NON-VANISHING OF GEOMETRIC WHITTAKER COEFFICIENTS FOR REDUCTIVE GROUPS

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Abstract. We prove that cuspidal automorphic D-modules have non-vanishing Whittaker coefficients, generalizing known results in the geometric Langlands program from GL_n to general reductive groups. The key tool is a microlocal interpretation of Whittaker coefficients.

We establish various exactness properties in the geometric Langlands context that may be of independent interest. Specifically, we show Hecke functors are t-exact on the category of tempered D-modules, strengthening a classical result of Gaitsgory (with different hypotheses) for GL_n. We also show that Whittaker coefficient functors are t-exact for sheaves with nilpotent singular support. An additional consequence of our results is that the tempered, restricted geometric Langlands conjecture must be t-exact.

We apply our results to show that for suitably irreducible local systems, Whittaker-normalized Hecke eigensheaves are perverse sheaves that are irreducible on each connected component of Bun_G.

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Date: March 17th, 2022. Last updated: May 14, 2024.

1. Introduction

1.1. **A mystery.** Although this paper is concerned with automorphic sheaves and not with automorphic forms, our motivation comes from phenomena more easily witnessed in the latter context. Therefore, we begin our story on the upper half plane.

1.1.1. *Background on modular forms.* We start with some elementary recollections about modular forms (say: holomorphic, of level 1, without nebentypus, of arbitrary weight).

Recall that such a modular form is a holomorphic function f on the upper half plane $\mathbf{H}$, satisfying a family of functional equations. Among these is the relation $f(\tau+1) = f(\tau)$ for $\tau \in \mathbf{H}$. Moreover, as a function on the analytic punctured disc function on:

$$\mathbf{H}/\{\tau \sim \tau+1\} \overset{\tau\mapsto\exp(2\pi i\cdot\tau)}{\simeq} \{0 < |q| < 1\}$$

there is a requirement that $f(q)$ extend to a holomorphic function over the puncture at $q = 0$. It follows that $f(q)$ can be expanded as a power series:

$$f(q) = \sum_{n\geq 0} a_n q^n.$$

We remind that the coefficients a_n in the q-expansion are the fundamental numerical invariants in the theory of modular forms.

Remark 1.1.1.1. Recall that f is a *cusp form* if $a_0 = 0$. It is manifest that for a non-zero cusp form f, there exists an $n \geq 1$ such that $a_n \neq 0$. More generally, non-constant modular forms have some $a_n \neq 0$ for $n \geq 1$.

1.1.2. *Adèlic interpretation.* It is not our purpose to review the construction of functions on adèlic groups from modular forms. However, we briefly state the outcomes.

We let $G = PGL_2$ and let $N = \mathbf{G}_a$ denote the radical of its standard Borel and $T = \mathbf{G}_m$ be its standard Cartan.

- For f as above, there is an associated function $\widetilde{f}$ on $G(\mathbf{A_Q})$.
- The function $\widetilde{f}$ is invariant under the left action of $G(\mathbf{Q})$ and the right action of $G(\mathbf{A}_{\mathbf{Q}}^{\mathrm{int}})$, where $\mathbf{A}_{\mathbf{Q}}^{\mathrm{int}} := \widehat{\mathbf{Z}} \subseteq \mathbf{A_Q}$ is the subring of integral adèles.

- The coefficient a_0 of f is the *constant term*:[1]

$$a_0 = \int_{N(\mathbf{Q})\backslash N(\mathbf{A_Q})} \widetilde{f}(\gamma)d\gamma.$$

- The coefficient a_n of f is *essentially* the *Whittaker coefficient*:

$$a_n \sim \int_{N(\mathbf{Q})\backslash N(\mathbf{A_Q})} \widetilde{f}(\gamma\alpha_n)\cdot\psi(-\gamma)d\gamma. \tag{1.1.1}$$

  Here $\alpha_n \in T(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}}) = \mathbf{A}_{\mathbf{Q}}^{\mathrm{fin},\times}$ is n (considered as a finite idèle, living in $PGL_2(\mathbf{A_Q})$), and ψ is the standard character of $\mathbf{A_Q} = N(\mathbf{A})$ vanishing on $\mathbf{Q}$ and $\widehat{\mathbf{Z}}$. The symbol $\sim$ indicates that we have omitted a normalizing factor of Archimedean nature; see [Del] Proposition 2.5.3.2 or [Gel] Lemma 3.6 for precise assertions.

We refer to [Del] and [Gel] for detailed derivations of the above dictionary.

1.1.3. *Generalization to reductive groups.* The notions from the previous section make sense for general reductive groups G over global fields F. There are automorphic forms, and they have constant terms indexed by proper parabolic subgroups of G. A cusp form is one with vanishing constant terms.

Similarly, there are Whittaker coefficients. When our automorphic forms are unramified at finite places, these coefficients are[2] indexed by divisors on[3] "Spec" of the ring of integers of the global field, where these divisors take values in the set $\check{\Lambda}^+$ of dominant coweights for G.[4]

Then there is a natural question, attempting to generalize the naive observation from §1.1.1:

Question. For a non-zero cusp form f on $G(\mathbf{A}_F)$, is some Whittaker coefficient of f non-zero?

The easy argument from §1.1.1 can readily be adapted to the adèlic setting to give a positive answer for $(P)GL_2$. This argument adapts more generally to GL_n: this is related to the strong multiplicity one theorem, and uses the special *mirabolic* subgroup of GL_n.

However, the answer is *no* for general G. It fails already for SL_2 for silly[5] reasons, and it fails for GSp_4 for serious reasons (see [HPS]). There is a conjecture due to Shahidi that a tempered L-packet of automorphic representations has a unique representative with non-zero Whittaker coefficients (see the discussion in the introduction to [Sha]).

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[1]More conceptually, the *constant term* of f should be thought of as a function on $T(\mathbf{Q})\backslash T(\mathbf{A_Q})/T(\mathbf{A}_{\mathbf{Q}}^{\mathrm{int}}) \simeq \mathbf{R}^{>0}$. It happens to be constant in the holomorphic case; but this good fortune does not occur for non-holomorphic modular forms, where one needs to consider the constant term as a function on $\mathbf{R}^{>0}$.

[2]This is not quite accurate; we can get away with it only because of the simplicity of holomorphic modular forms. One needs to also allow the insertion of elements of the group at infinite places in general; this is serious for Maass forms, or automorphic forms for other groups. The easiest correction is to consider Whittaker *functions* on $G(\mathbf{A})$ rather than mere Whittaker *coefficients* (which are values of the Whittaker function at particular adèlic points; the above formula gives the values when we put the identity at the infinite place of $\mathbf{Q}$).

To simplify the exposition (particularly since we will eventually be concerned with everywhere unramified automorphic forms/ sheaves over function fields, we do not further emphasize this (important) point.

[3]The scare quotes indicate that for a global field of positive characteristic, we take the corresponding smooth projective curve.

[4]For instance, for $G = PGL_2$, $\check{\Lambda}^+ = \mathbf{Z}^{\geq 0}$. Note that a $\mathbf{Z}^{\geq 0}$-valued (i.e., effective) divisor on Spec($\mathbf{Z}$) is equivalent data to a number $n \geq 1$: $D = \sum k_p[p]$ corresponds to $n = \prod p^{k_p}$.

[5]Namely, the torus does not act transitively on the set of characters.

Roughly speaking, one can think of this failure as the source of the many complications in the theory of automorphic forms for reductive groups beyond GL_n, and for much of our ignorance in the subject.

1.1.4. *A vague statement of the problem.* Broadly, the problem we consider is: *where do Whittaker coefficients of automorphic forms come from for $G \neq GL_n$?* In this paper, we completely settle the corresponding problem in the setting of *global geometric Langlands* in characteristic 0.

We describe the geometric context in more detail below. But here, we state one long-standing motto: *there are no L-packets in geometric Langlands.* There are various ways of arriving at this conclusion, but one is simply that it is forced by the geometric Langlands conjectures, as reviewed below.

So we may formulate the mystery stated above: why are there no L-packets geometrically? This question has been the subject of speculation in the geometric Langlands community for some time now, with many possible answers having been suggested. The purpose of this paper is to provide a first definite answer this question.

1.2. **The geometric setting.** We now survey the role of Whittaker coefficients in geometric Langlands and state some of our main results.

1.2.1. *Notation.* We work over a field k of characteristic zero. We fix X a smooth, geometrically connected projective curve over k.[6]

We let G be a split reductive group over k with Langlands dual group $\check{G}$. We let B be a Borel in G with unipotent radical N. We let e.g. Bun_G denote the moduli stack of G-bundles on X, and $\mathrm{LS}_{\check{G}}$ the moduli stack of $\check{G}$-bundles on X with connections, i.e., *de Rham* $\check{G}$-local systems on X.

1.2.2. *The geometric Langlands conjecture (after Beilinson-Drinfeld, Arinkin-Gaitsgory, and Gaitsgory).* Recall the statement of the geometric Langlands conjecture of Beilinson-Drinfeld (given in this form by Arinkin-Gaitsgory [AG]):

$$\mathbb{L}_G : D(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}) \simeq \mathsf{Ind}(\mathsf{Coh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}})).$$

Here we refer to [AG] for discussion of the right hand side; we simply say that $\mathsf{Coh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}) \subseteq \mathsf{Coh}(\mathrm{LS}_{\check{G}})$ is a certain subcategory of the DG (derived) category of bounded complexes of coherent sheaves; we are denoting by $\mathcal{N}ilp_{\mathrm{spec}} \subseteq T^*[-1]\,\mathrm{LS}_{\check{G}}$ the *spectral global nilpotent cone* considered in [AG].

1.2.3. There are many compatibilities the above equivalence is supposed to satisfy; see [Gai6] for an overdetermined list. Here is a key one, called the *Whittaker normalization.*

There is a functor:

$$\mathrm{coeff} : D(\mathrm{Bun}_G) \to \mathsf{Vect}$$

of *first*[7] *Whittaker coefficient*; we remind that in the categorical framework, vector spaces are considered analogues of numbers in the classical setting. This functor is a precise geometric analogue

[6] In the body of the paper, we assume at times that X admits a k-point $x \in X(k)$. This is a lazy crutch; our main theorems are verifiable after finite degree field extensions. Therefore, in the body of the paper, we sometimes allow ourselves to ignore the fact that this is a genuine additional hypothesis on X if k is not algebraically closed.

[7] We are ambivalent about the indexing here, but use this terminology in the introduction. See Remark 5.2.1.2.

of the integral (1.1.1) for $n = 1$. Roughly speaking, one pulls back to Bun_N, tensors with an Artin-Schreier/exponential sheaf (relative to a non-degenerate character), and then pushes forward to a point; we refer to §5 for details (including normalizations on cohomological shifts).

Then the diagram:

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[rr, "\mathbb{L}_G", "\simeq"'] \arrow[dr, "\mathrm{coeff}"'] & & \mathsf{IndCoh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}) \arrow[dl, "\Gamma(\mathrm{LS}_{\check{G}}, -)"] \\ & \mathsf{Vect} & \end{tikzcd}$$

is supposed to commute.

Remark 1.2.3.1. For an irreducible local system $\sigma \in \mathrm{LS}_{\check{G}}(k)$, let δ_σ denote the skyscraper sheaf at this point. Geometric Langlands predicts that there is an object $\mathrm{Aut}_\sigma \in D(\mathrm{Bun}_G)$ corresponding to it, which is the corresponding automorphic eigensheaf. The Whittaker normalization here should yield an isomorphism $\mathrm{coeff}(\mathrm{Aut}_\sigma) \simeq k \in \mathsf{Vect}$, pinning down all ambiguity of the choice of eigensheaf. This may be compared to the classical setting of modular forms, where one normalizes a cuspidal eigenform by requiring $a_1 = 1$.

1.2.4. In the geometric setting, there are additional Whittaker coefficients, analogous to the a_n for other n's. The reader may turn to §5 for the construction of functors $\mathrm{coeff}_D : D(\mathrm{Bun}_G) \to \mathsf{Vect}$ indexed by $\check{\Lambda}^+$-valued divisors D on X; for a smarter (and more conceptual) construction, see [Gai6] §5.8.

Below, we describe an alternative construction that is more easily stated.

Gaitsgory has shown [Gai2] that there is a canonical action of $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$ on $D(\mathrm{Bun}_G)$ refining the Hecke action; this is the *spectral decomposition* of the automorphic category $D(\mathrm{Bun}_G)$.

It follows that there is a unique $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$-linear functor functor:

$$\mathrm{coeff}^{\mathrm{enh}} : D(\mathrm{Bun}_G) \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$$

fitting into a commutative diagram:

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}^{\mathrm{enh}}"] \arrow[dr, "\mathrm{coeff}"] & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[d, "\Gamma"] \\ & \mathsf{Vect}. \end{tikzcd}$$

We provide details in §10.2. As in *loc. cit.*, $\mathrm{coeff}^{\mathrm{enh}}$ "knows" all Whittaker coefficients of automorphic sheaves simultaneously (while also encoding reciprocity laws between them).

We see that the geometric Langlands equivalence must fit into a commutative diagram:

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[rr, "\mathbb{L}_G", "\simeq"'] \arrow[dr, "\mathrm{coeff}^{\mathrm{enh}}"'] & & \mathsf{IndCoh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}) \arrow[dl, "\Psi"] \\ & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) & \end{tikzcd}$$

where the functor Ψ is *almost* an equivalence (see [Gai4], [AG]).

1.2.5. *A strategy for proving the geometric Langlands equivalence.* The functor Ψ displayed above is fully faithful on compact objects (it induces the embedding $\mathsf{Coh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}) \subseteq \mathsf{Coh}(\mathrm{LS}_{\check{G}}) \subseteq \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$).

Therefore, proving the geometric Langlands equivalence amounts to showing:

- $\mathrm{coeff}^{\mathrm{enh}}$ is fully faithful on the subcategory $D(\mathrm{Bun}_G)^c \subseteq D(\mathrm{Bun}_G)$ of compact objects.
- $\mathrm{coeff}^{\mathrm{enh}}$ maps compact objects of $D(\mathrm{Bun}_G)$ onto $\mathsf{Coh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}})$.

In [Gai6], Gaitsgory outlines a strategy for proving these claims for GL_n (cf. below), with a strategy that should adapt for general reductive G assuming some knowledge of Whittaker coefficients here. The ideas are complicated, involving degenerate Whittaker coefficients, Eisenstein series, and Kac-Moody representations. However, the basic idea in the strategy is that stated above.

Remark 1.2.5.1. The above strategy may be compared to Soergel's bimodule theory via the analogies:

Soergel theory	Geometric Langlands
The functor $\mathbb{V}$	The functor coeff
Endomorphismensatz	The existence of $\mathrm{coeff}^{\mathrm{enh}}$
Struktursatz	Fully faithfulness of $\mathrm{coeff}^{\mathrm{enh}}$ on $D(\mathrm{Bun}_G)^c$

We remark that the bottom right entry of this table remains conjectural (beyond GL_n, see below).

1.2.6. *The GL_n case.* It is known[8] (cf. [Gai6] "Quasi-Theorem" 8.2.10, [Ber1]) that $\mathrm{coeff}^{\mathrm{enh}}$ is fully faithful on the category $D_{\mathrm{cusp}}(\mathrm{Bun}_{GL_n})$ of cuspidal D-modules *in the GL_n case.* The argument imitates the proof of the multiplicity one theorem for GL_n for automorphic forms, going through the mirabolic subgroup.

1.2.7. *The general case.* However, as for automorphic forms, we have been unable to prove anything about Whittaker coefficients for general reductive groups G.

This failure has stood as a point of some concern. For instance, number theorists often express consternation that the geometric situation is conjectured to be so different from the arithmetic situation, where cuspidal automorphic representations commonly are non-generic. One imagines that if geometric Langlands fails, it fails because the nice predictions regarding Whittaker coefficients are incompatible with some pathological example for automorphic sheaves.

Our first main theorem states that this does not occur:

Theorem A. *The functor:*

$$D_{\mathrm{cusp}}(\mathrm{Bun}_G) \subseteq D(\mathrm{Bun}_G) \xrightarrow{\mathrm{coeff}^{\mathrm{enh}}} \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$$

is conservative. That is, if $\mathcal{F} \in D_{\mathrm{cusp}}(\mathrm{Bun}_G)$ with $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}) = 0$, then $\mathcal{F} = 0$.

Remark 1.2.7.1. Applying the definition of cuspidal D-modules (and the existence of the left adjoint Eisenstein functors $\mathrm{Eis}_!$), the above is equivalent to the assertion that $D(\mathrm{Bun}_G)$ is generated under colimits by Eisenstein series D-modules for proper parabolic subgroups and Poincaré series D-modules; we refer to [Gai6] for the definitions.

[8]In geometric Langlands, this style of argument has a long lineage that we do not survey here. We refer to [FGV2] as one key example.

In other words, Theorem A asserts that the geometric representation theorist's favorite methods for producing automorphic D-modules are in fact exhaustive.

Remark 1.2.7.2. To match our knowledge in the GL_n case, we would want to know that the functor of Theorem A is fully faithful; this paper does not settle that question.

1.2.8. *Tempered D-modules.* In fact, we prove a stronger result that Theorem A; it is more technical to state, but optimal.

Fix a point $x \in X(k)$. Using derived Satake, Arinkin-Gaitsgory defined a subcategory:

$$D(\mathrm{Bun}_G)^{\mathrm{anti-temp}} \subseteq D(\mathrm{Bun}_G)$$

of *anti-tempered* objects; the terminology is taken from [Ber3] §2. The embedding here admits a left adjoint. There is a certain quotient category $D(\mathrm{Bun}_G)^{\mathrm{temp}}$.

A priori, the above definitions depend on the point $x \in X(k)$; in [FR], we showed the category is actually independent of this choice in a strong sense; this justifies omitting x from the notation.

According to Arinkin-Gaitsgory, under geometric Langlands, the quotient $D(\mathrm{Bun}_G)^{\mathrm{temp}}$ should identify with the quotient $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$ of $\mathsf{IndCoh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}})$.

It is straightforward to see that $\mathrm{coeff}^{\mathrm{enh}}$ factors through $D(\mathrm{Bun}_G)^{\mathrm{temp}}$. Therefore, by the logic of §1.2.5, one expects the induced functor:

$$D(\mathrm{Bun}_G)^{\mathrm{temp}} \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$$

to be an equivalence; this is the *tempered geometric Langlands* conjecture.

We prove:

Theorem B. *The above functor*

$$\mathrm{coeff}^{\mathrm{enh}} : D(\mathrm{Bun}_G)^{\mathrm{temp}} \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$$

is conservative.

This result appears as Theorem 10.3.3.1 in the body of the paper.

