmetric monoidal structure is in fact unital with unit the empty set.

Remark 4.2.1. The notation \mathbf{fSet} is borrowed from [Gai4].

4.3. Let $\mathcal{G} \in \mathbf{Gpd}$ be fixed. We define the groupoids:

$$\begin{aligned} \mathbf{Ran}_{\mathcal{G}} &:= \operatorname{colim}_{I \in \mathbf{fSet}^{op}} \mathcal{G}^I \\ \mathbf{Ran}_{\mathcal{G}, \emptyset} &:= \operatorname{colim}_{I \in \mathbf{fSet}_{\emptyset}^{op}} \mathcal{G}^I \end{aligned}$$

Remark 4.3.1. $\mathbf{Ran}_{\mathcal{G}, \emptyset}$ is just $\mathbf{Ran}_{\mathcal{G}}$ with a disjoint basepoint adjoined. We denote this basepoint by \emptyset where convenient and unambiguous.

The (resp. non-unital) symmetric monoidal structure on the functor $I \mapsto \mathcal{G}^I$ from $\mathbf{fSet}_{\emptyset}$ (resp. \mathbf{fSet}) determines the structure of (resp. non-unital) commutative monoid on $\mathbf{Ran}_{\mathcal{G}, \emptyset}$ (resp. $\mathbf{Ran}_{\mathcal{G}}$), using that product in \mathbf{Cat} commute with colimits in each variable.

We denote the corresponding maps:

$$\begin{aligned} \mathbf{Ran}_{\mathcal{G}} \times \mathbf{Ran}_{\mathcal{G}} &\rightarrow \mathbf{Ran}_{\mathcal{G}} \\ \mathbf{Ran}_{\mathcal{G}, \emptyset} \times \mathbf{Ran}_{\mathcal{G}, \emptyset} &\rightarrow \mathbf{Ran}_{\mathcal{G}, \emptyset} \end{aligned}$$

both by add .

Example 4.3.2. Suppose that $\mathcal{G} \in \mathbf{Set} \subseteq \mathbf{Gpd}$. In this case, one can show that $\mathbf{Ran}_{\mathcal{G}}$ is actually a set as well, and that it identifies in the obvious way with the set of non-empty finite subsets of \mathcal{G} . Similarly, $\mathbf{Ran}_{\mathcal{G}, \emptyset}$ then identifies with the set of possibly empty finite subsets of \mathcal{G} .

Remark 4.3.3. Observe that $\mathcal{G} \mapsto \mathbf{Ran}_{\mathcal{G}}$ and $\mathcal{G} \mapsto \mathbf{Ran}_{\mathcal{G}, \emptyset}$ commute with sifted colimits in the variable \mathcal{G} . Indeed, colimits commute with colimits, and for I finite, $\mathcal{G} \mapsto \mathcal{G}^I$ commutes with sifted colimits by definition of sifted.

Therefore, we can recover the functors $\mathcal{G} \mapsto \mathbf{Ran}_{\mathcal{G}}$ and $\mathcal{G} \mapsto \mathbf{Ran}_{\mathcal{G}, \emptyset}$ as the left Kan extensions of their restrictions to $\mathbf{Set}_{<\infty}$.

4.4. Unital Ran categories. Let \mathcal{G} be a groupoid. We will give three perspectives on a certain category $\mathbf{Ran}_{\mathcal{G}}^{un}$.

4.5. Partial-ordering. In the first construction, suppose first that \mathcal{G} is a set. Recall that in this case $\mathbf{Ran}_{\mathcal{G}, \emptyset}$ is the set of finite subsets of \mathcal{G} . We consider this set as a partially-ordered set under inclusions.

We then declare $\mathbf{Ran}_{\mathcal{G}}^{un} := \mathbf{Poset}_{\mathbf{Ran}_{\mathcal{G}, \emptyset}}$ to be the category associated with this partially-ordered set. It is easy to see that this construction commutes with filtered colimits in the variable \mathcal{G} .

Following Remark 4.3.3, we then extend this definition to an arbitrary groupoid \mathcal{G} by declaring that it should commute with sifted colimits.

4.6. Unital Ran as a lax colimit. We now give a second construction of $\text{Ran}_{\mathcal{G}}^{un}$.

We will begin by defining a second groupoid $'\text{Ran}_{\mathcal{G}}^{un}$, and then in Corollary 4.6.2 we will show that $'\text{Ran}_{\mathcal{G}}^{un}$ is isomorphic to $\text{Ran}_{\mathcal{G}}^{un}$.

Consider the functor $\text{Set}_{<\infty}^{op} \rightarrow \mathbf{Gpd}$ defined by $I \mapsto \mathcal{G}^I$. We denote this functor temporarily by $\Psi_{\mathcal{G}}$.

We then form the Cartesian fibration $\text{coGroth}(\Psi_{\mathcal{G}}) \rightarrow \text{Set}_{<\infty}^{op}$, and define $'\text{Ran}_{\mathcal{G}}^{un}$ to be the result of inverting all arrows in $\text{coGroth}(\Psi_{\mathcal{G}})$ that are Cartesian and lie over a surjective morphism in $\text{Set}_{<\infty}$, i.e., a morphism in fSet^{op} .

Note that unions induce a canonical symmetric monoidal structure on $'\text{Ran}_{\mathcal{G}}^{un}$ (c.f. §5.15).

Proposition 4.6.1. (1) *The functor $\mathcal{G} \mapsto '\text{Ran}_{\mathcal{G}}^{un}$ commutes with sifted colimits.*
 (2) *For \mathcal{G} a set, the functor:*

$$\text{coGroth}(\Psi_{\mathcal{G}}) \rightarrow \text{Poset}_{\text{Ran}_{\mathcal{G},\emptyset}} \quad (4.6.1)$$

sending a datum $(I \in \text{Set}_{<\infty}, x \in \mathcal{G}^I)$ to⁸ $x \in \text{Ran}_{\mathcal{G},\emptyset}$ induces an equivalence:

$$' \text{Ran}_{\mathcal{G}}^{un} \xrightarrow{\simeq} \text{Poset}_{\text{Ran}_{\mathcal{G},\emptyset}}. \quad (4.6.2)$$

Corollary 4.6.2. *There is a functorial equivalence of $\text{Ran}_{\mathcal{G}}^{un} \simeq '\text{Ran}_{\mathcal{G}}^{un}$ of symmetric monoidal categories.*

Proof of Proposition 4.6.1. The first part follows easily from the fact that $\mathcal{G} \mapsto \mathcal{G}^I$ commutes with sifted colimits for I finite.

The map (4.6.1) sends Cartesian arrows over fSet^{op} to isomorphisms, and therefore induces the symmetric monoidal functor (4.6.2).

To prove that this functor is an equivalence (and in particular, that the left hand side is a (1,1)-category), we will explicitly construct an inverse.

For $I = \{x_1, \dots, x_n\}$ a finite subset of \mathcal{G} , we attach an object of $\text{coGroth}(\Psi_{\mathcal{G}})$ in the tautological way: a point of $\text{coGroth}(\Psi_{\mathcal{G}})$ is a pair of a finite set and a subset of \mathcal{G} indexed by that finite set, and we attach the finite set I with the tautological associated subset of \mathcal{G} . This operation is evidently functorial, and projecting to $'\text{Ran}_{\mathcal{G}}^{un}$ evidently provides an inverse. □

4.7. Unital Ran space via tuples of finite sets. We now give a final construction that more explicitly describes $\text{Ran}_{\mathcal{G}}^{un}$ as a category by essentially describing its objects and morphisms and composition law. More precisely, we will describe its complete Segal groupoid.

4.8. Recall that $[n]$ denotes the totally ordered set $\{0, 1, \dots, n\}$ of order $n + 1$.

Let $\text{fSet}_{\emptyset, [n]}^{\rightarrow}$ denote the (1, 1)-category whose objects are data:

$$I_0 \xrightarrow{\varkappa_1} I_1 \xrightarrow{\varkappa_2} \dots \xrightarrow{\varkappa_n} I_n$$

with each I_i a (possibly empty) finite set and \varkappa_i an arbitrary map of sets, and where morphisms are given by commutative diagrams:

⁸Here we are using that objects of $\text{Poset}_{\text{Ran}_{\mathcal{G},\emptyset}}$ are points of $\text{Ran}_{\mathcal{G},\emptyset}$.

$$\begin{array}{ccccccc}
I_0 & \xrightarrow{y_1} & I_1 & \xrightarrow{y_2} & \dots & \xrightarrow{y_n} & I_n \\
\downarrow & & \downarrow & & & & \downarrow \\
J_0 & \xrightarrow{\delta_1} & J_1 & \xrightarrow{\delta_2} & \dots & \xrightarrow{\delta_n} & J_n.
\end{array}$$

The data $[n] \mapsto \mathbf{fSet}_{\emptyset, [n]}^{\rightarrow}$ defines a simplicial category in the obvious way.

Example 4.8.1. For $n = 0$, we recover the category $\mathbf{fSet}_{\emptyset}$ by this construction. This is the reason we include \emptyset in the notation.

Variant 4.8.2. We let $\mathbf{fSet}_{[n]}^{\rightarrow}$ denote the subcategory of $\mathbf{fSet}_{\emptyset}^{\rightarrow}$ in which we only allow *non-empty* finite sets to appear.

4.9. For \mathcal{G} a groupoid, we obtain a functor:

$$\begin{array}{c}
\mathbf{fSet}_{\emptyset, [n]}^{\rightarrow, op} \rightarrow \mathbf{Gpd} \\
I_0 \xrightarrow{y_1} I_1 \xrightarrow{y_2} \dots \xrightarrow{y_n} I_n \mapsto \mathcal{G}^{I_n}.
\end{array}$$

We define $\mathbf{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow}$ as the corresponding colimit:

$$\mathbf{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow} := \operatorname{colim}_{(I_0 \xrightarrow{y_1} I_1 \xrightarrow{y_2} \dots \xrightarrow{y_n} I_n) \in \mathbf{fSet}_{\emptyset, [n]}^{\rightarrow, op}} \mathcal{G}^{I_n} \in \mathbf{Gpd}. \quad (4.9.1)$$

Example 4.9.1. For $n = 0$, we recover $\mathbf{Ran}_{\mathcal{G}, \emptyset}$ through this construction.

Variant 4.9.2. As in Remark 4.8.2, we also obtain groupoids $\mathbf{Ran}_{\mathcal{G}, [n]}^{\rightarrow}$ by forming the colimit (4.9.1) over $\mathbf{fSet}_{[n]}^{\rightarrow, op}$ instead of $\mathbf{fSet}_{\emptyset, [n]}^{\rightarrow, op}$.

Example 4.9.3. For \mathcal{G} a set, one can show as in Example 4.3.2 that $\mathbf{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow}$ is the set with elements data $S_0 \subseteq S_1 \subseteq \dots \subseteq S_n \subseteq \mathcal{G}$ with each S_i finite.

$\mathbf{Ran}_{\mathcal{G}, [n]}^{\rightarrow}$ is similar, but with each S_i additionally assumed non-empty.

4.10. We observe that the assignment $[n] \mapsto \mathbf{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow}$ defines a simplicial groupoid.

Indeed, for $p : [m] \rightarrow [n]$ a map in $\mathbf{\Delta}$, we are supposed to specify a map:

$$\mathbf{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow} \rightarrow \mathbf{Ran}_{\mathcal{G}, \emptyset, [m]}^{\rightarrow}. \quad (4.10.1)$$

We construct it explicitly below.

Recall that $[n] \mapsto \mathbf{fSet}_{\emptyset, [n]}^{\rightarrow}$ is functorial for $[n] \in \mathbf{\Delta}^{op}$. For p as above and $I_0 \xrightarrow{y_1} I_1 \xrightarrow{y_2} \dots \xrightarrow{y_n} I_n \in \mathbf{fSet}_{\emptyset, [n]}^{\rightarrow}$, the induced object of $\mathbf{fSet}_{\emptyset, [m]}^{\rightarrow}$ is:

$$I_{p(0)} \xrightarrow{y_{p(1)}} I_{p(1)} \xrightarrow{y_{p(2)}} \dots \xrightarrow{y_{p(m)}} I_{p(m)} \in \mathbf{fSet}_{\emptyset, [m]}^{\rightarrow}.$$

Observe that we have a corresponding map:

$$\mathcal{G}^{I_n} \rightarrow \mathcal{G}^{I_{p(m)}}.$$

Indeed, there is a canonical map $I_{p(m)} \rightarrow I_n$, and we restrict along it to obtain $\mathcal{G}^{I_n} = \operatorname{Hom}(I_n, \mathcal{G}) \rightarrow \operatorname{Hom}(I_{p(m)}, \mathcal{G}) = \mathcal{G}^{I_{p(m)}}$.

This gives a map:

$$\mathcal{G}^{I_n} \rightarrow \mathcal{G}^{I_{p(m)}} \rightarrow \mathbf{Ran}_{\mathcal{G}, \emptyset, [m]}^{\rightarrow}$$

inducing (4.10.1) as desired.

Example 4.10.1. In Example 4.9.3, this is the obvious simplicial structure.

4.11. One easily finds that the simplicial groupoid $[n] \mapsto \text{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow}$ is a complete Segal space, and therefore defines a category ${}''\text{Ran}_{\mathcal{G}}^{un}$.

Proposition 4.11.1. *''Ran $_{\mathcal{G}}^{un}$ is canonically identified with $\text{Ran}_{\mathcal{G}}^{un}$.*

Proof. For \mathcal{G} a set, this follows from Example 4.9.3. But one clearly has that $\mathcal{G} \mapsto \text{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow}$ commutes with sifted colimits. □

Remark 4.11.2. That $[n] \mapsto \text{Ran}_{\mathcal{G}, \emptyset, [n]}^{\rightarrow}$ is a simplicial commutative monoid gives rise to the symmetric monoidal structure on ${}''\text{Ran}_{\mathcal{G}}^{un}$. The above comparison with $\text{Ran}_{\mathcal{G}}^{un}$ evidently extends to match up these two symmetric monoidal structures.

4.12. Before moving on, we record for later use some notation for the most important cases of the constructions. The reader may safely skip this section and refer back to it as necessary.

First, we follow [Gai4] is using the notations:

$$\text{Ran}_{\mathcal{G}}^{\rightarrow} := \text{Ran}_{\mathcal{G}, [1]}^{\rightarrow} \text{Ran}_{\mathcal{G}, \emptyset}^{\rightarrow} := \text{Ran}_{\mathcal{G}, \emptyset, [1]}^{\rightarrow}.$$

Our simplicial structure gives rise to the following natural maps:

We have the *left* and *right forgetful maps*:

$$\begin{aligned} \text{Oblv}^{\leftarrow} &: \text{Ran}_{\mathcal{G}, \emptyset}^{\rightarrow} \rightarrow \text{Ran}_{\mathcal{G}, \emptyset} \\ \text{Oblv}^{\rightarrow} &: \text{Ran}_{\mathcal{G}, \emptyset}^{\rightarrow} \rightarrow \text{Ran}_{\mathcal{G}, \emptyset} \end{aligned}$$

normalized so that for \mathcal{G} a set, we have:

$$\begin{aligned} \text{Oblv}^{\leftarrow}(S \subseteq T \subseteq \mathcal{G}) &= S \\ \text{Oblv}^{\rightarrow}(S \subseteq T \subseteq \mathcal{G}) &= T. \end{aligned}$$

We also have the map:

$$\begin{aligned} \sigma &: \text{Ran}_{\mathcal{G}, \emptyset} \rightarrow \text{Ran}_{\mathcal{G}, \emptyset}^{\rightarrow} \\ (S \subseteq \mathcal{G}) &\mapsto (S \subseteq S \subseteq \mathcal{G}) \end{aligned}$$

(the formula being literally true for \mathcal{G} a set, and given the obvious meaning otherwise). Note that σ serves as a simultaneous section to both Oblv^{\leftarrow} and $\text{Oblv}^{\rightarrow}$.

4.13. **The disjoint loci.** It is convenient to record the following constructions before proceeding.

Recall that a *monomorphism* of groupoids is synonymous with “fully-faithful functor.” In other words $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a monomorphism if the morphism $\pi_0(\mathcal{G}_1) \rightarrow \pi_0(\mathcal{G}_2)$ is an injective morphism of sets, and the canonical morphism:

$$\mathcal{G}_1 \rightarrow \mathcal{G}_2 \times_{\pi_0(\mathcal{G}_2)} \pi_0(\mathcal{G}_1)$$

is an equivalence. Note that, for \mathcal{G}_2 fixed, the assignment $(\mathcal{G}_1 \mapsto \mathcal{G}_2) \mapsto \pi_0(\mathcal{G}_1) \subseteq \pi_0(\mathcal{G}_2)$ defines a bijection between monomorphisms $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ and subsets of $\pi_0(\mathcal{G}_2)$.

Returning to \mathcal{G} our fixed, groupoid, define the monomorphism:

$$[\text{Ran}_{\mathcal{G}} \times \text{Ran}_{\mathcal{G}}]_{disj} \rightarrow \text{Ran}_{\mathcal{G}} \times \text{Ran}_{\mathcal{G}}$$

by allowing those points in $\text{Ran}_{\mathcal{G}} \times \text{Ran}_{\mathcal{G}}$ whose class in:

$\pi_0(\mathrm{Ran}_{\mathfrak{G}} \times \mathrm{Ran}_{\mathfrak{G}}) = \mathrm{Ran}_{\pi_0(\mathfrak{G})} \times \mathrm{Ran}_{\pi_0(\mathfrak{G})} = \{S, T \subseteq \pi_0(\mathfrak{G}) \text{ pairs of finite subsets}\}$
 is given by a pair of disjoint subsets of $\pi_0(\mathfrak{G})$.

On the other hand, for I, J two non-empty finite sets, we also have the monomorphism:

$$[\mathfrak{G}^I \times \mathfrak{G}^J]_{disj} \rightarrow \mathfrak{G}^I \times \mathfrak{G}^J$$

defined in the same way, or equivalently, as:

$$[\mathfrak{G}^I \times \mathfrak{G}^J]_{disj} := (\mathfrak{G}^I \times \mathfrak{G}^J) \times_{\mathrm{Ran}_{\mathfrak{G}} \times \mathrm{Ran}_{\mathfrak{G}}} [\mathrm{Ran}_{\mathfrak{G}} \times \mathrm{Ran}_{\mathfrak{G}}]_{disj}.$$

We have the canonical morphism:

$$\mathrm{colim}_{I, J \in \mathrm{Set}^{op}} [\mathfrak{G}^I \times \mathfrak{G}^J]_{disj} \rightarrow [\mathrm{Ran}_{\mathfrak{G}} \times \mathrm{Ran}_{\mathfrak{G}}]_{disj}. \quad (4.13.1)$$

Lemma 4.13.1. *The morphism (4.13.1) is an equivalence.*

Proof. Immediate from the universality of colimits in PreStk . □

Variant 4.13.2. Because the 1-full subcategory of $\mathrm{Ran}_{\mathfrak{G}}^{un}$ formed by invertible morphisms identifies with $\mathrm{Ran}_{\mathfrak{G}, \emptyset}$, we obtain the corresponding full subcategory $[\mathrm{Ran}_{\mathfrak{G}}^{un} \times \mathrm{Ran}_{\mathfrak{G}}^{un}]_{disj}$ of $\mathrm{Ran}_{\mathfrak{G}}^{un} \times \mathrm{Ran}_{\mathfrak{G}}^{un}$.

4.14. Lax prestacks. We will digress temporarily to introduce the following convenient formalism.

Definition 4.14.1. A *lax prestack* is an (accessible) functor $\mathrm{AffSch}^{op} \rightarrow \mathrm{Cat}$.

We denote the 2-category of lax prestacks by PreStk^{lax} . We have an obvious embedding $\mathrm{PreStk} \hookrightarrow \mathrm{PreStk}^{lax}$ that admits a right adjoint we will denote by $\mathcal{Y} \mapsto \mathcal{Y}^{\mathrm{PreStk}}$. Note that for \mathcal{Y} a lax prestack and S an affine scheme, $\mathcal{Y}^{\mathrm{PreStk}}(S)$ is computed as the maximal subgroupoid of $\mathcal{Y}(S)$.

We say a lax prestack is *convergent* if it is obtained by right Kan extension from $>^{-\infty}\mathrm{AffSch}$, the category of eventually coconnective affine schemes. We say that a lax prestack is *locally almost of finite type* if it is convergent and for any n , its restriction to $\geq^{-n}\mathrm{AffSch}$ (the category of n -coconnective affine schemes) is left Kan extended from $\geq^{-n}\mathrm{AffSch}_{aft}$ (the category of n -coconnective affine schemes almost of finite type).

4.15. For any lax prestack \mathcal{Y} , we can make sense of $\mathrm{QCoh}(\mathcal{Y})$ as the category of natural transformations $\mathcal{Y} \rightarrow \mathrm{QCoh} : \mathrm{AffSch}^{op} \rightarrow \mathrm{Cat}$.

Remark 4.15.1. Because we require that \mathcal{Y} take values in small categories, $\mathrm{QCoh}(\mathcal{Y})$ is locally small.

If \mathcal{Y} is locally almost of finite type, then we similarly have categories $\mathrm{IndCoh}(\mathcal{Y})$ and $D(\mathcal{Y})$. Note that formation of QCoh , IndCoh and D are contravariant in \mathcal{Y} , and we denote restriction functors in the usual ways.

Note that if \mathcal{Y} is a usual prestack, i.e., \mathcal{Y} takes values in $\mathrm{Gpd} \subseteq \mathrm{Cat}$, then the above notions coincide with the usual ones.

4.16. Somewhat more explicitly, e.g. a quasi-coherent sheaf \mathcal{F} on a lax prestack \mathcal{Y} is an assignment:

$$\begin{aligned} (f : S \rightarrow \mathcal{Y}, S \in \mathrm{AffSch}) &\mapsto f^*(\mathcal{F}) \in \mathrm{QCoh}(S) \\ (T \xrightarrow{g} S \xrightarrow{f} \mathcal{Y}, S, T \in \mathrm{AffSch}) &\mapsto g^* f^*(\mathcal{F}) \simeq (f \circ g)^*(\mathcal{F}) \\ (\varepsilon : f \rightarrow g \in \mathcal{Y}(S)) &\mapsto f^*(\mathcal{F}) \rightarrow g^*(\mathcal{F}). \end{aligned} \quad (4.16.1)$$

4.17. The notion of sheaf of categories on a lax prestack is somewhat more subtle: some 2-categorical problems play a role.

Here is what we want to model:

As in §3.5.3, for \mathcal{Y} a lax prestack we want to define two categories $\mathrm{ShvCat}_{/\mathcal{Y}}^{\mathrm{naive}}$ and $\mathrm{ShvCat}_{/\mathcal{Y}}$ of sheaves of categories on \mathcal{Y} . The objects are the same, but $\mathrm{ShvCat}_{/\mathcal{Y}}^{\mathrm{naive}} \subseteq \mathrm{ShvCat}_{/\mathcal{Y}}$ is merely a 1-full subcategory.

Sheaves of categories on \mathcal{Y} admit a description as in (4.16.1). Then morphisms $C \rightarrow D$ in $\mathrm{ShvCat}_{/\mathcal{Y}}$ amount to the data:

$$\begin{aligned} (f : S \rightarrow \mathcal{Y}, S \in \mathrm{AffSch}) &\mapsto \eta_f : f^*(C) \rightarrow f^*(D) \\ (\varepsilon : f \rightarrow g \in \mathcal{Y}(S)) &\mapsto \begin{array}{ccc} f^*(C) & \longrightarrow & g^*(C) \\ \downarrow \eta_f & \nearrow & \downarrow \eta_g \\ f^*(D) & \longrightarrow & g^*(D) \end{array} \\ (T \xrightarrow{g} S \xrightarrow{f} \mathcal{Y}, S, T \in \mathrm{AffSch}) &\mapsto g^*(\eta_f) \simeq \eta_{f \circ g}. \end{aligned}$$

Here the notation on the second line means that we specify a 2-morphism between the compositions:

$$(f^*(C) \rightarrow f^*(D) \rightarrow g^*(D)) \Longrightarrow (f^*(C) \rightarrow g^*(C) \rightarrow g^*(D)).$$

A morphism as above is a morphism in $\mathrm{ShvCat}_{/\mathcal{Y}}^{\mathrm{naive}}$ if and only if these natural transformations are natural equivalences.

Example 4.17.1. For $C = D = \mathrm{QCoh}_{\mathcal{Y}}$, we have the canonical equivalence:

$$\mathrm{Hom}_{\mathrm{ShvCat}_{/\mathcal{Y}}}(\mathrm{QCoh}_{\mathcal{Y}}, \mathrm{QCoh}_{\mathcal{Y}}) = \mathrm{QCoh}(\mathcal{Y}).$$

Indeed, this is the main motivation for constructing $\mathrm{ShvCat}_{/\mathcal{Y}}$ as we have.

By comparison, if $\mathcal{Y}^{\mathrm{inv}}$ is the prestack obtained from \mathcal{Y} by termwise inverting all arrows, then we have:

$$\mathrm{Hom}_{\mathrm{ShvCat}_{/\mathcal{Y}}^{\mathrm{naive}}}(\mathrm{QCoh}_{\mathcal{Y}}, \mathrm{QCoh}_{\mathcal{Y}}) = \mathrm{QCoh}(\mathcal{Y}^{\mathrm{inv}}).$$

Here the induced functor $\mathrm{QCoh}(\mathcal{Y}^{\mathrm{inv}}) \rightarrow \mathrm{QCoh}(\mathcal{Y})$ is given by pullback along $\mathcal{Y} \rightarrow \mathcal{Y}^{\mathrm{inv}}$, and is fully-faithful.

Remark 4.17.2. We will give a precise construction of the above in what follows. The reader who can take the above on faith may safely skip ahead to §4.20.

4.18. **Lax functors.** Given a category⁹ \mathcal{C} and a 2-category \mathcal{D} , there is a 1-category $\mathrm{Hom}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D})$, the *category of lax functors* $\mathcal{C} \rightarrow \mathcal{D}$, described as follows. Objects of $\mathrm{Hom}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ are functors $F : \mathcal{C} \rightarrow \mathcal{D}$. Morphisms (alias: *lax natural transformations*) $\eta : F \rightarrow G$ are given by data of natural maps $\eta_X : F(X) \rightarrow G(X)$ defined for every $X \in \mathcal{C}$, plus for every $f : X \rightarrow Y$ in \mathcal{C} , we are given a 2-morphism in \mathcal{D} between the compositions:

⁹More generally, a 2-category can be allowed, but we will not use the construction in this generality.

$$\begin{array}{c}
\left(F(X) \xrightarrow{\eta_X} G(X) \xrightarrow{G(f)} G(Y) \right) \\
\Downarrow \\
\left(F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{\eta_Y} G(Y) \right).
\end{array} \tag{4.18.1}$$

For the identity map $\text{id}_X : X \rightarrow X$, this natural transformation should be the tautological 2-isomorphism. Of course, the data above are required to be natural in all variables, compatible with categorical operations (e.g., composition), all understood in the natural meaning given by higher category theory.

Let $\mathcal{D}^{1\text{-cat}}$ denote the 1-category underlying \mathcal{D} , in which we only allow invertible 2-morphisms. Note that $\text{Hom}^{\text{la}x}(\mathcal{C}, \mathcal{D})$ contains $\text{Hom}(\mathcal{C}, \mathcal{D}^{1\text{-cat}})$ as a 1-full subcategory, where objects are the same but morphisms require the 2-morphism (4.18.1) to be invertible.

Remark 4.18.1. If the morphism $f : X \rightarrow Y \in \mathcal{C}$ above is invertible, then the natural transformation (4.18.1) is necessarily invertible. Therefore, $\text{Hom}^{\text{la}x}(\mathcal{C}, \mathcal{D}) = \text{Hom}(\mathcal{C}, \mathcal{D}^{1\text{-cat}})$ if \mathcal{C} is a groupoid.

Remark 4.18.2. Formation of $\text{Hom}^{\text{la}x}(\mathcal{C}, \mathcal{D})$ is appropriately functorial in \mathcal{C} and \mathcal{D} . The best way to say this precisely is to use the definition for \mathcal{C} allowed to be a 2-category, and to say that we have a certain 2-category of 2-categories where the category of functors $\mathcal{C} \rightarrow \mathcal{D}$ is taken to be $\text{Hom}^{\text{la}x}(\mathcal{C}, \mathcal{D})$.

Remark 4.18.3. More generally, suppose that \mathcal{J} is an indexing category and consider objects $i \mapsto \mathcal{C}_i$ and $i \mapsto \mathcal{D}_i$ of $\text{Hom}(\mathcal{J}, 2\text{-Cat})$. Then we have a category $\text{Hom}^{\text{la}x}(\mathcal{C}, \mathcal{D})$ constructed in the same way as above, where roughly, objects of $\text{Hom}^{\text{la}x}(\mathcal{C}, \mathcal{D})$ are compatible functors $\mathcal{C}_i \rightarrow \mathcal{D}_i$, and morphisms are compatible systems of lax natural transformations.

One can alternatively recover this notion from the one presented above (in the case $\mathcal{J} = *$) by using the Grothendieck construction; we do not pursue this here.

4.19. In the framework of Remark 4.18.3, for \mathcal{Y} a lax prestack, we define $\text{ShvCat}_{/\mathcal{Y}}$ as the category of lax morphisms $\mathcal{Y} \rightarrow \text{ShvCat}_{/-}$, where $\text{ShvCat}_{/-}$ is the functor $\text{AffSch}^{\text{op}} \rightarrow 2\text{-Cat}$ sending S to $\text{ShvCat}_{/S}$.

We define $\text{ShvCat}_{/\mathcal{Y}}^{\text{naive}}$ as the category of usual functors $\mathcal{Y} \rightarrow \text{ShvCat}_{/-}$.

Remark 4.19.1. Tautologically, $\text{ShvCat}_{/\mathcal{Y}}$ contains $\text{ShvCat}_{/\mathcal{Y}}^{\text{naive}}$ as a 1-full subcategory with the same underlying groupoid, and therefore we may speak without hesitation about a sheaf of categories on $\mathcal{Y} \in \text{PreStk}^{\text{la}x}$: the only ambiguity is in speaking of morphisms of sheaves of categories. Of course, if \mathcal{Y} is a usual prestack then this issue disappears.

Example 4.19.2. We have the obvious sheaf of categories $\text{QCoh}_{\mathcal{Y}}$ on \mathcal{Y} .

Remark 4.19.3. Note that both $\text{ShvCat}_{/\mathcal{Y}}$ and $\text{ShvCat}_{/\mathcal{Y}}^{\text{naive}}$ admit obvious 2-categorical enhancements, and we will sometimes abuse notation by denoting the corresponding 2-categories by the same notation.

Even better, they both are enriched over $\text{DGCat}_{\text{cont}}$. We abuse notation in letting Hom also denote the enriched Hom over $\text{DGCat}_{\text{cont}}$.

By Example 4.17.1, for $\mathcal{C} \in \text{ShvCat}_{/\mathcal{Y}}$, we define $\Gamma(\mathcal{Y}, \mathcal{C}) \in \text{DGCat}_{\text{cont}}$ as:

$$\Gamma(\mathcal{Y}, \mathcal{C}) := \text{Hom}_{\text{ShvCat}_{/\mathcal{Y}}}(\text{QCoh}_{\mathcal{Y}}, \mathcal{C}).$$

4.20. For every lax prestack \mathcal{Y} , recall that $\mathcal{Y}^{\text{PreStk}}$ denotes the (non-lax) prestack underlying \mathcal{Y} . We have the following obvious lemma:

Lemma 4.20.1. *The functors:*

$$\begin{aligned} \text{QCoh}(\mathcal{Y}) &\rightarrow \text{QCoh}(\mathcal{Y}^{\text{PreStk}}) \\ \text{ShvCat}/_{\mathcal{Y}} &\rightarrow \text{ShvCat}/_{\mathcal{Y}^{\text{PreStk}}} \end{aligned}$$

of restriction along the map:

$$\mathcal{Y}^{\text{PreStk}} \rightarrow \mathcal{Y}$$

are conservative.

4.21. **Ran space for prestacks.** If \mathcal{X} is a prestack, then we obtain the prestack $\text{Ran}_{\mathcal{X}}$ defined by

$$\text{Ran}_{\mathcal{X}}(S) := \text{Ran}_{\mathcal{X}(S)} \in \text{Gpd}$$

for $S \in \text{AffSch}$, and similarly, we have the prestack $\text{Ran}_{\mathcal{X}, \emptyset} = \text{Ran}_{\mathcal{X}} \coprod *$ and the lax prestack $\text{Ran}_{\mathcal{X}}^{\text{un}}$.

Each of $\text{Ran}_{\mathcal{X}, \emptyset}$ and $\text{Ran}_{\mathcal{X}}^{\text{un}}$ admits a commutative monoid structure defined by add , and $\text{Ran}_{\mathcal{X}}$ admits a commutative semigroup structure.

Note that the prestack $\text{Ran}_{\mathcal{X}}^{\text{un, PreStk}}$ underlying $\text{Ran}_{\mathcal{X}}^{\text{un}}$ is $\text{Ran}_{\mathcal{X}, \emptyset}$.

Remark 4.21.1. We obtain prestacks $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ and $\text{Ran}_{\mathcal{X}, \emptyset}^{\rightarrow}$ by the same procedure, referring to §4.12 for the corresponding construction for groupoids. We use the notations Oblv^{\leftarrow} and $\text{Oblv}^{\rightarrow}$ in the same way as in *loc. cit.*

We recall that $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ should be thought of as parametrizing pairs $S \subseteq T \subseteq \mathcal{X}$ of finite sets, and that Oblv^{\leftarrow} is the forgetful map corresponding to the S -variable, while $\text{Oblv}^{\rightarrow}$ is the forgetful map corresponding to the T -variable.

4.22. By definition, a *unital quasi-coherent sheaf* on $\text{Ran}_{\mathcal{X}}$ is a quasi-coherent sheaf on $\text{Ran}_{\mathcal{X}}^{\text{un}}$. Similarly, we have the notion of *unital sheaf of categories* over $\text{Ran}_{\mathcal{X}}$.