By [Ber3], the composition:

$$D_{\mathrm{cusp}}(\mathrm{Bun}_G) \subseteq D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)^{\mathrm{temp}}$$

is fully faithful (more precisely, Beraldo shows $D_{\mathrm{cusp}}(\mathrm{Bun}_G)$ is left orthogonal to $D(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$).

Therefore, Theorem B implies Theorem A. We focus our attention on the latter result in the remainder of this introduction.

1.3. **Informal overview.** Our work contains other new, intermediate results of independent interest; we detail them later in the introduction. First, we informally describe their context, and the overall setting for our proof of Theorem B.

1.3.1. First, the recent work [AGKRRV1] provides a new set of tools for studying $D(\mathrm{Bun}_G)$. In effect, §20 of *loc. cit.* reduces the study of $D(\mathrm{Bun}_G)$ to its subcategory $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ of *D-modules with nilpotent singular support* on Bun_G.

The methods of [AGKRRV1] allow us to reduce Theorem B to instead showing:

Theorem C. *The composition:*

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \subseteq D(\mathrm{Bun}_G) \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$$

is conservative.

The reader can find the details on the reduction of Theorem B to Theorem C in §10.3.

A nice feature of $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ is that its objects are (colimits of) *holonomic D-modules with regular singularities.* These are the objects with which the Riemann-Hilbert correspondence is concerned, so one may use additional sheaf-theoretic tools to study them, see e.g. [KK], [Gin], and [KS]; broadly speaking, these additional tools are parts of *microlocal geometry.*

1.3.2. *Irregular singular support.* Let us return to the case where $G = PGL_2$. Recall from Remark 1.1.1.1 that modular forms with vanishing Whittaker coefficients are constant. In the geometric setting, one similarly can show that $D(\mathrm{Bun}_{PGL_2})^{\mathrm{anti-temp}}$ consists of objects with constant cohomologies, equivalently (by simple-connectivity in this case), with lisse cohomologies.

One can say this differently: $D(\mathrm{Bun}_{PGL_2})^{\mathrm{anti-temp}}$ is exactly the category of D-modules with singular support in the zero section.

Our starting point is the idea that this should generalize: for general G, $D(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$ should be the category with *irregular* singular support.

At the very least, we obtain a similar result in the nilpotent setting:

Theorem D. *The category* $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$ *coincides with* $\mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}(\mathrm{Bun}_G)$*, the subcategory of objects with* irregular nilpotent singular support.

We will refine this result in our later discussion. But, roughly speaking, the overall strategy is to connect both (anti-)temperedness and Whittaker coefficients to microlocal properties of sheaves, thereby proving Theorem B.

1.4. **Results for nilpotent sheaves.** We obtain some striking results for Whittaker coefficients of sheaves with nilpotent singular support, that we describe presently.

1.4.1. *Some motivating geometry.* Recall that there is a characteristic polynomial map $\chi : \mathfrak{g}/G \to \mathfrak{g}//G$; here the left hand side is the stack quotient and the right hand side is the GIT quotient. The nilpotent cone $\mathcal{N}$ is characterized by the formula $\mathcal{N}/G = \chi^{-1}(0)$. The Kostant slice defines a section of χ. Clearly the Kostant slice intersects $\mathcal{N}/G$ in a single point.

The above story has a global analogue. The role of $\mathfrak{g}/G$ is played by $\mathrm{Higgs}_G = T^* \mathrm{Bun}_G$, the space of Higgs bundles for G. The role of $\mathfrak{g}//G$ is played by the Hitchin base, while χ is replaced by the Hitchin fibration. The role of $\mathcal{N}/G$ is played by the *global nilpotent cone* $\mathcal{N}ilp \subseteq T^* \mathrm{Bun}_G$, which is by definition the zero fiber of the Hitchin fibration. The Kostant slice admits a global analogue (called $\mathfrak{g}$–0-opers in [BD] §3.1.14).

Therefore, the global Kostant slice intersects $\mathcal{N}ilp$ at a distinguished point, which we label f^{glob} in §2.5.6. One can show that this point lies in the smooth locus of $\mathcal{N}ilp$; therefore, there is a unique irreducible component $\mathcal{N}ilp^{\mathrm{Kos}}$ of $\mathcal{N}ilp$ containing f^{glob}.

We remark that when the genus of the curve is > 1, the Kostant slice and $\mathcal{N}ilp$ are both Lagrangians in $T^* \operatorname{Bun}_G$.

1.4.2. Finally, we make one more remark, connecting the above to Whittaker coefficients.

To state it precisely, we recall that it is not quite Bun_N that appears in the definition of coeff, but a twisted form of Bun_N that we denote $\operatorname{Bun}_N^\Omega$ in §5. This form of $\operatorname{Bun}_N^\Omega$ has a canonical map $\psi : \operatorname{Bun}_N^\Omega \to \mathbf{A}^1$, which is the *Whittaker character*. This defines a Lagrangian $d\psi : \operatorname{Bun}_N^\Omega \to T^* \operatorname{Bun}_N^\Omega$. We may compose this Lagrangian with the natural Lagrangian correspondence between $T^* \operatorname{Bun}_N^\Omega$ and $T^* \operatorname{Bun}_G$; tracing the definitions, the resulting Lagrangian in $T^* \operatorname{Bun}_G$ is the global Kostant slice.

In this sense, we may view the global Kostant slice as a microlocal shadow of the functor coeff; physicists would say that the global Kostant slice is the *brane* corresponding to the functor coeff.

1.4.3. *Statement of the main result.* We can now formulate:

Theorem E. *The cohomologically shifted functor of first Whittaker coefficient:*

$$\operatorname{coeff}[\dim \operatorname{Bun}_G] : \mathsf{Shv}_{\mathcal{N}ilp}(\operatorname{Bun}_G) \to \mathsf{Vect}$$

is t-exact and commutes with Verdier duality. Moreover, for constructible $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\operatorname{Bun}_G)$*, the Euler characteristic of* $\operatorname{coeff}[\dim \operatorname{Bun}_G](\mathcal{F})$ *equals the order of the characteristic cycle at* $\mathcal{N}ilp^{\mathrm{Kos}}$.

We hope the geometry described above adequately has motivated this result. It is obtained by combining Theorem 6.1.2.1, Theorem 8.0.0.1, and Theorem 8.2.1.1 in the body of the text.

1.4.4. *A more refined picture.* The above discussion can be made more precise as follows. Suppose $\mathcal{Y}^{\mathrm{an}}$ is a complex manifold, $\Lambda \subseteq T^*\mathcal{Y}^{\mathrm{an}}$ is a closed, conical holomorphic Lagrangian; let $\Lambda^{\mathrm{sm}} \subseteq \Lambda$ be the smooth locus.

Kashiwara-Schapira[9] [KS] associate to a sheaf $\mathcal{F}$ on $\mathcal{Y}^{\mathrm{an}}$ with singular support in Λ a certain local system $\mu_\Lambda(\mathcal{F})$ on Λ^{sm}, which is a form of the *microlocalization* of $\mathcal{F}$.

The fibers of $\mu_\Lambda(\mathcal{F})$ at points of Λ^{sm} are called *microstalks*, and may be computed by suitably transverse vanishing cycles of $\mathcal{F}$. In particular, formation of microstalks is t-exact (up to shift) and commutes with Verdier duality. In addition, the Euler characteristics of these fibers are the degrees of the characteristic cycle of $\mathcal{F}$ at the given point.

Therefore, our motivation is that for sheaves with nilpotent singular supports (which, we remind, are automatically regular singular, so have Betti cousins), coeff is the microstalk at f^{glob}.

Remark 1.4.4.1. This idea is quite natural, and indeed, when we began discussing this work with others, we learned that it had been considered some time ago by others: Drinfeld advertised the idea some 20 years ago, and Nadler advertised it some 10 years ago. We are not aware of any recorded source for it.

We understand that David Nadler and Jeremy Taylor have a proof of this precise assertion, directly proving that coeff is computed by the microstalk at f^{glob}, using topological methods; this is in contrast to our methods, which use special properties of the automorphic setting.[10]

[9] This assertion is difficult to track in the stated form. A close result is [KS] Corollary 7.5.7, as well as the subsequent remark.

[10] Since the first draft of this paper was circulated, the work of Nadler-Taylor appeared: see [NT]. Their work yields an alternative proof of Theorem E that is topological in nature and does not use [Lin].

Remark 1.4.4.2. We do not actually prove that Whittaker coefficients of nilpotent sheaves *actually* are microstalks. What we prove is Theorem E as stated, which informally asserts that the coefficient functor has all the same good properties as the microstalk functor would.

Although the geometric picture described above is quite simple and feels general, we use the specific tools of geometric representation theory; specifically, we use a recent result of Kevin Lin from [Lin]. See §8 for details.

1.4.5. *Comparison to arithmetic ideas.* We note that similar principles have appeared previously in harmonic analysis. See for example [Rod] §IV Remarque 2 or [MW]. The assertion is that (under favorable hypotheses), the multiplicity of the Fourier transform of a character of a representation of a p-adic reductive group at the regular nilpotent orbit is the dimension of its Whittaker model.

It is quite enticing to make this analogy more precise.

1.5. **A result for Hecke functors.** We now state another intermediate result of independent interest.

1.5.1. *Background.* Recall that Hecke functors are not t-exact on Bun_G. For instance, a Hecke functor for a representation V acting on the constant sheaf of Bun_G (which is in a single degree by smoothness of Bun_G) tensors this constant sheaf with a *cohomologically sheared* version of V.

However, Hecke functors are not too far from exact either. For instance, one classically expects cuspidal perverse eigensheaves for irreducible local systems (and see Theorem G below). By definition, Hecke functors transform such objects by tensoring with a (classical) vector space.

1.5.2. *Statement of the result.* The above discussion is suggestive of what the obstruction to exactness is in general:

Theorem F. *(1) There is a unique t-structure on $D(\mathrm{Bun}_G)^{\mathrm{temp}}$ for which the projection $D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)^{\mathrm{temp}}$ is t-exact.*

(2) Let $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ be a representation. Then for $x \in X$, the induced Hecke functor:

$$D(\mathrm{Bun}_G)^{\mathrm{temp}} \to D(\mathrm{Bun}_G)^{\mathrm{temp}}$$

is t-exact.

(3) More generally, for V as above, the parametrized *Hecke functor:*

$$H_V : D(\mathrm{Bun}_G)^{\mathrm{temp}} \to D(\mathrm{Bun}_G)^{\mathrm{temp}} \otimes D(X)$$

is t-exact (up to shift by $1 = \dim X$).

See Theorem 7.5.0.1 and Theorem 7.7.1.1 in the body of the paper.

The proofs of the first two statements are quite direct, but seem not to have been previously observed. The third is a minor variant, except it relies on the independence of point in the definition of temperedness (in other words: in (2), it is important that the implicit point $x \in X(k)$ in the definition of the tempered category be taken to be the same as where the Hecke functors are taken). The argument from [FR] works for $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ in the ℓ-adic setting, but we are not sure how to adapt it to the full category $\mathsf{Shv}(\mathrm{Bun}_G)$ in this setting. If so, our methods would yield a proof of Theorem F in the ℓ-adic case as well.

Remark 1.5.2.1. Besides the ℓ-adic issue raised above, our argument provides an alternative to [Gai1] §2.12. We highlight that the construction in *loc. cit.* applies only for GL_n, and is the major technical point in that paper.

More explicitly: the main result of [Gai1] is the formation of a certain quotient of $D(\mathrm{Bun}_{GL_n})$ with certain favorable properties, including that Hecke functors act exactly on it. The construction in *loc. cit.* does not make sense for general G. We have provided an alternative (genuinely different) construction of a quotient category with the same favorable properties. Moreover, our arguments are substantially more direct than those in [Gai1].

With that said, our argument in [FR] uses Gaitsgory's *generalized vanishing conjecture* from [Gai2]. As explained in [Gai2], this result immediately implies the vanishing conjecture considered in [Gai1]. For this reason, we cannot say that we have found a better understanding of the (not generalized) vanishing conjecture, only of the intermediate results used in [Gai1].

1.6. **A remark on tempered Langlands.** We now describe a surprising consequence of our work for the geometric Langlands equivalence.

Roughly speaking, one is commonly taught that geometric Langlands is an equivalence of *derived* categories, not abelian categories. We explain that this is in some sense wrong; *most* of geometric Langlands can actually be understood as an equivalence of *abelian* categories.

1.6.1. It is well-known that the geometric Langlands equivalence is emphatically not exact.

First, in geometric class field theory, one finds $D(\mathrm{Bun}_{\mathbf{G}_m}) \simeq \mathsf{QCoh}(\mathrm{LS}_{\mathbf{G}_m})$. The functor is a variant on Fourier-Mukai for abelian varieties; experimentally, one finds the latter is far from exact.

Second, for non-abelian G, the constant sheaf on Bun_G should map to an object of $\mathsf{IndCoh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}})$ concentrated in cohomological degree $-\infty$ (i.e., in degrees $\leq -n$ for all n).

However, one expects some exactness properties. For instance, there are supposed to exist *perverse* eigensheaves for irreducible local systems (and see Theorem G); these correspond (up to shift) to skyscraper sheaves at smooth, irreducible points of $\mathrm{LS}_{\check{G}}$.

1.6.2. We recall the setting of *restricted* geometric Langlands considered in [AGKRRV1].

Here we expect an equivalence:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}).$$

The space $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ is defined as in *loc. cit.*

The tempered analogue should instead find an equivalence:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{temp}} \overset{\mathbb{L}_G^{\mathrm{temp}}}{\simeq} \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

We claim that our results imply this (conjectural) equivalence must be t- exact up to shift. Perhaps more concretely, this means that the composition:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp_{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \xrightarrow{\Psi} \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

should be t-exact.

Indeed, let $x \in X(k)$ be a point; this defines a $\check{G}$-torsor on $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$. By [AGKRRV1] Theorem 1.4.5, the total space of this torsor is a union of ind-affine formal schemes by an action of $\check{G}$.

It follows that its functor $\Gamma_!$ to Vect (considered in [AGKRRV1] §7) is t-exact, and that an object $\mathcal{G} \in \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$ lies in degree 0 if and only if $\Gamma_!(\mathcal{G} \otimes \mathcal{E}_{V,x}) \in \mathsf{Vect}$ is in degree 0 for all V; here $\mathcal{E}_{V,x}$ is the vector bundle on $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ defined by the pair (V, x).

On the other hand, for $\mathcal{F} \in D(\mathrm{Bun}_G)^{\mathrm{temp},\heartsuit}$, we expect $\Gamma_!(\mathbb{L}_G^{\mathrm{temp}}(\mathcal{F}) \otimes \mathcal{E}_{V,x}) \simeq \mathrm{coeff}(H_{V,x} \star \mathcal{F})$; the latter lies purely in degree $\dim \mathrm{Bun}_G$ by Theorem E and Theorem F, so $\mathbb{L}_G^{\mathrm{temp}}(\mathcal{F})$ must lie in degree $\dim \mathrm{Bun}_G$ as well.

Remark 1.6.2.1. We similarly expect that the tempered *Betti* Langlands equivalence conjecture in [BZN] is t-exact; this corresponds to fact that the Betti moduli stack of local systems is the quotient of an affine scheme by an action of $\check{G}$.

1.7. **Application to Hecke eigensheaves.** We apply our results to show the following result, which can be thought of as an unconditional realization of the philosophy of §1.6.

Theorem G. *Let σ be an irreducible $\check{G}$-local system on X.*

(1) Any Whittaker-normalized[11] *Hecke eigensheaf $\mathcal{F}_\sigma$ with eigenvalue σ is perverse. Moreover, at least if k is algebraically closed, Whittaker-normalized Hecke eigensheaves exist.*
(2) If σ is very irreducible *in the sense of §11.1.2, the restriction of $\mathcal{F}_\sigma$ to each connected component of* Bun_G *is irreducible.*

The reader will find this result as Theorem 11.1.4.1; we also refer to Remark 11.1.4.3 where it is noted that Whittaker-normalized eigensheaves are semi-simple for any (possibly not very) irreducible local system.

We note that this answers an old question. Namely, from the point of view of the categorical geometric Langlands conjectures, it is not clear why eigensheaves should be perverse or irreducible. We show that this follows from exactness (and conservativeness) properties of the Whittaker functor.

We remark that the existence of Hecke eigensheaves stated above is proved via opers and is disjoint from the methods of our paper; we have relegated the argument to Appendix A. Our contributions are more to the structure of (normalized) eigensheaves, and we include the existence argument for the sake of completeness.

1.8. **Outline of the argument.** We now outline our argument for Theorem B. The details are provided in §10.

1.8.1. First, we reduce to the corresponding statement for $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ using the technology of [AGKRRV1].

1.8.2. Let $\mathcal{N}\mathring{i}lp \subseteq \mathcal{N}ilp$ denote the open of *generically regular* nilpotent Higgs bundles.

We prove:

Theorem H. *Any object $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ that does not lie in $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$ has $\mathrm{SS}(\mathcal{F}) \cap \mathcal{N}\mathring{i}lp \neq \emptyset$.*

This result is our Theorem 4.1.0.1. The proof reduces to a parallel statement for the flag variety of the finite dimensional group G. We translate this to a statement about Lie algebra representations via Beilinson-Bernstein. Finally, we apply a theorem of Loseu [Los] regarding associated varieties of $\mathfrak{g}$-modules (and proved using ideas reminiscent of microlocal differential operators).

We wish to be clear: this is not the proof of the theorem written in The Book (in the sense of Erdős); our argument is not geometric. It would be far better to have a proof that relies only on standard properties of singular support and adapts to the ℓ-adic setting.

[11]See Remark 11.1.4.4 for our precise convention.

1.8.3. Next, we have:

Theorem I. *Suppose* $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ *satisfies* $\mathrm{SS}(\mathcal{F}) \cap \mathring{\mathcal{N}ilp} \neq \emptyset$. *Then for a point* $x \in X(k)$, *there exists a representation* $V^{\check{\lambda}} \in \mathsf{Rep}(\check{G})^{\heartsuit}$ *such that* $\mathcal{N}ilp^{\mathrm{Kos}} \subseteq \mathrm{SS}(H_{V^{\check{\lambda}},x} \star \mathcal{F})$.

This result appears in our work as Theorem 9.1.0.1. The proof uses basic geometry of the global nilpotent cone and standard properties of singular support.

1.8.4. Finally, we deduce the claim as follows.

Suppose $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ does not lie in $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\text{anti–temp}}$. We need to show that $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}) \neq 0$.

By Theorem H, we have $\mathrm{SS}(\mathcal{F}) \cap \mathring{\mathcal{N}ilp} \neq \emptyset$. By definition, it suffices to show that $\mathrm{coeff}(H_{V^{\check{\lambda}},x} \star \mathcal{F}) \neq 0$ for some $V^{\check{\lambda}}$. Therefore, by Theorem I, we are reduced to showing that $\mathrm{coeff}(\mathcal{F}) \neq 0$ when $\mathcal{N}ilp^{\mathrm{Kos}} \subseteq \mathrm{SS}(\mathcal{F})$.

But now the argument follows from Theorem E: by t-exactness, we are reduced to the case where $\mathcal{F}$ is perverse (i.e., constructible and concentrated in cohomological degree 0). In that case, the Euler characteristic of $\mathrm{coeff}(\mathcal{F})$ equals the degree of $\mathrm{CC}(\mathcal{F})$ at $\mathcal{N}ilp^{\mathrm{Kos}}$, which is non-zero by assumption.

1.8.5. We remark that Theorem D follows easily, and therefore do not prove it in the body of the paper. Here is the argument:

- $\mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}(\mathrm{Bun}_G) \subseteq \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\text{anti–temp}}$ by Theorem H, and $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\text{anti–temp}}$ equals $\mathrm{Ker}(\mathrm{coeff}^{\mathrm{enh}}|_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)})$ by Theorem C.
- By the above, it suffices to show $\mathrm{Ker}(\mathrm{coeff}^{\mathrm{enh}}|_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}) \subseteq \mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}(\mathrm{Bun}_G)$. The proof of Corollary 10.1.1.2 shows exactly this.

1.8.6. *Regarding the ℓ-adic setting.* We expect analogues of each of the theorems above to hold for the setting of ℓ-adic sheaves considered in [AGKRRV1]. In particular, we believe our overall strategy is the right one.

However, for each[12] of the above theorems, we at some point in the argument use specifics of D-modules, particularly regular holonomic D-modules/Betti perverse sheaves. This is most egregious for Theorem H, but is true at some point for every one of these results.

1.9. **Acknowledgements.** We are happy to thank Dima Arinkin, Sasha Beilinson, David Ben-Zvi, Dario Beraldo, Gurbir Dhillon, Vladimir Drinfeld, Pavel Etingof, Tony Feng, Dennis Gaitsgory, Tom Gannon, David Kazhdan, Kevin Lin, Ivan Loseu, Victor Ginzburg, Sergey Lysenko, Ivan Mirkovic, David Nadler, Nick Rozenblyum, Yiannis Sakellaridis, Will Sawin, Jeremy Taylor, and Yasha Varshavsky for generously sharing their ideas, for related collaborations, and for their encouragement.

S.R. was supported by NSF grant DMS-2101984.

2. Notation

In this section, we set up some notation that will be used throughout the paper.

[12]The first two parts of Theorem F are an exception.

2.1. **Categories.** We freely use the language of ∞-categories and higher algebra, cf. [Lur1], [Lur2], [GR2], [GR3].

We understand DG categories as k-linear stable ∞-categories. We let $\mathsf{DGCat}_{\mathrm{cont}}$ denote the category of presentable (in particular: cocomplete) DG categories with morphisms being continuous DG functors. We freely use Lurie's symmetric monoidal structure $\otimes$ on $\mathsf{DGCat}_{\mathrm{cont}}$, and the associated duality formalism.

2.2. **Categories of sheaves.** Here we set up our notation and conventions for sheaves. We refer to [GKRV] Appendix A and [AGKRRV1] Appendices D and E for details and proofs of various assertions.

2.2.1. *D-modules.* For a prestack $\mathcal{Y}$ locally almost of finite type, we let $D(\mathcal{Y})$ denote the DG category of D-modules on $\mathcal{Y}$, defined as in [GR2]. For a map $f : \mathcal{Y} \to \mathcal{Z}$, we let $f^! : D(\mathcal{Z}) \to D(\mathcal{Y})$ denote the corresponding pullback functor. If f is ind-representable, we let $f_{*,\mathrm{dR}} : D(\mathcal{Y}) \to D(\mathcal{Z})$ denote the pushforward functor.

Where defined, we let $f_!$ (resp. $f^{*,\mathrm{dR}}$) denote the left adjoints to these functors.

2.2.2. For $\mathcal{Y}$ an algebraic stack and $\mathcal{F} \in D(\mathcal{Y})$, we say that $\mathcal{F}$ is *locally compact* if for every affine scheme S and every smooth map $f : S \to \mathcal{Y}$, $f^!(\mathcal{F}) \in D(S)$ is compact.

We remind that compact objects of $D(\mathcal{Y})$ are locally compact, but the converse does not hold. For example, the constant sheaf on $\mathbf{BG}_m$ is locally compact but not compact; the same applies for the constant sheaf on a non-quasi-compact scheme. More generally, any constructible object (defined as below) is locally compact.

2.2.3. *Ind-constructible sheaves.* For S an affine scheme of finite type, we let $\mathsf{Shv}(S)^c \subseteq D(S)$ denote the subcategory of compact objects that are holonomic with regular singularities. We then let $\mathsf{Shv}(S) = \mathsf{Ind}(\mathsf{Shv}(S)^c)$; this is a full subcategory of $D(S)$.

For $\mathcal{Y}$ an algebraic stack, we let $\mathsf{Shv}(\mathcal{Y}) := \lim_{S \to \mathcal{Y}} \mathsf{Shv}(S)$ and let $\mathsf{Shv}(\mathcal{Y})^{\mathrm{constr}} := \lim_{S \to \mathcal{Y}} \mathsf{Shv}(S)^c$. In both circumstances, the limits are taken over affine schemes S mapping to $\mathcal{Y}$ and the implied functors are upper-! functors. Standard arguments allow us to replace the limit by that over the subcategory of S's mapping *smoothly* to $\mathcal{Y}$. It follows that $\mathsf{Shv}(\mathcal{Y})$ has a natural t-structure, and $\mathsf{Shv}(\mathcal{Y})^{\mathrm{constr}}$ is closed under truncations.

We refer to objects of $\mathsf{Shv}(\mathcal{Y})$ as *ind-constructible sheaves* on $\mathcal{Y}$ and objects of $\mathsf{Shv}(\mathcal{Y})^{\mathrm{constr}}$ as *constructible sheaves* on $\mathcal{Y}$.

As in [AGKRRV1] §F.2.5, we have a well-defined Verdier duality equivalence $\mathsf{Shv}(\mathcal{Y})^{\mathrm{constr,op}} \simeq \mathsf{Shv}(\mathcal{Y})^{\mathrm{constr}}$ that we denote $\mathbf{D}^{\mathrm{Verdier}}$.

Remark 2.2.3.1. Our usage of the notation Shv here is slightly different than e.g. in [AGKRRV1], where it is meant to express a certain ambivalence about the specific choice of sheaf theory. We work in the context of D-modules in characteristic 0, so do not share the ambivalence of *loc. cit.* With that said, the notation is similar in spirit.

2.2.4. *Singular support.* When $\mathcal{Y}$ is an algebraic stack and $\Lambda \subseteq T^*\mathcal{Y}$ a closed, conical subscheme, we let $D_\Lambda(\mathcal{Y}) \subseteq D(\mathcal{Y})$ denote the full subcategory of sheaves with singular support in Λ.

Similarly, we let $\mathsf{Shv}_\Lambda(\mathcal{Y}) \subseteq \mathsf{Shv}(\mathcal{Y})$ denote the corresponding full subcategory.

We again refer to [GKRV] §A.3 for definitions.

For $\mathcal{F} \in D(\mathcal{Y})$, we let $\mathrm{SS}(\mathcal{F}) \subseteq T^*\mathcal{Y}$ denote the singular support of $\mathcal{F}$, which is ind-closed in $T^*\mathcal{Y}$; cf. [AGKRRV1] §H.1.2. On occasion, for $\mathcal{F}$ constructible, we let $\mathrm{CC}(\mathcal{F})$ denote the characteristic cycle of $\mathcal{F}$.

2.3. **Lie theory.**

2.3.1. Throughout the paper, G denotes a split reductive group over k. We choose opposing Borel subgroups $B, B^- \subseteq G$ with $B \cap B^- = T$ a fixed Cartan. We let N (resp. N^-) denote the unipotent radical of B (resp. B^-).

We let Λ denote the lattice of weights of G and let $\check{\Lambda}$ denote the lattice of coweights.[13] For $\check{\lambda} \in \check{\Lambda}$ and $\mu \in \Lambda$, we let $(\check{\lambda}, \mu) \in \mathbf{Z}$ denote the pairing of the two. We let $\Lambda^+ \subseteq \Lambda$ denote the subset of dominant weights, and similarly for $\check{\Lambda}^+$.

We let $\mathcal{I}_G$ denote the set of nodes for the Dynkin diagram of G. For $i \in \mathcal{I}_G$, we let α_i denote the corresponding simple root.

We let $2\rho \in \Lambda$ denote the sum of the positive roots, and similarly for $2\check{\rho} \in \check{\Lambda}$.

We let $\check{G}$ denote the Langlands dual group of G, considered as an algebraic group over k.

2.3.2. We let $\mathfrak{g}_{\mathrm{irreg}} \subseteq \mathfrak{g}$ denote the reduced closed subscheme consisting of irregular elements.

We let $\mathcal{N} \subseteq \mathfrak{g}$ denote the nilpotent cone. We let $\mathcal{N}_{\mathrm{irreg}} := \mathcal{N} \cap \mathfrak{g}_{\mathrm{irreg}}$ denote the subscheme of irregular nilpotent elements. We let $\mathring{\mathcal{N}} \subseteq \mathcal{N}$ denote the open complement to $\mathcal{N}_{\mathrm{irreg}}$, which parametrizes of regular nilpotent elements.

2.4. **Higgs bundles.**

2.4.1. We remind that a *Higgs bundle* (on X, for the group G) is a pair $(\mathcal{P}_G, \varphi)$ where $\mathcal{P}_G$ is a G-bundle and $\varphi \in \Gamma(X, \mathfrak{g}_{\mathcal{P}_G} \otimes \Omega^1_X)$.

Recall that Higgs bundles form an algebraic stack Higgs_G, which can be written as a mapping stack:

$$\mathrm{Higgs}_G := \mathcal{M}aps(X, \mathfrak{g}/G \times \mathbf{G}_m) \underset{\mathcal{M}aps(X, \mathbf{BG}_m)}{\times} \{\Omega^1_X\}.$$

We remind that our choice[14] of κ_0 induces an isomorphism:

$$T^* \mathrm{Bun}_G \simeq \mathrm{Higgs}_G,$$

which we take for granted in the sequel.

2.4.2. *Globalization.* For $\Lambda \subseteq \mathfrak{g}$ closed, conical, and stable under the G-action, it is convenient to denote:

$$\mathrm{Higgs}_{G,\Lambda} := \mathcal{M}aps(X, \Lambda/G \times \mathbf{G}_m) \underset{\mathcal{M}aps(X, \mathbf{BG}_m)}{\times} \{\Omega^1_X\}.$$

Clearly $\mathrm{Higgs}_{G,\Lambda}$ forms a closed substack of Higgs_G.

In the special case $\Lambda = \mathcal{N}$, we let:

$$\mathcal{N}ilp := \mathrm{Higgs}_{G,\mathcal{N}}.$$

[13] Our convention here is opposite to [AGKRRV1].

[14] More canonically, $T^* \mathrm{Bun}_G$ identifies with the variant of Higgs_G with $\mathfrak{g}^\vee$ replacing $\mathfrak{g}$ everywhere. For example, this applies as well to possibly non-reductive affine algebraic groups.

To reiterate:

$$\mathcal{N}ilp \subseteq \mathrm{Higgs}_G = T^* \mathrm{Bun}_G \text{ is the global nilpotent cone.}$$

(We highlight this in part in acknowledgement that the notation does not make it easy for the reader to remember which of $\mathcal{N}$ and $\mathcal{N}ilp$ has to do with $\mathfrak{g}$ and which has to do with Higgs_G.)

2.5. **Global nilpotent cone.** We now establish some notation relating to $\mathcal{N}ilp$.

2.5.1. *Irregular nilpotent Higgs bundles.* We define:

$$\mathcal{N}ilp_{\mathrm{irreg}} := \mathrm{Higgs}_{G,\mathcal{N}_{\mathrm{irreg}}} \subseteq \mathcal{N}ilp.$$

This stack parametrizes *irregular nilpotent* Higgs bundles. Clearly $\mathcal{N}ilp_{\mathrm{irreg}} \subseteq \mathcal{N}ilp$ is a closed substack.

Example 2.5.1.1. For $G = GL_2$, we have $\mathcal{N}ilp_{\mathrm{irreg}} = \mathrm{Bun}_G$, which is embedded in Higgs_G as the zero section.

2.5.2. *Generically regular nilpotent Higgs bundles.* We let:

$$\overset{\circ}{\mathcal{N}ilp} \subseteq \mathcal{N}ilp$$

denote the open complement to $\mathcal{N}ilp_{\mathrm{irreg}}$. This is the stack of *generically regular* Higgs bundles. (We use this terminology because a point $\varphi \in \mathcal{N}ilp$ lies in $\overset{\circ}{\mathcal{N}ilp}$ if and only if there is a dense open $U \subseteq X$ over which φ is regular nilpotent.)

Example 2.5.2.1. For $G = GL_2$, $\overset{\circ}{\mathcal{N}ilp}$ parametrizes pairs $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a rank 2 vector bundle and $\varphi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ is a non-zero Higgs field with $\varphi^2 = 0$.

2.5.3. *Mapping stack notation.* Let $\mathcal{Y}$ be a stack and let $\overset{\circ}{\mathcal{Y}} \subseteq \mathcal{Y}$ be an open substack.

We let $\mathcal{M}aps_{\mathrm{nondeg}}(X, \mathcal{Y} \supseteq \overset{\circ}{\mathcal{Y}})$ denote the prestack with S-points given by maps:

$$y : X_S := X \times S \to \mathcal{Y}$$

such that there exists an open $U \subseteq X_S$ such that:

- $U \subseteq X_S$ is schematically dense.[15]
- $U \to S$ is a (necessarily flat) cover.
- $y|_U$ factors through $\overset{\circ}{\mathcal{Y}}$.

(See e.g. [Ras3] §2.9 and [Sch] §2.2.1, where similar constructions are discussed.)

2.5.4. *A Springer construction.* In the above notation, we now clearly have:

$$\overset{\circ}{\mathcal{N}ilp} = \mathcal{M}aps_{\mathrm{nondeg}}(X, \mathcal{N}/G \supseteq \overset{\circ}{\mathcal{N}}/G).$$

Let $\mathring{\mathfrak{n}} \subseteq \mathfrak{n}$ denote the open subscheme of elements of $\mathfrak{n}$ that are regular as elements of $\mathfrak{g}$. For the reader's convenience, we remind that if we choose negative simple root vectors $f_i \in \mathfrak{n}^-$ (for $i \in \mathcal{I}_G$) and let $f_i^\vee := \kappa_0(f_i, -) : \mathfrak{n} \to \mathbf{A}^1$ denote the corresponding projection onto the simple root space, then $\mathring{\mathfrak{n}} = \cap_{i \in \mathcal{I}_G} \{f_i^\vee \neq 0\}$.

We now form:

$$\widetilde{\overset{\circ}{\mathcal{N}ilp}} := \mathcal{M}aps_{\mathrm{nondeg}}(X, \mathring{\mathfrak{n}}/B \subseteq \mathfrak{n}/B).$$

[15] I.e., for $U \subseteq Z \subseteq X_S$ with $Z \subseteq X_S$ closed, we necessarily have $Z = X_S$.

Observe that there are canonical maps:

$$\widetilde{\mathring{\mathcal{N}ilp}} \to \mathring{\mathcal{N}ilp}$$

and:

$$\widetilde{\mathring{\mathcal{N}ilp}} \to \mathrm{Bun}_B \to \mathrm{Bun}_T .$$

For a coweight $\check{\lambda}$, we let:

$$\widetilde{\mathring{\mathcal{N}ilp}}^{\check{\lambda}} \subseteq \widetilde{\mathring{\mathcal{N}ilp}}$$

denote the inverse image of[16] $\mathrm{Bun}_T^{\check{\lambda}}$ along the top map.

It is easy to see[17] that the projection:

$$\widetilde{\mathring{\mathcal{N}ilp}}^{\check{\lambda}} \to \mathring{\mathcal{N}ilp}$$

is a locally closed embedding. We therefore abuse notation in letting $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$ denote the corresponding locally closed subscheme of $\mathring{\mathcal{N}ilp}$. Note that every field-valued point of $\mathring{\mathcal{N}ilp}$ lifts to $\widetilde{\mathring{\mathcal{N}ilp}}$ because $\mathfrak{n}/B \to \mathcal{N}/G$ is proper and induces an isomorphism $\mathring{\mathfrak{n}}/B \xrightarrow{\simeq} \mathring{\mathcal{N}}/G$; therefore, the various $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$ define a stratification of $\mathring{\mathcal{N}ilp}$.

---

[16] To be explicit about our conventions, $\mathrm{Bun}_T^{\check{\lambda}}$ is the component of Bun_T parametrizing T-bundles $\mathcal{P}_T$ where $\deg(\mathcal{P}_T^{\mu}) = (\mu, \check{\lambda})$ for each weight μ; here $\mathcal{P}_T^{\mu}$ is the line bundle (aka $\mathbf{G}_m$-bundle) induced by the map $\mu : T \to \mathbf{G}_m$.

[17] Namely, suppose $(\mathcal{P}_G, \phi)$ is an S-point of $\mathring{\mathcal{N}ilp}$.

For $\lambda \in \Lambda^+$, let $\mathcal{P}_G^{\lambda}$ denote the induced vector bundle and let $\phi^{\lambda} : \mathcal{P}_G^{\lambda} \to \mathcal{P}_G^{\lambda} \otimes \Omega_X^1$ be the corresponding Higgs field.

Because ϕ is generically regular, note that ϕ^{λ} is nilpotent of order $(2\check{\rho}, \lambda) + 1$; indeed, this reduces to the corresponding fact for a principal nilpotent acting on V^{λ}, and this follows from $\mathfrak{sl}_2$-representation theory.