For X a scheme of finite type, we say a *unital D -module* on Ran_X is a quasi-coherent sheaf on $\text{Ran}_{X, dR}^{\text{un}} = (\text{Ran}_X^{\text{un}})_{dR}$, and similarly for *unital crystal of categories* on Ran_X .

Notation 4.22.1. For, say, \mathbb{C} a unital sheaf of categories on $\text{Ran}_{\mathcal{X}}$, we generally do not differentiate in our notation between the underlying sheaves of categories on $\text{Ran}_{\mathcal{X}}^{\text{un}}$ and $\text{Ran}_{\mathcal{X}, \emptyset}$, leaving the distinction to context or to some explicit signifier where necessary.

4.23. We will need the following general constructions with unital sheaves of categories on Ran space.

For such \mathbb{C} a unital sheaf of categories, we have a canonical *unit* or *fusion* morphism:

$$\mathfrak{fus} = \mathfrak{fus}_{\mathbb{C}} : \text{Oblv}^{\leftarrow, *}(\mathbb{C}) \rightarrow \text{Oblv}^{\rightarrow, *}(\mathbb{C}) \in \text{ShvCat}/_{\text{Ran}_{\mathcal{X}, \emptyset}^{\rightarrow}} \quad (4.23.1)$$

where the relevant notation was introduced in Remark 4.21.1.

Remark 4.23.1. Of course, such a map exists for unital quasi-coherent sheaves, D -modules, etc.

The following hypothesis is natural to require on the unit of a chiral category.

Definition 4.23.2. The sheaf of categories \mathbb{C} is *adj-unital* if the unit map $\mathfrak{fus}_{\mathbb{C}}$ admits a right adjoint in the 2-category $\text{ShvCat}/_{\text{Ran}_{\mathcal{X}, \emptyset}^{\rightarrow}}$.

4.24. For \mathbf{C} as above, let $\mathbf{C}_\emptyset \in \mathrm{DGCat}_{\mathrm{cont}}$ denote the fiber of \mathbf{C} along the map $\mathrm{Spec}(k) \xrightarrow{\emptyset} \mathrm{Ran}_X^{\mathrm{un}}$. Suppose that we are given an identification $\mathbf{C}_\emptyset \simeq \mathrm{Vect}$.

Applying the restriction functor for sheaves of categories on $\mathrm{Ran}_X^{\mathrm{un}}$ to Ran_X , the map $\mathfrak{Fus}_{\mathbf{C}}$ produces a canonical map:

$$\mathrm{QCoh}_{\mathrm{Ran}_X} \rightarrow \mathbf{C} \in \mathrm{ShvCat}_{/\mathrm{Ran}_X}$$

or equivalently, an object $\mathrm{unit}_{\mathbf{C}}$ of $\Gamma(\mathrm{Ran}_X, \mathbf{C})$.

Definition 4.24.1. The resulting object $\mathrm{unit}_{\mathbf{C}}$ is called the *unit object* of the unital sheaf of categories \mathbf{C} .

Terminology 4.24.2. According to Corollary 4.6.2, a unital sheaf of categories is equivalent to a system (in the homotopical sense) of sheaves of categories $\mathbf{C}_{\mathcal{X}^I} \in \mathrm{ShvCat}_{/\mathcal{X}^I}$ defined for every finite set I , plus compatible morphisms:

$$\Delta_f^*(\mathbf{C}_{\mathcal{X}^I}) \rightarrow \mathbf{C}_{\mathcal{X}^J}$$

for every $f : I \rightarrow J$, with $\Delta_f : \mathcal{X}^J \rightarrow \mathcal{X}^I$ the induced map, and such that when f is a surjection this map is an equivalence.

For a pair of finite sets I and J , the inclusion $I \hookrightarrow I \amalg J$ therefore defines a map:

$$\mathbf{C}_{\mathcal{X}^I} \boxtimes \mathrm{QCoh}_{\mathcal{X}^J} \rightarrow \mathbf{C}_{\mathcal{X}^I \amalg J}$$

that we will also refer to as a *unit functor*.

4.25. Let $\mathcal{Y} \in \mathrm{PreStk}^{\mathrm{lax}}$ be fixed. As in §A.4, we say that a functor $\mathbf{D} \rightarrow \mathbf{C}$ in $\mathrm{ShvCat}_{/\mathcal{Y}}$ is (locally) *fully-faithful* if for every affine scheme S and map $f : S \rightarrow \mathcal{Y}$ the corresponding functor $\Gamma(S, f^*(\mathbf{D})) \rightarrow \Gamma(S, f^*(\mathbf{C}))$ is fully-faithful.

The following lemma records the immediate consequences of the definition.

Lemma 4.25.1. *Let \mathcal{Y} be a lax prestack.*

- (1) *A morphism $\mathbf{D} \rightarrow \mathbf{C}$ in $\mathrm{ShvCat}_{/\mathcal{Y}}$ is fully-faithful if and only if its restriction to $\mathcal{Y}^{\mathrm{PreStk}}$ is.*
- (2) *Every fully-faithful functor is a monomorphism in the category $\mathrm{ShvCat}_{/\mathcal{Y}}$. Moreover, given $\mathbf{D} \rightarrow \mathbf{C}$ fully-faithful and a map $\varphi : \mathbf{E} \rightarrow \mathbf{C}$, to see if φ factors through \mathbf{D} it suffices to check this after restriction to $\mathcal{Y}^{\mathrm{PreStk}}$.*
- (3) *For $\mathbf{D} \rightarrow \mathbf{C}$ fully-faithful, the induced functor:*

$$\Gamma(\mathcal{Y}, \mathbf{D}) \rightarrow \Gamma(\mathcal{Y}, \mathbf{C})$$

is fully-faithful.

- (4) *Fully-faithful functors are preserved under pullbacks $\mathcal{Y}' \rightarrow \mathcal{Y}$.*
- (5) *Given $\mathbf{C} \in \mathrm{ShvCat}_{/\mathcal{Y}}$ with restriction $\bar{\mathbf{C}} \in \mathrm{ShvCat}_{/\mathcal{Y}^{\mathrm{PreStk}}}$, the datum of a fully-faithful functor $\mathbf{D} \rightarrow \mathbf{C}$ in $\mathrm{ShvCat}_{/\mathcal{Y}}^{\mathrm{naive}}$ is equivalent to the datum of a fully-faithful embedding:*

$$\bar{\mathbf{D}} \hookrightarrow \bar{\mathbf{C}} \in \mathrm{ShvCat}_{/\mathcal{Y}^{\mathrm{PreStk}}}$$

such that, for every test scheme S and pair of morphisms $f, g : S \rightarrow \mathcal{Y}$ with a 2-morphism $\varepsilon : f \rightarrow g \in \mathcal{Y}(S)$, the induced functor:

$$\Gamma(S, f^*(\bar{\mathbf{D}})) \rightarrow \Gamma(S, g^*(\bar{\mathbf{D}}))$$

maps $\Gamma(S, f^(\bar{\mathbf{D}}))$ to $\Gamma(S, g^*(\bar{\mathbf{D}}))$.*

4.26. Next, we give a general construction of unital sheaves of categories that is useful, for example, in dealing with the geometric Whittaker models. The reader without interest in such applications may safely skip this material and go ahead to §4.27.

The following result is somewhat technical and perhaps difficult to interpret. We present it in a more down-to-earth way in Remark 4.26.2.

Proposition-Construction 4.26.1. *Suppose that \mathcal{C} is an adj-unital sheaf of categories on $\text{Ran}_{\mathcal{X}}$, \mathcal{D} is a sheaf of categories on $\text{Ran}_{\mathcal{X},\emptyset}$, and we are given a fully-faithful functor:*

$$\mathcal{D} \hookrightarrow \mathcal{C} \in \text{ShvCat}/\text{Ran}_{\mathcal{X},\emptyset}.$$

Suppose that we have $\mathcal{D}_{\emptyset} \xrightarrow{\cong} \mathcal{C}_{\emptyset} \simeq \text{Vect}$, where the former is induced by the fully-faithful functor and the latter is an extra piece of structure.

Let:

$$\mathfrak{Fus}_{\mathcal{C}}^R : \text{Oblv}^{\rightarrow,*}(\mathcal{C}) \rightarrow \text{Oblv}^{\leftarrow,*}(\mathcal{C}) \in \text{ShvCat}/\text{Ran}_{\mathcal{X},\emptyset}^{\rightarrow}$$

denote the right adjoint to the functor $\mathfrak{Fus}_{\mathcal{C}}$ from (4.23.1).

Suppose that $\mathfrak{Fus}_{\mathcal{C}}^R$ sends $\text{Oblv}^{\rightarrow,}(\mathcal{D})$ into $\text{Oblv}^{\leftarrow,*}(\mathcal{D}) \subseteq \text{Oblv}^{\leftarrow,*}(\mathcal{C})$.*

Suppose, moreover, that the corresponding functor:

$$\text{Oblv}^{\rightarrow,*}(\mathcal{D}) \rightarrow \text{Oblv}^{\leftarrow,*}(\mathcal{D}) \in \text{ShvCat}/\text{Ran}_{\mathcal{X},\emptyset}^{\rightarrow}$$

admits a left adjoint $\mathfrak{Fus}_{\mathcal{D}}$.

Then \mathcal{D} inherits a canonical unital structure such that the functor $\mathcal{D} \rightarrow \mathcal{C}$ upgrades to a functor of unital sheaves of categories on $\text{Ran}_{\mathcal{X}}$. The unit for this structure is given by $\mathfrak{Fus}_{\mathcal{D}}$.

Remark 4.26.2. We use the notation of §3.5 to speak about unital sheaves of categories. For compatibility with *loc. cit.*, we use the notation X in place of \mathcal{X} , and \mathcal{C} and \mathcal{D} in place of \mathcal{C} and \mathcal{D} .

The question Proposition-Construction 4.26.1 addresses is, given \mathcal{C} a unital sheaf of categories and a (non-unital) subcategory \mathcal{D} , when does \mathcal{D} inherit a unital structure?

One easy answer: if the unit maps preserve \mathcal{D} . I.e., in our heuristic, this says that for every embedding:

$$\{x_1, \dots, x_n\} \subseteq \{x_1, \dots, x_n, x_{n+1}, \dots, x_m\} \subseteq X$$

we have:

$$\begin{array}{ccc} \mathcal{D}_{x_1, \dots, x_n} & \xrightarrow{\quad \quad \quad} & \mathcal{D}_{x_1, \dots, x_m} \\ \downarrow & & \downarrow \\ \mathcal{C}_{x_1, \dots, x_n} & \longrightarrow & \mathcal{C}_{x_1, \dots, x_m}. \end{array} \tag{4.26.1}$$

Proposition-Construction 4.26.1 gives a less obvious situation in which \mathcal{D} still inherits a unit structure.

It asks the following:

- The functors $\mathcal{C}_{x_1, \dots, x_n} \rightarrow \mathcal{C}_{x_1, \dots, x_m}$ should admit right adjoints.
- The right adjoints $\mathcal{C}_{x_1, \dots, x_m} \rightarrow \mathcal{C}_{x_1, \dots, x_n}$ should take $\mathcal{D}_{x_1, \dots, x_m}$ to $\mathcal{D}_{x_1, \dots, x_n}$, i.e., we ask for the mirror image of the diagram (4.26.1).
- The resulting functors $\mathcal{D}_{x_1, \dots, x_m} \rightarrow \mathcal{D}_{x_1, \dots, x_n}$ should admit left adjoints.

In this case, \mathcal{D} will admit a unit structure with unit maps:

$$\mathcal{D}_{x_1, \dots, x_n} \rightarrow \mathcal{D}_{x_1, \dots, x_m}$$

given by these left adjoints.

We emphasize that this does not at all force the diagram (4.26.1) to commute (and it will not for Whittaker sheaves!); this is exactly the difference between $\mathrm{ShvCat}_{/_-}$ and $\mathrm{ShvCat}_{/_-}^{\mathrm{naive}}$.

Warning 4.26.3. The heuristic of Remark 4.26.2 sweeps an important point under the rug: it is not enough to check these properties pointwise — one needs to verify them as the points move and are allowed to collide.

Proof of Proposition-Construction 4.26.1. We freely use the description of unital Ran space from §4.7. We also assume the 2-categorical formalism of [GR], which allows us to functorially pass to adjoints.

Let $\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}, \mathrm{op}}$ denote the lax prestack in which we take opposite categories at every point.

The adj-unital condition on \mathbf{C} produces a sheaf of categories $\tilde{\mathbf{C}}$ on $\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}, \mathrm{op}}$ with “fusion” given by (4.26.1).

Then Lemma 4.25.1 produces a sheaf of categories $\tilde{\mathbf{D}}$ on $\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}, \mathrm{op}}$ with a fully-faithful functor:

$$\tilde{\mathbf{D}} \rightarrow \tilde{\mathbf{C}} \in \mathrm{ShvCat}_{/\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}, \mathrm{op}}}^{\mathrm{naive}}.$$

Finally, passing to left adjoints, we obtain the desired result. \square

4.27. Define the prestack $[\mathcal{X} \times \mathcal{X}]_{\mathrm{disj}}$ as follows. An S -point is the data of a pair of maps $x_i : S \rightarrow \mathcal{X}$, $i = 1, 2$ with the property that the diagram:

$$\emptyset \longrightarrow S \begin{array}{c} \xrightarrow{x_1} \\ \xrightarrow{x_2} \end{array} \mathcal{X}$$

is an equalizer diagram (equivalently, $S \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} = \emptyset$). We have a tautological monomorphism $[\mathcal{X} \times \mathcal{X}]_{\mathrm{disj}} \hookrightarrow \mathcal{X} \times \mathcal{X}$.

More generally, for I and J two finite sets, we write $[\mathcal{X}^I \times \mathcal{X}^J]_{\mathrm{disj}}$ for the prestack whose S -points are maps $S \rightarrow \mathcal{X}^I \times \mathcal{X}^J$ such that for every $i \in I$ and $j \in J$, the induced point of $\mathcal{X} \times \mathcal{X}$ factors through $[\mathcal{X} \times \mathcal{X}]_{\mathrm{disj}}$.

We then define:

$$[\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{\mathrm{disj}} := \mathrm{colim}_{I, J \in \mathrm{Set}^{\mathrm{op}}} [\mathcal{X}^I \times \mathcal{X}^J]_{\mathrm{disj}}.$$

Finally, we obtain the lax prestack $[\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}} \times \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}]_{\mathrm{disj}}$ as in Variant 4.13.2.

Remark 4.27.1. We emphasize that for a test scheme S , $[\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}]_{\mathrm{disj}}(S) \rightarrow [\mathrm{Ran}_{\mathcal{X}(S)} \times \mathrm{Ran}_{\mathcal{X}(S)}]_{\mathrm{disj}}$ (the second term being defined by §4.13) is *not* an equivalence: the right hand side is not even functorial in S . We remark that the definition we give here is appropriate in any (∞) -topos.

4.28. Let $\mathrm{PreStk}_{\mathrm{corr}}$ and $\mathrm{PreStk}_{\mathrm{corr}}^{\mathrm{lux}}$ denote the categories of correspondences associated with the complete categories PreStk and $\mathrm{PreStk}^{\mathrm{lux}}$. We regard these categories as equipped with the usual symmetric monoidal structures computed objectwise by Cartesian products.

We have canonical non-unital commutative algebra structures on $\mathrm{Ran}_{\mathcal{X}}$ in $\mathrm{PreStk}_{\mathrm{corr}}$ and $\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$ in $\mathrm{PreStk}_{\mathrm{corr}}^{\mathrm{lux}}$, where the multiplication maps are defined by the correspondences:

$$\begin{array}{ccc}
[\mathrm{Ran}_\chi \times \mathrm{Ran}_\chi]_{disj} & & [\mathrm{Ran}_\chi^{un} \times \mathrm{Ran}_\chi^{un}]_{disj} \\
\swarrow & \searrow & \swarrow \quad \searrow \\
\mathrm{Ran}_\chi \times \mathrm{Ran}_\chi & \mathrm{Ran}_\chi & \mathrm{Ran}_\chi^{un} \times \mathrm{Ran}_\chi^{un} \quad \mathrm{Ran}_\chi^{un}
\end{array}$$

For Ran_χ^{un} , this commutative algebra structure is unital, with the obvious unit.

We let add_{disj} denote each of the right arrows in the correspondences above.

For emphasis, we will write Ran_χ^{ch} and $\mathrm{Ran}_\chi^{un,ch}$ for the resulting commutative algebras, referring to the multiplication as the *chiral product*.

We will also denote by $\mathrm{Ran}_{\chi,\emptyset}^*$ the commutative monoid in PreStk given by $\mathrm{Ran}_{\chi,\emptyset}$ with the multiplication add , and similarly for $\mathrm{Ran}_\chi^* \in \mathrm{ComAlg}_{non-unital}(\mathrm{PreStk})$ and $\mathrm{Ran}_\chi^{un,*} \in \mathrm{ComAlg}(\mathrm{PreStk}^{lax})$.

Remark 4.28.1. For a more detailed approach to the construction of the chiral product, see §8.7.

5. MULTIPLICATIVE SHEAVES AND CORRESPONDENCES

5.1. In this section, we provide a general language that we will apply in §6 to the Ran space to obtain the theory of chiral categories.

5.2. The material of this section is mostly a matter of organization of the type that is not typically needed outside of homotopical algebra.

Therefore, we give an extended introduction to its contents in §5.3-5.8.

5.3. **Algebras under correspondences.** Our basic format is a (lax) prestack \mathcal{S} with a *commutative* algebra structure under correspondences.

Concretely, this means that we are given *multiplication* and *unit* correspondences:

$$\begin{array}{ccc}
& \mathrm{mult}_{\mathcal{S}} & \\
m_1 \swarrow & & \searrow m_2 \\
\mathcal{S} \times \mathcal{S} & & \mathcal{S}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
& \mathrm{unit}_{\mathcal{S}} & \\
e_1 \swarrow & & \searrow e_2 \\
* & & \mathcal{S}
\end{array}$$

satisfying various associativity and commutativity conditions. E.g., commutativity here says that $\mathrm{mult}_{\mathcal{S}}$ is given a $\mathbb{Z}/2\mathbb{Z}$ -action with m_1 being $\mathbb{Z}/2\mathbb{Z}$ -equivariant with respect to switching the two factors of the target, and m_2 being $\mathbb{Z}/2\mathbb{Z}$ -equivariant with respect to the trivial action on \mathcal{S} .

Example 5.3.1. As in §3.9, $\mathrm{Ran}_{\chi,\emptyset}$ and Ran_χ^{un} admit this structure using the loci of disjoint pairs of points.

5.4. **Multiplicative sheaves of categories.** Given such a datum, we define in §5.21 the notion of *multiplicative sheaf of categories* on \mathcal{S} .

Up to homotopic problems, this means that we give a sheaf of categories Ψ on \mathcal{S} along with isomorphisms:

$$\begin{aligned}
m_1^*(\Psi) &\simeq m_2^*(\Psi) \in \mathrm{ShvCat}_{/\mathrm{mult}_{\mathcal{S}}} \\
\mathrm{QCoh}_{\mathrm{unit}_{\mathcal{S}}} &\simeq e_2^*(\Psi) \in \mathrm{ShvCat}_{/\mathrm{unit}_{\mathcal{S}}}
\end{aligned}$$

with these isomorphisms satisfying associativity and commutativity.

Remark 5.4.1. We also introduce a notion of *weakly* multiplicative sheaf of categories, where e.g. we are only required to specify a morphism:

$$m_1^*(\Psi) \rightarrow m_2^*(\Psi)$$

5.5. Multiplicative sheaves. Given Ψ a multiplicative sheaf of categories on \mathcal{S} , there is a notion of *multiplicative object* ψ of Ψ .

This is an object:

$$\psi \in \Gamma(\mathcal{S}, \Psi)$$

with isomorphisms:

$$\begin{aligned} m_1^*(\psi) &\simeq m_2^*(\psi) \in \Gamma(\text{mult}_{\mathcal{S}}, m_1^*(\Psi)) \simeq \Gamma(\text{mult}_{\mathcal{S}}, m_2^*(\Psi)) \\ \mathcal{O}_{\text{unit}_{\mathcal{S}}} &\simeq e_2^*(\psi) \in \text{QCoh}(\text{unit}_{\mathcal{S}}) \simeq \Gamma(\text{mult}_{\mathcal{S}}, e_2^*(\Psi)). \end{aligned}$$

Remark 5.5.1. As in Remark 5.4.1, there is a similar notion of weakly multiplicative object of a weakly multiplicative sheaf of categories.

5.6. Modules. There are variants of the above notions for modules. Let \mathcal{S} , Ψ , and ψ be as above.

A *module space* for \mathcal{S} is a (lax) prestack \mathcal{M} which is a module for \mathcal{S} under correspondences, so we are in particular given an *action* correspondence:

$$\begin{array}{ccc} & \text{act}_{\mathcal{M}} & \\ \text{act}_1 \swarrow & & \searrow \text{act}_2 \\ \mathcal{S} \times \mathcal{M} & & \mathcal{M} \end{array}$$

defining an associative and unital action of \mathcal{S} in the sense of correspondences.

We can then speak about Ψ -module categories on \mathcal{M} : this is the datum of a sheaf of categories Φ being a *module for* Ψ . This means that we are given isomorphisms:

$$\text{act}_1^*(\Psi \boxtimes \Phi) \simeq \text{act}_2^*(\Phi) \in \text{ShvCat}/_{\text{act}_{\mathcal{M}}}$$

satisfying associativity and unitality.

In this case, we can also speak about *modules for* ψ . Such a datum is an object $\phi \in \Gamma(\mathcal{M}, \Phi)$ equipped with associative and unital isomorphisms:

$$\text{act}_1^*(\psi \boxtimes \phi) \simeq \text{act}_2^*(\phi) \in \Gamma(\text{act}_{\mathcal{M}}, \text{act}_1^*(\Psi \boxtimes \Phi)) \simeq \Gamma(\text{act}_{\mathcal{M}}, \text{act}_2^*(\Phi)).$$

Remark 5.6.1. The above is an indication that multiplicative sheaves can be defined in much more generality: they can be defined for any colored operad. Then, e.g., taking the colored operad of choice to be the operad for a commutative algebra and a module over it, one recovers the above.

5.7. Finally, in §5.31-5.32 we mention that subcategories and quotients of multiplicative sheaves of categories inherit such structures when certain obvious conditions are satisfied: for subcategories, the multiplicative isomorphisms should induce an isomorphism between the subcategories, and for quotient categories, there is an ideal-type condition to be satisfied.

We refer to *loc. cit.*, where these conditions are spelled out completely (and in a way that should be easy to read given the above).

5.8. At this point, the reader may safely skip ahead to §6.

5.9. A Grothendieck construction among correspondences. The major technical tool we will use is the following construction:

Given a functor¹⁰ $F : \mathcal{J}^{op} \rightarrow \mathbf{Cat}_{pres}$, we will define a certain category $\mathbf{Groth}_{corr}(F)$, described below.

This construction will play a key role in setting up the theory of multiplicative sheaves in the correspondence setting. With that said, the reader should be fine understanding the heuristic description below and skipping ahead to §5.18 to see how it is actually used (which we do not to explain presently).

$\mathbf{Groth}_{corr}(F)$ has the following properties:

- Objects of $\mathbf{Groth}_{corr}(F)$ are pairs $i \in \mathcal{J}$ and $X_i \in F(i)$.
- Morphisms $(i, X_i) \rightarrow (j, X_j)$ in $\mathbf{Groth}_{corr}(F)$ are given by the data of a correspondence:

$$\begin{array}{ccc} & h & \\ \alpha \swarrow & & \searrow \beta \\ i & & j \end{array}$$

in \mathcal{J} , and a morphism:¹¹

$$\varphi_{ij} : \alpha(X_i) \rightarrow \beta(X_j) \in F(h).$$

- To compute compositions, we compose the correspondences in \mathcal{J} in the usual way:

$$\begin{array}{ccccc} & & h \times h' & & \\ & & j & & \\ \varepsilon \swarrow & & & \searrow \eta & \\ & h & & & h' \\ \alpha \swarrow & & \beta & \swarrow \gamma & \searrow \delta \\ i & & j & & k \end{array}$$

and take the induced map:

$$\varepsilon\alpha(X_i) \xrightarrow{\varepsilon(\varphi_{ij})} \varepsilon\beta(X_j) = \eta\gamma(X_j) \xrightarrow{\eta(\varphi_{jk})} \eta\delta(X_k)$$

in $F(h \times_j h')$.

Remark 5.9.1. In 5.15-5.16, we will explain that if \mathcal{J} is equipped with a symmetric monoidal structure and F is lax symmetric monoidal, then $\mathbf{Groth}_{corr}(F)$ inherits a natural symmetric monoidal structure.

5.10. Suppose that \mathcal{J} is a category equipped with a functor $F : \mathcal{J}^{op} \rightarrow \mathbf{Cat}_{pres}$, where we recall that \mathbf{Cat}_{pres} denotes the category of cocomplete categories under functors commuting with all colimits.

Lemma 5.10.1. *If \mathcal{J} admits fiber products, then the category $\mathbf{Groth}(F)$ admits pushouts. The functor $\mathbf{Groth}(F) \rightarrow \mathcal{J}^{op}$ commutes with pushouts.*

¹⁰The covariance of the functor F is for convenience: it is what occurs in practice for us, and the author personally finds the notation easier to follow this way.

¹¹Our notation follows the convention of §2.7 here.

Proof. This follows from the results in [Lur1] §4.3.1.

For completeness, we note that pushouts can be computed in the following manner. For a diagram:

$$\begin{array}{ccc} X_k & \longrightarrow & X_i \\ \downarrow & & \\ X_j & & \end{array}$$

in $\text{Groth}(F)$, one forms the pushout of the diagram:

$$\begin{array}{ccc} \varkappa(X_k) & \longrightarrow & \beta(X_j) \\ \downarrow & & \\ \alpha(X_i) & & \end{array}$$

in $\mathcal{J}_{i \times_k j}$, where α, β and \varkappa are the maps $i \times_k j \rightarrow i$, $i \times_k j \rightarrow j$ and $i \times_k j \rightarrow k$ in \mathcal{J} . □

Remark 5.10.2. The above can be generalized to any class of diagrams in place of pushouts. Moreover, we only need to require that F is a functor to the category of categories admitting colimits for these diagrams under functors preserving such.

5.11. For a category \mathcal{C} with pushouts, we let $\mathcal{C}_{op-corr}$ denote the category of correspondences for \mathcal{C}^{op} . We represent morphisms $X \rightarrow Y$ in $\mathcal{C}_{op-corr}$ by diagrams:

$$\begin{array}{ccc} X & & Y \\ & \searrow & \swarrow \\ & H & \end{array}$$

Remark 5.11.1. The category $\mathcal{C}_{op-corr}$, being a category of correspondences, admits a canonical 2-category enhancement $\mathcal{C}_{op-corr}^{2-cat}$. For clarity the sake of clarity, we note that this construction is normalized so that a 2-morphism:

$$\left(\begin{array}{ccc} X & & Y \\ & \searrow & \swarrow \\ & H_1 & \end{array} \right) \longrightarrow \left(\begin{array}{ccc} X & & Y \\ & \searrow & \swarrow \\ & H_2 & \end{array} \right)$$

is equivalent to a commutative diagram:

$$\begin{array}{ccc} X & & Y \\ & \searrow & \swarrow \\ & H_1 & \\ & \updownarrow & \\ & H_2 & \end{array}$$

5.12. Suppose that \mathcal{J} admits fiber products and $F : \mathcal{J}^{op} \rightarrow \text{Cat}_{pres}$ is a functor.

By Lemma 5.10.1, we may form the category $\text{Groth}(F)_{op-corr}$.

5.13. The category $\mathbf{Groth}(F)_{op-corr}$ may be described explicitly as follows.

The objects of $\mathbf{Groth}(F)_{op-corr}$ are pairs $i \in \mathcal{J}$, $X_i \in F(i)$. Morphisms $X_i \rightarrow X_j$ are given by the data of a hat:

$$\begin{array}{ccc} & h & \\ \alpha \swarrow & & \searrow \beta \\ i & & j \end{array} \quad (5.13.1)$$

in \mathcal{J} , an object $H_h \in F(h)$, and a diagram:

$$\begin{array}{ccc} \alpha(X_i) & & \beta(X_j) \\ & \searrow & \swarrow \\ & H_h & \end{array} \quad (5.13.2)$$

in $F(h)$. Composition of two morphisms $X_i \rightarrow X_j \rightarrow X_k$ is defined by forming the fiber product:

$$\begin{array}{ccccc} & h'' & \xlongequal{\quad} & h \times_j h' & \\ & \varepsilon \swarrow & & \searrow \eta & \\ & h & & h' & \\ \alpha \swarrow & & \beta & \xrightarrow{\quad} & \delta \searrow \\ i & & j & & k \end{array} \quad (5.13.3)$$

and then taking the induced diagram:

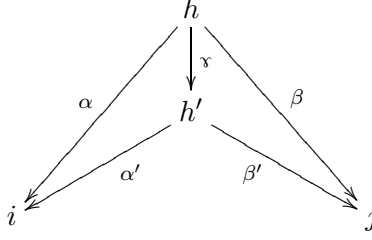
$$\begin{array}{ccccc} \varepsilon\alpha(X_i) & & \varepsilon\beta(X_j) \xlongequal{\quad} \eta\gamma(X_j) & & \eta\delta(X_k) \\ & \searrow & & \swarrow & \\ & \varepsilon(H_h) & & \eta(H_{h'}) & \\ & & \searrow & \swarrow & \\ & & H_{h''} & & \end{array}$$

5.14. Define the 1-full subcategory $\mathbf{Groth}_{corr}(F) \subseteq \mathbf{Groth}(F)_{op-corr}$ by allowing the same objects, but only allowing morphisms (5.13.2) in which the map $\beta(X_j) \rightarrow H_h$ is an equivalence in $F(h)$.

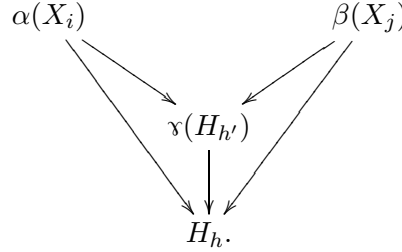
Note that $\mathbf{Groth}_{corr}(F)$ is equipped with a functor to \mathcal{J}_{corr} and the fiber of $\mathbf{Groth}_{corr}(F)$ over the 1-full subcategory \mathcal{J}^{op} of \mathcal{J}_{corr} is equivalent to $\mathbf{Groth}(F)$. Moreover, the fiber of $\mathbf{Groth}_{corr}(F)$ over any object $i \in \mathcal{J}$ is equivalent to $F(i)$.

Variant 5.14.1. As in Remark 5.11.1, $\mathbf{Groth}(F)_{op-corr}$ admits a canonical 2-categorical enhancement $\mathbf{Groth}(F)_{op-corr}^{2-cat}$. We will define a similar 2-categorical structure $\mathbf{Groth}_{corr}(F)^{2-cat}$ on $\mathbf{Groth}_{corr}(F)$.

In the explicit terms used above, 2-morphisms in $\mathbf{Groth}(F)_{op-corr}^{2-cat}$ between morphisms in $\mathbf{Groth}_{corr}(F)$ are represented by pairs of commutative diagrams:



and:



We will take the corresponding 2-categorical structure $\mathbf{Groth}_{corr}(F)^{2-cat}$ on $\mathbf{Groth}_{corr}(F)$ where we also require that the corresponding morphism $\gamma(H_{h'}) \rightarrow H_h$ is an equivalence.

Note that the corresponding morphism $\mathbf{Groth}_{corr}(F) \rightarrow \mathcal{J}_{corr}$ upgrades to a functor $\mathbf{Groth}_{corr}(F)^{2-cat} \rightarrow \mathcal{J}_{corr}^{2-cat}$ of 2-categories, because $\mathbf{Groth}(F)_{op-corr} \rightarrow \mathcal{J}_{corr}$ obviously does.

Remark 5.14.2. The reason for only allowing certain 2-morphisms in Variant 5.11.1 is so that the fiber product:

$$\mathbf{Groth}_{corr}(F)^{2-cat} \times_{\mathcal{J}_{corr}^{2-cat}} \mathcal{J}_{corr}$$

identifies with $\mathbf{Groth}_{corr}(F)$. Of course, here $\mathcal{J}_{corr} \rightarrow \mathcal{J}_{corr}^{2-cat}$ is the embedding of the 2-full subcategory where we only allow invertible 2-morphisms.

5.15. We digress to give a general construction from category theory.

Suppose that \mathcal{C} is a category equipped with a functor:

$$\Phi : \mathcal{C} \rightarrow \mathbf{Cat}.$$

Recall that objects of the base of the coCartesian fibration $\mathbf{Groth}(\Phi) \rightarrow \mathcal{C}$ may be described as pairs (Y, Z) consisting of $Y \in \mathcal{C}$ and $Z \in \Phi(Y)$.

Now suppose that \mathcal{C} is equipped with a symmetric monoidal structure \otimes and Φ is lax symmetric monoidal. For $Y_1, Y_2 \in \mathcal{C}$ we let $\varepsilon_{Y_1, Y_2} : F(Y_1) \times \Phi(Y_2) \rightarrow \Phi(Y_1 \otimes Y_2)$ denote the corresponding functor.

In this case, $\mathbf{Groth}(\Phi)$ is equipped with a canonical symmetric monoidal structure as well so that $\mathbf{Groth}(\Phi) \rightarrow \mathcal{C}$ is symmetric monoidal. E.g., the product is given pointwise by the formula:

$$(Y_1, Z_1) \otimes (Y_2, Z_2) = (Y_1 \otimes Y_2, \varepsilon_{Y_1, Y_2}(Z_1, Z_2)).$$

Remark 5.15.1. This construction generalizes to any colored operad. In particular, the above generalizes the the non-unital symmetric monoidal case and there is an obvious variant in the presence of a module category for \mathcal{C} with a (lax) compatible functor to \mathbf{Cat} .