We now form:

$$\mathrm{Image}((\phi^{\lambda})^{(2\check{\rho},\lambda)}) \subseteq \mathcal{P}_G^{\lambda} \otimes (\Omega_X^1)^{\otimes (2\check{\rho},\lambda)}$$

and:

$$\mathcal{L}^{\lambda} := \mathrm{Image}((\phi^{\lambda})^{(2\check{\rho},\lambda)}) \otimes (\Omega_X^1)^{\otimes -(2\check{\rho},\lambda)} \subseteq \mathcal{P}_G^{\lambda}.$$

If $(\mathcal{P}_G, \phi)$ lifts to a point of $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$, then each $\mathcal{L}^{\lambda}$ is a line bundle of degree $(\check{\lambda}, \lambda)$ and the quotient $\mathcal{P}_G^{\lambda}/\mathcal{L}^{\lambda}$ is also a vector bundle. Indeed, one verifies that in this case, $\mathcal{L}^{\lambda}$ is the line subbundle coming from the B-structure on $\mathcal{P}_G$ in the Plücker picture.

Conversely, we claim that if $\mathcal{P}_G^{\lambda}/\mathcal{L}^{\lambda}$ is S-flat, $(\mathcal{P}, \phi)$ lifts (on each connected component of S) to some $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$.

Indeed, this hypothesis implies that formation of $\mathrm{Image}((\phi^{\lambda})^{(2\check{\rho},\lambda)})$ commutes with further base change, i.e., that for every $T \to S$, we have:

$$\mathrm{Image}(((\phi^{\lambda})|_T)^{(2\check{\rho},\lambda)}) = \mathrm{Image}((\phi^{\lambda})^{(2\check{\rho},\lambda)})|_T.$$

When S is the spectrum of a field, it is easy to see $\mathcal{L}^{\lambda}$ is a torsion free sheaf of generic rank 1, i.e., a line bundle. Therefore, by the base-change property, every fiber of the coherent sheaf $\mathcal{L}^{\lambda}$ is 1-dimensional, so $\mathcal{L}^{\lambda}$ is a line bundle.

For dominant weights λ, μ, $\phi^{\lambda+\mu}$ equals (the restriction of) $\phi^{\lambda} \otimes \mathrm{id} + \mathrm{id} \otimes \phi^{\mu}$ under the natural map $\mathcal{P}_G^{\lambda} \otimes \mathcal{P}_G^{\mu} \hookrightarrow \mathcal{P}_G^{\lambda+\mu}$. Therefore, we see that:

$$(\phi^{\lambda+\mu})^{(2\check{\rho},\lambda+\mu)} = (\phi^{\lambda} \otimes \mathrm{id} + \mathrm{id} \otimes \phi^{\mu})^{(2\check{\rho},\lambda+\mu)} = (\phi^{\lambda})^{(2\check{\rho},\lambda)} \otimes (\phi^{\mu})^{(2\check{\rho},\mu)}$$

It then follows that the Plücker relations hold, so the $\mathcal{L}^{\lambda}$'s determine a reduction of $\mathcal{P}_G$ to B. Clearly the Higgs field ϕ is a section of $\mathfrak{n}_{\mathcal{P}_B} \otimes \Omega_X^1$ because $\phi^{\lambda}(\mathcal{L}^{\lambda}) = 0$ for each λ. So we have lifted our point to $\coprod_{\check{\lambda}} \mathring{\mathcal{N}ilp}^{\check{\lambda}}$.

It follows that $\coprod_{\check{\lambda}} \mathring{\mathcal{N}ilp}^{\check{\lambda}}$ is a flattening stratification for $\oplus \mathcal{P}_G^{\lambda}/\mathcal{L}^{\lambda} \in \mathsf{QCoh}(X \times \mathring{\mathcal{N}ilp})$ relative to the (projective) morphism $X \times \mathring{\mathcal{N}ilp} \to \mathring{\mathcal{N}ilp}$, or for the *coherent* sheaf $\oplus_{i=1}^{r} \mathcal{P}_G^{\lambda_i}/\mathcal{L}^{\lambda_i}$ for a spanning set of weights $\lambda_1, \ldots, \lambda_r$.

It then follows from the theory of flattening stratifications that for each connected component $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$, the embedding into $\mathring{\mathcal{N}ilp}$ is locally closed.

The following result describes the geometry in more detail:

Proposition 2.5.4.1. *Let $\check{\lambda} \in \check{\Lambda}$ be a coweight.*

(1) $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$ *is smooth of dimension* $\dim \mathrm{Bun}_G$.
(2) Suppose $(\alpha_i, \check{\lambda}) + 2g - 2 < 0$ *for some* $i \in \mathcal{I}_G$. *Then:*

$$\mathring{\mathcal{N}ilp}^{\check{\lambda}} = \emptyset.$$

(3) Suppose $(\alpha_i, \check{\lambda}) + 2g - 2 \geq 0$ *for all* $i \in \mathcal{I}_G$. *Then* $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$ *is non-empty and connected.*

This result is proved[18] in [BD] §2.10.3.

It follows from the proposition that the irreducible components of $\mathring{\mathcal{N}ilp}$ are exactly the closures of the strata $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$. Therefore, we have an injective map:

$$c : \mathrm{Irr}(\mathring{\mathcal{N}ilp}) \hookrightarrow \check{\Lambda} \tag{2.5.1}$$

where $\mathrm{Irr}(\mathring{\mathcal{N}ilp})$ is the set of irreducible components of $\mathring{\mathcal{N}ilp}$. We let $\check{\Lambda}^{\mathrm{rel}} \subseteq \check{\Lambda}$ denote the image of this map (the notation abbreviates *relevant*); explicitly, $\check{\lambda} \in \check{\Lambda}^{\mathrm{rel}}$ if and only if $(\alpha_i, \check{\lambda}) \geq 2 - 2g$ for all $i \in \mathcal{I}_G$.

Example 2.5.4.2. Suppose $G = GL_2$. Then for a coweight $\check{\lambda} = (d_1, d_2) \in \mathbf{Z}^2$, $\mathring{\mathcal{N}ilp}^{\check{\lambda}}$ parametrizes short exact sequences $0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$ plus a non-zero map $\overline{\varphi} : \mathcal{L}_2 \to \mathcal{L}_1 \otimes \Omega^1_X$ where $\deg \mathcal{L}_i = d_i$; the corresponding Higgs field is:

$$\mathcal{E} \to \mathcal{L}_2 \to \mathcal{L}_1 \otimes \Omega^1_X \hookrightarrow \mathcal{E} \otimes \Omega^1_X.$$

We remark that $\mathcal{L}_1 = \mathrm{Ker}(\varphi)$ and $\mathcal{L}_2 = \mathrm{Image}(\varphi) \otimes \Omega_X^{1,\otimes -1}$ can be recovered from the generically regular nilpotent Higgs field φ.

2.5.5. *Invariants.* For later reference, we attach two numerical data to a field-valued point $(\mathcal{P}_G, \varphi) \in \mathring{\mathcal{N}ilp}$.

First, we let:

$$c_1(\mathcal{P}_G, \varphi) \in \pi_1^{\mathrm{alg}}(G) := \check{\Lambda}/\mathbf{Z}\Delta$$

denote the first Chern class of $\mathcal{P}_G$ (cf. [BD] §2.1.1). We explicitly say: there is no dependence on φ, and this invariant behaves well in moduli (it is locally constant on Bun_G).

Second, we define the *discrepancy divisor* of $(\mathcal{P}_G, \varphi)$, which is a $\check{\Lambda}^+_{G^{\mathrm{ad}}}$-valued divisor $\mathrm{disc}(\mathcal{P}_G, \varphi)$ on X, (for $\check{\Lambda}_{G^{\mathrm{ad}}}$ being coweights of the adjoint group G^{ad} of G) as follows. First, as above, by generic regularity (and the valuative criterion of properness), this point lifts uniquely to a map $X \to \mathfrak{n}/(B \times \mathbf{G}_m)$, where the underlying map $X \to \mathbf{BG}_m$ is given by the canonical bundle. We have a projections $\mathfrak{n}/B \times \mathbf{G}_m \to \mathbf{A}^1/\mathbf{G}_m$ corresponding to projections to simple coroot spaces. By assumption, each induced map:

$$X \to \mathbf{A}^1/\mathbf{G}_m$$

[18] In fact, the result in [BD] is formulated in greater generality: it allows for non-regular nilpotent elements as well. We remark for the sake of comparison that in the notation of [BD] §2.10.3, $Y_C = Y_C^*$ for the regular nilpotent conjugacy class. Similarly, in this regular nilpotent case, the space M_C maps to Bun_T; its fiber over $\mathcal{P}_T \in \mathrm{Bun}_T^{\check{\lambda}}$ is $\prod_{i \in \mathcal{I}_G} \Gamma(X, \mathcal{P}_T^{\alpha_i} \otimes \Omega^1_X) \setminus 0$; in particular, the fibers are empty or connected, and the condition of some fiber being non-empty is exactly the numerical condition given in the proposition.

sends the generic point of X to the open point of $\mathbf{A}^1/\mathbf{G}_m$. Equivalently, we have obtained ($\mathbf{Z}$-valued) effective divisors $\mathrm{disc}_i(\mathcal{P}_G, \varphi)$ on X for each $i \in \mathcal{I}_G$. As the map:

$$\check{\Lambda}_{G^{\mathrm{ad}}} \xrightarrow{\check{\lambda} \mapsto (\check{\lambda}, \alpha_i)_{i \in \mathcal{I}_G}} \bigoplus_{i \in \mathcal{I}_G} \mathbf{Z}$$

is an isomorphism, we see that there is a unique divisor $\mathrm{disc}(\mathcal{P}_G, \varphi)$ as above such that $(\mathrm{disc}(\mathcal{P}_G, \varphi), \alpha_i) = \mathrm{disc}_i(\mathcal{P}_G, \varphi)$.

Finally, we observe that for $(\mathcal{P}_G, \varphi) \in \overset{\circ}{\mathcal{N}ilp}{}^{\check{\lambda}}$, we can explicitly compute the above invariants. Specifically, $c_1(\mathcal{P}_G, \varphi)$ is the image of $\check{\lambda}$ in $\pi_1^{\mathrm{alg}}(G)$. Second, we have:

$$\deg(\mathrm{disc}(\mathcal{P}_G, \varphi)) = \overline{\check{\lambda}} + (2g-2)\check{\rho} \in \check{\Lambda}^+_{G^{\mathrm{ad}}}. \tag{2.5.2}$$

Here $\overline{\check{\lambda}} \in \check{\Lambda}_{G^{\mathrm{ad}}}$ is the image of $\check{\lambda}$ under the natural map $\check{\Lambda} \to \check{\lambda}_{G^{\mathrm{ad}}}$.

Remark 2.5.5.1. As $\check{\Lambda} \to \pi_1^{\mathrm{alg}}(G) \times \check{\Lambda}_{G^{\mathrm{ad}}}$ is injective,[19] we can recover the invariant $\check{\lambda}$ of $(\mathcal{P}_G, \varphi)$ from $c_1(\mathcal{P}_G, \varphi)$ and $\deg(\mathrm{disc}(\mathcal{P}_G, \varphi))$.

Remark 2.5.5.2. The discrepancy divisor is a more natural indexing tool than $\check{\lambda}$ itself. For example, for $\check{\lambda} \in \check{\Lambda}$ to be relevant is equivalent to saying $\overline{\check{\lambda}} + (2g-2)\check{\rho}$ is a dominant coweight for G^{ad}.

Remark 2.5.5.3. Observe that $\deg(\mathrm{disc}(\mathcal{P}_G, \varphi))$ always lies in the image of $\check{\Lambda} \to \check{\Lambda}_{G^{\mathrm{ad}}}$; indeed, this follows from (2.5.2) as $(2g-2)\check{\rho} = (g-1) \cdot 2\check{\rho}$ lifts.

Example 2.5.5.4. In the setting of Example 2.5.4.2, $c_1(\mathcal{E}, \varphi)$ is the degree of $\mathcal{E}$, while the discrepancy divisor is the divisor of zeroes of the map $\overline{\varphi}$.

2.5.6. *Everywhere regular Higgs fields and the Kostant component.* We let $\overset{\circ}{\mathcal{N}ilp}{}^{\mathrm{reg}}$ denote the mapping space:

$$\mathcal{M}aps(X, \overset{\circ}{\mathcal{N}}/G).$$

Clearly $\overset{\circ}{\mathcal{N}ilp}{}^{\mathrm{reg}} \subseteq \overset{\circ}{\mathcal{N}ilp}$ is open.

Let $Z_f \subseteq G$ denote the stabilizer subgroup of some regular nilpotent $f \in \overset{\circ}{\mathcal{N}}$. Because G acts transitively on $\overset{\circ}{\mathcal{N}}$, we have $\overset{\circ}{\mathcal{N}ilp}{}^{\mathrm{reg}} = \mathrm{Bun}_{Z_f}$. Clearly the center Z_G of G embeds into Z_f; recall that Z_G maps isomorphically onto the reductive quotient of Z_f; therefore, $\overset{\circ}{\mathcal{N}ilp}{}^{\mathrm{reg}}$ is smooth.

The Kostant slice defines a base-point $f^{\mathrm{glob}} \in \overset{\circ}{\mathcal{N}ilp}{}^{\mathrm{reg}}$. We let $\mathcal{N}ilp^{\mathrm{Kos}} \subseteq \overset{\circ}{\mathcal{N}ilp}{}^{\mathrm{reg}}$ denote the corresponding connected component. Explicitly, we have:

$$\mathcal{N}ilp^{\mathrm{Kos}} = \overset{\circ}{\mathcal{N}ilp}{}^{(2-2g)\check{\rho}}.$$

The point f^{glob} is given by the G-bundle $\mathcal{P}_G^{\mathrm{can}}$ (induced from $\Omega^{\otimes \frac{1}{2}}$ via $-2\check{\rho} : \mathbf{G}_m \to T \to G$) with its natural regular nilpotent Higgs bundle.

Remark 2.5.6.1. The distinguished component $\mathcal{N}ilp^{\mathrm{Kos}}$ plays an outsized role in this work.

Remark 2.5.6.2. We remark that $(\mathcal{P}_G, \varphi) \in \overset{\circ}{\mathcal{N}ilp}$ lies in $\overset{\circ}{\mathcal{N}ilp}{}^{\mathrm{reg}}$ if and only if $\mathrm{disc}(\mathcal{P}_G, \varphi) = 0$. Assuming the center of G is connected, it additionally lies in $\overset{\circ}{\mathcal{N}ilp}{}^{\mathrm{Kos}}$ if and only if $c_1(\mathcal{P}_G, \varphi) = c_1(-\check{\rho}(\Omega^1_X))$

2.6. **Tempered D-modules.**

[19] It is an isomorphism on tensoring with $\mathbf{Q}$, and $\check{\Lambda}$ is of course torsion-free.

2.6.1. *Local formalism.* Fix $x \in X(k)$ a point. Let $\mathcal{H}_x^{\mathrm{sph}}$ denote the spherical Hecke category based at this point.

For $\mathcal{D}$ be a module category for $\mathcal{H}_x^{\mathrm{sph}}$. We refer to [Ber3] §2 for a definition of the categories $\mathcal{D}^{x-\text{anti-temp}}$ and $\mathcal{D}^{x-\text{temp}}$; our notation is the same as [Ber3] §2.4.1, except that we include the dependence on the point x in the notation. We remind that $\mathcal{D}^{x-\text{anti-temp}} \subseteq \mathcal{D}$ is a certain full subcategory, and the embedding admits a left adjoint. Then $\mathcal{D}^{x-\text{temp}}$ is the quotient $\mathcal{D}/\mathcal{D}^{x-\text{anti-temp}}$ in $\mathsf{DGCat}_{\mathrm{cont}}$; the projection $\mathcal{D} \to \mathcal{D}^{x-\text{temp}}$ admits a fully faithful left adjoint.

2.6.2. *Global setting.* The above discussion applies in particular for $\mathcal{D} = D(\mathrm{Bun}_G)$.

The main result of [FR] asserts that $D(\mathrm{Bun}_G)^{x-\text{anti-temp}}$ and $D(\mathrm{Bun}_G)^{x-\text{temp}}$ are independent of the choice of x (in a strong sense). Therefore, we often write $D(\mathrm{Bun}_G)^{\text{anti-temp}}$ and $D(\mathrm{Bun}_G)^{\text{temp}}$ to indicate this category.

On the other hand, we sometimes include the point x in the notation when we are performing a particular manipulation at the point.

We refer to objects of $D(\mathrm{Bun}_G)^{\text{anti-temp}}$ as *anti-tempered* D-modules on Bun_G, and objects of $D(\mathrm{Bun}_G)^{\text{temp}}$ as *tempered* D-modules on Bun_G.

Although we can think of $D(\mathrm{Bun}_G)^{\text{temp}}$ as a subcategory of $D(\mathrm{Bun}_G)$ (via the left adjoint referenced above), we generally consider it rather as a quotient category. Roughly speaking, the quotient functor is better behaved.

2.7. Normalizations regarding exponential sheaves and characters.

2.7.1. We let $\exp \in D(\mathbf{A}^1)$ denote the exponential D-module, normalized to live in cohomological degree -1, i.e., the same degree as the dualizing sheaf $\omega_{\mathbf{A}^1}$. This object is a multiplicative sheaf with respect to upper-! functors.

2.7.2. At various points in the text, we consider characters $\psi : N \to \mathbf{G}_a$, or loop group analogues with N replaced by $\mathring{I}$ (the radical of Iwahori) or the loop group $N(K)$. These are always assumed to be *non-degenerate* in the appropriate sense. Precisely:

- For N, the map $\mathrm{Lie}(\psi) : \mathfrak{n} \to k$ should send each Chevalley generator $e_i \in \mathfrak{n}$ ($i \in \mathcal{I}_G$) to a non-zero number.
- For $\mathring{I}$, the map $\mathring{I} \to \mathbf{G}_a$ should be the composition of the previous non-degenerate character $N \to \mathbf{G}_a$ with the projection $\mathring{I} \to N$ (so this is non-degenerate in the standard Whittaker sense, but not the affine Kac-Moody sense).
- For $N(K)$, the map $N(K) \to \mathbf{G}_a$ should have conductor 0 and be non-degenerate in the standard sense.

For $\mathcal{Y}$ with a G-action, we write $D(\mathcal{Y})^{N,\psi} \subseteq D(\mathcal{Y})$ to mean the category of D-modules that are twisted N-equivariant against $\psi^!(\exp)$; similarly for $\mathring{I}$ and $N(K)$.

Part 1. Singular support and temperedness

3. Irregular singular support in finite dimensions

In this section, we study a version of irregular singular support for G-spaces. Our main result is Theorem 3.1.2.1, which we prove by reduction to [Los].

This material is used in §4, and may safely be skipped at the first pass.