Remark 5.15.2. In the above setting, let $\text{coGroth}(\Phi) \rightarrow \mathcal{C}^{op}$ denote the corresponding Cartesian fibration. By duality, in the above setting $\text{coGroth}(\Phi)$ carries a canonical (resp. non-unital) symmetric monoidal structure such that $\text{coGroth}(\Phi) \rightarrow \mathcal{C}$ is symmetric monoidal.

5.16. Suppose now that \mathcal{J} is equipped with a symmetric monoidal structure and $F : \mathcal{J}^{op} \rightarrow \text{Cat}_{pres}$ is lax symmetric monoidal for the Cartesian monoidal structure on Cat_{pres} .

As in §5.15, $\text{Groth}_{corr}(F)$ carries a canonical symmetric monoidal structure such that the forgetful functor $\text{Groth}_{corr}(F) \rightarrow \mathcal{J}_{corr}$ is symmetric monoidal.

The same holds true with any operad replacing the commutative operad.

5.17. As in §A.3 and 4.19, we have a functor:

$$\text{ShvCat}_{/_-} : \text{PreStk}^{lax,op} \rightarrow \text{Cat}_{pres}$$

that assigns to every lax prestack \mathcal{Y} the category $\text{ShvCat}_{/\mathcal{Y}}$ of sheaves of categories on \mathcal{Y} .

The functor $\text{ShvCat}_{/_-}$ is lax symmetric monoidal relative to the Cartesian product monoidal structures, where for lax prestacks \mathcal{Y} and \mathcal{Z} the corresponding structure map is:

$$\boxtimes : \text{ShvCat}_{/\mathcal{Y}} \times \text{ShvCat}_{/\mathcal{Z}} \rightarrow \text{ShvCat}_{/\mathcal{Y} \times \mathcal{Z}}.$$

Remark 5.17.1. Note that for any lax prestacks \mathcal{Y}_1 and \mathcal{Y}_2 we have:

$$\text{QCoh}_{\mathcal{Y}} \boxtimes \text{QCoh}_{\mathcal{Z}} \xrightarrow{\cong} \text{QCoh}_{\mathcal{Y} \times \mathcal{Z}}.$$

The failure of Γ to send \boxtimes to \otimes accounts for the failure of the map $\text{QCoh}(\mathcal{Y}) \otimes \text{QCoh}(\mathcal{Z}) \rightarrow \text{QCoh}(\mathcal{Y} \times \mathcal{Z})$ to be an isomorphism in general.

5.18. We apply the above formalism to $\mathcal{J} = \text{PreStk}^{lax}$ and $F = \text{ShvCat}_{/_-}$.

We obtain the symmetric monoidal category $\text{Groth}_{corr}(\text{ShvCat}_{/_-})$ that we will denote by the shorthand $\text{PreStk}_{corr}^{lax, \text{ShvCat}}$. We consider objects of $\text{PreStk}_{corr}^{lax, \text{ShvCat}}$ as pairs of \mathcal{Y} a lax prestack and \mathcal{C} a sheaf of categories on \mathcal{Y} .

We let $\text{PreStk}_{corr}^{\text{ShvCat}}$ denote the subcategory of $\text{PreStk}_{corr}^{lax, \text{ShvCat}}$ in which we only allow usual prestacks, not lax prestacks.

Remark 5.18.1. Note that PreStk_{corr} and its relatives are not locally small categories. This fact will not cause any difficulties for us below.

5.19. **Digression: The 2-categorical structure on 2-categorical correspondences.** The following discussion will be used implicitly in the text, but may be skipped by the reader at first.

Let \mathcal{C} be a 2-category and let \mathcal{C}^{1-cat} denote its underlying 1-category. We propose a canonical 2-categorical enhancement \mathcal{C}_{corr} of $\mathcal{C}^{1-cat} := (\mathcal{C}^{1-cat})_{corr}$.

Note that there are two flavors of 2-morphism present: one coming from the correspondence structure, and one coming from \mathcal{C} .

Exactly as in [GR], one can construct a 2-category structure \mathcal{C}_{corr} on \mathcal{C}^{1-cat} so that objects are $X \in \mathcal{C}$, 1-morphisms $X \rightarrow Y$ in \mathcal{C}_{corr} are given by correspondences $(X \leftarrow H \rightarrow Y)$, and 2-morphisms:

$$\left(\begin{array}{ccc} & H_1 & \\ \swarrow & & \searrow \\ X & & Y \end{array} \right) \longrightarrow \left(\begin{array}{ccc} & H_2 & \\ \swarrow & & \searrow \\ X & & Y \end{array} \right)$$

are given by diagrams:

$$\begin{array}{ccc}
& H_1 & \\
& \swarrow \downarrow \searrow & \\
& H_2 & \\
& \swarrow \leftarrow \searrow & \\
X & & Y.
\end{array} \tag{5.19.1}$$

Here the notation indicates that we specify a 2-morphism:

$$(H_1 \rightarrow Y) \rightarrow (H_1 \rightarrow H_2 \rightarrow Y)$$

and that the left triangle of (5.19.1) is honestly commutative (i.e., there is an implicit *invertible* 2-morphism).

Remark 5.19.1. The purpose of imposing this restriction on 2-morphisms is so that the 1-full subcategory \mathcal{C}^{1-cat} of $\mathcal{C}_{corr}^{1-cat}$ inherits the 2-categorical structure \mathcal{C} .

When discussing the 2-categorical structure of $\text{PreStk}_{corr}^{lax}$, we will be implicitly referring to the 2-categorical structure coming from the above.

Remark 5.19.2. This discussion can be integrated with the discussion of Variant 5.14.1 in the obvious way. This is relevant for describing the 2-categorical structure on $\text{PreStk}_{corr}^{lax, \text{ShvCat}}$.

Note that in the framework above, there were two types of 2-morphisms; in this setting, there are three. There are those of correspondence nature, those that reflect the 2-categorical structure of the base of the “fibration,” and those that reflect the fact that the functor “ F ” takes values in 2-categories.

5.20. Let \mathcal{S} be a commutative algebra in $\text{PreStk}_{corr}^{lax} := (\text{PreStk}^{lax})_{corr}$.

Definition 5.20.1. A *weakly multiplicative sheaf of categories* on \mathcal{S} is a commutative algebra in $\text{PreStk}_{corr}^{lax, \text{ShvCat}}$ mapping to \mathcal{S} as a commutative algebra under the forgetful functor.

We let $\text{MultCat}^w(\mathcal{S})$ denote the category of weakly multiplicative sheaves of categories on \mathcal{S} , i.e., the appropriate category of commutative algebras.

Every weakly multiplicative sheaf of categories on \mathcal{S} has an underlying sheaf of categories $\Psi \in \text{ShvCat}_{/\mathcal{S}}$. We sometimes abuse terminology in saying that $\Psi \in \text{ShvCat}_{/\mathcal{S}}$ itself is a multiplicative sheaf of categories.

5.21. Let \mathcal{S} be a commutative algebra in $\text{PreStk}_{corr}^{lax}$ with correspondences:

$$\begin{array}{ccc}
& \text{mult}_{\mathcal{S}} & \\
& \swarrow m_1 \searrow & \\
\mathcal{S} \times \mathcal{S} & & \mathcal{S}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
& \text{unit}_{\mathcal{S}} & \\
& \swarrow e_1 \searrow & \\
* & & \mathcal{S}
\end{array} \tag{5.21.1}$$

defining the multiplication and unit operations for \mathcal{S} . Then a weakly multiplicative sheaf of categories $\Psi \in \text{ShvCat}_{/\mathcal{S}}$ has a “multiplication” map:

$$\eta_m : m_1^*(\Psi \boxtimes \Psi) \rightarrow m_2^*(\Psi) \in \text{ShvCat}_{/\text{mult}_{\mathcal{S}}} \tag{5.21.2}$$

and a “unit” map:

$$\eta_e : \text{QCoh}_{\text{unit}_{\mathcal{S}}} = e_1^*(\text{Vect}) \rightarrow e_2^*(\Psi) \in \text{ShvCat}_{/\text{unit}_{\mathcal{S}}}. \tag{5.21.3}$$

We have similar maps for the n -ary multiplications for all n .

Definition 5.21.1. A weakly multiplicative sheaf of categories Ψ is *multiplicative* if, for every $n \geq 0$, the corresponding structure map as above is an equivalence.

We let $\text{MultCat}(\mathcal{S}) \subseteq \text{MultCat}^w(\mathcal{S})$ denote the category of multiplicative sheaves of categories on \mathcal{S} .

Example 5.21.2. $\text{QCoh}_{\mathcal{S}}$ carries a canonical structure of multiplicative sheaf on any \mathcal{S} .

Remark 5.21.3. We made a choice earlier by using $\text{ShvCat}_{/-}$ in place of $\text{ShvCat}_{/-}^{\text{naive}}$. Had we used $\text{ShvCat}_{/-}^{\text{naive}}$ instead of $\text{ShvCat}_{/-}$, we would end up with different weakly multiplicative sheaves, because e.g the morphism (5.21.2) would have to be a morphism in $\text{ShvCat}_{/\text{mult}_{\mathcal{S}}}$. However, we would have the same multiplicative sheaves of categories, because the underlying groupoids of $\text{ShvCat}_{/-}$ and $\text{ShvCat}_{/-}^{\text{naive}}$ are the same.

However, while the objects would be the same, the morphisms allowed in $\text{MultCat}(\mathcal{S})$ are different by virtue of choosing $\text{ShvCat}_{/-}$.

5.22. More generally, for any colored operad \mathfrak{D} and any \mathfrak{D} -algebra \mathcal{S} in $\text{PreStk}_{\text{corr}}^{\text{lax}}$, we have the category $\text{MultCat}_{\mathfrak{D}}^w(\mathcal{S})$, and the full subcategory $\text{MultCat}_{\mathfrak{D}}(\mathcal{S})$ where the morphisms analogous to (5.21.2) corresponding to all operations are equivalences.

In particular, for \mathcal{S} a non-unital commutative algebra in $\text{PreStk}_{\text{corr}}^{\text{lax}}$, we have $\text{MultCat}_{\text{non-unital}}(\mathcal{S})$ the category of non-unital multiplicative sheaves of categories on \mathcal{S} .

5.23. Let \mathcal{C} be a symmetric monoidal 2-category and let $X, Y \in \mathcal{C}$ be commutative algebras.

Recall that in this case we have a notion *lax morphism of commutative algebras* $X \rightarrow Y$, which gives rise in particular to a morphism $X \rightarrow Y$ and a natural transformation between the compositions:

$$\begin{array}{c} (X \otimes X \rightarrow Y \otimes Y \rightarrow Y) \\ \Downarrow \\ (X \otimes X \rightarrow X \rightarrow Y) \end{array}$$

When \mathcal{C} is the 2-category of categories, this gives rise to the usual notion of lax symmetric monoidal functor between symmetric monoidal categories.

5.24. Note that $\text{PreStk}_{\text{corr}}^{\text{lax}, \text{ShvCat}}$ carries a canonical structure of 2-category as in §5.19. We see that the symmetric monoidal structure lifts to this enhancement as well.

Therefore, we obtain the category $\text{MultCat}^{w, \text{lax}}(\mathcal{S})$ where we allow lax morphisms (lying over the identity for \mathcal{S}). Then $\text{MultCat}^{w, \text{lax}}(\mathcal{S})$ contains $\text{MultCat}^w(\mathcal{S})$ as a 1-full subcategory with the same underlying groupoid.

Remark 5.24.1. We emphasize that the use of the term *lax* here is of different nature from that of lax prestack, and rather reflect a general categorical notion applied in two different circumstances. In particular, for a non-lax prestack \mathcal{S} with commutative algebra structure in $\text{PreStk}_{\text{corr}}$, there is a significant difference between the categories $\text{MultCat}^{w, \text{lax}}(\mathcal{S})$ and $\text{MultCat}^w(\mathcal{S})$.

Remark 5.24.2. Recall from Remark 5.19.2 that there are essentially three types of 2-morphisms in $\text{PreStk}_{\text{corr}}^{\text{lax}, \text{ShvCat}}$. Only the third from the list of *loc. cit.* plays a role in the above discussion: the two coming from the discussion in the beginning of §5.19 are irrelevant.

5.25. Let Ψ be a weakly multiplicative sheaf of categories on a commutative algebra $\mathcal{S} \in \text{PreStk}_{\text{corr}}^{\text{la}x}$.

Definition 5.25.1. A *weakly multiplicative object* ψ in Ψ is a morphism:

$$\text{QCoh}_{\mathcal{S}} \rightarrow \Psi \quad (5.25.1)$$

in the category $\text{MultCat}^{w,\text{la}x}(\mathcal{S})$.

We denote the category of weakly multiplicative objects in Ψ by $\text{Mult}^w(\Psi)$.

Notation 5.25.2. Any weakly multiplicative object ψ in Ψ has an underlying morphism $\text{QCoh}_{\mathcal{S}} \rightarrow \Psi$ in $\text{ShvCat}_{/\mathcal{S}}$, i.e., it defines an object of $\Gamma(\mathcal{S}, \Psi)$.

We denote this object also by ψ , and summarize the situation by saying that *the object* ψ is a weakly multiplicative object in Ψ .

5.26. Here is a convenient reformulation of the definition of weakly multiplicative object. The reader may skip this material and return to it where needed.

Recall that $\text{Groth}(\text{ShvCat}_{/-})$ denotes the coCartesian fibration over $\text{PreStk}^{\text{la}x, \text{op}}$ defined by the functor $\text{ShvCat}_{/-}$. We have the canonical functor:

$$\begin{aligned} \Gamma(-, -) : \text{Groth}(\text{ShvCat}_{/-}) &\rightarrow \text{DGCat}_{\text{cont}} \\ (\mathcal{Y}, \mathcal{C} \in \text{ShvCat}_{/\mathcal{Y}}) &\mapsto \Gamma(\mathcal{Y}, \mathcal{C}). \end{aligned}$$

As in §5.14, a variant of the Grothendieck construction defines a category for this section simply by \mathcal{G} , whose objects are triples:

$$\left(\mathcal{Y} \in \text{PreStk}^{\text{la}x}, \mathcal{C} \in \text{ShvCat}_{/\mathcal{Y}}, \mathcal{F} \in \Gamma(\mathcal{Y}, \mathcal{C}) \right) \quad (5.26.1)$$

and where morphisms:

$$\left(\mathcal{Y}_1 \in \text{PreStk}^{\text{la}x}, \mathcal{C}_1 \in \text{ShvCat}_{/\mathcal{Y}_1}, \mathcal{F}_1 \in \Gamma(\mathcal{Y}_1, \mathcal{C}_1) \right) \rightarrow \left(\mathcal{Y}_2 \in \text{PreStk}^{\text{la}x}, \mathcal{C}_2 \in \text{ShvCat}_{/\mathcal{Y}_2}, \mathcal{F}_2 \in \Gamma(\mathcal{Y}_2, \mathcal{C}_2) \right)$$

are defined by the data of a correspondence:

$$\begin{array}{ccc} & \mathcal{H} & \\ \alpha \swarrow & & \searrow \beta \\ \mathcal{Y}_1 & & \mathcal{Y}_2 \end{array}$$

in $\text{PreStk}^{\text{la}x}$, a morphism:

$$\eta : \alpha^*(\mathcal{C}_1) \rightarrow \beta^*(\mathcal{C}_2) \in \text{ShvCat}_{/\mathcal{H}}$$

and a morphism in $\Gamma(\mathcal{H}, \beta^*(\mathcal{C}_2))$ from the image of \mathcal{F}_1 to the image of \mathcal{F}_2 under the two morphisms:

$$\begin{aligned} \Gamma(\mathcal{Y}_1, \mathcal{C}_1) &\rightarrow \Gamma(\mathcal{H}, \alpha^*(\mathcal{C}_1)) \xrightarrow{\Gamma(\eta)} \Gamma(\mathcal{H}, \beta^*(\mathcal{C}_2)) \\ &\text{and } \Gamma(\mathcal{Y}_2, \mathcal{C}_2) \rightarrow \Gamma(\mathcal{H}, \beta^*(\mathcal{C}_2)). \end{aligned}$$

The category \mathcal{G} is canonically symmetric monoidal in the obvious way, and we have a symmetric monoidal functor:

$$\mathcal{G} \rightarrow \text{PreStk}_{\text{corr}}^{\text{la}x, \text{ShvCat}} \quad (5.26.2)$$

given by forgetting the third term in (5.26.1).

Then, tautologically, a weakly multiplicative object in a weakly multiplicative sheaf of categories $\Psi \in \text{MultCat}^w(\mathcal{S})$ is equivalent to a commutative algebra in \mathcal{G} mapping to Ψ under the forgetful functor (5.26.2).

5.27. In the notation of §5.21, a weakly multiplicative object $\psi \in \Psi$ defines a morphism:

$$\eta_m(m_1^*(\psi \boxtimes \psi)) \rightarrow m_2^*(\psi) \in \Gamma(\text{mult}_{\mathcal{S}}, m_2^*(\Psi))$$

and similarly for the unit operation, and general n -ary multiplication operations.

Definition 5.27.1. The object ψ is a *multiplicative object* in Ψ if these morphisms are isomorphisms.

Remark 5.27.2. Tautologically, one can rephrase the definition by asking that the morphism (5.25.1) be a morphism of commutative algebras and not a lax morphism, i.e., it should be a morphism in $\text{MultCat}^w(\mathcal{S})$.

We denote the resulting full subcategory of $\text{Mult}^w(\Psi)$ by $\text{Mult}(\Psi)$.

Example 5.27.3. In the setting of Example 5.21.2, the object $\mathcal{O}_{\mathcal{S}}$ carries a canonical multiplicative structure.

Remark 5.27.4. By Remark 5.21.3, the choice to use $\text{ShvCat}_{/-}$ in place of $\text{ShvCat}_{/-}^{\text{naive}}$ gives a different definition of multiplicative objects.

The key difference is explained in Example 4.17.1: we would not have “interesting” multiplicative sheaves, i.e., they would be insensitive to the non-invertibility of morphisms in the categories taken as values of \mathcal{S} .

Remark 5.27.5. The category $\text{Mult}(\mathcal{S})$ admits sifted colimits

5.28. In the setting of §5.22, for $\Psi \in \text{Mult}_{\mathfrak{D}}^w$ we obtain the categories $\text{Mult}_{\mathfrak{D}}^w(\Psi)$ and its full subcategory $\text{Mult}_{\mathfrak{D}}(\Psi)$.

For the sake of clarity: let us denote the category of colors underlying \mathfrak{D} by $\mathfrak{D}^{\heartsuit}$. For an \mathfrak{D} -algebra in $\text{PreStk}^{\text{la}x}$, we have in particular a rule assigning to $\xi \in \mathfrak{D}^{\heartsuit}$ a lax prestack \mathcal{S}_{ξ} . Then the role of $\text{QCoh}_{\mathcal{S}}$ from the symmetric monoidal case is played by the rule assigning to each \mathcal{S}_{ξ} the sheaf of categories $\text{QCoh}_{\mathcal{S}_{\xi}}$.

5.29. **Variant: Coalgebraic description.** Let \mathcal{S} be as above.

For any category \mathcal{C} with fiber products, we have the canonical equivalence $(\mathcal{C}_{\text{corr}})^{\text{op}} \simeq \mathcal{C}_{\text{corr}}$ given by “flipping” the correspondence. This construction allows us to view \mathcal{S} as a cocommutative coalgebra in $\text{PreStk}_{\text{corr}}^{\text{la}x}$.

We have the category $\text{MultCat}^{\text{op}-w}(\mathcal{S})$ of *op-weakly multiplicative sheaves of categories*: these are coalgebras in $\text{PreStk}_{\text{corr}}^{\text{la}x, \text{ShvCat}}$ lying over \mathcal{S} .

Any op-weakly multiplicative sheaf of categories has structure maps:

$$\begin{aligned} \tilde{\eta}_m : m_2^*(\Psi) &\rightarrow m_1^*(\Psi \boxtimes \Psi) \in \text{ShvCat}_{/\text{mult}_{\mathcal{S}}} \text{ and} \\ \tilde{\eta}_e : e_2^*(\Psi) &\rightarrow \text{QCoh}_{\text{unit}_{\mathcal{S}}} = e_1^*(\text{Vect}) \in \text{ShvCat}_{/\text{unit}_{\mathcal{S}}}. \end{aligned} \tag{5.29.1}$$

By general principles from [GR], the subcategory of $\text{MultCat}^{\text{op}-w}(\mathcal{S})$ where the maps in (5.29.1) are equivalences is canonically equivalent to $\text{MultCat}(\mathcal{S})$.

More generally, we have the following general result.

Proposition 5.29.1. *Let $\text{MultCat}^{w,l.adj}(\mathcal{S}) \subseteq \text{MultCat}^w(\mathcal{S})$ denote the full subcategory in which the arrows (5.21.2) and (5.21.3) admit left adjoint in the 2-category $\text{ShvCat}/_{\text{mult}_{\mathcal{S}}}$ and $\text{ShvCat}/_{\text{unit}_{\mathcal{S}}}$ respectively (equivalently: the analogous result for all n -ary operations in the commutative operad).*

Similarly, define $\text{MultCat}^{op-w,r.adj}(\mathcal{S})$ to be the full subcategory of $\text{MultCat}^{op-w}(\mathcal{S})$ in which the morphisms (5.29.1) admit right adjoints.

Then there is a canonical equivalence:

$$\text{MultCat}^{w,l.adj}(\mathcal{S}) \simeq \text{MultCat}^{op-w,r.adj}(\mathcal{S})$$

commuting with forgetful functors to $\text{ShvCat}/_{\mathcal{S}}$, defined by passing to the appropriate adjoints for all operations.

Remark 5.29.2. The roles of left and right could be interchanged in the statement of this proposition, but we will apply it with the normalizations above.

5.30. Similarly, we have the notion of op-weakly multiplicative object of an op-multiplicative sheaf of categories $\Psi \in \text{MultCat}^{op-w}(\mathcal{S})$. We denote the resulting category by $\text{Mult}^{op-w}(\Psi)$. In a multiplicative sheaf of categories Ψ , considered as an op-weakly multiplicative sheaf of categories as above, the corresponding notion of multiplicative object canonically identifies with the category $\text{Mult}(\Psi)$ as defined in the ‘‘covariant’’ setting above.

The op-multiplicative setting has the following advantages:

Lemma 5.30.1. *The categories $\text{MultCat}^{op-w}(\mathcal{S})$ and $\text{Mult}^{op-w}(\Psi)$ are cocomplete (even presentable) and the corresponding functors:*

$$\begin{aligned} \text{MultCat}^{op,w}(\mathcal{S}) &\rightarrow \text{ShvCat}(\mathcal{S}) \\ \text{Mult}^{op,w}(\Psi) &\rightarrow \Gamma(\mathcal{S}, \Psi) \end{aligned}$$

commute with colimits.

5.31. **Subcategories.** Suppose that \mathcal{S} is a commutative monoid in $\text{PreStk}_{\text{corr}}^{lax}$, Ψ is a weakly multiplicative sheaf of categories on \mathcal{S} , and $\Phi \hookrightarrow \Psi$ is a fully-faithful functor in $\text{ShvCat}/_{\mathcal{S}}$, in the sense of §4.26.

We say that Φ is *weakly compatible* with the weakly multiplicative structure on Ψ if the morphism η_m from (5.21.2) maps $m_1^*(\Phi \boxtimes \Phi)$ into $m_2^*(\Phi) \subseteq m_2^*(\Psi)$, and η_e from (5.21.3) factors through $e_2^*(\Phi) \subseteq e_2^*(\Psi)$.

In this case, Φ inherits a unique weakly multiplicative structure such that the morphism $\Phi \rightarrow \Psi$ upgrades to a morphism of weakly multiplicative sheaves of categories.

We say that Φ is *compatible* if the induced weakly multiplicative structure is multiplicative.

A variant of this discussion holds for general colored operads.

5.32. **Localizations.** Suppose that \mathcal{S} is a commutative monoid in $\text{PreStk}_{\text{corr}}^{lax}$, Ψ is an op-weakly multiplicative sheaf of categories on \mathcal{S} , and $\Phi \subseteq \Psi$ is a full subcategory.

As in §A.6, we can form the quotient sheaf of categories $\Psi/\Phi \in \text{ShvCat}/_{\mathcal{S}}$.

We say that Φ is a *weak ideal* subcategory of Ψ if the compositions:

$$\begin{aligned} m_2^*(\Phi) \hookrightarrow m_2^*(\Psi) \xrightarrow{\tilde{\eta}_m} m_1^*(\Psi \boxtimes \Psi) \rightarrow m_1^*\left((\Psi/\Phi) \boxtimes (\Psi/\Phi)\right) \text{ and} \\ e_2^*(\Phi) \hookrightarrow e_2^*(\Psi) \xrightarrow{\tilde{\eta}_e} \text{QCoh}_{\text{unit}_{\mathcal{S}}} \end{aligned}$$

are zero. Here the notations $\tilde{\eta}_m$ and $\tilde{\eta}_e$ are taken from (5.29.1).

In this case, the quotient Ψ/Φ inherits a canonical structure of op-weakly multiplicative sheaf of categories on \mathcal{S} .

If Ψ is a (non-weakly) multiplicative sheaf of categories on \mathcal{S} , we say that Φ is an *ideal* subcategory if induced op-weakly multiplicative structure on the quotient Ψ/Φ is multiplicative.

Again, this material generalizes in the appropriate way to an arbitrary colored operad.

5.33. Functoriality. Before discussing functoriality of multiplicative sheaves, we return to the general framework of §5.16, so \mathcal{I} is a symmetric monoidal category that admits fiber products and $F : \mathcal{I}^{op} \rightarrow \text{Cat}_{pres}$ is a lax symmetric monoidal functor.

Lemma 5.33.1. *Let \mathcal{D} be a colored operad, and denote also by \mathfrak{D} the underlying category in which we only allow 1-ary operations.*

Then the functor:

$$\text{Alg}_{\mathcal{D}}\left(\text{Groth}_{corr}(F)\right) \times_{\text{Hom}(\mathfrak{D}, \mathcal{I}_{corr})} \text{Hom}(\mathfrak{D}, \mathcal{I}^{op}) \rightarrow \text{Alg}_{\mathcal{D}}(\mathcal{I}_{corr}) \times_{\text{Hom}(\mathfrak{D}, \mathcal{I}_{corr})} \text{Hom}(\mathfrak{D}, \mathcal{I}^{op})$$

is a coCartesian fibration.

This result follows from the following more general categorical lemma.

Lemma 5.33.2. *Suppose that \mathcal{C} and \mathcal{J} are symmetric monoidal categories and $F : \mathcal{C} \rightarrow \mathcal{J}$ is a symmetric monoidal functor.*

Suppose that \mathcal{J}^0 is a symmetric monoidal 1-full subcategory of \mathcal{J} such that $\mathcal{C} \times_{\mathcal{J}} \mathcal{J}^0 \rightarrow \mathcal{J}^0$ is a coCartesian fibration, and arrows in \mathcal{C} coCartesian over \mathcal{J}^0 are coCartesian over all \mathcal{J} . Suppose moreover that arrows in \mathcal{C} coCartesian over \mathcal{J}^0 are preserved under tensor products in \mathcal{C} .

Suppose that we are given a symmetric monoidal category \mathcal{D} , symmetric monoidal functors $G_i : \mathcal{D} \rightarrow \mathcal{J}$, $i = 1, 2$ and morphism $\eta : G_1 \rightarrow G_2$ of symmetric monoidal functors, such that for every $X \in \mathcal{D}$ the morphism $G_1(X) \rightarrow G_2(X)$ is a morphism in \mathcal{J}^0 .

Then the functor:

$$\text{Hom}^{\otimes}(\mathcal{D}, \mathcal{C}) \times_{\text{Hom}^{\otimes}(\mathcal{D}, \mathcal{J})} \Delta^1 \rightarrow \Delta^1$$

is coCartesian, where the fiber is taken over η . Here Hom^{\otimes} denotes the category of symmetric monoidal functors. An arrow in $\text{Hom}^{\otimes}(\mathcal{D}, \mathcal{C}) \times_{\text{Hom}^{\otimes}(\mathcal{D}, \mathcal{J})} \Delta^1$ is coCartesian if and only if, for every $X \in \mathcal{D}$, the induced arrow in \mathcal{C} is coCartesian over \mathcal{J}^0 .

Remark 5.33.3. That we can reduce Lemma 5.33.1 to the symmetric monoidal case follows from the theory of monoidal envelopes in [Lur4]. However, this is not a serious point.

Proof (sketch). Using the description of symmetric monoidal categories in terms of coCartesian fibrations, reduce to the case where we deal with non-symmetric monoidal categories and functors, where it follows by an appropriate generalization of [Lur1] Proposition 3.1.2.1. □

Remark 5.33.4. The above material is stated in a somewhat abstract way. It amounts to the following. Suppose we are in the setting of Lemma 5.33.2, but let us omit the words “symmetric monoidal” everywhere. The lemma then says that, given $G_1 \rightarrow G_2$ as in *loc. cit.*, and a \tilde{G}_1 a lift of G_1 to a functor $\mathcal{D} \rightarrow \mathcal{C}$, then we obtain a functor \tilde{G}_2 lifting G_2 and equipped with a morphism $\tilde{G}_1 \rightarrow \tilde{G}_2$.

Naively: for $X \in \mathcal{D}$, define $\tilde{G}_2(X)$ as the tip of the coCartesian arrow in \mathcal{C} with source $\tilde{G}_1(X)$, and lying over the morphism $G_1(X) \rightarrow G_2(X)$ (which, by assumption, is an arrow in \mathcal{J}^0). Then, for a morphism $X \rightarrow Y$ in \mathcal{D} , we have the square:

$$\begin{array}{ccc} \tilde{G}_1(X) & \longrightarrow & \tilde{G}_2(X) \\ \downarrow & & \downarrow \text{dotted} \\ \tilde{G}_1(Y) & \longrightarrow & \tilde{G}_2(Y). \end{array}$$

The dotted arrow comes from the fact that $\tilde{G}_1(X) \rightarrow \tilde{G}_2(X)$ is a coCartesian arrow in \mathcal{C} , and from the morphism $\tilde{G}_1(X) \rightarrow \tilde{G}_2(Y)$ given by tracing out the lower edge of the diagram.

Variant 5.33.5. In the setting of Lemma 5.33.2, suppose that \mathcal{C} and \mathcal{J} are taken to be symmetric monoidal 2-categories instead, and \mathcal{J}^0 is again a 1-full subcategory with the same compatibility. Then the conclusion of Lemma 5.33.2 again holds, but in the 2-categorical sense. In fact, there are two formulations: we can allow lax or strict morphisms of symmetric monoidal functors, and the result holds in either setting.

Therefore, by Remark 5.14.2, we have a variant of Lemma 5.33.1 in which we use the 2-categorical enhancements $\text{Groth}_{\text{corr}}(F)^{2\text{-cat}}$ and $\mathcal{J}_{\text{corr}}^{2\text{-cat}}$.

5.34. Suppose that $f : \mathcal{S} \rightarrow \mathcal{T}$ is a morphism of commutative algebras (or \mathcal{D} -algebras) in $\text{PreStk}_{\text{corr}}^{\text{lax}}$ such that the underlying morphism in $\text{PreStk}_{\text{corr}}^{\text{lax}}$ is a morphism in the 1-full subcategory $\text{PreStk}^{\text{lax}}$.

By Lemma 5.33.1 we obtain pullback functors:

$$\begin{aligned} f^* : \text{MultCat}^w(\mathcal{T}) &\rightarrow \text{MultCat}^w(\mathcal{S}) \\ \text{Mult}^w(\Psi) &\rightarrow \text{Mult}^w(f^*(\Psi)) \end{aligned} \tag{5.34.1}$$

where $\Psi \in \text{MultCat}^w(\mathcal{T})$. These functors preserve the full subcategories MultCat and Mult respectively.

Moreover, the 2-categorical version of Lemma 5.33.1, applied to account for the 2-categorical structure on $\text{PreStk}^{\text{lax}}$, implies that if $\eta : f \rightarrow g$ is a 2-morphism of maps $f, g : \mathcal{S} \rightarrow \mathcal{T}$ of commutative algebras as above, then we obtain natural transformations of the corresponding functors (5.34.1).

5.35. **A variant.** We have the following variant of these definitions as well. Let \mathcal{S} be a commutative algebra in $\text{PreStk}_{\text{corr}}^{\text{lax}}$ as above.

Suppose that $\mathfrak{F} : \text{PreStk}^{\text{lax.op}} \rightarrow \text{Cat}$ (or valued in Cat_{pres}) is a lax symmetric monoidal functor. Then, exactly as in the definition of multiplicative sheaf of categories, we have a notion of *multiplicative sheaf on \mathcal{S} with values in \mathfrak{F}* .

Example 5.35.1. If $\mathfrak{F} = \text{ShvCat}_{/-}$, then we recover the notion of multiplicative sheaf of categories on \mathcal{S} .

If $\mathfrak{F} = \text{QCoh}(-)$ with the exterior product defining the lax symmetric monoidal structure, then we recover the notion of multiplicative object in the multiplicative sheaf of categories $\text{QCoh}_{\mathcal{S}}$.

Example 5.35.2. If \mathcal{C} is a symmetric monoidal category, then we may view \mathcal{C} as a lax symmetric monoidal functor $* \rightarrow \text{Cat}$ and therefore we obtain a lax symmetric monoidal functor:

$$\text{PreStk}^{\text{op}} \rightarrow * \rightarrow \text{Cat}.$$

Taking this composition as the functor \mathfrak{F} , we recover a notion of multiplicative sheaf with values in the symmetric monoidal category \mathcal{C} .

Example 5.35.3. One can use this framework to make sense of a factorizable monoidal category.

Again, this discussion carries over to a general colored operad.

6. CHIRAL CATEGORIES AND FACTORIZATION ALGEBRAS

6.1. In this section, we give the formalism of chiral categories and factorization algebras in them by applying the material of §5 to Ran space.

We fix a prestack \mathcal{X} throughout this section.

6.2. **Chiral categories and factorization algebras.** Here are the main definitions of this section.

Definition 6.2.1. A *chiral category* or *factorization category* \mathbf{C} on \mathcal{X} is a non-unital multiplicative category on the non-unital commutative algebra $\mathrm{Ran}_{\mathcal{X}}^{ch} \in \mathrm{PreStk}_{corr} \subseteq \mathrm{PreStk}_{corr}^{lax}$.

A *factorization algebra* \mathcal{A} in a factorization category \mathbf{C} is a multiplicative object of \mathbf{C} .

A *unital chiral category* or *unital factorization category* \mathbf{C} on \mathcal{X} is a multiplicative category on $\mathrm{Ran}_{\mathcal{X}}^{un, ch} \in \mathrm{PreStk}_{corr}^{lax}$.

A *unital factorization algebra* \mathcal{A} in a unital factorization category is a multiplicative object of \mathbf{C} .

We denote the respective categories by:

$$\begin{array}{cc} \mathrm{Cat}^{ch}(\mathcal{X}) & \mathrm{Cat}_{un}^{ch}(\mathcal{X}) \\ \mathrm{Alg}^{\mathrm{fact}}(\mathbf{C}) & \mathrm{Alg}_{un}^{\mathrm{fact}}(\mathbf{C}) \end{array}$$

for \mathbf{C} a (resp. unital) chiral category. We have forgetful functors:

$$\begin{array}{c} \mathrm{Cat}_{un}^{ch}(\mathcal{X}) \rightarrow \mathrm{Cat}^{ch}(\mathcal{X}) \\ \mathrm{Alg}_{un}^{\mathrm{fact}}(\mathbf{C}) \rightarrow \mathrm{Alg}^{\mathrm{fact}}(\mathbf{C}). \end{array}$$

for \mathbf{C} a unital factorization category.

Remark 6.2.2. We refer to §3 for more concrete descriptions of factorization categories.

Remark 6.2.3. One immediately sees that e.g. factorization categories on \mathcal{X} are equivalent to *unital* multiplicative categories on $\mathrm{Ran}_{\mathcal{X}, \emptyset}$.

Terminology 6.2.4. We will frequently abuse language by saying that $\mathbf{C} \in \mathrm{ShvCat}_{\mathrm{Ran}_{\mathcal{X}}}$ is a chiral category, or $\mathcal{A} \in \Gamma(\mathrm{Ran}_{\mathcal{X}}, \mathbf{C})$ is a factorization algebra in \mathbf{C} , and so on.

Notation 6.2.5. For $\mathbf{C} = \mathrm{QCoh}_{\mathrm{Ran}_{\mathcal{X}}}$, we write $\mathrm{Alg}^{\mathrm{fact}}(\mathcal{X})$ and $\mathrm{Alg}_{un}^{\mathrm{fact}}(\mathcal{X})$ in place of the notation above, and refer to objects of these categories merely as (unital) factorization algebras on \mathcal{X} .

Terminology 6.2.6. We refer to morphisms in $\mathrm{Cat}^{ch}(\mathcal{X})$ and $\mathrm{Cat}_{un}^{ch}(\mathcal{X})$ as *factorization functors* and *unital factorization functors* respectively.

Remark 6.2.7. The comparison with the theory of [FG1] is indirect, and therefore postponed to Remark 6.19.5.

Remark 6.2.8. By definition of multiplicative sheaf, given a factorization functor $\mathbf{C} \rightarrow \mathbf{D}$ we obtain a canonical morphism:

$$\mathrm{Alg}^{\mathrm{fact}}(\mathbf{C}) \rightarrow \mathrm{Alg}^{\mathrm{fact}}(\mathbf{D})$$

compatible with forgetful functors. The same holds in the unital setting.

Variante 6.2.9. A *weak chiral category* is a weakly multiplicative sheaf of categories on $\text{Ran}_{\mathcal{X}}^{ch}$. We let $\text{Cat}^{w,ch}(\mathcal{X})$ denote the category of weak chiral categories on \mathcal{X} . Similarly, we have the unital variant $\text{Cat}^{w,ch}(\mathcal{X})$. Recall that $\text{Cat}^{ch}(\mathcal{X})$ (resp. $\text{Cat}_{un}^{ch}(\mathcal{X})$) is tautologically a full subcategory of $\text{Cat}^{w,ch}(\mathcal{X})$ (resp. $\text{Cat}_{un}^{w,ch}(\mathcal{X})$).

6.3. The unit. Therefore, we may apply the discussion of §4.24, and we will use the terminology of *loc. cit.* freely.

We will show that $\text{unit}_{\mathbb{C}}$ admits a canonical unital factorization algebra structure.

The chiral product on $\text{Ran}_{\mathcal{X}}^{un}$ induces commutative algebra structures on $\text{Ran}_{\mathcal{X}}^{un} \times \text{Ran}_{\mathcal{X}}^{un}$ and $[\text{Ran}_{\mathcal{X}}^{un} \times \text{Ran}_{\mathcal{X}}^{un}]_{disj}$ as objects of $\text{PreStk}_{corr}^{lax}$.

Moreover, one sees first that the maps:

$$[\text{Ran}_{\mathcal{X}}^{un} \times \text{Ran}_{\mathcal{X}}^{un}]_{disj} \begin{array}{c} \xrightarrow{p_2} \\ \xrightarrow{\text{add}} \end{array} \text{Ran}_{\mathcal{X}}^{un}$$

are morphisms of commutative algebras in $\text{PreStk}_{corr}^{lax}$, and that the obvious 2-morphism:

$$[\text{Ran}_{\mathcal{X}}^{un} \times \text{Ran}_{\mathcal{X}}^{un}]_{disj} \begin{array}{c} \xrightarrow{p_2} \\ \Downarrow \\ \xrightarrow{\text{add}} \end{array} \text{Ran}_{\mathcal{X}}^{un}. \quad (6.3.1)$$

is compatible with the commutative algebra structures.

Restricting to $\text{Ran}_{\mathcal{X}}^{un} \times \{\emptyset\}$ and applying the discussion from §5.34 we see that $\text{unit}_{\mathbb{C}}$ inherits the canonical structure of unital factorization algebra.

Furthermore, we see that any $\mathcal{A} \in \text{Alg}_{un}^{fact}(\mathbb{C})$ admits a canonical map:

$$\text{unit}_{\mathbb{C}} \rightarrow \mathcal{A} \quad (6.3.2)$$

of unital factorization algebras. We refer to this map as the *unit map* for \mathcal{A} .

Remark 6.3.1. Given a unital factorization functor $F : \mathbb{C} \rightarrow \mathbb{D}$, there is not necessarily an identification $F(\text{unit}_{\mathbb{C}}) \simeq \text{unit}_{\mathbb{D}}$, but rather there is only a morphism:

$$\text{unit}_{\mathbb{D}} \rightarrow F(\text{unit}_{\mathbb{C}}) \quad (6.3.3)$$

of unital factorization algebras in \mathbb{D} .

Definition 6.3.2. A unital factorization functor is *strictly unital* if (6.3.3) is an equivalence.

We let $\text{Cat}_{un,str}^{ch}(\mathcal{X})$ denote the 1-full subcategory of $\text{Cat}_{un}^{ch}(\mathcal{X})$ consisting of unital chiral categories on \mathcal{X} under strictly unital morphisms.

Remark 6.3.3. We will sometimes say a general unital factorization functor is *lax unital* to emphasize that it may not be (or is not) strictly unital, but the word “lax” should be taken as redundant here.

Recalling that unital factorization algebras in \mathbb{C} are by definition unital factorization functors $\text{QCoh}_{\mathcal{X}} \rightarrow \mathbb{C}$, we see that this construction generalizes the construction of (6.3.2) presented above.

Remark 6.3.4. Remark 6.3.1 is a manifestation of the following general philosophy: under the analogy between chiral categories and monoidal categories, chiral functors correspond to *lax* monoidal functors (recall that in the setting of (unital) monoidal categories, it is natural to assume that lax monoidal functors are merely lax unital).

6.4. We now discuss a construction of unital factorization structures useful in [Ras2] and [Ras3].

Suppose that \mathcal{C} is a unital factorization category and $\mathcal{D} \hookrightarrow \mathcal{C}$ is a fully-faithful functor in $\mathbf{ShvCat}/_{\mathbf{Ran}_X^{un}}$.

Suppose that \mathcal{D} is compatible with the factorization structure in the sense that we have a (necessarily unique) factorization:

$$\begin{array}{ccc} (\mathcal{D} \boxtimes \mathcal{D})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} & \dashrightarrow & \mathbf{add}^*(\mathcal{D})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} \\ \downarrow & & \downarrow \\ (\mathcal{C} \boxtimes \mathcal{C})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} & \xlongequal{\quad} & \mathbf{add}^*(\mathcal{C})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} \end{array}$$

that is an equivalence, and moreover, the map:

$$\mathcal{D}_{\emptyset} \rightarrow \mathcal{C}_{\emptyset} \simeq \mathbf{Vect}$$

is an equivalence as well.

In this case, the discussion of §5.31 implies that \mathcal{D} inherits a canonical unital factorization structure.

Remark 6.4.1. Note that there is an analogous version of this discussion for non-unital factorization categories.

Moreover, in the unital setting, we observe that for factorization category \mathcal{C} and $\mathcal{D} \subseteq \mathcal{C} \in \mathbf{ShvCat}/_{\mathbf{Ran}_X^{un}}$ as above, it suffices to check the compatibility with the unital factorization structure by checking compatibility with the non-unital factorization structure by restriction to $\mathbf{Ran}_{X,\emptyset}$ (viewing non-unital factorization categories via 6.2.3).

Combining this discussion with Proposition-Construction 4.26.1, we obtain the following result:

Proposition 6.4.2. *Suppose that \mathcal{C} is a unital factorization category on \mathcal{X} that is adj-unital (as a mere unital sheaf of categories, i.e., ignoring the factorization structure).*

Suppose that \mathcal{D} is a factorization category on \mathcal{X} equipped with a factorization functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such that the underlying morphism in $\mathbf{ShvCat}/_{\mathbf{Ran}_X}$ is fully-faithful.

Let \mathcal{D} also denote the corresponding sheaf of categories on $\mathbf{Ran}_{X,\emptyset} = \mathbf{Ran}_X \coprod \mathbf{Spec}(k)$ where $\mathcal{D}_{\emptyset} := \mathbf{Vect}$.

Now suppose that the hypotheses of Proposition-Construction 4.26.1 are satisfied.

Then \mathcal{D} with its unital structure from Proposition-Construction 4.26.1 inherits a unique unital factorization structure such that the functor $\mathcal{D} \rightarrow \mathcal{C} \in \mathbf{ShvCat}/_{\mathbf{Ran}_X^{un}}$ upgrades to a functor of unital factorization categories.

6.5. **Localizations.** We now render the material of §5.32 to the setting of factorization categories.

Suppose that \mathcal{C} is a unital factorization category on \mathcal{X} and $\mathcal{D} \hookrightarrow \mathcal{C} \in \mathbf{ShvCat}/_{\mathbf{Ran}_X^{un}}$ is a unital subcategory with $\mathcal{D}_{\emptyset} = 0$ and such that the composition:

$$\begin{array}{ccc} \mathbf{add}^*(\mathcal{D})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} & \hookrightarrow & \mathbf{add}^*(\mathcal{C})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} \xrightarrow{\simeq} \\ (\mathcal{C} \boxtimes \mathcal{C})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} & \rightarrow & (\mathcal{C}/\mathcal{D} \boxtimes \mathcal{C}/\mathcal{D})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} \end{array}$$

is zero, and the induced map:

$$\mathbf{add}^*(\mathcal{C}/\mathcal{D})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}} \rightarrow (\mathcal{C}/\mathcal{D} \boxtimes \mathcal{C}/\mathcal{D})|_{[\mathbf{Ran}_X^{un} \times \mathbf{Ran}_X^{un}]_{disj}}$$

is an equivalence.

Then \mathcal{C}/D inherits a canonical structure of unital factorization category. Moreover, the structure morphism $\mathcal{C} \rightarrow \mathcal{C}/D$ is a morphism of unital factorization categories. Note that \mathcal{C}/D satisfies a universal property: to give a unital factorization functor $\mathcal{C}/D \rightarrow \mathcal{C}'$ is equivalent to give a functor $\mathcal{C} \rightarrow \mathcal{C}'$ sending D to 0.

This material renders to the non-unital setting with the appropriate changes in notation.

6.6. Module spaces. Next, we discuss factorization modules. We begin with the non-unital setting.

Definition 6.6.1. A *factorization module space* \mathcal{Z} for $\text{Ran}_{\mathcal{X}}$ is a (by necessity: non-unital) $\text{Ran}_{\mathcal{X}}^{ch}$ -module in PreStk_{corr} . An *augmented factorization module space* (over $\text{Ran}_{\mathcal{X}}$) is a factorization module space equipped with a morphism:

$$\varpi : \mathcal{Z} \rightarrow \text{Ran}_{\mathcal{X}}$$

of prestacks (not merely a correspondence), with ϖ equipped with a structure of morphism of $\text{Ran}_{\mathcal{X}}^{ch}$ -modules in PreStk_{corr} , where $\text{Ran}_{\mathcal{X}}^{ch}$ acts on itself by the chiral action.

Remark 6.6.2. To unwind this definition somewhat: a factorization module space \mathcal{Z} is, in particular, equipped with an *action* correspondence:

$$\begin{array}{ccc} & \mathcal{H}_{\mathcal{Z}} & \\ & \swarrow \quad \searrow & \\ \text{Ran}_{\mathcal{X}} \times \mathcal{Z} & & \mathcal{Z}. \end{array}$$

For an augmented factorization module space \mathcal{Z} , the morphism ϖ induces a map:

$$\begin{array}{ccccc} & & \mathcal{H}_{\mathcal{Z}} & & \\ & \swarrow & \downarrow & \searrow & \\ \text{Ran}_{\mathcal{X}} \times \mathcal{Z} & & [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} & & \mathcal{Z} \\ \text{id} \times \varpi \downarrow & \swarrow & & \searrow & \downarrow \varpi \\ \text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}} & & & & \text{Ran}_{\mathcal{X}} \end{array}$$

with the left square Cartesian.

Note that this means that if we are trying to define the structure of augmented factorization module space on $\mathcal{Z} \rightarrow \text{Ran}_{\mathcal{X}}$ over $\text{Ran}_{\mathcal{X}}$, we already know what $\mathcal{H}_{\mathcal{Z}}$ must be, and the content lies in defining the map:

$$\mathcal{H}_{\mathcal{Z}} = \left(\text{Ran}_{\mathcal{X}} \times \mathcal{Z} \right)_{\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}} \times [\text{Ran}_{\mathcal{X}} \times \text{Ran}_{\mathcal{X}}]_{disj} \rightarrow \mathcal{Z}$$

and its higher compatibilities.

Example 6.6.3. Suppose that $\mathcal{Z} \in \text{PreStk}$ admits an action (in PreStk) by $\text{Ran}_{\mathcal{X}}^* = (\text{Ran}_{\mathcal{X}}, \text{add})$, and a $\text{Ran}_{\mathcal{X}}$ -equivariant morphism:

$$\mathcal{Z} \rightarrow \text{Ran}_{\mathcal{X}}.$$

Then we claim that \mathcal{Z} admits a canonical structure of augmented factorization module space. Indeed, this follows in the same way that $\text{Ran}_{\mathcal{X}}$ inherits its chiral multiplication.

6.7. Examples of factorization module spaces. We have two key examples of factorization module spaces: $\text{Ran}_{\mathcal{X},I}$ introduced below for I a finite set, and $\text{Ran}_{\mathcal{X}}^{\rightarrow}$.

Let \mathbf{fSet}_I denote the category whose objects are arbitrary maps $I \rightarrow J$ and where morphisms are commutative diagrams:

$$\begin{array}{ccc} & & J \\ & \nearrow & \downarrow \\ I & & \\ & \searrow & \\ & & J' \end{array}$$

We define the \mathcal{X}^I -marked Ran space $\text{Ran}_{\mathcal{X},I}$ as the colimit:

$$\text{Ran}_{\mathcal{X},I} := \text{colim}_{(I \rightarrow J) \in \mathbf{fSet}_I^{\text{op}}} \mathcal{X}^J \in \text{PreStk}.$$

There is a canonical map $\text{Ran}_{\mathcal{X},I} \rightarrow \mathcal{X}^I$.

Remark 6.7.1. The reader should think of $\text{Ran}_{\mathcal{X},I}$ as the parameter space of a map $I \xrightarrow{i \rightarrow x_i} \mathcal{X}$ and an embedding $\{x_i\} \subseteq J \subseteq \mathcal{X}$ of finite subsets.

Then $\text{Ran}_{\mathcal{X},I}$ admits an obvious structure of $\text{Ran}_{\mathcal{X}}^*$ -module space, and therefore, by Example 6.6.3, $\text{Ran}_{\mathcal{X},I}$ obtains a canonical structure of augmented factorization module space.

Similarly, $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ admits a canonical $\text{Ran}_{\mathcal{X}}^*$ -module space structure.

Here we introduce the category $\mathbf{fSet}^{\rightarrow}$ whose objects are arbitrary maps $I \rightarrow J$ of non-empty finite sets, and where morphisms are commutative diagrams with termwise surjective maps. We remark that $\mathbf{fSet}^{\rightarrow}$ was introduced in 4.8 under the notation $\mathbf{fSet}_{[1]}^{\rightarrow}$.

Recall that we have:

$$\text{Ran}_{\mathcal{X}}^{\rightarrow} = \text{colim}_{(I \rightarrow J) \in \mathbf{fSet}^{\rightarrow, \text{op}}} \mathcal{X}^J \in \text{PreStk}.$$

The action of $\text{Ran}_{\mathcal{X}}$ on $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ is then defined using the maps:

$$\begin{aligned} \mathbf{fSet} \times \mathbf{fSet}^{\rightarrow} &\rightarrow \mathbf{fSet} \\ \left(K, (\gamma : I \rightarrow J) \right) &\mapsto (I \rightarrow J \coprod K). \end{aligned}$$

Notation 6.7.2. We use the notation:

$$\begin{aligned} \sigma_{\mathcal{X}^I} &: \mathcal{X}^I \rightarrow \text{Ran}_{\mathcal{X},I} \\ \sigma_{\text{Ran}_{\mathcal{X}}} &: \text{Ran}_{\mathcal{X}} \rightarrow \text{Ran}_{\mathcal{X}}^{\rightarrow} \end{aligned}$$

for the obvious sections.

6.8. Factorization modules. Let \mathcal{Z} be a factorization $\text{Ran}_{\mathcal{X}}$ -module space.

Definition 6.8.1. As in §5.6, for \mathbf{C} a chiral category on \mathcal{X} , we have a notion of chiral \mathbf{C} -module category \mathbf{M} over \mathcal{Z} . We denote the resulting category by $\text{ModCat}_{\mathcal{Z}}^{\text{ch}}(\mathbf{C})$.

Moreover, for \mathcal{A} a factorization algebra in \mathbf{C} and $\mathbf{M} \in \text{ModCat}_{\mathcal{Z}}^{\text{ch}}(\mathbf{C})$, §5.6 gives a notion of *factorization \mathcal{A} -module* in \mathbf{M} . We denote the resulting category by $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{M})$.

Remark 6.8.2. Our notation will frequently identify $M \in \text{ModCat}_{/\mathcal{Z}}^{ch}(\mathbb{C})$ with its underlying sheaf of categories on \mathcal{Z} , and $M \in \mathcal{A}\text{-mod}^{\text{fact}}(M)$ with the underlying object of $\Gamma(\mathcal{Z}, M)$.

Remark 6.8.3. Using the general stability results in [Lur1], one readily sees that $\mathcal{A}\text{-mod}^{\text{fact}}(M)$ is a cocomplete DG category.

Remark 6.8.4. Let \mathcal{Z} be a factorization $\text{Ran}_{\mathcal{X}}$ -module space. Suppose that we have \mathbb{C} and \mathbb{D} chiral categories on \mathcal{X} with chiral module categories $M \in \text{ModCat}_{/\mathcal{Z}}^{ch}(\mathbb{C})$, $N \in \text{ModCat}_{/\mathcal{Z}}^{ch}(\mathbb{D})$. Suppose that we have a morphism of factorization module data¹² from (\mathbb{C}, M) to (\mathbb{D}, N) with underlying functors:

$$\begin{aligned}\psi &: \mathbb{C} \rightarrow \mathbb{D} \\ \varphi &: M \rightarrow N.\end{aligned}$$

By Remark 6.2.8, there is an induced functor $\psi : \text{Alg}^{\text{fact}}(\mathbb{C}) \rightarrow \text{Alg}^{\text{fact}}(\mathbb{D})$, and as in *loc. cit.*, for $\mathcal{A} \in \text{Alg}^{\text{fact}}(\mathbb{C})$ we obtain a canonical functor:

$$\mathcal{A}\text{-mod}^{\text{fact}}(M) \rightarrow \varphi(\mathcal{A})\text{-mod}^{\text{fact}}(N). \quad (6.8.1)$$

Notation 6.8.5. When $\mathcal{Z} = \text{Ran}_{\mathcal{X}, I}$, we use the notation $\text{ModCat}_{/\mathcal{X}^I}^{ch}(\mathbb{C})$ in place of $\text{ModCat}_{/\text{Ran}_{\mathcal{X}, I}}^{ch}(\mathbb{C})$, and $\mathcal{A}\text{-mod}^{\text{fact}}(\sigma_{\mathcal{X}^I}^*(M))$ in place of $\mathcal{A}\text{-mod}^{\text{fact}}(M)$ when there is no risk for confusion. We refer to e.g. such chiral module categories as chiral module categories on \mathcal{X}^I (for \mathbb{C}). Note that in this setting, $\mathcal{A}\text{-mod}^{\text{fact}}(M)$ is a $\text{QCoh}(\mathcal{X}^I)$ -module category.

We remark that these notions were defined previously in the $I = *$ case in [BD2], and for higher order I in [Roz] and [FG1].

Example 6.8.6. The restriction $C_{\mathcal{X}^I}$ of \mathbb{C} to \mathcal{X}^I can be regarded as a factorization module category over \mathbb{C} on \mathcal{X}^I .

6.9. Unital modules. Next, we discuss the unital setting. The definitions are largely parallel to those in the non-unital setting, and therefore we indicate them only briefly.

6.10. A *unital factorization module space* for $\text{Ran}_{\mathcal{X}}$ is a lax prestack \mathcal{Z}^{un} with an action of $\text{Ran}_{\mathcal{X}}^{un, ch}$ in $\text{PreStk}_{\text{corr}}^{\text{lax}}$. Similarly, we have the notion of *augmented unital factorization module space*: we ask in addition for a $\text{Ran}_{\mathcal{X}}^{un, ch}$ -equivariant map $\varpi : \mathcal{Z}^{un} \rightarrow \text{Ran}_{\mathcal{X}}^{un}$ that is a morphism in the 1-full subcategory $\text{PreStk}^{\text{lax}}$ of $\text{PreStk}_{\text{corr}}^{\text{lax}}$.

Remark 6.10.1. Understanding these conditions explicitly works exactly as in the non-unital setting of Remark 6.6.2.

For \mathcal{Z}^{un} a unital factorization module space, we define $\mathcal{Z} := \mathcal{Z}^{un, \text{PreStk}} \in \text{PreStk}$ to be the underlying prestack. Clearly \mathcal{Z} carries a canonical structure of factorization module space for $\text{Ran}_{\mathcal{X}}$.

Remark 6.10.2. We alert the reader to a potential source of confusion in this notation: \mathcal{Z} is constructed from \mathcal{Z}^{un} , and not the other way around.

Terminology 6.10.3. We will sometimes abbreviate the situation by simply saying that \mathcal{Z} is a unital factorization module space for $\text{Ran}_{\mathcal{X}}$, with the structure of \mathcal{Z}^{un} being implicit.

As in Example 6.6.3, we can produce augmented unital factorization module spaces from augmented $\text{Ran}_{\mathcal{X}}^{un, *}$ -modules in $\text{PreStk}^{\text{lax}}$.

¹²Really, we mean a morphism of multiplicative sheaves of categories with respect to the colored operad controlling non-unital algebras with a left module.

Example 6.10.4. From this construction, one obtains lax prestacks $\text{Ran}_{\mathcal{X}}^{un,\rightarrow}$ and $\text{Ran}_{\mathcal{X},I}^{un}$ with unital factorization module space structures, and with underlying prestacks $\text{Ran}_{\mathcal{X}}^{\rightarrow}$ and $\text{Ran}_{\mathcal{X},I}$ respectively.

6.11. For \mathcal{Z}^{un} a unital factorization module space for $\text{Ran}_{\mathcal{X}}$, we define unital chiral module category \mathbb{M} for a unital chiral category \mathbb{C} as in the non-unital case.¹³

Similarly, we define unital factorization modules for a unital factorization algebra \mathcal{A} in a specified unital factorization module category.

We denote the resulting categories by:

$$\text{ModCat}_{un,\mathcal{Z}}^{ch}(\mathbb{C}) \text{ and } \mathcal{A}\text{-mod}_{un}^{\text{fact}}(\mathbb{M}).$$

The latter is a cocomplete DG category.

Notation 6.11.1. We will allow notations parallel to those from Notation 6.8.5 when $\mathcal{Z} = \text{Ran}_{\mathcal{X},I}$.

Remark 6.11.2. The obvious counterpart to Remark 6.8.4 holds in the unital setting just as well.

6.12. **External fusion.** Next, we discuss the *external fusion* construction. For definiteness, we take $\mathcal{X} = X_{dR}$. Let \mathbb{C} be a chiral category on \mathcal{X} and let \mathcal{A} be a factorization algebra in \mathbb{C} .

We give a description of what is expected from external fusion in this section, postponing its construction to 6.22.

For I a finite set, let $\mathbb{C}_{X_{dR}^I}$ denote the corresponding sheaf of categories on X_{dR}^I . As in Example 6.8.6, $\mathbb{C}_{X_{dR}^I}$ is a chiral module category for \mathbb{C} . Therefore, we obtain the category $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbb{C}_{X_{dR}^I})$ of chiral modules for \mathcal{A} on X_{dR}^I .

For I and J two finite sets, we form $[X_{dR}^I \times X_{dR}^J]_{disj}$ and let:

$$\mathbb{C}_{I,J,disj} \in \text{ShvCat}_{/[X_{dR}^I \times X_{dR}^J]_{disj}}$$

denote the restriction of $\mathbb{C}_{X_{dR}^{I \amalg J}}$, considered as a \mathbb{C} -chiral module category in the natural way.

The *external fusion* construction is a canonical functor:

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbb{C}_{X_{dR}^I}) \otimes \mathcal{A}\text{-mod}^{\text{fact}}(\mathbb{C}_{X_{dR}^J}) \rightarrow \mathcal{A}\text{-mod}^{\text{fact}}(\mathbb{C}_{I,J,disj}). \quad (6.12.1)$$

of $D(X^I) \otimes D(X^J)$ -module categories.

At the level of global sections on X_{dR}^I , X_{dR}^J and $[X_{dR}^I \times X_{dR}^J]_{disj}$, this construction is given by external product. We describe it completely at the module level in §6.22.

Remark 6.12.1. We do not expect (6.12.1) to be an equivalence in general: rather, we expect this only after an appropriate *renormalization*, and this depends on the specific factorization algebra under consideration. For the Kac-Moody factorization algebra, the appropriate notion of renormalization is explained over a point in [FG2].

Remark 6.12.2. The functoriality of this construction will be enhanced in §8.14.

6.13. **Modules for the unit factorization algebra.** A key slogan in the unital setting is that a unital module structure for the unit is no extra data. We make this precise below.

Let \mathbb{C} be a unital factorization category on \mathcal{X} and let I be a finite set.

¹³However, we emphasize that the colored operad we use is that controlling unital commutative algebras equipped with a unital module.

Construction 6.13.1. Form the diagram:

$$\begin{array}{ccc} \mathrm{Ran}_{\mathcal{X}, I, \emptyset} & \xrightarrow{p_2} & \mathrm{Ran}_{\mathcal{X}} \\ p_1 \downarrow & & \\ \mathcal{X}^I & & \end{array}$$

As in §6.3, the map $\mathfrak{Fus}_{\mathbb{C}}$ induces a functor:

$$p_1^*(\mathbb{C}_{\mathcal{X}^I}) \rightarrow p_2^*(\mathbb{C}).$$

As in *loc. cit.*, the material of §5.34 shows that the functor upgrades to give:

$$\Gamma(\mathcal{X}^I, \mathbb{C}_{\mathcal{X}^I}) \rightarrow \mathrm{unit}_{\mathbb{C}}\text{-mod}_{un}^{\mathrm{fact}}(\mathbb{C}_{\mathcal{X}^I}).$$

This functor is easily seen to be left adjoint to the obvious restriction functor.

Theorem 6.13.2. *For $\mathcal{X} = X_{dR}$ with X a finite type scheme, the restriction functor:*

$$\mathrm{unit}_{\mathbb{C}}\text{-mod}_{un}^{\mathrm{fact}}(\mathbb{C}_{X_{dR}^I}) \rightarrow \Gamma(X_{dR}^I, \mathbb{C}_{X_{dR}^I})$$

is an equivalence with inverse given by Construction 6.13.1.

Proof. The composition:

$$\Gamma(X_{dR}^I, \mathbb{C}_{X_{dR}^I}) \rightarrow \mathrm{unit}_{\mathbb{C}}\text{-mod}_{un}^{\mathrm{fact}}(\mathbb{C}_{X_{dR}^I}) \rightarrow \Gamma(X_{dR}^I, \mathbb{C}_{X_{dR}^I})$$

is obviously the identity functor.

One easily constructs (for general \mathcal{X}) a canonical natural transformation:

$$\begin{array}{c} \left(\mathrm{unit}_{\mathbb{C}}\text{-mod}_{un}^{\mathrm{fact}}(\mathbb{C}_{X_{dR}^I}) \rightarrow \Gamma(X_{dR}^I, \mathbb{C}_{X_{dR}^I}) \rightarrow \mathrm{unit}_{\mathbb{C}}\text{-mod}_{un}^{\mathrm{fact}}(\mathbb{C}_{X_{dR}^I}) \right) \\ \Downarrow \\ \mathrm{id}_{\mathrm{unit}_{\mathbb{C}}\text{-mod}_{un}^{\mathrm{fact}}(\mathbb{C}_{X_{dR}^I})} \end{array}$$

using fusion.

But this natural transformation is immediately seen to be an equivalence over strata in $\mathrm{Ran}_{X_{dR}, I}$ by exploiting factorization, and then the fact that we are dealing with D -modules means that this map is an equivalence. □

6.14. In §6.14-6.20, we compare our definition of factorization algebra with that of [FG1] in the case $\mathcal{X} = X_{dR}$.

This material is a bit digressive, and the reader may safely skip it and refer back to it as necessary.

We fix X a separated scheme of finite type through §6.20.

Remark 6.14.1. We follow [FG1] closely in our definitions here.

Remark 6.14.2. What follows is, by necessity, entirely in the non-unital setting.

6.15. We begin with a construction in the general framework as in §5: let \mathcal{S} be a commutative algebra in $\text{PreStk}_{\text{corr}}$. We use the notation (5.21.1) for the correspondences defining the multiplication and unit operations.

Under this hypothesis, Corollary A.12.1 implies that $\text{ShvCat}/_{\mathcal{S}}$ carries a canonical symmetric monoidal structure with monoidal product the composition:

$$\text{ShvCat}/_{\mathcal{S}} \times \text{ShvCat}/_{\mathcal{S}} \xrightarrow{-\boxtimes-} \text{ShvCat}/_{\mathcal{S} \times \mathcal{S}} \xrightarrow{m_1^*} \text{ShvCat}/_{\text{mult}_{\mathcal{S}}} \xrightarrow{m_{2,*}} \text{ShvCat}/_{\mathcal{S}}.$$

We will denote the tensor product for this symmetric monoidal structure by:

$$\Psi * \Phi := m_{2,*} m_1^*(\Psi \boxtimes \Phi).$$

Remark 6.15.1. Observe that the functor:

$$\Gamma(\mathcal{S}, -) : \text{ShvCat}/_{\mathcal{S}} \rightarrow \text{DGCat}_{\text{cont}}$$

is lax symmetric monoidal relative to the symmetric monoidal structure $*$ and the tensor product of cocomplete DG categories, respectively. The structure maps are given by the tautological map:

$$\begin{aligned} \Gamma(\mathcal{S}, \Psi) \otimes \Gamma(\mathcal{S}, \Phi) &\rightarrow \Gamma(\mathcal{S} \times \mathcal{S}, \Psi \boxtimes \Phi) \rightarrow \Gamma(\text{mult}_{\mathcal{S}}, m_1^*(\Psi \boxtimes \Phi)) = \\ &\Gamma(\mathcal{S}, m_{2,*} m_1^*(\Psi \boxtimes \Phi)) =: \Gamma(\mathcal{S}, \Psi * \Phi). \end{aligned}$$

Recall that we have defined $\text{MultCat}^{op-w}(\mathcal{S})$ in §5.29. The following result follows from the theory of correspondences.

Proposition 6.15.2. *There is a canonical equivalence of categories:*

$$\text{MultCat}^{op-w}(\mathcal{S}) \simeq \text{ComCoalg}^{lax} \left((\text{ShvCat}/_{\mathcal{S}}, *) \right).$$

Here the right hand side of the equality is the category of commutative coalgebras under lax morphisms, as in §5.23.