3.1. G-irregularity.

3.1.1. Let $\mathcal{Y}$ be an algebraic stack locally of finite type and equipped with a G-action.

We obtain a moment map $\mu : T^*\mathcal{Y} \to \mathfrak{g}^\vee \simeq \mathfrak{g}$. We define:

$$D_{G-\mathrm{irreg}}(\mathcal{Y}) := D_{\mu^{-1}(\mathfrak{g}_{\mathrm{irreg}})}(\mathcal{Y}).$$

In other words, a D-module $\mathcal{F}$ lies in $D_{G-\mathrm{irreg}}(\mathcal{Y})$ if for every point $(y, \xi) \in \mathrm{SS}(\mathcal{F})$, $\mu(y, \xi)$ is an irregular element of $\mathfrak{g}$.

We let $\mathsf{Shv}_{G-\mathrm{irreg}}(\mathcal{Y}) := \mathsf{Shv}(\mathcal{Y}) \cap D_{G-\mathrm{irreg}}(\mathcal{Y})$.

Remark 3.1.1.1. We use the notation $G-$irreg here rather than simply irreg to reserve the latter for the *global* context considered in §4.

3.1.2. We can now state:

Theorem 3.1.2.1. *Let $\mathcal{Y}$ be an algebraic stack locally of finite type with a G-action. Then the Whittaker averaging functor:*

$$\mathsf{Shv}_{G-\mathrm{irreg}}(\mathcal{Y})^{B^-} \to D(\mathcal{Y})^{B^-} \xrightarrow{\mathrm{Av}_!^\psi} D(\mathcal{Y})^{N,\psi}$$

is identically zero.

3.2. A statement for $\mathfrak{g}$-modules.

Below, we fix a G-invariant isomorphism $\mathfrak{g} \simeq \mathfrak{g}^\vee$ for convenience.

3.2.1. *Singular support.* Let $M \in \mathfrak{g}\text{–}\mathsf{mod}^\heartsuit$ be finitely generated. Recall that we may choose a good filtration on M (relative to the PBW filtration on $U(\mathfrak{g})$); the reduced support of its associated graded is a well-defined closed conical subscheme $\mathrm{SS}(M) \subseteq \mathfrak{g}^\vee \simeq \mathfrak{g}$, which we call the *singular support* of M. We remind that this construction is often instead called the *associated variety* of M.

3.2.2. Let $Z(\mathfrak{g}) \subseteq U(\mathfrak{g})$ denote the center, and let $\chi_0 : Z(\mathfrak{g}) \to k$ be the homomorphism defined by the trivial representation of $\mathfrak{g}$.[20] Let $U(\mathfrak{g})_0 = U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathrm{Ker}(\chi_0)$, so $U(\mathfrak{g})_0$ is quotient of $U(\mathfrak{g})$ whose modules of those $\mathfrak{g}$-modules with the same central character as the trivial representation. We sometimes use the notation $\mathfrak{g}\text{–}\mathsf{mod}_0$ to denote the DG category of $U(\mathfrak{g})_0$-modules.

Recall that $U(\mathfrak{g})_0$ carries a filtration induced from the PBW filtration, and $\mathrm{gr}_\bullet U(\mathfrak{g})_0 = \Gamma(\mathcal{N}, \mathcal{O}_\mathcal{N})$. Therefore, for any $M \in \mathfrak{g}\text{–}\mathsf{mod}_0^\heartsuit$ finitely generated, we have $\mathrm{SS}(M) \subseteq \mathcal{N} \subseteq \mathfrak{g}$.

3.2.3. *Whittaker localization.* Note that:

$$\mathsf{Hom}_{G\text{–}\mathsf{mod}}(\mathfrak{g}\text{–}\mathsf{mod}, D(G)^{N,\psi}) \simeq D(G)^{(N,\psi),(G,w)} = \mathfrak{g}\text{–}\mathsf{mod}^{N,\psi} \simeq Z(\mathfrak{g})\text{–}\mathsf{mod}$$

where in the last equality we have used Skryabin's theorem. We let:

$$\mathrm{Loc}^\psi : \mathfrak{g}\text{–}\mathsf{mod} \to D(G)^{N,\psi}$$

denote the G-equivariant functor corresponding to $Z(\mathfrak{g}) \in Z(\mathfrak{g})\text{–}\mathsf{mod}$ under the above identification. It is not hard to see that Loc^ψ is t-exact up to shift, but we do not use this result below. Explicitly,

[20] In what follows, all our results about $\mathfrak{g}$ generalize to the case where χ_0 is replaced by any central character $\chi_\lambda : Z(\mathfrak{g}) \to k$. We use χ_0 to simplify the notation, and to focus on our case of interest.

the fiber of $\mathrm{Loc}^{\psi}(M)$ at $1 \in G$ is computed via Lie algebra homology $\mathrm{C}_\bullet(\mathfrak{n}, M \otimes \psi)$, and its fiber at $g \in G$ is computed similarly but we first twist the action of $\mathfrak{g}$ on M via $\mathrm{Ad}_g : \mathfrak{g} \xrightarrow{\simeq} \mathfrak{g}$.

We also use a variant of this construction; let:

$$\mathrm{Loc}_0^{\psi} : \mathfrak{g}\text{–}\mathsf{mod}_0 \to D(G)^{N,\psi}$$

denote the composition:

$$\mathfrak{g}\text{–}\mathsf{mod}_0 \to \mathfrak{g}\text{–}\mathsf{mod} \xrightarrow{\mathrm{Loc}^{\psi}} D(G)^{N,\psi}.$$

Equivalently, this is the G-equivariant functor corresponding to:

$$k \in \mathsf{Vect} \simeq \mathfrak{g}\text{–}\mathsf{mod}_0^{N,\psi} \simeq \mathsf{Hom}_{G\text{–}\mathsf{mod}}(\mathfrak{g}\text{–}\mathsf{mod}_0, D(G)^{N,\psi}).$$

3.2.4. We will show:

Theorem 3.2.4.1. *Suppose $M \in \mathfrak{g}\text{–}\mathsf{mod}_0^{\heartsuit}$ is a finite length module with $\mathrm{SS}(M) \subseteq \mathcal{N}_{\mathrm{irreg}}$.*[21] *Then $\mathrm{Loc}_0^{\psi}(M) = 0 \in D(G)^{N,\psi}$.*

Remark 3.2.4.2. Probably the theorem is true for finitely generated modules, not simply finite length ones. The restriction to finite length modules corresponds to our reference to [Los].[22]

3.2.5. *Preliminary comments.* Our proof of Theorem 3.2.4.1 relies on some results about primitive ideals. We collect some results from the literature here.

First, we have the following theorem of Loseu:

Theorem 3.2.5.1 (Loseu, [Los])**.** *Let $M \in \mathfrak{g}\text{–}\mathsf{mod}_0^{\heartsuit}$ be a faithful $U(\mathfrak{g})_0$-module; i.e., suppose that the map $U(\mathfrak{g})_0 \to \mathrm{End}_{\mathsf{Vect}^{\heartsuit}}(M)$ is injective. Then $\mathrm{SS}(M) \cap \overset{\circ}{\mathcal{N}} \neq \emptyset$.*

Indeed, this is a special case of [Los] Theorem 1.1 (1).

Remark 3.2.5.2. In our setting, we can actually avoid the strength of Theorem 3.2.5.1. Namely, suppose $M \in \mathfrak{g}\text{–}\mathsf{mod}_0^{\heartsuit}$ has the property that the D-module $\mathcal{F} := \mathrm{Loc}(M) \in D(G/B)$ corresponding to M under Beilinson-Bernstein is *holonomic*; the reader will readily see that we only apply the theorem in this case. Under these hypotheses, the theorem is more elementary.

Namely, as in §3.3.5, $\mathrm{SS}(M)$ is the image of $\mathrm{SS}(\mathcal{F})$ along the Springer map $\pi : T^*(G/B) = \widetilde{\mathcal{N}} \to \mathcal{N}$. By [Los] Lemma A.1 (ii) (which is elementary), $\pi(\mathrm{SS}(\mathcal{F})) \cap \mathbf{O}$ is isotropic for every nilpotent orbit $\mathbf{O} \subseteq \mathcal{N}$. In particular, this intersection has dimension $\leq \frac{1}{2}\dim \mathbf{O} \leq \frac{1}{2}\dim \mathcal{N}$, with the latter inequality being an equality only for $\mathbf{O} = \overset{\circ}{\mathcal{N}}$.

We see that if $\mathrm{SS}(M) \cap \overset{\circ}{\mathcal{N}} = \emptyset$, then $\dim(\mathrm{SS}(M)) < \frac{1}{2}\dim \mathcal{N}$. However, [KL] Theorem 9.11 (due to Gabber) and Proposition 6.6 assert that $\frac{1}{2}\dim \mathrm{SS}(U(\mathfrak{g})_0/\mathrm{Ann}(M)) \leq \dim \mathrm{SS}(M)$, so we see that $\mathrm{Ann}(M) \neq 0$ under the above hypotheses.

[21] Here we remind that $\mathrm{SS}(M)$ is by definition reduced; so e.g., such an inclusion can be checked on field-valued points.

[22] Since the first version of this paper appeared, this question was settled in [DF] Theorem 3.0.2.1.

3.2.6. Next, let $\mathbf{M}_0^\psi \in \mathfrak{g}\text{–}\mathsf{mod}_0^\heartsuit$ be the object corresponding to:

$$k \in \mathsf{Vect} \simeq \mathfrak{g}\text{–}\mathsf{mod}_0^{N,\psi} \subseteq \mathfrak{g}\text{–}\mathsf{mod}_0.$$

Explicitly, we have:

$$\mathbf{M}_0^\psi = \mathrm{ind}_{\mathfrak{n}}^{\mathfrak{g}}(\psi) \underset{Z(\mathfrak{g})}{\otimes} k.$$

Here $\psi \in \mathfrak{n}\text{–}\mathsf{mod}^\heartsuit$ denotes the 1-dimensional module defined by the character ψ, and we consider k as a $Z(\mathfrak{g})$-module via χ_0.

We recall the following basic fact:

Proposition 3.2.6.1 (Kostant, [Kos] Theorem D)**.** $\mathbf{M}_0^\psi$ *is a faithful* $U(\mathfrak{g})_0$*-module.*

We deduce:

Corollary 3.2.6.2. *Any non-zero object of* $\mathfrak{g}\text{–}\mathsf{mod}_0^{(N,\psi),\heartsuit}$ *is a faithful* $U(\mathfrak{g})_0$*-module.*

Indeed, by Skryabin's equivalence, such an object has the form $V \otimes \mathbf{M}_0^\psi$ for some non-zero vector space V, so Proposition 3.2.6.1 yields the assertion.

3.2.7. *Proof of the theorem for Lie algebras.* With the preliminary results above completed, we are now in a position to prove the theorem.

Proof of Theorem 3.2.4.1.

Step 1. We are obviously reduced to the case where M is a simple module (with $\mathrm{SS}(M) \subseteq \mathcal{N}_{\mathrm{irreg}}$).

Let $I \subseteq U(\mathfrak{g})_0$ be the two-sided ideal annihilating M. By Loseu's theorem (Theorem 3.2.5.1), $I \neq 0$. Set $A = U(\mathfrak{g})_0/I$; this is a classical associative algebra.

Step 2. Note that A receives a Lie algebra homomorphism $i : \mathfrak{g} \to A$ via $\mathfrak{g} \to U(\mathfrak{g}) \to U(\mathfrak{g})_0 \to A$. Moreover, the adjoint[23] action of $\mathfrak{g}$ on A integrates to an action of G on A; indeed, this follows from the corresponding property of $U(\mathfrak{g})$.

In other words, A receives a canonical Harish-Chandra datum for G.

It follows that there is a (strong) G-action on $A\text{–}\mathsf{mod}$ such that the forgetful functor $A\text{–}\mathsf{mod} \to \mathfrak{g}\text{–}\mathsf{mod}_0$ is G-equivariant.

Step 3. Next, we claim that $A\text{–}\mathsf{mod}^{N,\psi} = 0$.

It suffices to check this at the abelian categorical level (the t-structure on $A\text{–}\mathsf{mod}^{N,\psi}$ is obviously separated).

Any object of $A\text{–}\mathsf{mod}^\heartsuit$ maps to a non-faithful module of $U(\mathfrak{g})_0$ (as $I \neq 0$). Therefore, any object of $A\text{–}\mathsf{mod}^{N,\psi,\heartsuit}$ maps to an object of $\mathfrak{g}\text{–}\mathsf{mod}_0^{N,\psi,\heartsuit}$ that is not faithful as a $U(\mathfrak{g})_0$-module; by Corollary 3.2.6.2, the object must be zero.

Step 4. Finally, we note that (as in §3.2.3), we have:

$$\mathsf{Hom}_{G\text{–}\mathsf{mod}}(A\text{–}\mathsf{mod}, D(G)^{N,\psi}) \simeq A\text{–}\mathsf{mod}^{N,\psi} = 0.$$

Therefore, the composition:

$$A\text{–}\mathsf{mod} \to \mathfrak{g}\text{–}\mathsf{mod}_0 \xrightarrow{\mathrm{Loc}_0^\psi} D(G)^{N,\psi}$$

[23] I.e., the action defined by the formula $\xi \star a := [i(\xi), a]$ for $\xi \in \mathfrak{g}$ and $a \in A$.

is zero.

As $M \in \mathfrak{g}\text{–}\mathsf{mod}_0^\heartsuit$ lifts to $A\text{–}\mathsf{mod}^\heartsuit$ by definition, the result follows.

□

3.3. **Proof of Theorem 3.1.2.1.** We now prove the theorem. We proceed by steps.

Below, we fix $\mathcal{F} \in \mathsf{Shv}_{G\text{–irreg}}(\mathcal{Y})^{B^-}$.

3.3.1. *Reduction to the quasi-compact case.* Note that the stack $\mathcal{Y}/G$ is a union of its quasi-compact open substacks. Replacing $\mathcal{Y}$ by the preimage of each such open, we are reduced to the case where $\mathcal{Y}$ itself is quasi-compact.

3.3.2. *Reduction to the simple case.* First, recall from [BBM] (see also [Ras2] Appendix A) that the functor:

$$\mathrm{Av}_!^\psi : D(\mathcal{Y})^{B^-} \to D(\mathcal{Y})^{N,\psi}$$

is t-exact up to shift. Therefore, as $\mathsf{Shv}_{G\text{–irreg}}(\mathcal{Y})^{B^-} \subseteq D(\mathcal{Y})^{B^-}$ is closed under truncations, we may assume $\mathcal{F} \in \mathsf{Shv}_{G\text{–irreg}}(\mathcal{Y})^{B^-,\heartsuit}$.

Moreover, as $\mathsf{Shv}_{G\text{–irreg}}(\mathcal{Y})^{B^-,\heartsuit} \subseteq \mathsf{Shv}(\mathcal{Y})^{B^-,\heartsuit}$ is stable under taking subobjects, we may assume $\mathcal{F}$ is also perverse (i.e., "small") by quasi-compactness of $\mathcal{Y}$. In particular, $\mathcal{F}$ has finite length. Finally, we are clearly reduced to the case where $\mathcal{F} \in \mathsf{Shv}_{G\text{–irreg}}(\mathcal{Y})^{B^-,\heartsuit}$ is simple; we explicitly remark that it is equivalent to say that it is simple in $D(\mathcal{Y})^{B^-,\heartsuit}$.

3.3.3. *Reduction to the case where $\mathcal{Y} = G \times \mathcal{Y}_0$ where G acts trivially on $\mathcal{Y}_0$.* Consider $G \times \mathcal{Y}$ as acted on by G via the action on the first factor alone. Next, we have the G-equivariant map:

$$G \times \mathcal{Y} \xrightarrow{\mathrm{act}} \mathcal{Y}.$$

As act is smooth, $\mathrm{act}^![-\dim G]$ is t-exact, conservative, preserves simples, and maps $\mathsf{Shv}_{G\text{–irreg}}(\mathcal{Y})$ to $\mathsf{Shv}_{G\text{–irreg}}(G \times \mathcal{Y})$. By G-equivariance, $\mathrm{act}^!$ commutes with finite Whittaker averaging.

Therefore, we are clearly reduced to case where $\mathcal{Y}$ has the stated form.

3.3.4. *Reduction to the case $\mathcal{Y} = G$.* Recall that we have $\mathcal{Y} = G \times \mathcal{Y}_0$ as above.

Observe that $\mathrm{Av}_!^\psi(\mathcal{F})$ is a compact, holonomic object (because $\mathcal{F}$ is). Therefore, it suffices to check that for every field extension k'/k of finite degree and every pair $(g, y) \in (G \times \mathcal{Y}_0)(k')$, the !-restriction of $\mathrm{Av}_!^\psi(\mathcal{F})$ to (g, y) vanishes. Up to a finite extension of the ground field, we may assume $k' = k$; we do so in what follows to simplify the notation.

Let $y \in \mathcal{Y}_0(k)$ be fixed. Let $i_y : \mathrm{Spec}(k) \to \mathcal{Y}_0$ be the corresponding map. The map $G \xrightarrow{\mathrm{id} \times i_y} G \times \mathcal{Y}_0$ is obviously G-equivariant (as G acts trivially on $\mathcal{Y}_0$), so we have:

$$(\mathrm{id} \times i_y)^! \,\mathrm{Av}_!^\psi(\mathcal{F}) = \mathrm{Av}_!^\psi(\mathrm{id} \times i_y)^!(\mathcal{F}) \in D(G)^{N,\psi}.$$

Here we also remind that $\mathrm{Av}_!^\psi$ equals Av_*^ψ up to a cohomological shift, see [BBM] once again.

Now we claim:

Lemma 3.3.4.1. *The object $(\mathrm{id} \times i_y)^!(\mathcal{F}) \in \mathsf{Shv}(G)^{B^-}$ lies in $\mathsf{Shv}_{G\text{–irreg}}(G)^{B^-}$.*

Clearly this result follows from the general result:

Proposition 3.3.4.2 (Kashiwara-Schapira)**.** *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be smooth schemes, and let $\Lambda \subseteq T^*\mathcal{Z}_1$ be closed and conical. Let $f : \mathcal{Z}_3 \to \mathcal{Z}_2$ be a given map with $\mathcal{Z}_3$ smooth (but f possibly non-smooth).*

Then the functor:

$$(\mathrm{id} \times f)^! : \mathsf{Shv}(\mathcal{Z}_1 \times \mathcal{Z}_2) \to \mathsf{Shv}(\mathcal{Z}_1 \times \mathcal{Z}_3)$$

maps $\mathsf{Shv}_{\Lambda \times T^*\mathcal{Z}_2}(\mathcal{Z}_1 \times \mathcal{Z}_2)$ to $\mathsf{Shv}_{\Lambda \times T^*\mathcal{Z}_3}(\mathcal{Z}_1 \times \mathcal{Z}_3)$.

Proof. This follows immediately from the estimates[24] on singular support of pullbacks from [KS] Corollary 6.4.4.