6.16. We will need the following material about the equivalence of Proposition 6.15.2.

Let:

$$\text{ComCoalg}^{r.adj} \left((\text{ShvCat}/_{\mathcal{S}}, *) \right) \subseteq \text{ComCoalg} \left((\text{ShvCat}/_{\mathcal{S}}, *) \right)$$

denote the full subcategory consisting of commutative coalgebras \mathbb{C} for which the maps:

$$\Psi \rightarrow \Psi * \Psi \text{ and } \Psi \rightarrow \text{QCoh}_{\mathcal{S}}$$

admit right adjoints in the category $\text{ShvCat}/_{\mathcal{S}}$ (equivalently: we can ask this for all n -ary operations). Define the full subcategory:

$$\text{ComAlg}^{l.adj} \left((\text{ShvCat}/_{\mathcal{S}}, *) \right) \subseteq \text{ComAlg} \left((\text{ShvCat}/_{\mathcal{S}}, *) \right)$$

similarly, with left adjoints replacing the role of right adjoints.

By the theory [GR] of 2-categories, we obtain an equivalence:

$$\text{ComAlg}^{l.adj} \left((\text{ShvCat}/_{\mathcal{S}}, *) \right) \simeq \text{ComCoalg}^{r.adj} \left((\text{ShvCat}/_{\mathcal{S}}, *) \right) \quad (6.16.1)$$

given by passing to adjoints in our operations.

Observe that, by Proposition A.9.1 (3), if m_2 and e_2 are quasi-compact quasi-separated schematic morphisms, then the category $\text{ComCoalg}^{r.adj}(\text{ShvCat}/\mathcal{S}, *)$ contains

$$\text{MultCat}^{op-w,r.adj}(\mathcal{S}) \subseteq \text{MultCat}^{op-w}(\mathcal{S})$$

under the equivalence of Proposition 6.15.2. In particular, it contains $\text{MultCat}(\mathcal{S})$.

6.17. We now give a version of Proposition 6.15.2 for multiplicative sheaves.

Given $\Psi \in \text{ComAlg}(\text{ShvCat}/\mathcal{S}, *)$, the category $\Gamma(\mathcal{S}, \Psi)$ inherits a canonical symmetric monoidal structure, coming from the lax symmetric monoidal structure of Remark 6.15.1.

Suppose that m_2 and e_2 are quasi-compact quasi-separated schematic morphisms. Proposition 6.15.2, the conclusion of §6.16, and (6.16.1) imply that for $\Psi \in \text{MultCat}^{op-w,r.adj}(\mathcal{S})$, $\Gamma(\mathcal{S}, \Psi)$ inherits a canonical symmetric monoidal structure. We will denote the symmetric monoidal product here by $*$ as well.

Using the perspective of §5.26, we obtain the following counterpart to Proposition 6.15.2.

Proposition 6.17.1. *For $\Psi \in \text{MultCat}^{op-w,r.adj}$, there is a canonical equivalence of categories:*

$$\text{Mult}^{op-w}(\Psi) \simeq \text{ComAlg}(\Gamma(\mathcal{S}, \Psi), *).$$

6.18. We now specialize to the case of Ran space.

We have the following lemma.

Lemma 6.18.1. *The morphism:*

$$\text{add} : [\text{Ran}_{X_{dR}} \times \text{Ran}_{X_{dR}}]_{disj} \rightarrow \text{Ran}_{X_{dR}}$$

is schematic and a quasi-compact étale morphism.

Proof. First, note that tautologically we have $\text{Ran}_{X_{dR}} = (\text{Ran}_X)_{dR}$.

Let S be an affine test scheme. As in Example 4.3.2, a morphism $\varphi : S \rightarrow [\text{Ran}_{X_{dR}} \times \text{Ran}_{X_{dR}}]_{disj}$ is equivalent to giving two finite sets:

$$\{\varphi_1^1, \dots, \varphi_n^1\} \text{ and } \{\varphi_1^2, \dots, \varphi_m^2\}$$

where each φ_i^j is a map $S^{cl,red} \rightarrow X$, and such that, for every $1 \leq i \leq n$ and $1 \leq i' \leq m$, the map $\varphi_i^1 \times \varphi_{i'}^2 : S^{cl,red} \rightarrow X \times X$ factors through the open $X \times X \setminus \Delta(X)$.

Moreover, a map $\psi : S \rightarrow \text{Ran}_{X_{dR}}$ is equivalent to giving a finite collection of maps $\psi_1, \dots, \psi_r : S^{cl,red} \rightarrow X$. Therefore, we see that the fiber over such a map is the coproduct of spaces:

$$S \times_{X_{dR}^r} [X_{dR}^n \times X_{dR}^m]_{disj}$$

with the coproduct taken over positive integers with $n + m = r$. This evidently gives the result. \square

6.19. By Lemma 6.18.1, $\mathcal{S} := \text{Ran}_{X_{dR}, \emptyset}$ satisfies the requirements of the discussion in §6.15-6.17. Therefore, for $\mathbf{C} \in \text{Cat}^{ch}(X_{dR})$, the category:

$$\Gamma(\text{Ran}_{X_{dR}, \emptyset}, \mathbf{C}) = \text{Vect} \oplus \Gamma(\text{Ran}_{X_{dR}}, \mathbf{C})$$

inherits a symmetric monoidal structure. More precisely, $\Gamma(\text{Ran}_{X_{dR}}, \mathbf{C})$ carries a non-unital commutative algebra structure in DGCat_{cont} , and this unital symmetric monoidal structure arises by formally adding a unit (in DGCat_{cont}).

We refer to this (resp. non-unital) symmetric monoidal structure as the *chiral tensor product* on $\Gamma(\text{Ran}_{X_{dR}, \emptyset}, \mathbb{C})$ (resp. $\Gamma(\text{Ran}_{X_{dR}}, \mathbb{C})$). We denote the resulting binary product by $-\overset{ch}{\otimes}-$.

Definition 6.19.1. A *chiral coalgebra* in \mathbb{C} is a non-unital commutative coalgebra in $(\Gamma(\text{Ran}_{X_{dR}}, \mathbb{C}), \overset{ch}{\otimes})$. We denote the resulting category by $\text{Coalg}^{\text{ch}}(\mathbb{C})$.

Remark 6.19.2. The category $\text{Coalg}^{\text{ch}}(\mathbb{C})$ is cocomplete.

Remark 6.19.3. We can identify $\text{Coalg}^{\text{ch}}(\mathbb{C})$ with the full subcategory of unital coalgebras in $(\Gamma(\text{Ran}_{X_{dR}, \emptyset}, \mathbb{C}), \overset{ch}{\otimes})$ consisting of those coalgebras such that the counit map becomes an isomorphism after applying the projection:

$$\left(\Gamma(\text{Ran}_{X_{dR}, \emptyset}, \mathbb{C}), \overset{ch}{\otimes}\right) = \text{Vect} \oplus \Gamma(\text{Ran}_{X_{dR}}, \mathbb{C}) \rightarrow \text{Vect}.$$

The following results from Proposition 6.17.1.

Proposition 6.19.4. *There is a canonical equivalence:*

$$\text{Mult}_{\text{non-unital}}^{\text{op-w}}(\mathbb{C}) \simeq \text{Coalg}^{\text{ch}}(\mathbb{C}).$$

Here, as in §6.2, the subscript “non-unital” indicates that we take the operad controlling non-unital commutative algebras.

Remark 6.19.5. This proposition implies that for X a separated scheme of finite type, the category $\text{Alg}^{\text{fact}}(X_{dR})$ coincides with the category of factorization algebras as defined in [FG1]. A variant of the above material with general colored operads allows us to put the theory of chiral modules from [FG1] into our framework as well.

6.20. Let \mathbb{C} be a factorization category on X_{dR} .

Definition 6.20.1. We define the category $\text{LieAlg}^{\text{ch}}(\mathbb{C})$ of *chiral Lie algebras* in \mathbb{C} as the category of Lie algebras in $(\Gamma(\text{Ran}_{X_{dR}}, \mathbb{C}), \overset{ch}{\otimes})$.

We define the full subcategory $\text{Alg}^{\text{ch}}(\mathbb{C}) \subseteq \text{LieAlg}^{\text{ch}}(\mathbb{C})$ of *chiral algebras* in \mathbb{C} to consist of those chiral Lie algebras whose underlying object lies in the full subcategory:

$$\Gamma(X_{dR}, \mathbb{C}|_{X_{dR}}) \subseteq \Gamma(\text{Ran}_{X_{dR}}, \mathbb{C}).$$

6.21. Fix $\mathbb{C} \in \text{Cat}^{\text{ch}}(X_{dR})$, and let $\mathcal{C} = \Gamma(\text{Ran}_{X_{dR}}, \mathbb{C})$ be considered a non-unital algebra in $\text{DGCat}_{\text{cont}}$ through the chiral tensor product.

As in [FG1], we have the following result:

Theorem 6.21.1. *The Koszul duality functor:*

$$\text{LieAlg}^{\text{ch}}(\mathbb{C}) := \text{LieAlg}(\mathcal{C}) \rightarrow \text{ComCoalg}(\mathcal{C}) =: \text{Coalg}^{\text{ch}}(\mathbb{C})$$

is an equivalence.

This equivalence identifies the full subcategories $\text{Alg}^{\text{ch}}(\mathbb{C})$ and $\text{Alg}^{\text{fact}}(\mathbb{C})$.

Warning 6.21.2. We remind that this functor does not commute with forgetful functors to \mathcal{C} : rather, the composition $\text{LieAlg}(\mathcal{C}) \rightarrow \text{ComCoalg}(\mathcal{C}) \xrightarrow{\text{Oblv}} \mathcal{C}$ is given by the (reduced) homological Chevalley complex.

Remark 6.21.3. We have a theory of chiral modules $\mathcal{A}\text{-mod}^{ch}(\mathbb{C}_{X_{dR}^I})$ on X_{dR}^I for a chiral Lie algebra $\mathcal{A} \in \text{Alg}^{ch} \mathbb{C}$, as in [FG1], and for $\mathcal{B} \in \text{Alg}^{fact} \mathbb{C}$ the Koszul dual to \mathcal{A} , we again have a Koszul duality equivalence:

$$\mathcal{A}\text{-mod}^{ch}(\mathbb{C}_{X_{dR}^I}) \simeq \mathcal{B}\text{-mod}^{fact}(\mathbb{C}_{X_{dR}^I})$$

6.22. Construction of external fusion. As promised in §6.12, we now carefully describe the external fusion construction.

Remark 6.22.1. The construction imitates the construction of the tensor product of modules as the geometric realization of the bar construction.

We give two (tautologically the same) descriptions of external fusion, in §6.23 and §6.24. The former is somewhat more hands on while the latter is somewhat more conceptual.

6.23. First, we describe external fusion explicitly in terms of descent data.

Recall the prestack $\text{Ran}_{X_{dR},I}$ (resp. $\text{Ran}_{X_{dR},J}$) from Example 6.7. Let σ_I (resp. σ_J) denote the structure map to $\text{Ran}_{X_{dR}}$. Let $\text{Ran}_{X_{dR},I,J,disj}$ denote the variant of $\text{Ran}_{X_{dR},I} \amalg \text{Ran}_{X_{dR},J}$ where we require our points in $X_{dR}^I \times X_{dR}^J$ to lie $[X_{dR}^I \times X_{dR}^J]_{disj} \xrightarrow{\text{add}} \text{Ran}_{X_{dR}}$.

Let $M \in \mathcal{A}\text{-mod}^{fact}(\mathbb{C}_{X_{dR}^I})$ and $N \in \mathcal{A}\text{-mod}^{fact}(\mathbb{C}_{X_{dR}^J})$. Let $\tilde{M} \in \Gamma(\text{Ran}_{X_{dR},I}, \sigma_I^*(\mathbb{C}))$ be the object defining the factorization module structure for M , and let \tilde{N} be defined similarly.

We form the augmented simplicial object:

$$\dots \rightrightarrows [\text{Ran}_{X_{dR},I} \times \text{Ran}_{X_{dR},\emptyset} \times \text{Ran}_{X_{dR},J}]_{disj} \rightrightarrows [\text{Ran}_{X_{dR},I} \times \text{Ran}_{X_{dR},J}]_{disj} \longrightarrow \text{Ran}_{X_{dR},I,J,disj}$$

where e.g. $[\text{Ran}_{X_{dR},I} \times \text{Ran}_{X_{dR},J}]_{disj}$ denotes the locus where the corresponding points of $\text{Ran}_{X_{dR}} \times \text{Ran}_{X_{dR}}$ are disjoint, and $[\text{Ran}_{X_{dR},I} \times \text{Ran}_{X_{dR}} \times \text{Ran}_{X_{dR},J}]_{disj}$ denotes the locus where the triple of points of $\text{Ran}_{X_{dR}}$ are pairwise disjoint, etc. The two horizontal maps in the above simplicial object are given by the action maps for $\text{Ran}_{X_{dR},I}$ and $\text{Ran}_{X_{dR},J}$ respectively.

We form a compatible sheaf of categories on this simplicial diagram by pullback of \mathbb{C} from $\text{Ran}_{X_{dR}}$. Indeed, the factorization of \mathbb{C} allows us to form this construction.

Then the structure of module on M and N allows us to form a compatible system of global sections here, where on the first term we take $\tilde{M} \boxtimes \tilde{N}$ (i.e., its restriction to the disjoint locus), and on the second term we take $\tilde{M} \boxtimes \mathcal{A} \boxtimes \tilde{N}$, $\tilde{M} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \boxtimes \tilde{N}$, etc.

Observe that our augmented simplicial object above is an étale hypercovering of $\text{Ran}_{X_{dR},I,J,disj}$ (c.f. Lemma 6.18.1). Therefore, by étale hyperdescent, this defines an object $\widetilde{M \boxtimes N}$ on $\text{Ran}_{X_{dR},I,J,disj}$. One easily verifies that it carries a canonical structure of \mathcal{A} -module as desired.

Remark 6.23.1. The above works in the unital setting as well, showing that the external fusion of unital modules is naturally a unital module as well.

6.24. We now reinterpret external fusion as a kind of cotensor product of comodules over a commutative coalgebra.

6.25. In some generality, let \mathcal{D} be a commutative algebra in DGCat_{cont} and let \mathcal{M} and \mathcal{N} be two \mathcal{D} -modules in DGCat_{cont} . Suppose $A \in \mathcal{D}$ is a commutative coalgebra and $M \in \mathcal{M}$ and $N \in \mathcal{N}$ are two A -comodules.

Then we can form the *cotensor product*:

$$M \boxtimes_A N \in \mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}.$$

Indeed, the object $M \boxtimes N \in \mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}$ carries a canonical comodule structure for $A \boxtimes A \in \mathcal{D} \otimes \mathcal{D}$, and the cotensor product is then the right adjoint to the restriction functor from A -comodules to $A \boxtimes A$ -comodules. It can be computed as the totalization of the cobar complex, whose terms are of the usual form $A^{\otimes n} \otimes (M \boxtimes N) \in \mathcal{M} \otimes_{\mathcal{D}} \mathcal{N}$.

6.26. We apply the above framework as follows.

We have an action of $\text{Ran}_{X_{dR}, \emptyset}^{ch}$ on $\text{Ran}_{X_{dR}, I}$ in PreStk_{corr} : as usual, we use the correspondence encoding disjointness of pairs of points to define the action map.

We have a further correspondence of similar nature between $\text{Ran}_{X_{dR}, I} \times \text{Ran}_{X_{dR}, J}$ and $\text{Ran}_{X_{dR}, I, J, disj}$. These two constructions are compatible, so that the morphism:

$$\Gamma(\text{Ran}_{X_{dR}, I}, \sigma_I^*(\mathbb{C})) \otimes \Gamma(\text{Ran}_{X_{dR}, J}, \sigma_J^*(\mathbb{C})) \rightarrow \Gamma(\text{Ran}_{X_{dR}, I, J, disj}, \sigma_{I, J, disj}^*(\mathbb{C}))$$

is a morphism of $(\mathcal{C}, \otimes^{ch}) \otimes (\mathcal{C}, \otimes^{ch})$ -module categories, where the target obtains this structure through the action of $(\mathcal{C}, \otimes^{ch})$ by restriction along the map $\otimes^{ch}: \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$.

It then follows that for $\mathcal{A} \in \text{Coalg}^{ch}(\mathbb{C})$ and for modules M and N as above, we can form their cotensor product in $\Gamma(\text{Ran}_{X_{dR}, I, J, disj}, \sigma_{I, J, disj}^*(\mathbb{C}))$. Moreover, it follows tautologically that this agrees with the descent construction given above.

Remark 6.26.1. This format makes clear what the version of this construction for chiral Lie algebras is; we do not emphasize this perspective, since it does not translate so easily to the unital setting.

7. COMMUTATIVE CHIRAL CATEGORIES

7.1. In this section, we develop a theory of commutative chiral categories and commutative factorization algebras, following [BD2].

7.2. Let \mathcal{X} be a fixed prestack.

Recall that $\text{Ran}_{\mathcal{X}}^*$ denotes the prestack $\text{Ran}_{\mathcal{X}}$ considered with the non-unital commutative monoid structure of addition.

Definition 7.2.1. A *commutative weak chiral category* is a multiplicative sheaf of categories on $\text{Ran}_{\mathcal{X}}^*$.

The identity morphism for $\text{Ran}_{\mathcal{X}}$ obviously upgrades to a lax morphism:

$$\text{Ran}_{\mathcal{X}}^{ch} \rightarrow \text{Ran}_{\mathcal{X}}^*$$

of non-unital commutative algebras in the 2-category PreStk_{corr} (see §5.23 for the notion of lax morphism of monoids in a 2-category). Using this structure, one constructs a canonical restriction functor from commutative weak chiral categories to weak chiral categories.

Definition 7.2.2. A *commutative chiral category* is a commutative weak chiral category whose underlying weak chiral category is a chiral category.

Similarly, a *commutative factorization algebra* in a commutative chiral category \mathbb{C} is a weakly multiplicative sheaf over $\text{Ran}_{\mathcal{X}}^*$ whose underlying weakly multiplicative sheaf over $\text{Ran}_{\mathcal{X}}^{ch}$ is a multiplicative sheaf.

Remark 7.2.3. Roughly, a commutative chiral category is a sheaf of categories \mathbf{C} on $\mathrm{Ran}_{\mathcal{X}}$ with a morphism:

$$\kappa_{\mathbf{C}} : \mathbf{C} \boxtimes \mathbf{C} \rightarrow \mathrm{add}^*(\mathbf{C}) \in \mathrm{ShvCat}(\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}})$$

that is an isomorphism over the disjoint locus (and satisfying higher compatibilities).

A commutative factorization algebra in \mathbf{C} is an object $\mathcal{A} \in \Gamma(\mathrm{Ran}_{\mathcal{X}}, \mathbf{C})$ with morphisms:

$$\kappa_{\mathbf{C}}(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \mathrm{add}^*(\mathcal{A}) \in \Gamma(\mathrm{Ran}_{\mathcal{X}} \times \mathrm{Ran}_{\mathcal{X}}, \mathrm{add}^*(\mathbf{C}))$$

that is an isomorphism over the disjoint locus.

Remark 7.2.4. It is obvious that $\mathrm{QCoh}_{\mathcal{X}}$ is a commutative chiral category. In this case, our notion of commutative factorization algebra contains as a special case the same-named notion from [BD2], and provides a derived version of the latter.

7.3. Unital setting. We have obvious unital analogues of the above, with $\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$ replacing $\mathrm{Ran}_{\mathcal{X}}$ everywhere.

7.4. Notation. We use the notations $\mathrm{ComCat}^{w, \mathrm{ch}}$, $\mathrm{ComCat}^{\mathrm{ch}}$, $\mathrm{ComAlg}^{\mathrm{fact}}$ (resp. $\mathrm{ComCat}_{\mathrm{un}}^{w, \mathrm{ch}}$, $\mathrm{ComCat}_{\mathrm{un}}^{\mathrm{ch}}$, $\mathrm{ComAlg}_{\mathrm{un}}^{\mathrm{fact}}$) for the above notions.

7.5. Commutative chiral categories coming from symmetric monoidal category. Suppose that \mathcal{D} is a commutative monoid in $\mathrm{DGCat}_{\mathrm{cont}}$, and let X be a scheme of finite type.

In §7.6-7.13, we will associate a commutative unital chiral category $\mathrm{Loc}_{X_{dR}}(\mathcal{D})$ over X_{dR} .

7.6. Suppose at first that \mathcal{X} is a general prestack.

We observe that there is a functor:¹⁴

$$\mathrm{ComAlg}(\mathrm{ShvCat}/\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}) \rightarrow \mathrm{ComCat}^{w, \mathrm{ch}}(\mathcal{X})$$

commuting with forgetful functors to $\mathrm{ShvCat}/\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$. This functor is constructed as follows.

Suppose $\mathbf{C} \in \mathrm{ComAlg}(\mathrm{ShvCat}/\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}})$. From (6.3.1), we obtain morphisms:

$$p_i^*(\mathbf{C}) \rightarrow \mathrm{add}^*(\mathbf{C}) \in \mathrm{ComAlg}(\mathrm{ShvCat}/\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}} \times \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}), i = 1, 2$$

where the $p_i : \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}} \times \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}} \rightarrow \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$ are the projections.

We obtain a map from the coproduct in $\mathrm{ComAlg}(\mathrm{ShvCat}/\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}} \times \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}})$:

$$\mathbf{C} \boxtimes \mathbf{C} = p_1^*(\mathbf{C}) \underset{\mathrm{QCoh}_{\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}} \times \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}}}{\otimes} p_2^*(\mathbf{C}) \rightarrow \mathrm{add}^*(\mathbf{C}).$$

It is straightforward to extend this construction to give the whole datum (as required by the notion of commutative algebra in homotopical algebra) of weak chiral category on \mathbf{C} .

¹⁴We emphasize that ComAlg is taken here with respect to the tensor product symmetric monoidal structure on $\mathrm{ShvCat}/\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$, i.e., the one that exists on $\mathrm{ShvCat}/\mathcal{Y}$ for \mathcal{Y} any lax prestack.

7.7. We will need the following intermediate construction.

For $I \in \mathbf{Set}_{<\infty}$, let $\mathrm{Ran}_{\mathcal{X}, \mathrm{dis}j}^{\mathrm{un}, I}$ denote the disjoint locus in the I -fold product $\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}, I}$ of $\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$ with itself.

Define $\mathbf{Set}_{<\infty} \rightarrow \mathbf{PreStk}^{\mathrm{ lax}}$ by $I \mapsto \mathrm{Ran}_{\mathcal{X}, \mathrm{dis}j}^{\mathrm{un}, I}$. For $f : I \rightarrow J$, the corresponding map is given by coordinate-wise for $j \in J$ by:

$$\mathrm{Ran}_{\mathcal{X}, \mathrm{dis}j}^{\mathrm{un}, I} \subseteq \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}, I} \rightarrow \prod_{i \in I_j} \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}, I} \rightarrow \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}.$$

where the first map is projection and the second map is given by addition. Here I_j is the fiber of j in I . We remark that if $I_j = \emptyset$, this coordinate is just projection onto the empty set considered as a point in $\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$.

Let \mathcal{Z} be the lax prestack obtained by applying the Grothendieck construction to the above diagram, i.e., S -points are obtained by applying the Grothendieck construction termwise.

We have a tautological map $\mathcal{Z} \rightarrow \mathbf{Set}_{\infty}$, considering the latter as a constant lax prestack. We also have a map $\mathcal{Z} \rightarrow \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$ induced by the addition maps $\mathrm{Ran}_{\mathcal{X}, \mathrm{dis}j}^{\mathrm{un}, I} \rightarrow \mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}$.

7.8. From the above correspondence, we obtain a functor:

$$\mathrm{ShvCat}_{\mathbf{Set}_{<\infty}}^{\mathrm{naive}} \rightarrow \mathrm{ShvCat}_{\mathrm{Ran}_{\mathcal{X}}^{\mathrm{un}}}^{\mathrm{naive}}.$$

using pushforward and pullback, as in §A.10. This functor is tautologically lax symmetric monoidal, where we use the pointwise tensor product.

We have a canonical symmetric monoidal functor:

$$\mathrm{ComAlg}(\mathrm{DGCat}_{\mathrm{cont}}) \rightarrow \mathrm{ShvCat}_{\mathbf{Set}_{<\infty}}^{\mathrm{naive}} = \mathrm{Hom}(\mathbf{fSet}^{\triangleleft}, \mathrm{DGCat}_{\mathrm{cont}})$$

sending \mathcal{D} to the functor $I \mapsto \mathcal{D}^{\otimes I}$. This obviously upgrades to a functor:

$$\mathrm{ComAlg}(\mathrm{DGCat}_{\mathrm{cont}}) \rightarrow \mathrm{ComAlg}(\mathrm{ShvCat}_{\mathbf{Set}_{<\infty}}^{\mathrm{naive}})$$

and then using the above, we obtain the desired construction of $\mathrm{Loc}_{\mathcal{X}}(\mathcal{D})$ as a weak commutative unital chiral category on \mathcal{X} .

7.9. We now specialize to the case where $\mathcal{X} = X_{dR}$.¹⁵

7.10. Here is a heuristic version of the computations that follow. The reader may safely skip this description.

Suppose $x = \{x_i\}_{i \in I}$ is a finite subset of X . To compute the fiber of $\mathrm{Loc}_{X_{dR}}(\mathcal{D})$ at this point, we are supposed to form the undercategory whose points are data of $J \in \mathbf{Set}_{<\infty}$, a point $(y^j)_{j \in J} \in \mathrm{Ran}_{X_{dR}, \mathrm{dis}j}^{\mathrm{un}, J}$ ($y^j = \{y_k^j\}_{k \in K_j} \subseteq X$), such that $x \subseteq \cup_j y^j \subseteq X$, and take a limit along here.

This category has an obvious initial object: namely, $J = I$ and each y^j is the singleton $\{x_i\}$. Therefore, the above limit is just the fiber at this point. But we set the constant sheaf of categories corresponding to $\mathcal{D}^{\otimes J}$ over $\mathrm{Ran}_{X_{dR}, \mathrm{dis}j}^{\mathrm{un}, J}$ in the above construction, and therefore we see that this fiber is $\mathcal{D}^{\otimes J}$ as desired.

This concludes the heuristic, and what follows is a more precise treatment.

¹⁵This is probably just out laziness; it seems likely that the computations below apply for a general prestack, but we cannot rely on the set-theoretic interpretation of Ran 's space to compute fiber products then.

7.11. Let S be an affine scheme and let $\varphi : S \rightarrow \mathrm{Ran}_{X_{dR}}^{un}$ be given. This is equivalent to a set $\{\varphi_i\}_{i \in I}$ of maps $S^{red} \rightarrow X$.

Our main technical tool is the following.

Lemma 7.11.1. *The map:*

$$\mathcal{Z} \times_{\mathrm{Ran}_{X_{dR}}^{un}} S \rightarrow \mathcal{Z}_{\varphi/} \quad (7.11.1)$$

is op-cofinal in the sense of Definition A.10.3.

Proof. Let T be a test affine scheme.

Unwinding the definitions, a T -point of $\mathcal{Z}_{\varphi/}$ is given by a datum of a map $\sigma : T \rightarrow S$, a finite set $\psi = \{\psi_j\}_{j \in J}$ of maps $T^{red} \rightarrow X$, and a map¹⁶ $p : J \rightarrow K$, with two properties: 1) for every $p(j) \neq p(j')$ the map $(\psi_j, \psi_{j'}) : T^{red} \rightarrow X^2$ factors through the complement to the diagonal, and 2) we have $\{\varphi_i \circ \sigma^{red}\} \subseteq \{\psi_j\} \subseteq X(T^{red})$. Indeed, this exactly corresponds to a T -point of $\mathrm{Ran}_{X_{dR}, disj}^{un, K}$.

From this data, we construct a certain point of the left hand side above.

Let $I_0 = \{\varphi_i \circ \sigma^{red}\}$. We have an obvious surjection $I \rightarrow I_0$ (this measures which of the maps φ_i become equal when evaluated on T^{red}). We have an obvious canonical map $I_0 \hookrightarrow J$.

In this way, we obtain a T -point of $\mathcal{Z} \times_{\mathrm{Ran}_{X_{dR}}^{un}} S$: the map $T \rightarrow S$ is σ , and the map $T \rightarrow \mathcal{Z}$ is

given by the datum of the set K , and the corresponding point of $\mathrm{Ran}_{X_{dR}, disj}^{un, K}$ is defined by the map $I_0 \rightarrow K$ (and the maps $\varphi_i \sigma^{red}$).

This point is obviously final among all T -points of $\mathcal{Z} \times_{\mathrm{Ran}_{X_{dR}}^{un}} S$ mapping to our given one in $\mathcal{Z}_{\varphi/}$. This gives the op-cofinality as desired. \square

7.12. We now give a more explicit description of the restriction of $\mathrm{Loc}_{X_{dR}}(\mathcal{D})$ to X_{dR}^I .

Let \mathcal{U}_I denote the lax prestack whose points are a surjection $p : I \rightarrow J$ and a point of $U(p) \subseteq X_{dR}^I$. We have an obvious canonical map $\pi : \mathcal{U}_I \rightarrow X_{dR}^I$.

Note that π lifts over $\mathrm{Ran}_{X_{dR}}^{un}$ to a map $\mathcal{U}_I \rightarrow \mathcal{Z}$: for an S -point $(p : I \rightarrow J, x \in U(p)(S))$ of \mathcal{U}_I , we associate the obvious point of $\mathrm{Ran}_{X_{dR}, disj}^{un, J}$.

Corollary 7.12.1. *The map $\mathcal{U}_I \rightarrow \mathcal{Z} \times_{\mathrm{Ran}_{X_{dR}}^{un}} X_{dR}^I$ is op-cofinal.*

Proof. The right hand side has S -points that are maps $x = (x_i) \in X_{dR}^I(S)$ and a map $p : I \rightarrow J$ so that $(x_i, x_{i'}) \in (X^2 \setminus X)(S^{red})$ for $p(i) \neq p(i')$. The left hand side is the same, but $I \rightarrow J$ should be surjective. This makes the claim obvious. \square

Corollary 7.12.2. *The restriction of $\mathrm{Loc}_{X_{dR}}(\mathcal{D})$ to $\emptyset \in \mathrm{Ran}_{X_{dR}}^{un}$ is canonically equivalent to Vect , and its restriction to X_{dR} is canonically isomorphic to $\mathcal{D} \otimes \mathrm{QCoh}_{X_{dR}}$.*

7.13. Finally, we show that $\mathrm{Loc}_{X_{dR}}(\mathcal{D})$ is actually a commutative (unital) chiral category, not merely a weak one.

Let I_1 and I_2 be two finite sets with $I = I_1 \coprod I_2$. By the above, it is enough to show the following:

¹⁶For clarity's sake, note that p may not be surjective: this corresponds to allowing \emptyset in some coordinates.

Lemma 7.13.1. *The map:*

$$\mathcal{U}_{I_1} \times \mathcal{U}_{I_2} \times_{X_{dR}^{I_1} \times X_{dR}^{I_2}} [X_{dR}^{I_1} \times X_{dR}^{I_2}]_{disj} \rightarrow \mathcal{U}_I \times_{X_{dR}^I} [X_{dR}^{I_1} \times X_{dR}^{I_2}]_{disj}$$

is op-cofinal.

Proof. An S -point of the right hand side is a map $p : I \rightarrow J$ and a point $x = (x_i)_{i \in I} \in X^I(S^{red})$ that lies in $U(p) \cap [X_{dR}^{I_1} \times X_{dR}^{I_2}]_{disj}$.

Define J_1 (resp. J_2) to be the image of I_1 (resp. I_2) in J , and let $p_1 : I_1 \rightarrow J_1$ (resp. $p_2 : I_2 \rightarrow J_2$) denote the corresponding map. We have $U(p) \cap [X_{dR}^{I_1} \times X_{dR}^{I_2}]_{disj} = U(p_1 \amalg p_2)$.

We obtain the points $(x_{i_1})_{i_1 \in I_1} \in U(p_1)(S)$ and $(x_{i_2})_{i_2 \in I_2} \in U(p_2)(S)$, defining a point in:

$$\mathcal{U}_{I_1} \times \mathcal{U}_{I_2} \times_{X_{dR}^{I_1} \times X_{dR}^{I_2}} [X_{dR}^{I_1} \times X_{dR}^{I_2}]_{disj}(S).$$

Moreover, this point is obviously final among all points mapping to $(p : I \rightarrow J, x \in U(p)(S)) \in \mathcal{U}_I \times_{X_{dR}^I} [X_{dR}^{I_1} \times X_{dR}^{I_2}]_{disj}(S)$. □

This concludes our construction of $\text{Loc}_{X_{dR}^I}(\mathcal{D})$.

7.14. Next, we discuss the material from §6.14 in the case of a commutative chiral category.

7.15. We need some general material about crystalline sheaves of categories on *pseudo-indschemes*. We follow [Gai4] in using the following (somewhat clunky) terminology:

Definition 7.15.1. A *pseudo-indscheme* Y is a pair of an indexing category \mathcal{J} and a \mathcal{J} -diagram $i \mapsto Y_i$ of schemes of finite type such that all structure maps $Y_i \rightarrow Y_j$ are proper.

The *prestack underlying* Y is the colimit of this diagram $i \mapsto Y_i$ in PreStk . Where there is no risk for confusion, we denote this colimit also by Y .

Remark 7.15.2. The implicit notion of morphism:

$$Y = (\mathcal{J}, i \mapsto Y_i) \rightarrow Z = (\mathcal{J}, j \mapsto Z_j) \tag{7.15.1}$$

of pseudo-indschemes is that of a functor $F : \mathcal{J} \rightarrow \mathcal{J}$ and compatible morphisms $Y_i \rightarrow Z_{F(i)}$.

Remark 7.15.3. Our notion differs slightly from that of [Gai4]: in *loc. cit.*, pseudo-indschemes are defined as a full subcategory of PreStk obtained as colimits of diagrams of the above type. However, in many constructions in *loc. cit.*, pseudo-indschemes are assumed to be given by such a diagram and morphisms are assumed to be of the above type.

Definition 7.15.4. We say a morphism (7.15.1) of pseudo-indschemes is pseudo-indproper if each morphism $Y_i \rightarrow Z_{F(i)}$ is proper.

For a pseudo-indscheme Y , we let $Y_{dR} \in \text{PreStk}$ denote the de Rham space of the prestack underlying Y .

Proposition-Construction 7.15.5. *Let $f : Y \rightarrow Z$ be a map of pseudo-indschemes and let \mathcal{C} be a sheaf of categories on Z_{dR} . There is a canonical morphism:*

$$f_{*,dR,\mathcal{C}} : \Gamma(Y_{dR}, f^*(\mathcal{C})) \rightarrow \Gamma(Z_{dR}, \mathcal{C})$$

of de Rham pushforward, and that is canonically left adjoint to the pullback map if f is pseudo-indproper, and functorial for morphisms of pseudo-indschemes over Z . admits a left adjoint

Proof. For $Z = \operatorname{colim}_j Z_j$, let ψ_j denote the structure map $Z_j \rightarrow Z$. Then we tautologically have:

$$\Gamma(Z_{dR}, \mathbf{C}) = \lim_{j \in \mathcal{J}^{op}} \Gamma(Z_{j,dR}, \psi_j^*(\mathbf{C})).$$

However, because the structure maps $Z_j \rightarrow Z_{j'}$ are proper, and because each $Z_{j,dR}$ is 1-affine, we see that the structure maps in this limit admit left adjoints (given by tensoring with the de Rham pushforward functors $D(Z_j) \rightarrow D(Z_{j'})$). Therefore, we obtain an expression:

$$\operatorname{colim}_{j \in \mathcal{J}} \Gamma(Z_{j,dR}, \psi_j^*(\mathbf{C}))$$

for $\Gamma(Z_{dR}, \mathbf{C})$, the colimit taking place in $\operatorname{DGCat}_{cont}$.

We have a similar expression for $\Gamma(Y_{dR}, f^*(\mathbf{C}))$, and the de Rham pushforward functor is then constructed using the compatible maps:

$$\Gamma(Y_{i,dR}, \varphi_i^*(\mathbf{C})) \rightarrow \Gamma(Z_{F(i),dR}, \psi_i^*(\mathbf{C}))$$

(with $\varphi_i : Y_i \rightarrow Y$ the structure map). This obviously satisfies the desired properties. \square

7.16. Now observe that Ran_X is canonically a pseudo-indscheme, since $\operatorname{Ran}_X = \operatorname{colim}_{I \in \mathbf{fSet}^{op}} X^I$. Moreover, the map:

$$\operatorname{add} : \operatorname{Ran}_X \times \operatorname{Ran}_X \rightarrow \operatorname{Ran}_X$$

is canonically a morphism of pseudo-indschemes (considering the left hand side with the product pseudo-ind structure), using the maps:

$$\begin{aligned} \mathbf{fSet}^{op} \times \mathbf{fSet}^{op} &\rightarrow \mathbf{fSet}^{op} \\ (I, J) &\mapsto I \amalg J \\ X^I \times X^J &\xrightarrow{\operatorname{id}} X^I \amalg X^J. \end{aligned}$$

We immediately see that add is pseudo-indproper.

Of course, this discussion holds for higher products of Ran_X with itself and for higher operations in the non-unital commutative operad.

7.17. We fix \mathbf{C} a commutative chiral category on X_{dR} in what follows, and let $\mathcal{C} := \Gamma(\operatorname{Ran}_{X,dR}, \mathbf{C})$.

Observe that \mathcal{C} carries a canonical non-unital symmetric monoidal structure in $\operatorname{DGCat}_{cont}$ called the \star -tensor product, and denoted $-\overset{\star}{\otimes}-$. It is computed termwise as:

$$\begin{aligned} \Gamma(\operatorname{Ran}_{X,dR}, \mathbf{C}) \otimes \Gamma(\operatorname{Ran}_{X,dR}, \mathbf{C}) &\rightarrow \Gamma(\operatorname{Ran}_{X,dR} \times \operatorname{Ran}_{X,dR}, \mathbf{C} \boxtimes \mathbf{C}) \rightarrow \\ &\Gamma(\operatorname{Ran}_{X,dR} \times \operatorname{Ran}_{X,dR}, \operatorname{add}^*(\mathbf{C})) \rightarrow \Gamma(\operatorname{Ran}_{X,dR}, \mathbf{C}) \end{aligned}$$

where the last arrow is the de Rham pushforward functor from Proposition-Construction 7.15.5. We note that this functor is left adjoint to the obvious map by *loc. cit.*

We leave the remaining details of this construction to the reader.

Note that the identity functor for \mathcal{C} upgrades to a lax symmetric monoidal functor:

$$(\mathcal{C}, \overset{\star}{\otimes}) \rightarrow (\mathcal{C}, \overset{ch}{\otimes}).$$

Example 7.17.1. Suppose that \mathcal{D} is a non-unital symmetric monoidal category, and let $\text{Loc}_{X_{dR}}(\mathcal{D})$ denote the corresponding factorization category over X_{dR} .

Then the pushforward functor along $X \hookrightarrow \text{Ran}_X$ defines a colax symmetric monoidal functor:

$$\mathcal{D} \xrightarrow{\text{id} \otimes \omega_X} \mathcal{D} \otimes D(X) \rightarrow \Gamma(X_{dR}, \text{Loc}_{X_{dR}}(\mathcal{D})_{X_{dR}}) \rightarrow \Gamma(\text{Ran}_{X_{dR}}, \text{Loc}_{X_{dR}}(\mathcal{D}))$$

where the latter is considered with its $\overset{\star}{\otimes}$ symmetric monoidal structure.

7.18. We now observe that the theory of Lie- \ast algebras from [FG1] generalizes to this general setting.

Definition 7.18.1. A *generalized Lie- \ast algebra* in \mathcal{C} is a Lie algebra object in $(\mathcal{C}, \overset{\star}{\otimes})$. A *Lie- \ast algebra* in \mathcal{C} is a generalized Lie- \ast algebra supported on X , i.e., that lives in the subcategory:

$$\Gamma(X_{dR}, \mathcal{C}|_{X_{dR}}) \subseteq \Gamma(\text{Ran}_{X_{dR}}, \mathcal{C}) = \mathcal{C}.$$

There is an obvious forgetful functor from chiral Lie algebras to generalized Lie- \ast algebras. As in [FG1] §6.4, it admits a left adjoint, and this left adjoint sends Lie- \ast algebras to chiral algebras in \mathcal{C} . This functor, denoted U^{ch} , is called *chiral enveloping algebra*.

Remark 7.18.2. By the same argument as in [FG1], the appropriate version of the chiral PBW theorem holds in this generality.

7.19. **Lie- \ast modules.** Next, we briefly discuss modules in the above setting.

We have an action of $\text{Ran}_{X_{dR}}^*$ on $\text{Ran}_{X_{dR}, I}$ by addition, and this allows us to speak about Lie- \ast modules supported on $X^I \subseteq \text{Ran}_{X_{dR}}^*$. For L a Lie- \ast algebra in \mathcal{C} , we let $L\text{-mod}_{X^I}$ denote the resulting DG category.

We have a continuous restriction functor:

$$\text{Oblv} : U^{ch}(L)\text{-mod}^{ch}(X^I) \rightarrow L\text{-mod}_{X^I}$$

commuting with forgetful functors to $\Gamma(X_{dR}^I, \mathcal{C}_{X_{dR}^I})$. Moreover, Oblv admits a left adjoint Ind^{ch} .

We have a version of the chiral PBW theorem for modules in this setting, as in [BD2] §3.7.23. It says the following:

For $U^{ch}(L)^{\text{fact}}$ the factorization algebra corresponding to $U^{ch}(L)$, let $\text{Ind}^{ch}(M)^{\text{fact}}$ denote the corresponding factorization module on X_{dR}^I . Then the chiral PBW theorem says that the object of:

$$\Gamma(\text{Ran}_{X_{dR}, I}, \sigma_I^*(\mathcal{C}))$$

underlying $\text{Ind}^{ch}(M)^{\text{fact}}$ has a filtration indexed with subquotients:

$$(L[1])^{\overset{\star}{\otimes} n} \overset{\star}{\otimes} M$$

where we also use $\overset{\star}{\otimes}$ to indicate the action of $(\mathcal{C}, \overset{\star}{\otimes})$ on $\Gamma(\text{Ran}_{X_{dR}, I}, \sigma_I^*(\mathcal{C}))$.

8. CHIRAL CATEGORIES VIA PARTITIONS

8.1. In this section, we give an alternative approach to the theory of chiral categories and factorization algebras using categories of partitions.

This approach is a much more faithful realization of the heuristic of §1.7. In particular, it gives a theory of chiral categories on a finite type scheme that uses only finite-dimensional geometry, i.e., the Ran space is not explicitly mentioned.

After developing this material, the author found that the main idea of this approach independently appears already in a preprint of [Rei].

We fix a prestack \mathcal{X} throughout this section.

Remark 8.1.1. In this section, we prove a result that says that giving a factorization category is equivalent to giving data:

$$\mathbf{C}_{\mathcal{X}^I} \in \mathbf{ShvCat}/_{\mathcal{X}^I}$$

and equivalences:

$$\mathbf{C}_{\mathcal{X}^I} \boxtimes \mathbf{C}_{\mathcal{X}^J} |_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}} \simeq \mathbf{C}_{\mathcal{X}^I \amalg J} |_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}} \in \mathbf{ShvCat}/_{[\mathcal{X}^I \times \mathcal{X}^J]_{disj}}$$

satisfying further compatibilities.

The reader willing to take such statements on faith, or who believes this to be a tautology given our earlier material, is advised to skip this section entirely.

Remark 8.1.2. For the reader who has continued reader past Remark 8.1.1, we note what technical issues occur.

By definition, a multiplicative sheaf on a prestack with a multiplicative structure in the correspondence category (say, associative but not assumed commutative, for simplicity of terminology) is an algebra in a certain correspondence category.

Roughly, in higher algebra, an algebra somewhere is something like a simplicial object. A priori, if one thinks out what a simplicial object in a correspondence category is in terms of the original category, it appears to be a very large quantity of data.

This is exactly what we are trying to do here: to give a definition of chiral category that does not mention Ran space or correspondences, we need to give an alternative description of algebras in correspondence categories.

This is exactly what is done in the appendix Appendix B: we give a workable perspective on simplicial objects in correspondence categories, or more generally, on any functor into a correspondence category.

This is the main technique that is exploited in this section; the remainder consists of details.

8.2. We begin by defining certain combinatorial categories of partitions.

Define the (1,1)-category \mathbf{Part} of partitions as the category with objects surjections $(p : I \twoheadrightarrow J)$ of non-empty finite sets and with morphisms from $(p_1 : I_1 \twoheadrightarrow J_1)$ to $(p_2 : I_2 \twoheadrightarrow J_2)$ defined by commutative diagrams:

$$\begin{array}{ccc} I_1 & \twoheadrightarrow & I_2 \\ \downarrow p_1 & & \downarrow p_2 \\ J_1 & \twoheadleftarrow & J_2 \end{array} \quad (8.2.1)$$

under the obvious compositions.

Similarly, define \mathbf{Part}_{un} as the category whose objects are (arbitrary) maps $p : I \rightarrow J$ of (possibly empty) finite sets and in which morphisms $(p_1 : I_1 \rightarrow J_1) \rightarrow (p_2 : I_2 \rightarrow J_2)$ are commutative diagrams:

$$\begin{array}{ccc} I_1 & \longrightarrow & I_2 \\ \downarrow p_1 & & \downarrow p_2 \\ J_1 & \longleftarrow & J_2 \end{array}$$

Remark 8.2.1. One can think of such a map $p : I \rightarrow J$ as a partition of I indexed by J , where the associated partition is $I = \coprod_{j \in J} I_j$, $I_j := p^{-1}(j)$. Allowing non-surjective maps in \mathbf{Part}_{un} then translates into allowing partitions into possibly empty sets.

Remark 8.2.2. Note that \mathbf{Part} contains \mathbf{fSet} as the full subcategory of partitions indexed by a singleton set. The functor $\mathbf{fSet}^{op} \rightarrow \mathbf{Part}^{op}$ is cofinal. There is a canonical splitting $\mathbf{Part} \rightarrow \mathbf{fSet}$ of this functor sending $(p : I \rightarrow J) \in \mathbf{Part}$ to I .

The same remarks hold with \mathbf{Part}_{un} replacing \mathbf{Part} and $\mathbf{Set}_{<\infty}$ replacing \mathbf{fSet} .

We have a non-unital symmetric monoidal structure on \mathbf{Part} given by disjoint union of (pairs of) sets. We denote the corresponding product by \coprod , although it is not the coproduct on this category.

Remark 8.2.3. In the notation of Appendix B, we have $\mathbf{Part} = \mathbf{Tw}(\mathbf{fSet})$ and $\mathbf{Part}_{un} = \mathbf{Tw}(\mathbf{Set}_{<\infty})$, compatibly with (non-unital for \mathbf{Part}) symmetric monoidal structures.

8.3. For $(p : I \rightarrow J) \in \mathbf{Part}_{un}$, define $U(p) \in \mathbf{PreStk}$ as the open subprestack of \mathcal{X}^I defined for an affine test scheme S by:

$$U(p)(S) = \left\{ \varphi = (\varphi_i)_{i \in I} : S \rightarrow \mathcal{X}^I \mid \begin{array}{l} \text{for every } i_1, i_2 \in I \text{ with } p(i_1) \neq p(i_2) \text{ the map} \\ (\varphi_{i_1}, \varphi_{i_2}) : S \rightarrow \mathcal{X}^2 \text{ factors through } [\mathcal{X} \times \mathcal{X}]_{disj} \end{array} \right\}. \quad (8.3.1)$$

Example 8.3.1. For p the identity map $\{1, 2\} \rightarrow \{1, 2\}$ we have $U(p) = [\mathcal{X} \times \mathcal{X}]_{disj}$.

Given a map $\varepsilon : (p_1 : I_1 \rightarrow J_1) \rightarrow (p_2 : I_2 \rightarrow J_2)$ in \mathbf{Part}_{un} , we obtain a map $U(\varepsilon) : U(p_2) \rightarrow U(p_1)$ induced by the diagram:

$$\begin{array}{ccc} U(p_2) & \xrightarrow{U(\varepsilon)} & U(p_1) \\ \downarrow & & \downarrow \\ \mathcal{X}^{I_2} & \longrightarrow & \mathcal{X}^{I_1}. \end{array}$$

This gives a functor:

$$U : \mathbf{Part}_{un}^{op} \rightarrow \mathbf{PreStk} \quad (8.3.2)$$

sending p to $U(p)$. It is naturally colax symmetric monoidal relative to the Cartesian product on the target, i.e., we have natural maps:

$$U(p \coprod q) \hookrightarrow U(p) \times U(q). \quad (8.3.3)$$

Remark 8.3.2. We will also denote the restriction of the functor (8.3.2) to \mathbf{Part}^{op} by U .

8.4. Main result. We imitate the earlier constructions to obtain the lax symmetric monoidal functor:

$$\begin{aligned} \mathrm{ShvCat}/U &: \mathrm{Part}_{un} \rightarrow \mathrm{Cat}_{pres} \\ (p : I \rightarrow J) &\mapsto \mathrm{ShvCat}/U(p). \end{aligned}$$

and thereby (c.f. §5.15) the symmetric monoidal functor of symmetric monoidal categories:

$$\mathrm{Groth}(\mathrm{ShvCat}/U) \rightarrow \mathrm{Part}_{un}.$$

The main construction of this section is given by the following.

Proposition-Construction 8.4.1. (1) *The category $\mathrm{Cat}^{ch}(\mathcal{X})$ is equivalent to the category of symmetric monoidal¹⁷ sections:*

$$\begin{array}{ccc} & \mathrm{Groth}(\mathrm{ShvCat}/U) & \\ & \nearrow \text{dotted arrow} & \downarrow \\ \mathrm{Part} & \xrightarrow{\quad} & \mathrm{Part}_{un} \end{array} \quad (8.4.1)$$

sending all arrows in Part to $coCartesian$ arrows.

(2) *The category $\mathrm{Cat}_{un, str}^{ch}(\mathcal{X})$ (see Remark 6.3.1 for the notation) is canonically equivalent to the category of symmetric monoidal sections:*

$$p \mapsto C_{U_{un}(p)} \in \mathrm{ShvCat}/U_{un}(p) \quad (8.4.2)$$

of $\mathrm{Groth}(\mathrm{ShvCat}/U) \rightarrow \mathrm{Part}_{un}$ such that:

- (a) *Arrows in Part map to $coCartesian$ arrows.*
- (b) *Arrows in:*

$$\mathrm{Set}_{<\infty}^{op} = \{p : \emptyset \rightarrow I\} \subseteq \mathrm{Part}_{un}$$

map to $coCartesian$ arrows.

Remark 8.4.2. It will follow from the construction that Proposition-Construction 8.4.1 satisfies the following compatibilities.

- The non-unital symmetric monoidal functor $\mathrm{Part} \hookrightarrow \mathrm{Part}_{un}$ induces the restriction functor:

$$\mathrm{Cat}_{un, str}^{ch}(\mathcal{X}) \rightarrow \mathrm{Cat}^{ch}(\mathcal{X}).$$

- Restricting a functor (8.4.1) to:

$$\mathrm{fSet} = \{I \rightarrow *\} \subseteq \mathrm{Part}$$

(necessarily forgetting the symmetric monoidal structure), we obtain a compatible system of sheaves of categories on the \mathcal{X}^I for $I \in \mathrm{fSet}$, i.e., a sheaf of categories on $\mathrm{Ran}_{\mathcal{X}}$. This corresponds to the restriction:

$$\mathrm{Cat}^{ch}(\mathcal{X}) \rightarrow \mathrm{ShvCat}/\mathrm{Ran}_{\mathcal{X}}.$$

¹⁷Necessarily understood in the sense of non-unital symmetric monoidal categories and functors.

- Restricting a functor (8.4.2) to:

$$\mathbf{Set}_{<\infty} = \{I \rightarrow *\} \subseteq \mathbf{Part}_{un}$$

we obtain a lax compatible system of sheaves of categories on the \mathcal{X}^I for $I \in \mathbf{Set}_{<\infty}$, and that is strictly compatible with respect to morphisms in \mathbf{fSet} . By Corollary 4.6.2, this amounts to a sheaf of categories on \mathbf{Ran}_X^{un} .

This construction then corresponds to the restriction:

$$\mathbf{Cat}_{un, str}^{ch}(\mathcal{X}) \rightarrow \mathbf{ShvCat}/\mathbf{Ran}_X^{un}.$$

Remark 8.4.3. The reader who runs through the definitions should be convinced that Proposition-Construction 8.4.1 is essentially tautological. The only difficulties arising below are of the usual sort in higher category theory: we just provide the necessary categorical language for the obvious constructions.

Remark 8.4.4. The technical perspective on chiral categories provided by Proposition-Construction 8.4.1 differs from the one provided in §6 in that \mathbf{Ran} space is not explicitly mentioned. This is somewhat convenient for constructing chiral categories from geometry, but is somewhat complicates developing the theory of §6. Moreover, working with non-strict unital chiral functors is not technically convenient in the partition framework.

One can readily develop much of the language of (unital and non-unital) chiral algebras and their modules in this framework.

8.5. We will develop a minimal working theory of operadic right Kan extensions, similar to the theory of operadic left Kan extensions in [Lur4] §2. This material can be significantly generalized, but we take a more pedestrian approach.

The main result here is the following.

Proposition 8.5.1. *Suppose that we are given a symmetric monoidal functor $\Psi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ of symmetric monoidal categories such that for every $X, Y \in \mathcal{C}_2$ the tensor product functor:*

$$\mathcal{C}_{1, X/} \times \mathcal{C}_{1, Y/} \rightarrow \mathcal{C}_{1, X \otimes Y/}$$

is op-cofinal. Here, e.g., $\mathcal{C}_{1, X/}$ is the associated undercategory.

Suppose that \mathcal{D} is a symmetric monoidal category that is complete as a category.

Then the functor:

$$\mathbf{Hom}^{\otimes, \text{lax}}(\mathcal{C}_2, \mathcal{D}) \rightarrow \mathbf{Hom}^{\otimes, \text{lax}}(\mathcal{C}_1, \mathcal{D})$$

admits a right adjoint. At the level of mere functors, this right adjoint is computed as the right Kan extension.

Proof. Suppose that F is a lax symmetric monoidal functor $\mathcal{C}_1 \rightarrow \mathcal{D}$. Let $F^{\otimes} : \mathcal{C}_1^{\otimes} \rightarrow \mathcal{D}^{\otimes}$ denote the corresponding functor of categories coCartesian over Segal's category Γ , in the notation of [Lur4].

Standard arguments show that our hypotheses imply that the relative right Kan extension of F^{\otimes} , taken relative to Γ , exists, and preserves the appropriate coCartesian arrows to define a lax symmetric monoidal functor. This functor obviously computes the desired right adjoint. Moreover, by [Lur1] Corollary 4.3.1.16, we see that this relative right Kan extension restricts to the usual right Kan extension over $\{*\} \in \Gamma$, as desired.

□

Remark 8.5.2. As the proof shows, we do not need to assume that \mathcal{D} is complete: for a fixed lax symmetric monoidal functor $F : \mathcal{C}_1 \rightarrow \mathcal{D}$ we only need to assume that the relevant limits exist for the right adjoint to be defined on F .

Remark 8.5.3. In more down-to-earth terms, let $\tilde{F} : \mathcal{C}_2 \rightarrow \mathcal{D}$ be the right Kan extension of a lax symmetric monoidal functor $F : \mathcal{C}_1 \rightarrow \mathcal{D}$. For $X, Y \in \mathcal{C}_2$, we have a diagram:

$$\begin{array}{ccc} \tilde{F}(X) \otimes \tilde{F}(Y) := \lim_{\substack{X' \in \mathcal{C}_1 \\ X \rightarrow \Psi(X')}} F(X') \otimes \lim_{\substack{Y' \in \mathcal{C}_1 \\ Y \rightarrow \Psi(Y')}} F(Y') & \longrightarrow & \lim_{\substack{X' \in \mathcal{C}_1, X \rightarrow \Psi(X') \\ Y' \in \mathcal{C}_1, Y \rightarrow \Psi(Y')}} F(X') \otimes F(Y') \\ & & \uparrow \\ & & \lim_{\substack{Z \in \mathcal{C}_1 \\ X \otimes Y \rightarrow \Psi(Z)}} F(Z) =: \tilde{F}(X \otimes Y). \end{array}$$

Moreover, the vertical arrow at right is an equivalence by the cofinality assumption. Therefore, we obtain a canonical map:

$$\tilde{F}(X) \otimes \tilde{F}(Y) \rightarrow \tilde{F}(X \otimes Y)$$

as desired.

8.6. Let \mathcal{C} be a symmetric monoidal category. Then a unital commutative algebra in \mathcal{C} is equivalent to a symmetric monoidal functor $\mathbf{Set}_{<\infty} \rightarrow \mathcal{C}$, and a non-unital commutative algebra is equivalent to a non-unital symmetric monoidal functor $\mathbf{fSet} \rightarrow \mathcal{C}$ (see [Lur4] §2.2.4).

Therefore, a unital commutative algebra in \mathcal{C}_{corr} is equivalent to a symmetric monoidal functor $\mathbf{Set}_{<\infty} \rightarrow \mathcal{C}_{corr}$. By Appendix B, this data is equivalent to a symmetric monoidal functor $\mathbf{Part}_{un} = \mathbf{Tw}(\mathbf{Set}_{<\infty}) \rightarrow \mathcal{C}$ such that, for $I \xrightarrow{p} J \xrightarrow{q} K$ in $\mathbf{Set}_{<\infty}$ the diagram:

$$\begin{array}{ccc} (I \xrightarrow{q \circ p} K) & \longrightarrow & (J \xrightarrow{q} K) \\ \downarrow & & \downarrow \\ (I \xrightarrow{p} J) & \longrightarrow & (J \xrightarrow{\text{id}_J} J) \end{array} \tag{8.6.1}$$

maps to a Cartesian diagram.

Similarly, a non-unital commutative algebra in \mathcal{C}_{corr} is equivalent to a non-unital symmetric monoidal functor $\mathbf{Part} \rightarrow \mathcal{C}$ sending the appropriate squares to Cartesian squares.

More explicitly: suppose we are given a non-unital symmetric monoidal functor $F : \mathbf{Part} \rightarrow \mathcal{C}$. We have the following correspondence in \mathbf{Part} :

$$\begin{array}{ccc} & (\{1, 2\} \rightarrow *) & \\ & \swarrow \quad \searrow & \\ (\{1, 2\} \rightarrow \{1, 2\}) & & (* \rightarrow *) \end{array}$$

and its image under F defines a correspondence:

$$\begin{array}{ccc}
 & F(\{1, 2\} \rightarrow *) & \\
 \swarrow & & \searrow \\
 A \otimes A & & A
 \end{array}$$

for $A := F(* \rightarrow *)$, and this correspondence defines the multiplication for A in \mathcal{C}_{corr} . The condition on fiber squares is relevant for considering associativity.

8.7. As in the framework of §4, let \mathcal{G} be a groupoid.

For $(p : I \rightarrow J) \in \mathbf{Part}$, define the full subgroupoid $\mathbf{Ran}_{\mathcal{G}, p-disj}^I \subseteq \mathbf{Ran}_{\mathcal{G}}^I$ by only allowing objects in $\pi_0(\mathbf{Ran}_{\mathcal{G}}^I) = \mathbf{Ran}_{\pi_0(\mathcal{G})}^I$ corresponding to I -tuples:

$$(S_i \subseteq \pi_0(\mathcal{G}) \text{ non-empty and finite})_{i \in I}$$

such that, for every $i_1 \neq i_2 \in I$ with $p(i_1) = p(i_2)$, the point $(S_{i_1}, S_{i_2}) \in \mathbf{Ran}_{\mathcal{G}} \times \mathbf{Ran}_{\mathcal{G}}$ lies in $[\mathbf{Ran}_{\mathcal{G}} \times \mathbf{Ran}_{\mathcal{G}}]_{disj}$.

Example 8.7.1. For $I = \{1, 2\}$ and $J = *$, we have:

$$\mathbf{Ran}_{\mathcal{G}, p-disj}^I = [\mathbf{Ran}_{\mathcal{G}} \times \mathbf{Ran}_{\mathcal{G}}]_{disj}.$$

On the other hand, for p a bijection we have $\mathbf{Ran}_{\mathcal{G}, p-disj}^I = \mathbf{Ran}_{\mathcal{G}}^I$.

In general, one writes I as the disjoint union of the sets $I_j := p^{-1}(j)$ and then $\mathbf{Ran}_{\mathcal{G}, p-disj}^I$ is the product over J of the loci $[\mathbf{Ran}_{\mathcal{G}}^{I_j}]_{disj}$ in $\mathbf{Ran}_{\mathcal{G}}^{I_j}$ where all collections of points in \mathcal{G} are pairwise disjoint.

We claim that this construction extends to a symmetric monoidal functor:

$$\begin{array}{ccc}
 \mathbf{Part} & \rightarrow & \mathbf{Gpd} \\
 (p : I \rightarrow J) & \mapsto & \mathbf{Ran}_{\mathcal{G}, p-disj}^I.
 \end{array} \tag{8.7.1}$$

Indeed, first note that we have a canonical symmetric monoidal functor $\mathbf{Part} \rightarrow \mathbf{Gpd}$ sending $I \rightarrow \mathbf{Ran}_{\mathcal{G}}^I$ factoring through the projection $\mathbf{Part} \rightarrow \mathbf{fSet}$ and encoding the non-unital commutative algebra structure from §4.3. One immediately verifies that this induces the functor (8.7.1) in the obvious way.

Remark 8.7.2. The functor (8.7.1) has the following special property: given morphisms $I \xrightarrow{p} J \xrightarrow{q} K$ in \mathbf{fSet} , the square (8.6.1) maps to a Cartesian square of groupoids. Therefore, our functor defines the structure of non-unital commutative algebra on $\mathbf{Ran}_{\mathcal{G}}$ in \mathbf{Gpd}_{corr} , and this is exactly the chiral product.

The above constructions have obvious analogues when we replace groupoids by prestacks.

8.8. We will need the following combinatorial digression.

Define the category \mathbf{Trip} (of “triples”) to consist of diagrams:

$$I \twoheadrightarrow J \twoheadrightarrow K$$

of non-empty finite sets. For morphisms, we take surjective morphisms that are contravariant in the I and K -variables and covariant in J . That is, morphisms are given by commutative diagrams:

$$\begin{array}{ccccc}
I_1 & \xrightarrow{p_1} & J_1 & \xrightarrow{q_1} & K_1 \\
\uparrow \alpha & & \downarrow \beta & & \uparrow \gamma \\
I_2 & \xrightarrow{p_2} & J_2 & \xrightarrow{q_2} & K_2
\end{array} \tag{8.8.1}$$

Note that Trip is a non-unital symmetric monoidal category under disjoint unions.

Notation 8.8.1. For $(I \xrightarrow{p} J \xrightarrow{q} K) \in \text{Trip}$ and $k \in K$, we define:

$$\begin{aligned}
I_k &:= (q \circ p)^{-1}(k) \\
J_k &:= q^{-1}(k) \\
(p_k : I_k \rightarrow J_k) &:= p|_{I_k}.
\end{aligned}$$

Similarly, for $j \in J$ we let $I_j := p^{-1}(j)$.

Suppose that we are given a morphism (8.8.1) in Trip . Fix $k' \in K_2$ and let $k := \gamma(k') \in K_1$. We will construct a canonical map:

$$U(p_{1,\gamma(k')}) = U(p_{1,k}) \rightarrow U(p_{2,k'}) \tag{8.8.2}$$

First, note that we can write $p_{1,k} : I_{1,k} \rightarrow J_{1,k}$ as a disjoint union of terms $p_{1,\kappa} : I_{1,\kappa} \rightarrow J_{1,\kappa}$ over $\kappa \in \gamma^{-1}(k)$, where e.g. $I_{1,\kappa}$ is the fiber over κ of the map $I_1 \rightarrow K_2$ defined by the diagram (8.8.2).

Therefore, by the colax symmetric monoidal structure on (8.3.2), we obtain a canonical morphism:

$$U(p_{1,k}) \rightarrow \prod_{\kappa \in \gamma^{-1}(k)} U(p_{1,\kappa}) \rightarrow U(p_{1,k'})$$

where this second morphism is the projection.

Now the commutative diagram:

$$\begin{array}{ccc}
I_{1,k'} & \xrightarrow{\quad} & J_{1,k'} \\
\uparrow & & \downarrow \\
I_{2,k'} & \xrightarrow{\quad} & J_{2,k'}
\end{array}$$

gives a morphism:

$$U(p_{1,k'}) \rightarrow U(p_{2,k'})$$

inducing (8.8.2) as desired.

This defines a symmetric monoidal functor:

$$\begin{aligned}
\Psi^{\text{Trip}} : \text{Trip} &\rightarrow \text{PreStk} \\
(I \rightarrow J \rightarrow K) &\mapsto \prod_{k \in K} U(p_k).
\end{aligned} \tag{8.8.3}$$

where for (8.8.1), the functoriality is defined by the morphism:

$$\prod_{k \in K_1} U(p_{1,k}) \rightarrow \prod_{k' \in K_2} U(p_{2,k'})$$

given on a coordinate $k' \in K_2$ by:

$$\prod_{k \in K_1} U(p_{1,k}) \rightarrow U(p_{1,\mathfrak{y}(k')}) \xrightarrow{(8.8.2)} U(p_{2,k'}).$$

Remark 8.8.2. Trip^{op} is the non-unital monoidal envelope of Part in the sense of [Lur4], and the functor Ψ^{Trip} is induced by the functor $U : \text{Part}^{op} \rightarrow \text{PreStk}$ in this way.

8.9. We have a symmetric monoidal functor:

$$\begin{aligned} \text{Trip} &\rightarrow \text{Part} \\ (I \twoheadrightarrow J \twoheadrightarrow K) &\mapsto (J \twoheadrightarrow K). \end{aligned} \tag{8.9.1}$$

Therefore, we obtain a second symmetric functor:

$$\Phi^{\text{Trip}} : \text{Trip} \rightarrow \text{PreStk}$$

by composing (8.7.1) with (8.9.1).

We have a canonical natural transformation of symmetric monoidal functors:

$$\eta^{\text{Trip}} : \Psi^{\text{Trip}} \rightarrow \Phi^{\text{Trip}}$$

evaluated termwise at $(I \twoheadrightarrow J \twoheadrightarrow K) \in \text{Trip}$ as:

$$\begin{array}{ccc} \Psi^{\text{Trip}}(I \twoheadrightarrow J \twoheadrightarrow K) & \xlongequal{\quad} & \prod_{k \in K} U(p_k) \cdots \twoheadrightarrow \prod_{j \in J} [\text{Ran}_{\mathcal{X}}^{J_k}]_{disj} \xlongequal{\quad} & \Phi^{\text{Trip}}(I \twoheadrightarrow J \twoheadrightarrow K) \\ \downarrow & & \downarrow & \\ \prod_{k \in K} \mathcal{X}^{I_k} & & \prod_{j \in J} \text{Ran}_{\mathcal{X}}^{J_k} & \\ \parallel & & \parallel & \\ \prod_{j \in J} \mathcal{X}^{I_j} & \longrightarrow & \prod_{j \in J} \text{Ran}_{\mathcal{X}} & \end{array}$$

Remark 8.9.1. We will revisit the construction of η^{Trip} in §8.11 below.

8.10. We will need the following technical observation in what follows.

Fix $(J_\varepsilon \twoheadrightarrow K_\varepsilon) \in \text{Part}$, $\varepsilon = 1, 2$. Form the overcategory:

$$\text{Trip}_{/(J_1 \coprod J_2 \twoheadrightarrow K_1 \coprod K_2)}$$

with respect to (8.9.1).

We claim that the functor of disjoint union:

$$\text{Trip}_{/(K_1 \twoheadrightarrow J_1)} \times \text{Trip}_{/(K_2 \twoheadrightarrow J_2)} \rightarrow \text{Trip}_{/(J_1 \coprod J_2 \twoheadrightarrow K_1 \coprod K_2)}$$

is an equivalence.