□

Remark 3.3.4.3. One may also appeal to [Gin] in place of [KS]. In addition, we remark that the proof of [BG1] Proposition 2.1.2 essentially (up to a quasi-projectvity assumption) shows more generally that $\mathsf{Shv}_{G-\mathrm{irreg}}$ is preserved under sheaf-theoretic operations for G-equivariant maps. Indeed, *loc. cit.* considers a similar setting to ours, but with $0 \in \mathfrak{g}$ replacing $\mathcal{N}_{\mathrm{irreg}} \subseteq \mathfrak{g}$; more generally, the argument in *loc. cit.* applies for any closed, conical, G-equivariant subscheme of $\mathfrak{g}$.

Remark 3.3.4.4. We remind that the results from [KS] and [Gin] fail for general holonomic D-modules; regularity is crucial assumption.

We now continue with the reduction: by Lemma 3.3.4.1, $(\mathrm{id} \times i_y)^!(\mathcal{F}) \in \mathsf{Shv}_{G-\mathrm{irreg}}(G)^{B^-}$; therefore, if we know the result for $\mathcal{Y}_0 = \mathrm{Spec}(k)$, we obtain $\mathrm{Av}_!^{\psi}(\mathrm{id} \times i_y)^!(\mathcal{F}) = 0$, which yields the claim for general $\mathcal{Y}_0$ by our earlier discussion.

3.3.5. *Proof for $\mathcal{Y}_0 = \mathrm{Spec}(k)$.* We will use Beilinson-Bernstein to reduce to Theorem 3.2.4.1.

We remind the setting: $\mathcal{F} \in D(B^-\backslash G)^{\heartsuit}$ is a simple[25] D-module on the flag variety with singular support in:

$$\widetilde{\mathcal{N}}_{\mathrm{irreg}} :- \widetilde{\mathcal{N}} \underset{\mathcal{N}}{\times} \mathcal{N}_{\mathrm{irreg}} \subseteq \widetilde{\mathcal{N}} \simeq T^*(B^-\backslash G).$$

where $\widetilde{\mathcal{N}}$ is the Springer resolution of the nilpotent cone. We wish to show that:

$$\mathrm{Av}_!^{\psi}(\mathcal{F}) \in D(G)^{N,\psi}$$

is zero (where (N, ψ) invariants are taken for the left action).

We have a diagram:

$$\begin{tikzcd} D(B^-\backslash G) \arrow[d, "\Gamma(B^-\backslash G, -)"'] \arrow[dr, "\mathrm{Av}_!^{\psi}"] & \\ \mathfrak{g}\text{–}\mathsf{mod}_0 \arrow[r, "\mathrm{Loc}^{\psi}"] & D(G)^{N,\psi}. \end{tikzcd}$$

that commutes up to a cohomological shift. Indeed, by G-equivariance of the functors involved, it suffices to check that the images of the δ D-module at the base point of the flag variety are mapped to the same object (up to shift), and this is straightforward.

[24]We fill in some details in the reference here, referring to [KS] for notation.
In our setting, for $\mathcal{G} \in \mathsf{Shv}_{\Lambda \times T^*\mathcal{Z}_2}(\mathcal{Z}_1 \times \mathcal{Z}_2)$, recall that [KS] Corollary 6.4.4 (ii) bounds $\mathrm{SS}((\mathrm{id} \times f)^!(\mathcal{G}))$ by something denoted $(\mathrm{id} \times f)^{\sharp}(\Lambda \times T^*\mathcal{Z}_2)$. It is straightforward to see $(\mathrm{id} \times f)^{\sharp}(\Lambda \times T^*\mathcal{Z}_2) \subseteq \Lambda \times T^*\mathcal{Z}_2$; e.g., one can reduce to the case where f is a closed embedding and then the description from [KS] Remark 6.2.8 (i) is convenient.

[25]At this point, regularity, and even holonomicity, are no longer essential hypotheses.

Now for our $\mathcal{F}$, let $M := \Gamma(B^-\backslash G, \mathcal{F}) \in \mathfrak{g}\text{–}\mathsf{mod}_0$. By the above, it suffices to show that $\mathrm{Loc}^\psi(M) = 0$.

By Beilinson-Bernstein localization [BB1], $M \in \mathfrak{g}\text{–}\mathsf{mod}_0^\heartsuit$ is simple. Therefore, by Theorem 3.2.4.1, it suffices to show that $\mathrm{SS}(M) \subseteq \mathcal{N}_{\mathrm{irreg}}$. This is a standard compatibility: by the Corollary in §1.9 of [BB2], $\mathrm{SS}(M)$ is the image of $\mathrm{SS}(\mathcal{F})$ along the projection map:

$$\widetilde{\mathcal{N}} \to \mathcal{N}.$$

4. Irregular singular support on Bun_G

4.1. **Formulation of the main result.** Let $D_{\mathrm{irreg}}(\mathrm{Bun}_G) := D_{\mathrm{Higgs}_{G,\mathfrak{g}_{\mathrm{irreg}}}}(\mathrm{Bun}_G)$; see §2.4.2 for the notation. We similarly have $D_{\mathcal{N}ilp_{\mathrm{irreg}}}(\mathrm{Bun}_G) = \mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}(\mathrm{Bun}_G)$.

The purpose of this section is to prove:

Theorem 4.1.0.1. *Any object of* $\mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}(\mathrm{Bun}_G)$ *is anti-tempered.*

We will prove this result by reduction to Theorem 3.1.2.1.

Throughout, we remind that we have fixed a point $x \in X(k)$. We let K denote the field of Laurent series based at x and $O \subseteq K$ the subring of Taylor series. We sometimes use t for a coordinate at x. We let e.g. $G(K)$ and $G(O)$ denote the loop and arc groups for G. We freely use the formalism of (strong) loop group actions on objects of $\mathsf{DGCat}_{\mathrm{cont}}$.

Remark 4.1.0.2. Using the full results of this paper, we were able to show $D_{\mathrm{irreg}}(\mathrm{Bun}_G) \subseteq D(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$. We conjecture that this is an equivalence for any G. This is a folklore statement for $G = PGL_2$; see [Ber3] Corollary 4.2.6 for a variant of the assertion in this case.

4.2. **Anti-temperedness and Whittaker averaging.** We have the following general characterization of anti-temperedness.

Suppose the loop group $G(K)$ acts on $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$. Recall that we have the anti-tempered subcategory $\mathcal{C}^{G(O),x\text{–anti–temp}} \subseteq \mathcal{C}^{G(O)}$, cf. §1.2.8.

By [Ber3] Theorem 1.4.8, we have:

Lemma 4.2.0.1. $\mathcal{C}^{G(O),x\text{–anti–temp}}$ *is the kernel of the Whittaker* !*-averaging functor:*

$$\mathcal{C}^{G(O)} \xrightarrow{\mathrm{Av}_!^\psi} \mathsf{Whit}(\mathcal{C}).$$

Remark 4.2.0.2. We make a brief philosophical point. Ultimately, we are interested in Whittaker coefficients of tempered D-modules on Bun_G. If we think heuristically of Bun_G as a double quotient $G(k(X))\backslash G(\mathbf{A})/G(\mathbf{O})$, this involves integrating on the *left* with respect to $N(\mathbf{A})$ (twisted by a character, of course). On the other hand, temperedness involves the derived Satake action, which occurs *on the right*. Therefore, when we apply the present lemma to $\mathcal{C} = D(\mathrm{Bun}_G^{\mathrm{level},x})$, the Whittaker averaging in question should be thought of as occurring on the *right*. Moreover, the Whittaker integral on the right occurs at a single point, while the one on the left involves all points simultaneously.

Ultimately, the reader may think that singular support translates between left and right Whittaker conditions.

4.3. **Baby Whittaker.** Next, we record the following well-known result.

Let $I \subseteq G(O)$ (resp. I^-) denote the Iwahori subgroup corresponding to $B \subseteq G$ (resp. $B^- \subseteq G$). Let $\mathring{I}$ denote the prounipotent radical of I. Let $\mathring{I}_1 := \mathrm{Ad}_{-\check{\rho}(t)}(\mathring{I})$ and let $I_1^- := \mathrm{Ad}_{-\check{\rho}(t)}(I^-)$. We abuse notation in letting ψ denote the restriction of the canonical character of $N(K)$ to $\mathring{I}_1$. Finally, we let $\mathcal{K}_1$ denote $\mathrm{Ad}_{-\check{\rho}(t)}$ applied to the first congruence subgroup of $G(O)$; note that $\mathcal{K}_1 = \mathring{I}_1 \cap I_1^-$. Note that $\mathring{I}_1/\mathcal{K}_1 = N$ and $I_1^-/\mathcal{K}_1 = B^-$.

Following [Ras4], for $\mathcal{C} \in G(K)\text{–}\mathsf{mod}$, we use the notation $\mathsf{Whit}(\mathcal{C}) := \mathcal{C}^{N(K),\psi}$ and $\mathsf{Whit}^{\leq 1}(\mathcal{C}) := \mathcal{C}^{\mathring{I}_1,\psi}$. We let $\iota^!_{1,\infty} : \mathsf{Whit}(\mathcal{C}) \to \mathsf{Whit}^{\leq 1}(\mathcal{C})$ denote the $*$-averaging functor.

Finally, we can state:

Lemma 4.3.0.1. *Let $\mathcal{C} \in G(K)\text{–}\mathsf{mod}$ be given. Then the functor $\iota^!_{1,\infty} : \mathsf{Whit}(\mathcal{C}) \to \mathsf{Whit}^{\leq 1}(\mathcal{C})$ admits a fully faithful left adjoint $\iota_{1,\infty,!}$; moreover, there is a commutative diagram:*

$$\begin{tikzcd}
\mathcal{C}^{G(O)} \arrow[r, "\mathrm{Oblv}"] \arrow[drrr, "\mathrm{Av}_!^\psi"'] & \mathcal{C}^I \arrow[r, "\mathrm{Av}_*"] & \mathcal{C}^{I_1^-} \arrow[r, equal] & (\mathcal{C}^{\mathcal{K}_1})^{B^-} \arrow[r, "\mathrm{Av}_!^\psi"] & (\mathcal{C}^{\mathcal{K}_1})^{N,\psi} \arrow[r, equal] & \mathsf{Whit}^{\leq 1}(\mathcal{C}) \arrow[dll, "\iota_{1,\infty,!}"] \\
& & & \mathsf{Whit}(\mathcal{C}). & &
\end{tikzcd}$$

Proof. The existence of the fully faithful functor $\iota_{1,\infty,!}$ is a special case of [Ras4] Theorem 2.3.1. The existence of the commutative diagram is standard: it follows from the fact that the functor $\mathrm{Av}_* : \mathcal{C}^{I^-} \to \mathcal{C}^{I_1^-}$ is an equivalence (with inverse the natural !-averaging functor).

□

Combining this result with Lemma 4.2.0.1, we find that $\mathcal{C}^{G(O),x\text{–anti–temp}}$ is the kernel of the composition:

$$\mathcal{C}^{G(O)} \xrightarrow{\mathrm{Av}_*} \mathcal{C}^{I_1^-} \xrightarrow{\mathrm{Av}_!^\psi} \mathcal{C}^{\mathring{I}_1,\psi}. \tag{4.3.1}$$

4.4. **Parahoric bundles.** Let $P \subseteq G(K)$ be a compact open subgroup. Let $\mathrm{Bun}_G^{P\text{–lvl}} := \mathrm{Bun}_G^{\mathrm{level},x}/P$. For example, $\mathrm{Bun}_G^{G(O)\text{–lvl}} = \mathrm{Bun}_G$. We remark that $\mathrm{Bun}_G^{P\text{–lvl}}$ is always an Artin stack locally almost of finite type.

Remark 4.4.0.1. We care about exactly four cases: when $P = G(O)$, I^-, I_1^-, or $\mathcal{K}_1$. If $\check{\rho}$ is an (integral) coweight for G, e.g. the subgroups I^- and I_1^- are conjugate, so we can identify $\mathrm{Bun}_G^{I^-\text{–lvl}}$ and $\mathrm{Bun}_G^{I_1^-\text{–lvl}}$; in general, the latter can be thought of as a twisted form of the former, with only a mild technicality separating them.

4.4.1. *Ramified Higgs bundles.* Let P be a compact open subgroup as above. Let $(\mathcal{P}_G, \tau) \in \mathrm{Bun}_G^{P\text{–lvl}}$ be a point (the notation indicates that $\mathcal{P}_G$ is a G-bundle on $X \setminus x$ and τ is a reduction of the $G(K)$-bundle $\mathcal{P}_G|_{\mathring{\mathcal{D}}_x}$ to P).

The standard Serre duality argument shows that the cotangent space $T^*_{(\mathcal{P}_G,\tau)} \mathrm{Bun}_G^{P\text{–lvl}}$ consists of Higgs bundle $\varphi \in \Gamma(X \setminus x, \mathfrak{g}_{\mathcal{P}_G} \otimes \Omega^1_X)$ satisfying the condition:

$$(\mathcal{P}_G, \tau, \varphi|_{\mathring{\mathcal{D}}_x}) \in \mathrm{Lie}(P)^\perp/P \subseteq \mathfrak{g}((t))dt/P = \mathfrak{g}((t))^\vee/P.$$

We refer to such data as a *(P-)ramified Higgs bundles.*

4.4.2. We say that such a ramified Higgs bundle is *irregular* if the Higgs bundle $(\mathcal{P}_G|_{X\setminus x}, \varphi)$ on $X \setminus x$ is so. Note that irregular ramified Higgs bundles form a closed conical substack of $T^* \operatorname{Bun}_G^{P-\mathrm{lvl}}$. Therefore, we obtain the categories $D_{\mathrm{irreg}}(\operatorname{Bun}_G^{P-\mathrm{lvl}})$ and $\mathsf{Shv}_{\mathrm{irreg}}(\operatorname{Bun}_G^{P-\mathrm{lvl}})$ as in §4.1.

4.4.3. *Compatibility with the Hecke action.* Now suppose $P_1, P_2 \subseteq G(K)$ are a pair of parahoric subgroups. We can form the category $D(P_1\backslash G(K)/P_2)$ of (P_1, P_2)-equivariant D-modules on $G(K)$. There is a natural convolution action:

$$D(P_1\backslash G(K)/P_2) \otimes D(\operatorname{Bun}_G^{P_2-\mathrm{lvl}}) \to D(\operatorname{Bun}_G^{P_1-\mathrm{lvl}}).$$

Proposition 4.4.3.1. *The above convolution functor induces a functor:*

$$D(P_1\backslash G(K)/P_2) \otimes D_{\mathrm{irreg}}(\operatorname{Bun}_G^{P_2-\mathrm{lvl}}) \to D_{\mathrm{irreg}}(\operatorname{Bun}_G^{P_1-\mathrm{lvl}}).$$

In other words, in the parahoric setting, convolution preserves irregular singular support.

The proof of the proposition is identical to the proof of the Nadler-Yun theorem in [GKRV] Theorem B.5.2; we particularly refer to *loc. cit.* §B.6.6. We remark that the argument[26] in *loc. cit.* uses nothing about nilpotence. To be completely explicit: the argument in [GKRV] applies for any closed G-invariant conical subscheme $\Lambda \subseteq \mathfrak{g}$, where in *loc. cit.* $\Lambda = \mathcal{N}$, and for us here, $\Lambda = \mathfrak{g}_{\mathrm{irreg}}$.

Finally, we remark that only P_2 needs to be parahoric; we need ind-properness of $G(K)/P_2$ to control the singular support of the convolution.

4.5. **Mixing the ingredients.** We now prove Theorem 4.1.0.1.

4.5.1. *Step 1.* By (4.3.1), our goal is to show that the composition:

$$\mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}(\operatorname{Bun}_G) \subseteq D(\operatorname{Bun}_G) \to D(\operatorname{Bun}_G^{I_1^- -\mathrm{lvl}}) = D(\operatorname{Bun}_G^{\mathcal{K}_1-\mathrm{lvl}})^{B^-} \to D(\operatorname{Bun}_G^{\mathcal{K}_1-\mathrm{lvl}})^{N,\psi}$$

is zero.

By Proposition 4.4.3.1, the first functor maps $\mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}$ into $\mathsf{Shv}_{\mathrm{irreg}}(\operatorname{Bun}_G^{I_1^- -\mathrm{lvl}})$. (The right hand side may be replaced with $\mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}$ just as well, but this level of precision is not needed below.)

4.5.2. *Step 2.* Consider $\operatorname{Bun}_G^{\mathcal{K}_1-\mathrm{lvl}}$ as an algebraic stack acted on by G. The moment map:

$$\mu : T^* \operatorname{Bun}_G^{\mathcal{K}_1-\mathrm{lvl}} \to \mathfrak{g}^\vee \simeq \mathfrak{g}$$

sends a ramified Higgs bundle $(\mathcal{P}, \tau, \varphi)$ to $t \operatorname{Ad}_{\check{\rho}(t)}(\varphi) \bmod t$, where we note that $\varphi \in t^{-1} \operatorname{Ad}_{-\check{\rho}(t)} \mathfrak{g}[[t]]dt/\mathcal{K}_1$. Because irregularity is a closed condition, we see that any irregular φ maps into $\mathfrak{g}_{\mathrm{irreg}}$.

Therefore, it follows that we have an embedding:

$$\mathsf{Shv}_{\mathrm{irreg}}(\operatorname{Bun}_G^{\mathcal{K}_1-\mathrm{lvl}}) \subseteq \mathsf{Shv}_{G-\mathrm{irreg}}(\operatorname{Bun}_G^{\mathcal{K}_1-\mathrm{lvl}}).$$

Here for clarity, we say that the left hand side is defined in §4.4.2, and the right hand side is defined using the G-action as in §3.1.1.

[26] To be clear: we mean when the point is fixed, as in §B.6.6 in *loc. cit.* What follows there, regarding what is denoted ξ_X in *loc. cit.* and concerns variation in the point x, crucially uses nilpotence. But this is irrelevant for our purpose.

4.5.3. *Step 3.* We now conclude the argument.

By the above, we have a composition:

$$\mathsf{Shv}_{\mathcal{N}ilp_{\mathrm{irreg}}}(\mathrm{Bun}_G) \to \mathsf{Shv}_{\mathrm{irreg}}(\mathrm{Bun}_G^{\mathcal{K}_1^- -\mathrm{lvl}})^{B^-} \subseteq \mathsf{Shv}_{G-\mathrm{irreg}}(\mathrm{Bun}_G^{\mathcal{K}_1^- -\mathrm{lvl}})^{B^-} \to D(\mathrm{Bun}_G^{\mathcal{K}_1-\mathrm{lvl}})^{N,\psi}$$

and the latter functor is zero by Theorem 3.1.2.1. This concludes the argument.