By definition, $\text{Trip}_{/(K_1 \coprod K_2 \twoheadrightarrow J_1 \coprod J_2)}$ is the category of diagrams:

$$(I' \twoheadrightarrow J' \twoheadrightarrow K') \in \text{Trip}, \text{ plus } \begin{array}{ccc} J_1 \coprod J_2 & \twoheadrightarrow & K_1 \coprod K_2 \\ \downarrow & & \uparrow \\ J' & \twoheadrightarrow & K' \end{array}$$

under appropriate functoriality. Given such a datum, for $\varepsilon = 1, 2$ we define $I'_\varepsilon, J'_\varepsilon, K'_\varepsilon$ as the inverse images of K_ε under the map to $K_1 \coprod K_2$. This functor defines the desired inverse.

8.11. Given §8.10, we can apply the dual version of Proposition 8.5.1 to see that the left Kan extension of Ψ^{Trip} along $\text{Trip} \xrightarrow{(8.9.1)} \mathbf{Part}$ is a colax symmetric monoidal functor. Moreover, one immediately verifies that this left Kan extension is actually a symmetric monoidal functor and that it is computed as the functor (8.7.1).

Moreover, the natural transformation η^{Trip} now arises via the universal property from Proposition 8.5.1.

8.12. We can now give Proposition-Construction 8.4.1 (1), i.e., the non-unital case of *loc. cit.*

By definition, a chiral category on \mathcal{X} is a multiplicative sheaf of categories on $\text{Ran}_{\mathcal{X}}^{ch}$. Therefore, we will prove the following variant of *loc. cit.*

(*): There is a canonical equivalence of categories between $\text{MultCat}^w(\text{Ran}_{\mathcal{X}}^w)$ and the category of lax symmetric monoidal functors (8.4.1) sending all arrows in \mathbf{Part} to coCartesian arrows.

It will follow from the construction that this equivalence identifies the subcategory of chiral categories with the subcategory of usual (i.e., non-lax) symmetric monoidal functors.

Step 1. First, recall from Variant 6.2.9 that *weak* chiral categories (alias: weakly multiplicative sheaves of categories on $\text{Ran}_{\mathcal{X}}^{ch}$) are defined as commutative algebras in $\text{PreStk}_{corr}^{\text{ShvCat}}$ lifting $\text{Ran}_{\mathcal{X}}^{ch} \in \text{PreStk}_{corr}$. Here the notation $\text{PreStk}_{corr}^{\text{ShvCat}}$ was defined in §5.18. We recall that it is defined as a certain 1-full subcategory of:

$$\left((\text{Groth}(\text{ShvCat}_{/-}))^{op} \right)_{corr}. \quad (8.12.1)$$

By §8.6, such a datum is equivalent to a symmetric monoidal functor:

$$\mathbf{Part} \rightarrow (\text{Groth}(\text{ShvCat}_{/-}))^{op} \quad (8.12.2)$$

lifting the functor (8.7.1), sending squares (8.6.1) to Cartesian squares, and satisfying a certain property encoding that the corresponding functor to (8.12.1) should map into $\text{PreStk}_{corr}^{\text{ShvCat}}$.

Precisely, this last property is readily checked to say that every arrow in \mathbf{Part} inducing isomorphisms on the J -terms (i.e., in (8.2.1), $J_2 \xrightarrow{\cong} J_1$; in Appendix B, such arrows were called horizontal) should map to a coCartesian arrow (that is, when considered as an arrow in $\text{Groth}(\text{ShvCat}_{/-})$).

We then see that the condition that squares (8.6.1) map to Cartesian squares is actually redundant: it is subsumed by the condition that horizontal arrows map to coCartesian arrows by applying Remark 8.7.2 and (the proof of) Lemma 5.10.1.

Step 2. We will make implicit use of the following observation below:

We have a tautological Cartesian square:

$$\begin{array}{ccc} \text{Groth}(\text{ShvCat}_{/U}) & \longrightarrow & \text{Groth}(\text{ShvCat}_{/-}) \\ \downarrow & & \downarrow \\ \mathbf{Part} & \xrightarrow{U^{op}} & \text{PreStk}^{op}. \end{array}$$

Step 3. Suppose we are given a lax symmetric monoidal section (8.4.1) sending all arrows to coCartesian arrows.

As in Remark 8.8.2, we obtain a symmetric monoidal functor:

$$F : \mathbf{Trip} \rightarrow \mathbf{Groth}(\mathbf{ShvCat}_{/-})^{op}$$

lifting $\Psi^{\mathbf{Trip}, op}$.

The fact that (8.4.1) sends all arrows to coCartesian arrows implies that the left Kan extension of F along $\mathbf{Trip} \rightarrow \mathbf{Part}$ exists, and by Proposition 8.5.1, it carries a canonical structure of colax symmetric monoidal functor.

One readily verifies that it is actually symmetric monoidal, lifts (8.7.1) and satisfies the conditions articulated in Step 1. Therefore, this functor defines a weakly multiplicative sheaf of categories as desired.

Step 4. Suppose we have a functor (8.12.2) defining a weakly multiplicative sheaf of categories. Restricting along $\mathbf{Trip} \rightarrow \mathbf{Part}$, we obtain a similar functor with source \mathbf{Trip} .

Applying the coCartesian condition and the symmetric monoidal natural transformation $\eta^{\mathbf{Trip}}$, we obtain a symmetric monoidal functor $\mathbf{Trip} \rightarrow \mathbf{Groth}(\mathbf{ShvCat}_{/-})$ lifting $\Psi^{\mathbf{Trip}}$. Applying Remark 8.8.2 again, we obtain a lax symmetric monoidal functor $\mathbf{Part}^{op} \rightarrow \mathbf{Groth}(\mathbf{ShvCat}_{/-})$ of the desired type.

8.13. This completes the treatment of the non-unital case. The unital case is treated in exactly the same way, though the category \mathbf{Trip} should of course be replaced with a category \mathbf{Trip}_{un} with arbitrary maps of finite sets replacing surjections.

One may describe chiral module categories and factorization modules in similar terms. The formulation and the details of the comparison are left to the interested reader.

8.14. **External fusion redux.** Suppose that $\mathcal{X} = X_{dR}$ for X a scheme of finite type, as in §6.12. Let \mathbf{C} be a chiral category on X_{dR} and let $\mathcal{A} \in \mathbf{Alg}^{\mathbf{fact}}(\mathbf{C})$. As in *loc. cit.*, let $\mathbf{C}_{X_{dR}^I} \in \mathbf{ShvCat}(X_{dR}^I)$ denote the sheaf of categories underlying \mathbf{C} .

Enhancing the external fusion construction of §6.22, one can upgrade the construction $I \mapsto \mathcal{A}\text{-mod}^{\mathbf{fact}}(\mathbf{C}_{X_{dR}^I})$ to a functor (8.12.2) satisfying the hypotheses spelled out in Step 1 (found in §8.12 above). Indeed, it is clear how to extend the format of §6.24 to give the required data.

Therefore, we obtain a weak chiral category $\mathcal{A}\text{-mod}^{\mathbf{fact}}(\mathbf{C})$ on X_{dR} , where the morphisms (5.21.2) and (6.12.1) identify (upon passing to the limit for the latter).

Similarly, if \mathcal{A} and \mathbf{C} are unital, then $\mathcal{A}\text{-mod}_{un}^{\mathbf{fact}}(\mathbf{C})$ is a weak unital chiral category.

We can formulate this more precisely in the following proposition.

Proposition 8.14.1. (1) *External fusion defines functors:*

$$\begin{aligned} \{\mathbf{C} \in \mathbf{Cat}^{ch}(X_{dR}), \mathcal{A} \in \mathbf{Alg}^{\mathbf{fact}}(\mathbf{C})\} &\rightarrow \mathbf{Cat}^{w, ch} \\ (\mathbf{C}, \mathcal{A} \in \mathbf{Alg}^{\mathbf{fact}}(\mathbf{C})) &\mapsto \mathcal{A}\text{-mod}^{\mathbf{fact}}(\mathbf{C}) \end{aligned}$$

and:

$$\begin{aligned} \{\mathbf{C} \in \mathbf{Cat}_{un}^{ch}(X_{dR}), \mathcal{A} \in \mathbf{Alg}_{un}^{\mathbf{fact}}(\mathbf{C})\} &\rightarrow \mathbf{Cat}_{un}^{w, ch} \\ (\mathbf{C}, \mathcal{A} \in \mathbf{Alg}_{un}^{\mathbf{fact}}(\mathbf{C})) &\mapsto \mathcal{A}\text{-mod}_{un}^{\mathbf{fact}}(\mathbf{C}). \end{aligned}$$

(2) *The induced functor:*

$$\begin{aligned} \mathbf{Cat}_{un}^{w, ch}(X_{dR}) &\rightarrow \mathbf{Cat}_{un}^{w, ch}(X_{dR}) \\ \mathbf{C} &\mapsto \mathbf{unit}_{\mathbf{C}}\text{-mod}_{un}^{\mathbf{fact}}(\mathbf{C}) \end{aligned}$$

is (canonically identified with) the canonical embedding of unital chiral categories into weak unital chiral categories, and in a manner compatible with Theorem 6.13.2.

Remark 8.14.2. In the above, for example the somewhat ambiguous notation $\{\mathbf{C} \in \mathbf{Cat}_{un}^{ch}(X_{dR}), \mathcal{A} \in \mathbf{Alg}_{un}^{\text{fact}}(\mathbf{C})\}$ is properly understood using the formalism of §5. We note that the category is designed so that morphisms:

$$(\mathbf{C}_1, \mathcal{A}_1) \rightarrow (\mathbf{C}_2, \mathcal{A}_2)$$

are given by pairs of a morphism $\varphi : \mathbf{C}_1 \rightarrow \mathbf{C}_2$ of chiral categories and a morphism $\eta : \varphi(\mathcal{A}_1) \rightarrow \mathcal{A}_2$ of factorization algebras, where $\varphi(\mathcal{A}_1)$ is understood as a factorization algebra in \mathbf{C}_2 using the discussion of §5.34.

APPENDIX A. SHEAVES OF CATEGORIES

A.1. The purpose of this appendix is to recall the rudiments of the theory of sheaves of categories on prestacks, and the theory of 1-affineness from [Gai6].

A.2. **Linear categories.** We begin with a quick review of the theory of sheaves of categories from [Gai6] and [Lur3].

Recall that \mathbf{DGCat}_{cont} denotes the category of cocomplete DG categories under continuous functors, and that \mathbf{DGCat}_{cont} is equipped with a symmetric monoidal structure \otimes with unit \mathbf{Vect} , and whose tensor product commutes with colimits in each variable.

Let A be a commutative algebra. An A -linear category is an A -mod-module category in \mathbf{DGCat}_{cont} . A functor of A -linear categories is A -linear if it is a continuous functor of A -mod-module categories. When A is connective, we denote the category of A -linear categories under A -linear functors by $\mathbf{ShvCat}/_{\text{Spec}(A)}$.

Remark A.2.1. Note that $\mathbf{ShvCat}/_{\text{Spec}(A)}$ is a symmetric monoidal category with tensor product $(\mathbf{C}, \mathbf{D}) \mapsto \mathbf{C} \otimes_{A\text{-mod}} \mathbf{D}$. This symmetric monoidal structure has unit A -mod.

For $A \rightarrow B$ a map of commutative rings, we have the symmetric monoidal functor:

$$A\text{-mod} \rightarrow B\text{-mod} \tag{A.2.1}$$

sending $M \rightarrow M \otimes_A B$ and therefore we obtain the adjoint functors:

$$(A\text{-mod})\text{-mod}(\mathbf{DGCat}_{cont}) \overset{\mathbf{C} \mapsto \mathbf{C} \otimes_{A\text{-mod}} B\text{-mod}}{\rightleftarrows} (B\text{-mod})\text{-mod}(\mathbf{DGCat}_{cont}) \tag{A.2.2}$$

where the right adjoint is restriction along (A.2.1). Each of these functors commutes with arbitrary colimits.

Remark A.2.2. According to [Gai5], rigidity of A -mod implies that B -mod is dualizable as an A -mod-module category. Therefore, the left adjoint in (A.2.2) commutes with limits as well.

Lemma A.2.3. For a morphism $A \rightarrow B$ of commutative algebras and for an A -linear category \mathbf{C} , the tautological functor:

$$\mathbf{C} \otimes_{A\text{-mod}} B\text{-mod} \rightarrow \mathbf{C}$$

is conservative and admits an A -linear left adjoint.

Notation A.2.4. In the setting of Lemma A.2.3, we denote this left adjoint by:

$$X \mapsto X \underset{A}{\otimes} B.$$

Proof of Lemma A.2.3. The existence of a left adjoint follows from the existence of the adjoint A -linear functors:

$$A\text{-mod} \rightleftarrows B\text{-mod}$$

It suffices to see that this left adjoint generates the category $\mathbf{C} \underset{A\text{-mod}}{\otimes} B\text{-mod}$ under colimits. Because B generates $B\text{-mod}$ under colimits and shifts, it suffices to see that the essential image of the (non-exact) functor:

$$\mathbf{C} \times B\text{-mod} \rightarrow \mathbf{C} \underset{A\text{-mod}}{\otimes} B\text{-mod}$$

generates under colimits. But this is immediate from the universal property of the tensor product of categories. □

A.3. Sheaves of categories. We consider $\text{ShvCat}/_{\text{Spec}(-)}$ as a functor $\text{AffSch}^{op} \rightarrow \text{DGCat}_{cont}$ via the left adjoint functor in (A.2.1). We let $\text{ShvCat}/_{-} : \text{PreStk}^{op} \rightarrow \text{DGCat}_{cont}$ denote the right Kan extension of this functor.

For any prestack \mathcal{Y} , $\text{ShvCat}/_{\mathcal{Y}}$ is a symmetric monoidal category with tensor product computed “locally” using Remark A.2.1. We denote the tensor product by:

$$(\mathbf{C}, \mathbf{D}) \mapsto \mathbf{C} \underset{\text{QCoh}_{\mathcal{Y}}}{\otimes} \mathbf{D}.$$

For a prestack \mathcal{Y} we refer to objects of $\text{ShvCat}/_{\mathcal{Y}}$ as *sheaves of categories* on \mathcal{Y} . For a sheaf of categories \mathbf{C} on \mathcal{Y} we let

$$\Gamma(\mathcal{Y}, \mathbf{C}) \in \text{DGCat}_{cont}$$

denote the global sections of the category. We let $\text{QCoh}_{\mathcal{Y}} \in \text{ShvCat}/_{\mathcal{Y}}$ denote the canonical object with global sections $\text{QCoh}(\mathcal{Y})$. For $\mathbf{C} \in \text{ShvCat}/_{\mathcal{Y}}$ the category $\Gamma(\mathcal{Y}, \mathbf{C})$ is canonically a $\text{QCoh}(\mathcal{Y})$ -module category.

For $\mathbf{C} \in \text{ShvCat}/_{\mathcal{Y}}$ and $f : \mathcal{Y}' \rightarrow \mathcal{Y}$ we use both notations $\mathbf{C}_{\mathcal{Y}'}$ and $f^*(\mathbf{C})$ for the pullback of \mathbf{C} to \mathcal{Y}' . Note that if f is an affine (schematic) morphism then the functor $f^* : \text{ShvCat}/_{\mathcal{Y}} \rightarrow \text{ShvCat}/_{\mathcal{Y}'}$ admits a continuous right adjoint f_* computed “locally” using (A.2.2).

Remark A.3.1. By Remark A.2.2, limits in $\text{ShvCat}/_{\mathcal{Y}}$ are computed locally, i.e., pullbacks of sheaves of categories commute with limits.

A.4. Fully-faithful functors. For \mathcal{Y} a prestack, we say that a morphism $\mathbf{D} \rightarrow \mathbf{C} \in \text{ShvCat}/_{\mathcal{Y}}$ is *locally fully-faithful*, or simply *fully-faithful*,¹⁸ if, for every affine scheme S with a morphism $f : S \rightarrow \mathcal{Y}$, the induced functor:

$$\Gamma(S, \mathbf{D}) \rightarrow \Gamma(S, \mathbf{C})$$

is fully-faithful.

¹⁸This terminology is justified by Proposition A.4.3.

Example A.4.1. If $\mathcal{D} \rightarrow \mathcal{C}$ admits a right (resp. left) adjoint in the 2-category $\text{ShvCat}_{/\mathcal{Y}}$ with unit (resp. counit) an equivalence, then this functor is locally fully-faithful.

Terminology A.4.2. We sometimes simply summarize the situation in saying that \mathcal{D} is a full subcategory of \mathcal{C} , and write $\mathcal{D} \subseteq \mathcal{C}$.

The following result helps to identify locally fully-faithful functors.

Proposition A.4.3. For $\mathcal{Y} = \text{Spec}(A)$, a functor $F : \mathcal{D} \rightarrow \mathcal{C}$ of A -linear categories is locally fully-faithful if and only if it is fully-faithful as a mere functor.

Proof. It suffices to show that for every morphism $A \rightarrow B$ of commutative algebras, the induced functor:

$$F_B : \mathcal{D} \otimes_{A\text{-mod}} B\text{-mod} \rightarrow \mathcal{C} \otimes_{A\text{-mod}} B\text{-mod}$$

is fully-faithful.

Let $\text{Oblv}_{\mathcal{D}}^B$ denote the forgetful functor:

$$\mathcal{D} \otimes_{A\text{-mod}} B\text{-mod} \rightarrow \mathcal{D}$$

and similarly for \mathcal{C} .

By Lemma A.2.3, it suffices to show that, for $X \in \mathcal{D}$ and $Y \in \mathcal{D} \otimes_{A\text{-mod}} B\text{-mod}$, the morphism:

$$\text{Hom}_{\mathcal{D} \otimes_{A\text{-mod}} B\text{-mod}}(X \otimes_A B, Y) \rightarrow \text{Hom}_{\mathcal{C} \otimes_{A\text{-mod}} B\text{-mod}}(F_B(X \otimes_A B), F_B(Y)) \quad (\text{A.4.1})$$

is an equivalence.

Note that both operations $\text{Oblv}_{\mathcal{D}}^B$ and $- \otimes_A B$ commute with A -linear functors. Moreover, under the identifications:

$$\text{Hom}_{\mathcal{D} \otimes_{A\text{-mod}} B\text{-mod}}(X \otimes_A B, Y) = \text{Hom}_{\mathcal{D}}(X, \text{Oblv}_{\mathcal{D}}^B(Y))$$

and:

$$\begin{aligned} \text{Hom}_{\mathcal{C} \otimes_{A\text{-mod}} B\text{-mod}}(F_B(X \otimes_A B), F_B(Y)) &= \text{Hom}_{\mathcal{C} \otimes_{A\text{-mod}} B\text{-mod}}(F(X) \otimes_A B, F_B(Y)) = \\ &= \text{Hom}_{\mathcal{C}}(F(X), \text{Oblv}_{\mathcal{C}}^B(F_B(Y))) = \text{Hom}_{\mathcal{C}}(F(X), F \circ \text{Oblv}_{\mathcal{D}}^B(Y)) \end{aligned}$$

the morphism (A.4.1) is given by the canonical map:

$$\text{Hom}_{\mathcal{D}}(X, \text{Oblv}_{\mathcal{D}}^B(Y)) \rightarrow \text{Hom}_{\mathcal{C}}(F(X), F \circ \text{Oblv}_{\mathcal{D}}^B(Y))$$

so that the result follows from the hypothesis that F is fully-faithful. \square

We also note the following basic stability.

Proposition A.4.4. Given an \mathcal{J} -shaped diagram of fully-faithful functors $\mathcal{C}_i \rightarrow \mathcal{D}_i \in \text{ShvCat}_{/\mathcal{Y}}$, the induced functor:

$$\lim_{i \in \mathcal{J}} \mathcal{C}_i \rightarrow \lim_{i \in \mathcal{J}} \mathcal{D}_i$$

is fully-faithful as well.

This follows immediately from the corresponding statement for DG categories.

Corollary A.4.5. Given a system of subcategories $i \mapsto C_i \subseteq C$ indexed by a contractible category \mathcal{J} (i.e., the groupoid obtained by inverting all arrows is contractible), the induced functor $\lim_{i \in \mathcal{J}} C_i \rightarrow C$ is fully-faithful as well.

Proof. Apply Proposition A.4.4 to the functors:

$$C_i \hookrightarrow C$$

and note that contractibility of \mathcal{J} implies that $\lim_{i \in \mathcal{J}} C \xrightarrow{\cong} C$. □

A.5. Let $F : C \rightarrow D$ be a morphism of A -linear categories. We define $F(C)$ as the subcategory of D generated under colimits by objects $F(X)$, $X \in C$. Note that $F(C)$ is an A -linear subcategory since $A\text{-mod}$ is generated under colimits by A .

Lemma A.5.1. $A \rightarrow B$ be a morphism of commutative algebras and let $F : C \rightarrow D$ be an A -linear morphism of A -linear categories. Let F^B denote the induced functor:

$$F^B : C \otimes_{A\text{-mod}} B\text{-mod} \rightarrow D \otimes_{A\text{-mod}} B\text{-mod}.$$

Then the canonical functor:

$$F(C) \otimes_{A\text{-mod}} B\text{-mod} \rightarrow F^B(C \otimes_{A\text{-mod}} B\text{-mod}) \tag{A.5.1}$$

is an equivalence.

Proof. The morphism $F(C) \rightarrow D$ is fully-faithful, so by Proposition A.4.3 the morphism:

$$F(C) \otimes_{A\text{-mod}} B\text{-mod} \rightarrow D \otimes_{A\text{-mod}} B\text{-mod}$$

is as well. Therefore, it remains to show essential surjectivity of (A.5.1).

By Lemma A.2.3, $C \otimes_{A\text{-mod}} B\text{-mod}$ is generated under colimits by objects induced from C , giving the result. □

By the lemma, for $F : C \rightarrow D$ a morphism of sheaves of categories on $\mathcal{Y} \in \text{PreStk}$, we can make sense of $F(C)$ so that its formation commutes with base-change. Note that $F(C) \rightarrow D$ is locally fully-faithful.

A.6. Localizations. Let A be a fixed commutative algebra. Let C be a A -linear category, and let $D \subseteq C$ be a subcategory closed under colimits. As above, since $A\text{-mod}$ is generated under colimits by A , D is an $(A\text{-mod})$ -submodule category.

In this case, we can form the quotient category C/D , that is computed as a pushout:

$$\begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C/D \end{array} \tag{A.6.1}$$

in the category of A -linear categories.

Lemma A.6.1. Given $B \rightarrow A$ a map of commutative algebras, the induced restriction functor:

$$\{A\text{-linear categories}\} \rightarrow \{B\text{-linear categories}\}$$

commutes with formation of quotients.

Proof. Indeed, this functor commutes with arbitrary colimits, since it is the a restriction functor for modules in DGCat_{cont} from $A\text{-mod}$ to $B\text{-mod}$ (c.f. [Lur4] 4.2.3.5). □

More generally, one can form quotients for locally fully-faithful functors of sheaves of categories on an arbitrary prestack, defined also as a pushout. This operation tautologically commutes with pullback of sheaves of categories, and then can be computed “locally” using Lemma A.6.1.

A.7. For \mathcal{Y} a prestack and F a morphism $F : \mathcal{C} \rightarrow \mathcal{D} \in \mathrm{ShvCat}/_{\mathcal{Y}}$, the *kernel* $\mathrm{Ker}(F)$ of F is by definition the fiber product $\mathcal{C} \times_{\mathcal{D}} 0$. By Remark A.3.1, formation of kernels commutes with base-change.

Note that the natural morphism $\mathrm{Ker}(F) \rightarrow \mathcal{C}$ is always locally fully-faithful. Indeed, this tautologically reduces to the case where \mathcal{Y} is an affine scheme, where it is obvious.

Definition A.7.1. A morphism $F : \mathcal{C} \rightarrow \mathcal{D} \in \mathrm{ShvCat}/_{\mathcal{Y}}$ is a *localization functor* in $\mathrm{ShvCat}/_{\mathcal{Y}}$ if the natural morphism:

$$\mathcal{C}/\mathrm{Ker}(F) \rightarrow \mathcal{D}$$

is an equivalence.

We have the following equivalence between subcategories and localization functors.

Proposition A.7.2. Let \mathcal{C} be a sheaf of categories on a prestack \mathcal{Y} , and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory.

- (1) The kernel of the functor $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{C}^0$ is \mathcal{C}^0 .
- (2) The functor $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{C}^0$ is a localization functor.

Proof. The first statement immediately reduces to the affine case, where it is well-known, and the second statement follows tautologically from the first. □

Proposition A.7.3. Suppose that $\mathcal{C} = \mathrm{colim}_{i \in \mathcal{J}} \mathcal{C}_i \in \mathrm{ShvCat}/_{\mathcal{Y}}$, and suppose that \mathcal{J} is filtered and each structure map $\mathcal{C}_i \rightarrow \mathcal{C}_j$ is a localization functor.

Then for every $i_0 \in \mathcal{J}$, the functor $\mathcal{C}_{i_0} \rightarrow \mathcal{C}$ is a localization functor.

We first need the following lemma, which is obvious in the affine case and therefore in general.

Lemma A.7.4. Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a (not necessarily fully-faithful) morphism of sheaves of categories and let \mathcal{C}/\mathcal{D} denote the corresponding pushout. Then $\mathcal{C}/\mathcal{D} = \mathcal{C}/F(\mathcal{D})$. In particular, $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{D}$ is a localization functor.

Proof of Proposition A.7.3. We can assume i_0 is initial in \mathcal{J} by filteredness. A functor $\mathcal{C} \rightarrow \mathcal{D}$ is equivalent to compatible functors $\mathcal{C}_i \rightarrow \mathcal{D}$, which in turn are equivalent to functors $\mathcal{C}_{i_0} \rightarrow \mathcal{D}$ mapping $\mathrm{Ker}(\mathcal{C}_{i_0} \rightarrow \mathcal{C}_i)$ to 0. But this is obviously equivalent to giving a functor $\mathcal{C}_{i_0} \rightarrow \mathcal{D}$ mapping $\mathrm{colim}_i \mathrm{Ker}(\mathcal{C}_{i_0} \rightarrow \mathcal{C}_i)$ to 0, so the result follows from Lemma A.7.4. □

A.8. **1-affineness.** We follow [Gai6] in saying a prestack \mathcal{Y} is *1-affine* if the morphism:

$$\Gamma : \mathrm{ShvCat}_{/\mathcal{Y}} \rightarrow \mathrm{QCoh}(\mathcal{Y})\text{-mod}(\mathrm{DGCat}_{\mathrm{cont}})$$

is an equivalence.

The following useful results are proved in [Gai6].

Theorem A.8.1. (1) Any quasi-compact quasi-separated scheme is 1-affine.

(2) If T is a quasi-compact quasi-separated scheme, S is a closed subscheme with quasi-compact complement, and T_S^\wedge is the (indscheme) formal completion, then T_S^\wedge is 1-affine.

(3) For S an almost finite type scheme, S_{dR} is 1-affine.

We also need a relative version: we say that a morphism $f : \mathcal{Y} \rightarrow \mathcal{Z}$ of prestacks is *1-affine* if for every affine scheme S and map $S \rightarrow \mathcal{Z}$, the prestack $\mathcal{Y} \times_{\mathcal{Z}} S$ is 1-affine.

We immediately deduce from Theorem A.8.1 the following:

Proposition A.8.2. Any quasi-compact quasi-separated morphism is 1-affine.

Remark A.8.3. It is not tautological that a 1-affine prestack \mathcal{Y} has 1-affine structure map $\mathcal{Y} \rightarrow \mathrm{Spec}(k)$. However, we will prove this in §A.11 below.

A.9. **Pushforwards.** Next, we discuss the pushforward construction for sheaves of categories.

Proposition A.9.1. Let $f : \mathcal{Y} \rightarrow \mathcal{Z}$ be a morphism of prestacks.

(1) The functor:

$$f^* : \mathrm{ShvCat}_{/\mathcal{Z}} \rightarrow \mathrm{ShvCat}_{/\mathcal{Y}}$$

admits a right adjoint f_* compatible with arbitrary base-change.

(2) If f is 1-affine, then $f_* : \mathrm{ShvCat}_{/\mathcal{Y}} \rightarrow \mathrm{ShvCat}_{/\mathcal{Z}}$ commutes with arbitrary colimits and satisfies the projection formula in the sense that it is a morphism of $\mathrm{ShvCat}_{/\mathcal{Z}}$ -module categories.

(3) If f is quasi-compact quasi-separated, then for every $C \in \mathrm{ShvCat}_{/\mathcal{Z}}$ the unit map:

$$C \rightarrow f_* f^*(C)$$

admits a right adjoint in the 2-category $\mathrm{ShvCat}_{/\mathcal{Z}}$. This right adjoint commutes with base-change in the natural sense.

Corollary A.9.2. Let $f : \mathcal{Y} \rightarrow \mathcal{Z}$ be a quasi-compact quasi-separated schematic morphism of prestacks. Then for every $C \in \mathrm{ShvCat}_{/\mathcal{Z}}$ the morphism:

$$f_C^* : \Gamma(\mathcal{Z}, C) \rightarrow \Gamma(\mathcal{Y}, f^*(C)) = \Gamma(\mathcal{Z}, f_* f^*(C))$$

admits a continuous right adjoint $f_{C,*}$.

This right adjoint commutes with base-change in the sense that for every Cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{f_1} & \mathcal{Z}_1 \\ \downarrow \varphi & & \downarrow \psi \\ \mathcal{Y}_2 & \xrightarrow{f_2} & \mathcal{Z}_2 \end{array}$$

with f_2 quasi-compact quasi-separated and schematic and every $C \in \mathrm{ShvCat}_{/\mathcal{Z}}$ the natural morphism:

$$\psi_C^* \circ f_{2,C,*} \rightarrow f_{1,C_{\mathcal{Y}_2},*} \circ \varphi_{C_{\mathcal{Y}_2}}^*$$

is an equivalence.

Proof of Proposition A.9.1. We begin with (1).

Suppose first that $\mathcal{Z} = S$ is an affine scheme. Now the functor:

$$\begin{aligned} \mathrm{QCoh}(S)\text{-mod}(\mathrm{DGCat}_{cont}) &\rightarrow \mathrm{QCoh}(\mathcal{Y})\text{-mod}(\mathrm{DGCat}_{cont}) \\ \mathcal{M} &\mapsto \mathcal{M} \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(\mathcal{Y}) \end{aligned} \quad (\mathrm{A.9.1})$$

obviously admits a right adjoint given by restriction along $\mathrm{QCoh}(S) \rightarrow \mathrm{QCoh}(\mathcal{Y})$. This functor commutes with colimits by [Lur4] 4.2.3.5 and tautologically satisfies the projection formula.

We then see that the right adjoint $f_* : \mathrm{ShvCat}_{/\mathcal{Y}} \rightarrow \mathrm{QCoh}(S)\text{-mod}(\mathrm{DGCat}_{cont})$ is computed as the composition:

$$\mathrm{ShvCat}_{/\mathcal{Y}} \xrightarrow{\Gamma(\mathcal{Y}, -)} \mathrm{QCoh}(\mathcal{Y})\text{-mod}(\mathrm{DGCat}_{cont}) \xrightarrow{\text{restriction}} \mathrm{QCoh}(S)\text{-mod}(\mathrm{DGCat}_{cont}) = \mathrm{ShvCat}_{/S}.$$

We now verify the base-change property of this functor. Suppose first that we are given a Cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{\varphi} & \mathcal{Y} \\ \downarrow f' & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array} \quad (\mathrm{A.9.2})$$

with S' and S affine schemes. Then for $\mathbf{C} \in \mathrm{ShvCat}_{/\mathcal{Y}}$, we compute:

$$\begin{aligned} \Gamma(S', g^* f_*(\mathbf{C})) &= \Gamma(\mathcal{Y}, \mathbf{C}) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S') = \left(\lim_{\substack{\alpha: T \rightarrow \mathcal{Y} \\ T \in \mathrm{AffSch}^{op}}} \Gamma(T, \alpha^*(\mathbf{C})) \right) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S') = \\ & \lim_{\substack{\alpha: T \rightarrow \mathcal{Y} \\ T \in \mathrm{AffSch}^{op}}} \left(\Gamma(T, \alpha^*(\mathbf{C})) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S') \right) = \Gamma \left(\mathrm{colim}_{\substack{\alpha: T \rightarrow \mathcal{Y} \\ T \in \mathrm{AffSch}^{op}}} T \times_S S', p_1^* \alpha^*(\mathbf{C}) \right) = \Gamma(\mathcal{Y}', \varphi^*(\mathbf{C})). \end{aligned}$$

This verifies base-change for the Cartesian diagram (A.9.2), when S' is assumed affine; the case when S' is allowed to be an arbitrary prestack immediately reduces to this one.

We obtain the existence of a right adjoint in (1) compatible with base-change by an immediate reduction to the case when \mathcal{Z} is affine.

The claims of (2) follow from the observations we have already made about (A.9.1).

Using the same dévissage we obtain (3), using that in the quasi-compact quasi-separated case with affine target S we have the continuous right adjoint $f_* : \mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{QCoh}(S)$ satisfying the projection formula. □

A.10. Lax prestacks. We now digress to discuss pushforwards for morphisms of *lax* prestacks. The notion of lax prestack is defined in §4. We refer to *loc. cit.* for the basics of sheaves of categories in this setting.

The reader may safely skip this section as it is not needed in the rest of this appendix.

Given a diagram $\mathcal{X} \xrightarrow{z} \mathcal{Z} \xleftarrow{f} \mathcal{Y}$ of lax prestacks, define a lax prestack $\mathcal{Y}_{z/}$ by setting $\mathcal{Y}_{z/}(S)$ as the category parametrizing maps $x : S \rightarrow \mathcal{X}$ and $y : S \rightarrow \mathcal{Y}$, plus a morphism $z \circ x \rightarrow f \circ y \in \mathcal{Z}(T)$.

Example A.10.1. If \mathcal{Z} is a usual prestack, then this is just the usual fiber product.

Proposition A.10.2. For $f : \mathcal{Y} \rightarrow \mathcal{Z}$ a morphism of lax prestacks, the pullback $f^* : \mathrm{ShvCat}_{/\mathcal{Z}}^{\mathrm{naive}} \rightarrow \mathrm{ShvCat}_{/\mathcal{Y}}^{\mathrm{naive}}$ admits a right adjoint f_* . Formation of this right adjoint commutes base-change in the sense that for the commutative diagram:

$$\begin{array}{ccc} \mathcal{Y}_{z/} & \xrightarrow{p} & \mathcal{Y} \\ \downarrow f_x & & \downarrow f \\ \mathcal{X} & \xrightarrow{z} & \mathcal{Z} \end{array}$$

the natural morphism:

$$z^* f_* \rightarrow f_{x,*} p^*$$

is an equivalence.

Proof. The case where \mathcal{Z} and \mathcal{X} are an affine schemes proceeds exactly as in Proposition A.9.1. As in *loc. cit.*, this treats the general case where \mathcal{Z} and \mathcal{X} are (usual) prestacks.

For $S \in \mathrm{AffSch}$ and $z : S \rightarrow \mathcal{Z}$, we have the canonical morphisms $f_S : \mathcal{Y}_{z/} \rightarrow S$, $p : \mathcal{Y}_{z/} \rightarrow \mathcal{Y}$. We define $z^* f_*(\mathbf{C})$ as $f_{S,*} p^*(\mathbf{C})$. This defines a sheaf of categories on \mathcal{Z} by the case we treated above. It's then immediate to construct the unit and counit of the adjunction to verify that this actually is the right adjoint. □

In this setting, one also needs the following notion.

Definition A.10.3. We say a morphism $f : \mathcal{Y} \rightarrow \mathcal{Z}$ of lax prestacks is *op-cofinal* (over $\mathrm{Spec}(k)$) if for every affine scheme S , the map $\mathcal{Y}(S) \rightarrow \mathcal{Z}(S)$ is op-cofinal.

More generally, suppose:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{Z} \\ \searrow \psi & & \swarrow \varphi \\ & \mathcal{X} & \end{array}$$

is a commutative diagram of lax prestacks. We say that f is *op-cofinal over \mathcal{X}* if for every affine scheme S and map $x : S \rightarrow \mathcal{X}$, the map:

$$\mathcal{Y}_{x/} \rightarrow \mathcal{Z}_{x/}$$

is op-cofinal.

Proposition A.10.4. If f as above is op-cofinal, then for every $\mathbf{C} \in \mathrm{ShvCat}_{/c\mathcal{Z}}^{\mathrm{naive}}$, the morphism:

$$\phi_*(\mathbf{C}) \rightarrow \phi_* f_* f^*(\mathbf{C}) = \psi_* f^*(\mathbf{C})$$

is an equivalence.

Proof. Suppose $x : S \rightarrow \mathcal{X}$ is a map from an affine scheme. It suffices to show that for each such datum, the map:

$$x^* \phi_*(\mathbf{C}) \rightarrow x^* \psi_* f^*(\mathbf{C})$$

is an equivalence. Replacing \mathcal{X} by S , \mathcal{Y} by $\mathcal{Y}_{x/}$ and \mathcal{Z} by $\mathcal{Z}_{x/}$ and applying Proposition A.10.2, we reduce to the case where \mathcal{X} is an affine scheme S .

Let $\Gamma^{naive} : \mathrm{ShvCat}_{/-}^{naive} \rightarrow \mathrm{DGCat}_{cont}$ denote the “naive” global sections functor, i.e., the pushforward to the point in the $\mathrm{ShvCat}_{/-}^{naive}$ -setting. It then suffices to show that \mathbf{C} and $f^*(\mathbf{C})$ have the same global sections.

We compute:

$$\Gamma^{naive}(\mathcal{Y}, f^*(\mathbf{C})) = \lim_{T \in \mathrm{AffSch}^{op}} \lim_{g \in \mathcal{Y}(T)} \Gamma(T, g^* f^*(\mathbf{C})) \stackrel{\text{op-cofinality}}{=} \lim_{T \in \mathrm{AffSch}^{op}} \lim_{h \in \mathcal{Z}(T)} \Gamma(T, h^* \mathbf{C}) = \Gamma^{naive}(\mathcal{Z}, \mathbf{C})$$

as desired. □

A.11. We will prove the following technical result.

Proposition A.11.1. The composition of 1-affine morphisms is 1-affine.

We will prove the following more precise form of Proposition A.11.1.

Lemma A.11.2. If $f : \mathcal{Y} \rightarrow \mathcal{Z}$ is a 1-affine morphism of prestacks with \mathcal{Z} a 1-affine prestack, then \mathcal{Y} is 1-affine.

Proof of Proposition A.11.1 given Lemma A.11.2. Suppose $\mathcal{Y} \rightarrow \mathcal{Z} \rightarrow \mathcal{S}$ are 1-affine morphisms. To show that the composition is 1-affine, we reduce to showing that in the case when \mathcal{S} is an affine scheme, \mathcal{Y} is 1-affine. But in this case, \mathcal{Z} is a 1-affine prestack, so the result follows from Lemma A.11.2. □

We need the following result first.

Lemma A.11.3. For $f : \mathcal{Y} \rightarrow \mathcal{Z}$ a 1-affine morphism, the pushforward $f_ : \mathrm{ShvCat}_{/\mathcal{Y}} \rightarrow \mathrm{ShvCat}_{/\mathcal{Z}}$ is conservative.*

Proof. Suppose that \mathbf{C} and \mathbf{D} are two sheaves of categories on \mathcal{Y} and $\varphi : \mathbf{C} \rightarrow \mathbf{D}$ is a map such that $f_*(\varphi)$ is an equivalence. We will show that φ is an equivalence.

Let S be an affine scheme with a map $g : S \rightarrow \mathcal{Y}$. It suffices to show that for every such datum, $g^*(\varphi)$ is an equivalence.

We form the commutative diagram:

$$\begin{array}{ccccc} S & \longrightarrow & \mathcal{Y} \times_{\mathcal{Z}} S & \longrightarrow & \mathcal{Y} \\ & \searrow & \downarrow & & \downarrow f \\ & \mathrm{id}_S & S & \xrightarrow{f \circ g} & \mathcal{Z}. \end{array}$$

Note that pushforward along $\mathcal{Y} \times_{\mathcal{Z}} S \rightarrow S$ is conservative because:

$$\Gamma(\mathcal{Y} \times_{\mathcal{Z}} S, -)$$

is conservative by 1-affineness of f . But now base-change and this conservativity imply that the pullback of φ to $\mathcal{Y} \times_{\mathcal{Z}} S$ is an equivalence, giving the result after further restriction to S . □

Proof of Lemma A.11.2. Because f_* commutes with arbitrary colimits by Proposition A.9.1 and is conservative by Lemma A.11.3, Barr-Beck implies that we have:

$$f_* f^* \text{-mod}(\text{ShvCat}/\mathcal{Z}) \simeq \text{ShvCat}/\mathcal{Y}.$$

Therefore, we deduce:

$$\begin{aligned} \text{ShvCat}/\mathcal{Y} &= f_*(\text{QCoh}_{\mathcal{Y}})\text{-mod}(\text{ShvCat}/\mathcal{Z}) \stackrel{\Gamma}{\simeq} \text{QCoh}(\mathcal{Y})\text{-mod}(\text{QCoh}(\mathcal{Z})\text{-mod}(\text{DGCat}_{\text{cont}})) = \\ &\quad \text{QCoh}(\mathcal{Y})\text{-mod} \end{aligned}$$

as desired. □

Corollary A.11.4. For any pair of 1-affine prestacks \mathcal{Y} and \mathcal{Z} , the product $\mathcal{Y} \times \mathcal{Z}$ is 1-affine.

Proof. It suffices to show that the projection $\mathcal{Y} \times \mathcal{Z} \rightarrow \mathcal{Y}$ is 1-affine. By the definition, we reduce showing that in the case where S is an affine scheme, $S \times \mathcal{Z}$ is a 1-affine prestack.

Note that the morphism $S \times \mathcal{Z} \rightarrow \mathcal{Z}$ is affine and therefore 1-affine, so the result follows from Lemma A.11.2. □

Corollary A.11.5. A prestack \mathcal{Y} is 1-affine if and only if the structure map $\mathcal{Y} \rightarrow \text{Spec}(k)$ is 1-affine.

Proposition A.11.6. Given a commutative diagram of prestacks:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g} & \mathcal{Y} \\ & \searrow & \swarrow f \\ & \mathcal{Z} & \end{array}$$

$f \circ g$

with f and $f \circ g$ 1-affine, and such that the diagonal $\Delta_f : \mathcal{Y} \rightarrow \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ is 1-affine, the morphism g is 1-affine.

Proof. Applying base-change by any map to \mathcal{Z} from an affine scheme, we reduce to showing in the case $\mathcal{Z} = S \in \text{AffSch}$ that $g : \mathcal{W} \rightarrow \mathcal{Y}$ is 1-affine.

The graph morphism $\mathcal{W} \rightarrow \mathcal{W} \times_S \mathcal{Y}$ is obtained by base-change along $\mathcal{W} \rightarrow \mathcal{Y}$ from the diagonal $\mathcal{Y} \rightarrow \mathcal{Y} \times_S \mathcal{Y}$, and therefore by assumption is 1-affine. But the morphism g factors as:

$$\mathcal{W} \rightarrow \mathcal{W} \times_S \mathcal{Y} \rightarrow \mathcal{Y}$$

and the second morphism is 1-affine since it is obtained by base-change from $\mathcal{W} \rightarrow S$. □

A.12. Correspondences. Let $\text{PreStk}_{\text{corr}; \text{all}, 1\text{-aff}}$ denote the category of prestacks under correspondences of the form:

$$\begin{array}{ccc} & \mathcal{H} & \\ \alpha \swarrow & & \searrow \beta \\ \mathcal{Y} & & \mathcal{Y}' \end{array} \tag{A.12.1}$$

where β is a 1-affine morphism.

We consider $\text{PreStk}_{\text{corr}; \text{all}, 1\text{-aff}}$ as a symmetric monoidal category using the Cartesian monoidal structure on PreStk .

From the Gaitsgory-Rozenblyum theory [GR] of correspondences, we obtain the following result from Proposition A.9.1 (1)

Corollary A.12.1. *There is a canonical lax symmetric monoidal functor:*

$$\mathrm{ShvCat}_{/-}^{\mathrm{enh}} : \mathrm{PreStk}_{\mathrm{corr}; \mathrm{all}, 1\text{-aff}} \rightarrow \mathrm{Cat}$$

sending a prestack \mathcal{Y} to $\mathrm{ShvCat}_{/\mathcal{Y}}$ and sending:

$$\left(\begin{array}{ccc} & \mathcal{H} & \\ \alpha \swarrow & & \searrow \beta \\ \mathcal{Y} & & \mathcal{Y}' \end{array} \right) \mapsto (\beta_* \alpha^* : \mathrm{ShvCat}_{/\mathcal{Y}} \rightarrow \mathrm{ShvCat}_{/\mathcal{Y}'}).$$

The lax symmetric monoidal structure is given by exterior products.

A.13. Dualizability for sheaves of categories. Let \mathcal{Y} be a fixed prestack. As in §A.3, $\mathrm{ShvCat}_{/\mathcal{Y}}$ is a symmetric monoidal category with unit $\mathrm{QCoh}_{\mathcal{Y}}$.

We will say that a sheaf of categories \mathbf{C} on \mathcal{Y} is *dualizable* if it is dualizable as an object of the symmetric monoidal category $\mathrm{ShvCat}_{/\mathcal{Y}}$. For \mathbf{C} dualizable, we let $\mathbf{C}^\vee \in \mathrm{ShvCat}_{/\mathcal{Y}}$ denote its dual.

Proposition A.13.1. *The sheaf of categories $\mathbf{C} \in \mathrm{ShvCat}_{/\mathcal{Y}}$ is dualizable if and only if for every $f : S \rightarrow \mathcal{Y}$ a map from an affine scheme S , the category $\Gamma(S, \mathbf{C}_S)$ is dualizable as a DG category.*

Proof. Let S be an affine scheme. By [Gai5] a sheaf of categories:

$$\mathbf{D} \in \mathrm{ShvCat}_{/S} = \mathrm{QCoh}(S)\text{-mod}(\mathrm{DGCat}_{\mathrm{cont}})$$

is dualizable if and only if $\Gamma(S, \mathbf{D})$ is dualizable as an object of $\mathrm{DGCat}_{\mathrm{cont}}$.

Restriction functors for sheaves of categories are symmetric monoidal and therefore preserve dualizability and canonically commute with passage to the dual. Therefore, we see that dualizability for $\mathbf{C} \in \mathrm{ShvCat}_{/\mathcal{Y}}$ can be tested after pullback to any affine scheme, and now the result follows from the above. □

Lemma A.13.2. *For any dualizable $\mathbf{C} \in \mathrm{ShvCat}_{/\mathcal{Y}}$ the functor:*

$$\mathbf{C} \otimes_{\mathrm{QCoh}_{\mathcal{Y}}} - : \mathrm{ShvCat}_{/\mathcal{Y}} \rightarrow \mathrm{ShvCat}_{/\mathcal{Y}}$$

commutes with limits.

Proof. Combining Remark A.3.1 with Proposition A.13.1, we immediately reduce to the affine case, which is contained in [Gai5]. □

Construction A.13.3. Let $i \mapsto \mathbf{C}_i$ be an \mathcal{I} -shaped diagram of dualizable sheaves of categories on \mathcal{Y} with each \mathbf{C}_i dualizable. Let $\mathbf{C} := \mathrm{colim}_{i \in \mathcal{I}} \mathbf{C}_i$ and let $\bar{\mathbf{C}} := \lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathbf{C}_i^\vee$, where the limit is taken over the duals to the structure functors.

Then there is a canonical pairing:

$$\mathbf{C} \otimes_{\mathrm{QCoh}_{\mathcal{Y}}} \bar{\mathbf{C}} \rightarrow \mathrm{QCoh}_{\mathcal{Y}} \tag{A.13.1}$$

constructed as:

$$\left(\lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathbf{C}_i^\vee \right) \otimes_{\mathrm{QCoh}_{\mathcal{Y}}} \left(\mathrm{colim}_{j \in \mathcal{I}} \mathbf{C}_j \right) = \mathrm{colim}_{j \in \mathcal{I}} \left(\left(\lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathbf{C}_i^\vee \right) \otimes_{\mathrm{QCoh}_{\mathcal{Y}}} \mathbf{C}_j \right) \rightarrow \mathrm{colim}_{j \in \mathcal{I}} \left(\mathbf{C}_j^\vee \otimes_{\mathrm{QCoh}_{\mathcal{Y}}} \mathbf{C}_j \right) \rightarrow \mathrm{QCoh}_{\mathcal{Y}}.$$

Here the latter map is defined by compatible family of evaluation maps for each \mathbf{C}_i .

The following result is taken from [Gai5].

Proposition A.13.4. *Let $i \mapsto C_i$, C and \bar{C} be as in Construction A.13.3.*

- (1) *If C is dualizable, then (A.13.1) realizes \bar{C} as the dual of C .*
- (2) *C is dualizable if and only if, for every $D \in \text{ShvCat}_{\mathcal{Y}}$, the tautological map:*

$$D \otimes_{\text{QCoh}_{\mathcal{Y}}} \bar{C} \rightarrow \lim_{i \in \mathcal{I}^{op}} \left(D \otimes_{\text{QCoh}_{\mathcal{Y}}} C_i^\vee \right) \quad (\text{A.13.2})$$

is an equivalence.

- (3) *If each functor $C_i \rightarrow C_j$ admits a right adjoint in $\text{ShvCat}_{\mathcal{Y}}$, then C is dualizable.*

Proof. Suppose first that C is dualizable.

For every $i \in \mathcal{I}$ the coevaluation for C_i gives the canonical map:

$$\text{QCoh}_{\mathcal{Y}} \rightarrow C_i^\vee \otimes_{\text{QCoh}_{\mathcal{Y}}} C_i \rightarrow C_i^\vee \otimes_{\text{QCoh}_{\mathcal{Y}}} C.$$

These maps are compatible as i varies, and therefore we obtain the map:

$$\text{QCoh}_{\mathcal{Y}} \rightarrow \lim_{i \in \mathcal{I}^{op}} \left(C_i^\vee \otimes_{\text{QCoh}_{\mathcal{Y}}} C \right). \quad (\text{A.13.3})$$

Because C is dualizable, Lemma A.13.2 gives:

$$\left(\lim_{i \in \mathcal{I}^{op}} C_i^\vee \right) \otimes_{\text{QCoh}_{\mathcal{Y}}} C \xrightarrow{\cong} \lim_{i \in \mathcal{I}^{op}} \left(C_i^\vee \otimes_{\text{QCoh}_{\mathcal{Y}}} C \right) \quad (\text{A.13.4})$$

so (A.13.3) gives a coevaluation map, which one easily sees defines a duality datum alongside the evaluation pairing above. This completes the proof of (1).

For (2), suppose first that C is dualizable. By (1), we have $\bar{C} = C^\vee$. Therefore, we see that for any $D_1, D_2 \in \text{ShvCat}_{\mathcal{Y}}$, we have:

$$\begin{aligned} \text{Hom}(D_2, D_1 \otimes_{\text{QCoh}_{\mathcal{Y}}} \bar{C}) &= \text{Hom}(D_2 \otimes_{\text{QCoh}_{\mathcal{Y}}} C, D_1) = \text{Hom} \left(\text{colim}_{i \in \mathcal{I}} (D_2 \otimes_{\text{QCoh}_{\mathcal{Y}}} C_i), D_1 \right) = \\ &= \lim_{i \in \mathcal{I}^{op}} \text{Hom}(D_2 \otimes_{\text{QCoh}_{\mathcal{Y}}} C_i, D_1) = \lim_{i \in \mathcal{I}^{op}} \text{Hom}(D_2, D_1 \otimes_{\text{QCoh}_{\mathcal{Y}}} C_i^\vee) \end{aligned}$$

as desired.

For (3), note that each $C_i^\vee \rightarrow C_j^\vee$ then admits a left adjoint, and the limit defining C can be computed as the colimit of these categories. Now the hypothesis (2) is obviously satisfied. \square

APPENDIX B. THE TWISTED ARROW CONSTRUCTION AND CORRESPONDENCES

B.1. This appendix explains how to map into a category \mathcal{C}_{corr} of correspondences in \mathcal{C} . The desired answer is that giving a functor $\mathcal{D} \rightarrow \mathcal{C}_{corr}$ is the same as giving a functor from the *twisted arrow category* $\text{Tw}(\mathcal{D})$ of \mathcal{D} to \mathcal{C} with a certain property (formulated in §B.9 below).

However, there is a slight annoyance here: such a result should be formulated as an adjunction, and the domain and codomain of these functors needs to be treated carefully: correspondences are defined only for categories with fiber products, while $\text{Tw}(\mathcal{C})$ generally does not have fiber products, even if \mathcal{C} does (it needs to have pushouts as well).

Fortunately, this problem is essentially solved in [GR]. We describe their solution and construct this adjunction in what follows.

Presumably this material is well-known to specialists, but we are unaware of a reference. The main construction of this section was found independently by Nick Rozenblyum.

Remark B.1.1. This material plays a purely technical role; it is only used in the main construction of §8.

B.2. Twisted arrows. Let \mathcal{C} be a category.

We define a simplicial groupoid $[n] \mapsto \mathbf{Tw}_{[n]}(\mathcal{C})$ by taking n -simplices the groupoid of diagrams:

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & \dots & \longrightarrow & X_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_1 & \longleftarrow & \dots & \longleftarrow & Y_n \end{array}$$

in \mathcal{C} , as equipped with its obvious simplicial structure.

More precisely: for a finite totally ordered set I , let I^{op} denote the same set with the opposite ordering. We have a functor:

$$\begin{aligned} \Delta^{op} &\rightarrow \Delta^{op} \\ I &\mapsto I * I^{op} \end{aligned}$$

with the operation $*$ being the join (alias: concatenation) of two ordered sets.

The twisted arrow construction is more often given as composition with this endofunctor. This construction defines a complete Segal space $\mathbf{Tw}(\mathcal{C})$.

Remark B.2.1. One can show that $\mathbf{Tw}(\mathcal{C})$ coincides with the twisted arrow category of \mathcal{C} as defined in [Lur2].

Remark B.2.2. Note that the groupoid $\mathbf{Tw}_{[n]}(\mathcal{C})$ is canonically equivalent to the groupoid of composable morphisms:

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n \rightarrow Y_n \rightarrow Y_{n-1} \rightarrow \dots \rightarrow Y_0$$

in \mathcal{C} .

B.3. Categories with directions. We will need the following notion from [GR].

A *category with directions* is a category \mathcal{C} equipped with two classes (*hor*, *vert*) of morphisms in \mathcal{C} , called *horizontal* and *vertical* respectively, such that:

- (1) Equivalences are both horizontal and vertical.
- (2) Horizontal and vertical morphisms are closed under compositions.
- (3) Given $X \rightarrow Y$ horizontal and $Z \rightarrow Y$ vertical, their Cartesian product $X \times_Y Z$ exists, with the map $X \times_Y Z \rightarrow Z$ (resp. $X \times_Y Z \rightarrow X$) horizontal (resp. vertical).

Categories with directions form a category \mathbf{Cat}_{dir} with morphisms functors preserving horizontal and vertical arrows and preserving Cartesian products of diagrams $X \rightarrow Z \leftarrow Y$ with $X \rightarrow Y$ horizontal and $Z \rightarrow Y$ vertical.

Example B.3.1. Any category can be regarded as a category with directions in which horizontal arrows are allowed to be arbitrary and vertical arrows are required to be equivalences. This construction defines a fully-faithful functor $\mathbf{Cat} \hookrightarrow \mathbf{Cat}_{dir}$.

Example B.3.2. If \mathcal{C} admits fiber products, we can take horizontal and vertical maps to both be arbitrary morphisms in \mathcal{C} .

B.4. Let \mathcal{C} be a category. We will construct on $\text{Tw}(\mathcal{C})$ a canonical structure of category with directions.

We say that a morphism:

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_1 \end{array}$$

in $\text{Tw}(\mathcal{C})$ is *horizontal* if $Y_1 \rightarrow Y_0$ is an equivalence, and *vertical* if $X_0 \rightarrow X_1$ is an equivalence.

We claim that such a choice of horizontal and vertical maps in $\text{Tw}(\mathcal{C})$ define the structure of category with directions on \mathcal{C} .

The only non-trivial condition is the base-change one, so let us verify that one. Suppose that we are given a diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\text{id}_X} & X & \longleftarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longleftarrow & Y & \xrightarrow{\text{id}_Y} & Y \end{array}$$

in \mathcal{C} (equivalently: morphisms $W \rightarrow X \rightarrow Y \rightarrow Z$), which we regard as a diagram:

$$\left(X \rightarrow Z \right) \xrightarrow{\text{vert}} \left(X \rightarrow Y \right) \xleftarrow{\text{hor}} \left(W \rightarrow Y \right)$$

in $\text{Tw}(\mathcal{C})$. Then one immediately verifies that $W \rightarrow Z$ is the resulting fiber product.

Indeed, giving compatible maps $(A \rightarrow B)$ to $(X \rightarrow Z)$ and $(W \rightarrow Y)$ translates to giving a diagram:

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & & \searrow & \\ X & \xrightarrow{\text{id}_X} & X & \longleftarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longleftarrow & Y & \xrightarrow{\text{id}_Y} & Y \\ & \swarrow & & \searrow & \\ & & B & & \end{array}$$

which is obviously the same as giving compatible maps $A \rightarrow W$ and $Z \rightarrow B$.

We therefore see that Tw upgrades to a functor:

$$\text{Tw} : \text{Cat} \rightarrow \text{Cat}_{\text{dir}}.$$

B.5. **Grids.** We now recall the construction of correspondences following [GR].

Define the $(1, 1)$ -category $\text{Grid}_{[n]}$ to be the category associated with the partially ordered set of convex subsets of $[n]$.

Explicitly: objects of $\text{Grid}_{[n]}$ are indexed by pairs of integers (i, j) with $0 \leq i \leq j \leq n$, where i is the infimum of the corresponding subset of $[n]$ and j is its supremum. There is a (unique) morphism $(i, j) \rightarrow (i', j')$ if and only if $i' \leq i$ and $j \leq j'$.

An inclusion $S \subseteq T \subseteq [n]$ is said to be *horizontal* if $\inf(S) = \inf(T)$ and *vertical* if $\sup(S) = \sup(T)$ (see (B.6.1) for the reason).

B.6. Fix a category with directions $(\mathcal{C}, \text{hor}, \text{vert})$.

Define the groupoid $\text{Grid}_{[n]; \text{hor}, \text{vert}}^w(\mathcal{C})$ of *weak n -grids* in \mathcal{C} as the groupoid of functors $\text{Grid}_{[n]}^{\text{op}} \rightarrow \mathcal{C}$ sending horizontal arrows in $\text{Grid}_{[n]}$ to horizontal arrows in \mathcal{C} , and similarly for vertical arrows.

Weak n -grids can be identified with diagrams:

$$\begin{array}{ccccccc}
 X_{0,n} & \longrightarrow & X_{1,n} & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & X_{n,n} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 X_{0,n-1} & \longrightarrow & X_{1,n-1} & \longrightarrow & \cdots & \longrightarrow & X_{n-1,n-1} & & \\
 \downarrow & & \downarrow & & \downarrow & & & & \\
 \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & & & & \\
 \downarrow & & \downarrow & & \downarrow & & & & \\
 X_{0,1} & \longrightarrow & X_{1,1} & & & & & & \\
 \downarrow & & & & & & & & \\
 X_{0,0} & & & & & & & &
 \end{array} \tag{B.6.1}$$

in \mathcal{C} with the graphically horizontal arrows horizontal in \mathcal{C} and similarly for vertical arrows.

We say that a weak n -grid is an *n -grid* if each of the $(1 + \dots + (n-1))$ -commutative squares in (B.6.1) is Cartesian. We denote the groupoid of n -grids by $\text{Grid}_{[n]; \text{hor}, \text{vert}}(\mathcal{C})$.

As in [GR], $[n] \mapsto \text{Grid}_{[n]}(\mathcal{C})$ is a complete Segal space: the Segal condition is clear, and completeness translates to the statement that a correspondence is an equivalence if and only if each of its horizontal and vertical components are equivalences in \mathcal{C} . We will denote this category by $\mathcal{C}_{\text{corr}; \text{hor}, \text{vert}}$.

Example B.6.1. In Example B.3.1, we obtain the category \mathcal{C} again. In Example B.3.2, we obtain the category $\mathcal{C}_{\text{corr}}$.

B.7. Let \mathcal{C} be a category with directions. We will construct a canonical functor:

$$\text{Tw}(\mathcal{C}_{\text{corr}; \text{hor}, \text{vert}}) \rightarrow \mathcal{C} \tag{B.7.1}$$

of categories with directions.

We will do this at the level of Segal groupoids. As in Remark B.2.2, the n -simplices of $\text{Tw}(\mathcal{C}_{\text{corr}; \text{hor}, \text{vert}})$ are given by diagrams:

$$\begin{array}{ccccccc}
 X_{0,2n+1} & \longrightarrow & X_{1,2n+1} & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & X_{2n+1,2n+1} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 X_{0,2n} & \longrightarrow & X_{1,2n} & \longrightarrow & \dots & \longrightarrow & X_{2n,2n} & & \\
 \downarrow & & \downarrow & & \downarrow & & & & \\
 \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & & & & \\
 \downarrow & & \downarrow & & \downarrow & & & & \\
 X_{0,1} & \longrightarrow & X_{1,1} & & & & & & \\
 \downarrow & & & & & & & & \\
 X_{0,0} & & & & & & & &
 \end{array} \tag{B.7.2}$$

with all graphically horizontal arrows horizontal, similarly for vertical arrows, and all squares Cartesian. We then map this diagram to the n -composable arrows in \mathcal{C} :

$$X_{0,2n+1} \rightarrow X_{1,2n} \rightarrow \dots \rightarrow X_{n,n+1}.$$

One easily sees that this is compatible with simplicial structures as desired and therefore defines the desired functor (B.7.1).

Let us check that this functor is actually a functor of categories with directions.

An arrow:

$$\left(\begin{array}{ccc} H_1 & \xrightarrow{hor} & Y_1 \\ \downarrow vert & & \\ X_1 & & \end{array} \right) \rightarrow \left(\begin{array}{ccc} H_2 & \xrightarrow{hor} & Y_2 \\ \downarrow vert & & \\ X_1 & & \end{array} \right)$$

in $\text{Tw}(\mathcal{C}_{corr;hor,vert})$ is the datum of a diagram:

$$\begin{array}{ccccc}
 X_1 & \xleftarrow{vert} & H_1 & \xrightarrow{hor} & Y_1 \\
 \uparrow vert & & & & \uparrow hor \\
 W & & & & Z \\
 \downarrow hor & & & & \downarrow vert \\
 X_2 & \xleftarrow{vert} & H_2 & \xrightarrow{hor} & Y_2
 \end{array} \tag{B.7.3}$$

plus an isomorphism:

$$H_1 \simeq W \times_{X_2} H_2 \times_{Y_2} Z \tag{B.7.4}$$

as objects over both X_1 and Y_1 .

We draw the diagram (B.7.3) as in (B.7.2):

$$\begin{array}{ccccccc}
H_1 & \longrightarrow & H_2 \times Z & \longrightarrow & Z & \longrightarrow & Y_1. \\
\downarrow & & \downarrow Y_2 & & \downarrow & & \\
W \times_{X_2} H_2 & \longrightarrow & H_2 & \longrightarrow & Y_2 & & \\
\downarrow & & \downarrow & & & & \\
W & \longrightarrow & X_2 & & & & \\
\downarrow & & & & & & \\
X_1 & & & & & &
\end{array}$$

We see that this diagram maps to the map $H_1 \rightarrow H_2$ in \mathcal{C} . Note that the map $H_1 \rightarrow H_2$ is defined by (B.7.4).

Now, the diagram (B.7.3) is horizontal if the correspondence Z is an equivalence, i.e., if both maps $Z \rightarrow Y_1$ and $Z \rightarrow Y_2$ are equivalences.

Then we have an isomorphism $H_1 \simeq W \times_{X_2} H_2$. Therefore, we see that the morphism $H_1 \rightarrow H_2$ is horizontal in this case, since $W \rightarrow X_2$ is horizontal and we are base-changing along the vertical map $H_2 \rightarrow X_2$.

B.8. Next, we will construct a canonical map:

$$\mathcal{C} \rightarrow \mathrm{Tw}(\mathcal{C})_{\mathrm{corr};\mathrm{hor},\mathrm{vert}} \quad (\mathrm{B}.8.1)$$

with *hor* and *vert* defined as in §B.4, i.e., for any twisted arrow category.

We map n -composable arrows:

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n$$

in \mathcal{C} to the diagram (B.6.1) with $X_{i,j}$ the induced morphism $(X_i \rightarrow X_j) \in \mathrm{Tw}(\mathcal{C})$, i.e., the diagram:

$$\begin{array}{ccccccc}
(X_0 \rightarrow X_n) & \longrightarrow & (X_1 \rightarrow X_n) & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & (X_n \xrightarrow{\mathrm{id}} X_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
(X_0 \rightarrow X_{n-1}) & \longrightarrow & (X_1 \rightarrow X_{n-1}) & \longrightarrow & \dots & \longrightarrow & (X_{n-1} \xrightarrow{\mathrm{id}} X_{n-1}) & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & & & & \\
\downarrow & & \downarrow & & & & & & \\
(X_0 \rightarrow X_1) & \longrightarrow & (X_1 \xrightarrow{\mathrm{id}} X_1) & & & & & & \\
\downarrow & & & & & & & & \\
(X_0 \xrightarrow{\mathrm{id}} X_0) & & & & & & & &
\end{array}$$

in $\mathrm{Tw}(\mathcal{C})$. Note that all the graphically horizontal maps here are actually horizontal in $\mathrm{Tw}(\mathcal{C})$, and similarly for vertical maps.

This construction is compatible with simplicial structures and therefore defines the desired functor (B.8.1).

B.9. Note that the morphisms (B.7.1) and (B.8.1) are functorial in \mathcal{C} . One readily verifies that they define the unit and counit of an adjunction:

$$\mathrm{Cat} \begin{array}{c} \xrightarrow{\mathrm{Tw}(-)} \\ \xleftarrow{(-)_{\mathrm{corr};\mathrm{hor},\mathrm{vert}}} \end{array} \mathrm{Cat}_{\mathrm{dir}}.$$

In particular, we see that for a category \mathcal{C} with fiber products and a category \mathcal{D} , we have canonical identifications of the category of functors $\mathcal{D} \rightarrow \mathcal{C}_{\mathrm{corr}}$ and the category of functors $\mathrm{Tw}(\mathcal{D}) \rightarrow \mathcal{C}$ such that, for every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{D} , the square:

$$\begin{array}{ccc} (X \xrightarrow{g \circ f} Z) & \longrightarrow & (Y \xrightarrow{g} Z) \\ \downarrow & & \downarrow \\ (X \xrightarrow{f} Y) & \longrightarrow & (Y \xrightarrow{\mathrm{id}_Y} Y) \end{array}$$

in $\mathrm{Tw}(\mathcal{D})$ maps to a Cartesian square in \mathcal{C} . Indeed, unwinding the definitions, we find that this condition is equivalent to the requirement that those Cartesian squares in $\mathrm{Tw}(\mathcal{D})$ that are the base-change of a horizontal map by a vertical map should map to Cartesian squares in \mathcal{C} .

Remark B.9.1. The functors obviously commute with products of categories (where the product of categories with directions is a category with directions in the obvious way), and therefore we have similar endofunctors e.g. of the category of symmetric monoidal categories, and a similar adjunction.

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