Part 2. Microlocal properties of Whittaker coefficients

5. Background on coefficient functors

In this section, we review the classical geometric theory of Whittaker coefficients from [FGV1] and establish notation. This section houses no new results.

5.1. Moduli spaces.

5.1.1. We fix $\Omega_X^{\frac{1}{2}}$ a square root of the canonical sheaf on X. We obtain $\check{\rho}(\Omega_X^1) := (2\check{\rho})(\Omega_X^{\frac{1}{2}}) \in \mathrm{Bun}_T$.

We let Bun_N^{Ω} denote the fiber product:

$$\mathrm{Bun}_N^{\Omega} := \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \mathrm{Spec}(k)$$

where $\mathrm{Spec}(k) \to \mathrm{Bun}_T$ is $\check{\rho}(\Omega_X^1)$.

5.1.2. There is a standard character:

$$\psi : \mathrm{Bun}_N^{\Omega} \to \mathbf{A}^1.$$

We refer to [FGV1] §4.1.3 for its definition.

5.1.3. We also denote the canonical projection by:

$$\mathfrak{p} : \mathrm{Bun}_N^{\Omega} \to \mathrm{Bun}_G\,.$$

5.1.4. More generally, let D be a $\check{\Lambda}$-valued divisor on X.

We have a corresponding point $\mathcal{O}_X(D) \in \mathrm{Bun}_T$: it is characterized by the fact that for every weight $\lambda : T \to \mathbf{G}_m$, $\lambda(\mathcal{O}_X(D)) = \mathcal{O}_X(\lambda(D))$, where we note that $\lambda(D)$ is a usual ($\mathbf{Z}$-valued) divisor on X.

For a T-bundle $\mathcal{P}_T$, we let $\mathcal{P}_T(D)$ denote the image of $(\mathcal{P}_T, \mathcal{O}_X(D))$ under the multiplication $\mathrm{Bun}_T \times \mathrm{Bun}_T \to \mathrm{Bun}_T$.

We let $\mathrm{Bun}_N^{\Omega(D)}$ denote the fiber product:

$$\mathrm{Bun}_N^{\Omega(D)} := \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \mathrm{Spec}(k)$$

where this time we use the T-bundle $\check{\rho}(\Omega_X^1)(D)$.

5.1.5. For D a $\check{\Lambda}^+$-valued divisor, there is a canonical character:

$$\psi_D : \mathrm{Bun}_N^{\Omega(-D)} \to \mathbf{A}^1.$$

We again refer to [FGV1] §4.1.3 for its definition.

5.1.6. In this context, we let:

$$\mathfrak{p}_D : \mathrm{Bun}_N^{\Omega(-D)} \to \mathrm{Bun}_G$$

denote the canonical projection.

5.1.7. *Large divisors.* For later use, we record the following bounds.

Definition 5.1.7.1. We say D is *sufficiently large* if $\deg(D) \in \check{\Lambda}^+$ satisfies:

$$(\deg(D), \alpha) > g - 1 \tag{5.1.1}$$

for every positive[27] root $\alpha > 0$. (Here g is the genus of X.)

By reduction to the case of $\mathbf{G}_a$, it is immediate that:

Lemma 5.1.7.2. *For D sufficiently large, $\mathrm{Bun}_N^{\Omega(-D)}$ is an affine scheme.*

5.2. **Coefficient functors.**

5.2.1. *The primary Whittaker coefficient.* We define the *primary* Whittaker coefficient functor is the functor:

$$\mathrm{coeff} : D(\mathrm{Bun}_G) \to \mathsf{Vect}$$

defined by:

$$\mathcal{F} \mapsto \mathrm{C}_{\mathrm{dR}}\left(\mathrm{Bun}_N^{\Omega}, \mathfrak{p}^!(\mathcal{F}) \overset{!}{\otimes} \psi^!(\exp)\right)[-\dim \mathrm{Bun}_N^{\Omega}].$$

Remark 5.2.1.1. We include the above shift in the definition to make various formulae work out more nicely.

Remark 5.2.1.2. The above functor is usually called the *first* Whittaker coefficient. The terminology is borrowed from modular forms, where the above corresponds to the term a_1, cf. §1.1.1. This multiplicative normalization can be confusing in the geometric context, where *zeroth* Whittaker coefficient would be the more natural convention (since we index by divisors rather than their norms).

5.2.2. *Other Whittaker coefficients.* Now suppose D is a $\check{\Lambda}^+$-valued divisor on X. Define the functor:

$$\mathrm{coeff}_D : D(\mathrm{Bun}_G) \to \mathsf{Vect}$$

by the formula:

$$\mathcal{F} \mapsto \mathrm{C}_{\mathrm{dR}}\left(\mathrm{Bun}_N^{\Omega(-D)}, \mathfrak{p}_D^!(\mathcal{F}) \overset{!}{\otimes} \psi_D^!(\exp)\right)[-\dim \mathrm{Bun}_N^{\Omega(-D)}].$$

5.3. **The Casselman-Shalika formula.** We now recall the primary classical result in the subject.

Let D be a $\check{\Lambda}^+$-valued divisor on X as above. There is an associated object $\mathcal{V}^D$ of $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$; if $D = \sum \check{\lambda}_i \cdot x_i$ for some finite set of distinct points x_i, then $\mathcal{V}^D = \oplus V^{\check{\lambda}_{x_i}} \otimes \delta_{x_i}$.

We recall the following result from [FGV1], which is a geometric analogue[28] of the Casselman-Shalika formula from [CS].

[27]More economically, this condition for α simple obviously implies it for α positive.

[28]See [FGKV] for more discussion of the relationship between the results of [FGV1] and [CS].

Theorem 5.3.0.1 (Frenkel-Gaitsgory-Vilonen)**.** *There is a canonical isomorphism of functors:*

$$\mathrm{coeff}_D \simeq \mathrm{coeff}(\mathcal{V}^D \star -) : D(\mathrm{Bun}_G) \to \mathsf{Vect}.$$

Proof. For the sake of completeness, we include an argument deducing this result from [FGV1].

As above, suppose $D = \sum_{i=1}^n \check{\lambda}_i \cdot x_i$. Let $\mathbf{x}$ denote the collection of points x_i. As in [FGV1], we have usual the ind-stack:[29]

$$\overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}}.$$

There is a natural map $\mathfrak{p}_{\infty\cdot\mathbf{x}} : \overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}} \to \mathrm{Bun}_G$. There is a natural Hecke action of $\mathsf{Rep}(\check{G})^{\otimes n}$ on $D(\overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}})$ compatible with the Hecke action on Bun_G (corresponding to the points $x_1, \ldots, x_n$). We also note that $\mathcal{V}^D$ can evidently be considered as an object of $\mathsf{Rep}(\check{G})^{\otimes n}$.

There are natural locally closed embeddings:

$$\jmath : \mathrm{Bun}_N^{\Omega} \to \overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}}$$
$$\jmath_D : \mathrm{Bun}_N^{\Omega(-D)} \to \overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}}$$

compatible with the maps to Bun_G.

Let $\mathcal{V}^{D,\vee} = \mathcal{V}^{-w_0(D)}$ be the dual to $\mathcal{V}^D$ in $\mathsf{Rep}(\check{G})^{\otimes n}$. By [FGV1] Theorem 4 and Theorem 3 (2), we have:[30]

$$\mathcal{V}^{D,\vee} \star \jmath_{*,\mathrm{dR}}(\psi^!(\exp))[-\dim \mathrm{Bun}_N^{\Omega}] \simeq \jmath_{D,*,\mathrm{dR}}(\psi^!_D(\exp))[-\dim \mathrm{Bun}_N^{\Omega(-D)}].$$

Therefore, for $\mathcal{F} \in D(\mathrm{Bun}_G)$, we obtain:

$$\mathrm{coeff}_D(\mathcal{F}) := \mathrm{C}_{\mathrm{dR}}\left(\mathrm{Bun}_N^{\Omega(-D)}, \mathfrak{p}^!_D(\mathcal{F}) \overset{!}{\otimes} \psi^!_D(\exp)\right)[-\dim \mathrm{Bun}_N^{\Omega(-D)}] =$$
$$\mathrm{C}_{\mathrm{dR}}\left(\overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}}, \mathfrak{p}^!_{\infty\cdot\mathbf{x}}(\mathcal{F}) \overset{!}{\otimes} \jmath_{D,*,\mathrm{dR}}(\psi^!_D(\exp))\right)[-\dim \mathrm{Bun}_N^{\Omega(-D)}] =$$
$$\mathrm{C}_{\mathrm{dR}}\left(\overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}}, \mathfrak{p}^!_{\infty\cdot\mathbf{x}}(\mathcal{F}) \overset{!}{\otimes} \left(\mathcal{V}^{D,\vee} \star \jmath_{*,\mathrm{dR}}(\psi^!(\exp))\right)\right)[-\dim \mathrm{Bun}_N^{\Omega}] =$$
$$\mathrm{C}_{\mathrm{dR}}\left(\overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}}, \left(\mathcal{V}^D \star \mathfrak{p}^!_{\infty\cdot\mathbf{x}}(\mathcal{F})\right) \overset{!}{\otimes} \jmath_{*,\mathrm{dR}}(\psi^!(\exp))\right)[-\dim \mathrm{Bun}_N^{\Omega}] =$$
$$\mathrm{C}_{\mathrm{dR}}\left(\overline{\mathrm{Bun}}_N^{\Omega,\infty\cdot\mathbf{x}}, \mathfrak{p}^!_{\infty\cdot\mathbf{x}}(\mathcal{V}^D \star \mathcal{F}) \overset{!}{\otimes} \jmath_{*,\mathrm{dR}}(\psi^!(\exp))\right)[-\dim \mathrm{Bun}_N^{\Omega}] =$$
$$\mathrm{C}_{\mathrm{dR}}\left(\mathrm{Bun}_N^{\Omega}, \mathfrak{p}^!(\mathcal{V}^D \star \mathcal{F})\right) \overset{!}{\otimes} \jmath_{*,\mathrm{dR}}(\psi^!(\exp)))[-\dim \mathrm{Bun}_N^{\Omega}] =: \mathrm{coeff}(\mathcal{V}^D \star \mathcal{F}).$$

□

Remark 5.3.0.2. The above assertions admit natural generalizations where the divisors are along to vary in moduli, even over Ran space; we omit the statement here as we do not need it.

5.4. **More notation.** In the remainder of this section, we briefly introduce more notation.

[29]It is defined in [FGV1] §2.3, where it would be denoted ${}_{\mathbf{x},\infty}\mathrm{Bun}_N^{\check{\rho}(\Omega^1_X)}$.

[30]In fact, there is a sign choice to be made once and for all in defining the Hecke action on $D(\mathrm{Bun}_G)$; roughly speaking, the difference is whether we think of the Hecke category as acting naturally on the left or right, where in the latter case we use the inversion automorphism to change a right module structure to a left module structure. We have implicitly pinned down this choice in the statement of the formula. By contrast, the reader who looks in [FGV1] will find statements for each of the two possible choices.

5.4.1. *The Poincaré sheaf.* We let $\mathcal{P}oinc_! \in D(\mathrm{Bun}_G)$ be the object corepresenting coeff. Explicitly, we have:

$$\mathcal{P}oinc_! = \mathfrak{p}_!((-\psi)^*(\exp[-2]))[\dim \mathrm{Bun}_N^\Omega]. \tag{5.4.1}$$

In other words, we take the character sheaf (or its inverse) on Bun_N^Ω, cohomologically normalized to be perverse, and !-push it forward to Bun_G; this makes sense by holonomicity.

Convention 5.4.1.1. We use the subscript ! to remind that a lower-! functor appears in the formation of $\mathcal{P}oinc_!$. We use similar notation in related settings without further mention. We remark that coeff would be denoted coeff_* in this regime; we omit the $*$ for brevity, given how often the coeff functor is used in this work.

5.4.2. *The !-coefficient functor.* Let $D_{\mathrm{hol}}(\mathrm{Bun}_G) \subseteq D(\mathrm{Bun}_G)$ denote the category of (ind-)holonomic D-modules on Bun_G, i.e., D-modules $\mathcal{F} \in D(\mathrm{Bun}_G)$ such that for any $\pi : S \to \mathrm{Bun}_G$ with S affine, $\pi^!(\mathcal{F}) \in D(S)$ is (ind-)holonomic. We remark that $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \subseteq D_{\mathrm{hol}}(\mathrm{Bun}_G)$.

Then we have a functor:

$$\mathrm{coeff}_! : D_{\mathrm{hol}}(\mathrm{Bun}_G) \to \mathsf{Vect}$$
$$\mathcal{F} \mapsto C_{\mathrm{dR},c}\big(\mathrm{Bun}_N^\Omega, \mathfrak{p}^*(\mathcal{F}) \overset{*}{\otimes} \psi^*(\exp[-2])\big)[\dim \mathrm{Bun}_N^\Omega].$$

In other words, $\mathrm{coeff}_!$ is the Verdier conjugate to coeff. That is, we have:

Lemma 5.4.2.1. *For $\mathcal{F} \in D_{\mathrm{hol}}(\mathrm{Bun}_G)$ be locally compact with Verdier dual*[31] $\mathbf{D}^{\mathrm{Verdier}}\mathcal{F} \in D_{\mathrm{hol}}(\mathrm{Bun}_G)$, *we have:*

$$\mathrm{coeff}(\mathcal{F}) = \mathrm{coeff}_!(\mathbf{D}^{\mathrm{Verdier}}\mathcal{F})^\vee.$$

5.4.3. Similarly, we have functors:

$$\mathrm{coeff}_{D,!} : D_{\mathrm{hol}}(\mathrm{Bun}_G) \to \mathsf{Vect}.$$

The Verdier dual to Theorem 5.3.0.1 asserts:

Corollary 5.4.3.1. *There is a canonical isomorphism of functors:*

$$\mathrm{coeff}_{D,!} \simeq \mathrm{coeff}_!(\mathcal{V}^D \star -) : D_{\mathrm{hol}}(\mathrm{Bun}_G) \to \mathsf{Vect}.$$

This follows from the good properties of Hecke functors: see [AGKRRV3] §1.2.3.

6. The index formula

6.1. **Statement of the theorem.**

[31]Unlike e.g. [AGKRRV2], we consider Verdier duality as mapping locally compact D-modules on Bun_G to locally compact D-modules; it can be computed smooth locally by usual Verdier duality on schemes. By contrast, *loc. cit.* considers a smarter construction, sending compact D-modules to compact objects of $D(\mathrm{Bun}_G)^\vee$. The smarter construction from *loc. cit.* recovers ours after applying the functor $\mathrm{Id}^{\mathrm{naive}} : D(\mathrm{Bun}_G)^\vee \to D(\mathrm{Bun}_G)$ from *loc. cit.*

6.1.1. *Kostant invariant.* Let $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{constr}}$.

Let $\mathrm{Irr}(\mathcal{N}ilp)$ denote the set of irreducible components of $\mathcal{N}ilp$. Recall that $\mathcal{F}$ has a characteristic cycle:

$$\mathrm{CC}(\mathcal{F}) = \sum_{\alpha \in \mathrm{Irr}(\mathcal{N}ilp)} c_{\alpha,\mathcal{F}}[\alpha]$$

for $c_{\alpha,\mathcal{F}} \in \mathbf{Z}$; here $[\alpha]$ is the class of the component α in the group of cycles.[32]

For $\alpha = \mathcal{N}ilp^{\mathrm{Kos}}$, we use the abbreviation:

$$c_{\mathrm{Kos},\mathcal{F}} := c_{\mathcal{N}ilp^{\mathrm{Kos}},\mathcal{F}} \in \mathbf{Z}$$

for the multiplicity of the characteristic cycle of $\mathcal{F}$ at the Kostant component (see §2.5.6).

6.1.2. *Index formula.* For $\mathcal{F}$ constructible as above, we may form $\mathrm{coeff}(\mathcal{F}) \in \mathsf{Vect}$. Because $\mathcal{F}$ is constructible, this object is compact, so it has a well-defined Euler characteristic $\chi(\mathrm{coeff}(\mathcal{F})) \in \mathbf{Z}$.

The purpose of this section is to prove the following result.

Theorem 6.1.2.1. *There is a sign $\epsilon = \epsilon_{G,X} \in \{1, -1\}$ (depending only on G and the genus of the curve X) such that for $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{constr}}$, we have the equality of integers:*

$$\chi(\mathrm{coeff}(\mathcal{F})) = \epsilon \cdot c_{\mathrm{Kos},\mathcal{F}}.$$

Specifically, the sign ϵ is:

$$\varepsilon = (-1)^{\dim \mathrm{Bun}_G}.$$

We will prove this theorem using filtered D-modules, as is natural for a problem on characteristic cycles.

The argument we give passes through D-modules with filtrations that are not good, so is unlikely to have an easy analogue in other sheaf-theoretic settings.

6.2. **Filtered D-modules on stacks.**

6.2.1. We briefly develop the theory here, for lack of a good reference. Let $\mathcal{Y}$ be a smooth algebraic stack below.

6.2.2. Recall from [GR3] Example 2.4.3 that there is a canonical prestack[33] $\mathcal{Y}_{\mathrm{dR},\hbar}$ with a map $\pi : \mathcal{Y}_{\mathrm{dR},\hbar} \to \mathbf{A}^1_\hbar/\mathbf{G}_m$ so that $\pi^{-1}(1) \simeq \mathcal{Y}_{\mathrm{dR}}$ and $\pi^{-1}(0) = \mathbf{B}_{\mathcal{Y}}(T(\mathcal{Y})^\wedge_0)$ is the classifying stack for the tangent space of $\mathcal{Y}$ formally completed along its zero section. Filtered D-modules on $\mathcal{Y}$ are by definition ind-coherent sheaves on $\mathcal{Y}_{\mathrm{dR},\hbar}$; we denote the category by $\mathsf{Fil}\, D(\mathcal{Y})$. For a filtered D-module $F_\bullet\mathcal{F} \in \mathsf{Fil}\, D(\mathcal{Y})$, its underlying object $\mathcal{F}$ is the fiber at the open point $1 \in \mathbf{A}^1_\hbar/\mathbf{G}_m$, using the equivalence (or definition) $\mathsf{IndCoh}(\mathcal{Y}_{\mathrm{dR}}) \simeq D(\mathcal{Y})$. We may form the associated graded $\mathrm{gr}_\bullet\, \mathcal{F} \in \mathsf{QCoh}(T^*\mathcal{Y})$ by taking the fiber at $\hbar = 0$ and applying Koszul duality[34] $\mathsf{IndCoh}(\mathbf{B}_{\mathcal{Y}}(T(\mathcal{Y})^\wedge_0)) \simeq \mathsf{QCoh}(T^*\mathcal{Y})$.[35] We remind that $T^*\mathcal{Y}$ is a derived stack in general. For $\mathcal{Y}$ a smooth scheme, the comparison results in [GR3] show that the above notion corresponds to the usual notion.

[32]Technically, our group of cycles is completed here: it is the inverse limit over U of free abelian groups on cycles in T^*U for $U \subseteq \mathrm{Bun}_G$ a quasi-compact open. In particular, the infinite sum displayed above has a clear meaning.

[33]It is denotes $(\mathcal{Y}_{\mathrm{dR}})_{\mathrm{scaled}}$ in [GR3].

[34]Smoothness of $\mathcal{Y}$ is needed here.

[35]For a slower introduction to this circle of ideas, we refer to [Ras4] Appendix A.

6.2.3. For a morphism $f : \mathcal{Y} \to \mathcal{Z}$, we form the usual correspondence:

$$\begin{tikzcd} & T^*\mathcal{Z} \underset{\mathcal{Z}}{\times} \mathcal{Y} \arrow[dl, "Df"'] \arrow[dr, "\pi"] & \\ T^*\mathcal{Y} & & T^*\mathcal{Z}. \end{tikzcd}$$

Then for $\mathcal{F} \in \mathsf{Fil}\, D(\mathcal{Y})$ (resp. $\mathcal{G} \in \mathsf{Fil}\, D(\mathcal{Z})$), $f_{*,\mathrm{dR}}(\mathcal{F})$ (resp. $f^!(\mathcal{G})$) inherits a canonical filtration, and there are natural identifications:

$$\begin{aligned} \mathrm{gr}_\bullet\, f_{*,\mathrm{ren}}(\mathcal{F}) &\simeq \pi_* Df^*(\mathrm{gr}_\bullet\, \mathcal{F}). \\ \mathrm{gr}_\bullet\, f^!(\mathcal{G}) &\simeq Df_* \pi^{!,\mathsf{QCoh}}(\mathrm{gr}_\bullet\, \mathcal{G}). \end{aligned} \tag{6.2.1}$$

We remark that as $\mathcal{Y}$ and $\mathcal{Z}$ are smooth, π is a quasi-smooth morphism, so $\pi^{!,\mathsf{QCoh}}$ is defined. In the above, $f_{*,\mathrm{ren}}$ is the *renormalized* de Rham pushforward from [DG1]. We remind here that renormalized pushforward coincides with de Rham pushforward for morphisms representable in stacks with only unipotent stabilizers; this is the only case in which we will consider this construction.

6.2.4. *Good filtrations.* We say a filtration $F_\bullet\mathcal{F}$ on $\mathcal{F} \in D(\mathcal{Y})$ is a *good filtration* if for any $p : U \to \mathcal{Y}$ with p smooth and U affine, the induced filtration on $p^!\mathcal{F}$ is a good filtration (equivalently: the filtration is bounded from below and $\mathrm{gr}_\bullet\, p^!\mathcal{F} \in \mathsf{Perf}(T^*U)$). By (6.2.1), this is equivalent to the filtration being bounded from below with $\mathrm{gr}_\bullet\, \mathcal{F} \in \mathsf{Coh}(T^*\mathcal{Y})$.

Clearly if $\mathcal{F}$ admits a good filtration, it is locally compact (cf. §2.2.2). Conversely, we have:

Lemma 6.2.4.1. *Suppose $\mathcal{Y}$ is QCA and $\mathcal{F} \in D(\mathcal{Y})^\heartsuit$ is locally compact. Then $\mathcal{F}$ admits a good filtration.*

Proof. By [DG1] Theorem 0.4.5, there exists $\mathcal{G} \in \mathsf{Coh}(\mathcal{Y})^\heartsuit$ and a map $\mathrm{ind}(\mathcal{G}) \to \mathcal{F} \in D(\mathcal{Y})$ that is an epimorphism on H^0 (where $\mathrm{ind} : \mathsf{IndCoh}(\mathcal{Y}) \to D(\mathcal{Y})$ is the D-module induction functor). We immediately see that $\mathrm{ind}(\mathcal{G})$ admits a good filtration, which we denote by $F_\bullet\,\mathrm{ind}(\mathcal{G})$. By adjunction, we then obtain a map:

$$\alpha : F_\bullet\,\mathrm{ind}(\mathcal{G}) \to \mathcal{F}^{\mathrm{cnst}} \in \mathsf{Fil}\, D(\mathcal{Y})$$

where the right hand side denotes $\mathcal{F}$ with the "constant" filtration (informally: $F_i\mathcal{F} = \mathcal{F}$ for all $i \in \mathbf{Z}$).

Now observe that $\mathsf{Fil}\, D(\mathcal{Y})$ has a natural t-structure such that the forgetful functor to $\mathsf{IndCoh}(\mathcal{Y} \times \mathbf{A}^1_\hbar/\mathbf{G}_m)$ is t-exact.[36]

We can then form the image of $H^0(\alpha)$ in $\mathsf{Fil}\, D(\mathcal{Y})^\heartsuit$. By exactness of the functor $\mathsf{Fil}\, D(\mathcal{Y}) \to D(\mathcal{Y})$ (forgetting the filtration), $H^0(\alpha)$ is a filtration on $\mathcal{F}$. It is immediate to see that the induced filtration on $\mathcal{F}$ is good (and in fact: $\mathrm{gr}_\bullet\, \mathcal{F}$ lies in degree 0, i.e., it is a filtration in the *abelian* categorical sense, not just the derived categorical sense).

□

6.3. **Twisted Hodge-de Rham spectral sequences.**

[36]This t-structure may be constructed directly from the case of smooth schemes. Alternatively, one may observe that $\mathcal{Y} \times \mathbf{A}^1_\hbar/\mathbf{G}_m \to \mathcal{Y}_{\mathrm{dR},\hbar}$ has a connective relative tangent complex; therefore, by [GR3] §9, the corresponding monad on $\mathsf{IndCoh}(\mathcal{Y} \times \mathbf{A}^1_\hbar/\mathbf{G}_m)$ is right t-exact, and the claim follows on general grounds.

6.3.1. We now discuss Hodge filtrations adapted to exponential cohomology. As the proof shows, the construction is by the (standard) Kazhdan-Kostant technique. Unlike usual Hodge filtrations, there are subtleties about convergence in the twisted setting.

Lemma 6.3.1.1. *Fix an integer $r > 0$. Below, we always consider $\mathbf{A}^1$ as equipped with the $\mathbf{G}_m$-action that is the rth power of the action by homotheties.*

Let $F_\bullet \mathcal{F} \in \mathsf{Fil}\, D(\mathbf{A}^1)^{\mathbf{G}_m, w}$ be a filtered D-module on $\mathbf{A}^1$ with a compatible $\mathbf{Z}$-grading.

(1) $\mathrm{C}_{\mathrm{dR}}(\mathbf{A}^1, \mathcal{F} \overset{!}{\otimes} \exp) \in \mathsf{Vect}$ *has a canonical filtration $F_\bullet^{\mathrm{KK}}$ such that:*
$$\mathrm{gr}_\bullet^{\mathrm{KK}} \mathrm{C}_{\mathrm{dR}}(\mathbf{A}^1, \mathcal{F} \overset{!}{\otimes} \exp) \simeq \Gamma(\mathbf{A}^1, dt^* \operatorname{gr}_\bullet \mathcal{F})$$
where dt is considered as a section $\mathbf{A}^1 \to T^*\mathbf{A}^1$ and dt^* indicates pullback along this morphism.

(2) *Observe that $\Gamma(\mathbf{A}^1, 0^*(\operatorname{gr}_\bullet \mathcal{F}))$ carries a natural grading coming from that of $F_\bullet \mathcal{F}$. We write the ith graded piece of this complex as:*
$$\Gamma(\mathbf{A}^1, 0^*(\operatorname{gr}_\bullet(\mathcal{F})))_i.$$
We sometimes refer to this as the secondary *grading, which should not be confused with the grading $\operatorname{gr}_\bullet \mathcal{F} = \oplus_i \operatorname{gr}_i \mathcal{F}$.*

Suppose that:

- *The filtration $F_\bullet \mathcal{F}$ is bounded below, i.e., $F_i \mathcal{F} = 0$ for $i \ll 0$.*
- *The secondary grading is bounded from below, i.e., there exists an integer N such that:*[37]
$$\Gamma\big(\mathbf{A}^1, 0^*(\operatorname{gr}_\bullet \mathcal{F})\big)_i = 0 \tag{6.3.1}$$
for all $i < -N$.

Then the the filtration $F_\bullet^{\mathrm{KK}}$ is bounded from below, i.e., we have:
$$F_i^{\mathrm{KK}} \mathrm{C}_{\mathrm{dR}}(\mathbf{A}^1, \mathcal{F} \overset{!}{\otimes} \exp) = 0 \tag{6.3.2}$$
for $i \ll 0$.

(3) *In the setting of* (2)*, suppose instead that:*

- *The filtration $F_\bullet \mathcal{F}$ is bounded below.*
- *For every integer n, the second grading on:*
$$\tau^{>-n}\Gamma\big(\mathbf{A}^1, 0^*(\operatorname{gr}_\bullet \mathcal{F})\big) \tag{6.3.3}$$
is bounded from below, i.e., for $i \ll 0$ (with bound depending on n), we have:
$$\tau^{>-n}\Gamma\big(\mathbf{A}^1, 0^*(\operatorname{gr}_\bullet \mathcal{F})\big)_i = 0.$$
- $\mathrm{gr}_\bullet^{\mathrm{KK}}$ *is bounded from below, i.e., for $i \ll 0$, we have:*
$$\mathrm{gr}_i^{\mathrm{KK}} \mathrm{C}_{\mathrm{dR}}(\mathbf{A}^1, \mathcal{F} \overset{!}{\otimes} \exp) = 0.$$

Then the same conclusion holds: the filtration $F_\bullet^{\mathrm{KK}}$ is bounded below.

Proof. The existence of the filtration in (1) is part of the general formalism of Kazhdan-Kostant filtrations, cf. [Ras4] §A.5. For the reader's convenience, we make this explicit in our specific setting.

Namely, we consider $\mathcal{F}$ as a left module over the Weyl algebra, which has generators t and ∂_t. We normalize signs so that $\deg(t) = r$ and $\deg(\partial_t) = -r$. We also write t and ∂_t for the corresponding

[37]In fact, the proof shows a weaker condition suffices: we can assume $\Gamma\big(\mathbf{A}^1, 0^*(\operatorname{gr}_j \mathcal{F})\big)_i = 0$ for every $j < -i - N$, remarking that $\operatorname{gr}_j = 0$ for $j \ll 0$ by assumption on the filtration on $\mathcal{F}$. But in examples, it appears this condition is always verified in the form stated in the lemma.

functions on $T^*\mathbf{A}^1$, expecting confusion will not arise. Ignoring higher coherences (which are not actually needed for us), that $\mathcal{F}$ is a weakly $\mathbf{G}_m$-equivariant filtered D-module means the following. First, $\mathcal{F}$ has a grading:

$$\mathcal{F} = \underset{j\in\mathbf{Z}}{\oplus} \mathcal{F}_j.$$

This grading is compatible with the filtration in the sense that it refines to gradings:

$$F_i\mathcal{F} = \underset{j\in\mathbf{Z}}{\oplus} (F_i\mathcal{F})_j.$$

Moreover, the actions of t and ∂_t on $\mathcal{F}$ refine to maps:

$$t : (F_i\mathcal{F})_j \to (F_i\mathcal{F})_{j+r}$$
$$\partial_t : (F_i\mathcal{F})_j \to (F_{i+1}\mathcal{F})_{j-r}.$$

Note that $\mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp)$ is explicitly realized as the homotopy cokernel (i.e., cone):

$$\mathrm{Coker}(\mathcal{F} \xrightarrow{\partial_t - \mathrm{id}} \mathcal{F}).$$

The *Kazhdan-Kostant filtration* is defined by the following formula:

$$F_i^{\mathrm{KK}} \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) := \mathrm{Coker}\left(\underset{q\in\mathbf{Z}}{\oplus} (F_{\lfloor \frac{i-q}{r} \rfloor}\mathcal{F})_q \xrightarrow{\partial_t - \mathrm{id}} \underset{q\in\mathbf{Z}}{\oplus} (F_{\lfloor \frac{i-q}{r} \rfloor}\mathcal{F})_q \right).$$

Here we note that ∂_t maps $(F_{\lfloor \frac{i-q}{r} \rfloor}\mathcal{F})_q$ to $(F_{\lfloor \frac{i-q}{r} \rfloor+1}\mathcal{F})_{q-r}$, which is one of the summands of the rightmost term.

We comment briefly on the definition. One can rewrite:

$$\underset{q\in\mathbf{Z}}{\oplus} (F_{\lfloor \frac{i-q}{r} \rfloor}\mathcal{F})_q$$

as:

$$\underset{(p,q)\in\mathbf{Z}^2, rp+q\leq i}{\mathrm{colim}} (F_p\mathcal{F})_q \tag{6.3.4}$$

to remove the floor function; here the indexing category is a poset with $(p,q) \leq (p',q') \Leftrightarrow p \leq p', q = q'$. We also note that for $r = 1$, the above formula takes the simpler form:

$$F_i^{\mathrm{KK}} \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) := \mathrm{Coker}\left(\underset{q\in\mathbf{Z}}{\oplus} (F_{i-q}\mathcal{F})_q \xrightarrow{\partial_t - \mathrm{id}} \underset{q\in\mathbf{Z}}{\oplus} (F_{i-q}\mathcal{F})_q \right).$$

We clearly have:

$$\mathrm{gr}_i^{\mathrm{KK}} \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) = \mathrm{Coker}\left(\underset{q\in i+r\mathbf{Z}}{\oplus} (\mathrm{gr}_{\frac{i-q}{r}} \mathcal{F})_q \xrightarrow{\partial_t - \mathrm{id}} \underset{q\in i+r\mathbf{Z}}{\oplus} (\mathrm{gr}_{\frac{i-q}{r}} \mathcal{F})_q \right) =$$
$$\mathrm{Coker}\left(\underset{p\in\mathbf{Z}}{\oplus} (\mathrm{gr}_p \mathcal{F})_{i-rp} \xrightarrow{\partial_t - \mathrm{id}} \underset{p\in\mathbf{Z}}{\oplus} (\mathrm{gr}_p \mathcal{F})_{i-rp} \right)$$

so that:

$$\mathrm{gr}_\bullet^{\mathrm{KK}} \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) = \mathrm{Coker}\left(\mathrm{gr}_\bullet \mathcal{F} \xrightarrow{\partial_t - \mathrm{id}} \mathrm{gr}_\bullet \mathcal{F} \right) = \Gamma(\mathbf{A}^1, dt^* \mathrm{gr}_\bullet \mathcal{F})$$

yielding (1).

We now turn to the question of *spectral sequence convergence*, i.e., to (2) and (3). We focus first on (2).

First, we note that we can shift the filtration to assume that $F_j\mathcal{F} = 0$ for $j < 0$. Similarly, we can shift the grading to assume the integer N equals zero. We will then prove that (6.3.2) holds for $i < -r$.

Observe that in the above notation, (6.3.1) explicitly says that:

$$\partial_t : (\mathrm{gr}_j \mathcal{F})_i \to (\mathrm{gr}_{j+1} \mathcal{F})_{i-r} \tag{6.3.5}$$

is an isomorphism for $i < -N = 0$.

Now fix an index i. We define a further filtration:

$$\widetilde{F}_j F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp)$$

by the formula:

$$\widetilde{F}_j F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) := \mathrm{Coker}\Big(\bigoplus_{q \in \mathbf{Z}} (F_{\min\{j, \lfloor \frac{i-q}{r} \rfloor\}} \mathcal{F})_q \xrightarrow{\partial_t - \mathrm{id}} \bigoplus_{q \in \mathbf{Z}} (F_{\min\{j+1, \lfloor \frac{i-q}{r} \rfloor\}} \mathcal{F})_q \Big) =$$
$$\mathrm{Coker}\Big(\underset{\substack{(p,q) \in \mathbf{Z}^2, rp+q \leq i \\ p \leq j}}{\mathrm{colim}} (F_p\mathcal{F})_q \to \underset{\substack{(p,q) \in \mathbf{Z}^2, rp+q \leq i \\ p \leq j+1}}{\mathrm{colim}} (F_p\mathcal{F})_q \Big)$$

where the last expression is as in (6.3.4). This is clearly a filtration, i.e., the colimit over j yields $F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp)$. Moreover, the above term vanishes for $j < -1$ as the filtration on $\mathcal{F}$ is non-negative. Therefore, it suffices to show that when $i < -r$, the structure maps:

$$\alpha_{i,j} : \widetilde{F}_{j-1} F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) \to \widetilde{F}_j F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp)$$

are isomorphisms for all j.

We clearly have:

$$\begin{aligned} \widetilde{\mathrm{gr}}_j F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) &:= \mathrm{Coker}(\alpha_{i,j}) = \\ \bigoplus_{q \leq i - rj} \Big(\mathrm{Coker}\big((\mathrm{gr}_j \mathcal{F})_q &\xrightarrow{\partial_t} (\mathrm{gr}_{j+1} \mathcal{F})_{q-r} \big) \Big). \end{aligned} \tag{6.3.6}$$

By assumption, this complex is acyclic for $i - rj < 0$ (cf. (6.3.5)). Otherwise, $rj \leq i$. As $i < -r$, this means that $j + 1$ is negative, so $\mathrm{gr}_j \mathcal{F} = \mathrm{gr}_{j+1} \mathcal{F} = 0$, clearly yielding that $\widetilde{\mathrm{gr}}_j = 0$, and so completing the argument.

We now turn to (3). We again normalize so that $F_i\mathcal{F} = 0$ for $i < 0$ and $\mathrm{gr}_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) = 0$ for $i < 0$.

We aim to show that:

$$F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) = 0$$

when $i < 0$. By assumption, we have isomorphisms:

$$\ldots \xrightarrow{\simeq} F_{-2}^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) \xrightarrow{\simeq} F_{-1}^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp). \tag{6.3.7}$$

Therefore, it suffices to show that for every integer n, there is an index $i \ll 0$ such that:

$$F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) \in \mathsf{Vect}^{\leq -n}.$$

Indeed, then the stabilized complex in (6.3.7) is in $\cap_n \mathsf{Vect}^{\leq -n} = 0$.

By our assumption, there exists an integer M such that for $i < -M$, we have:

$$\operatorname{Coker}\left(\partial_t : (\operatorname{gr}_j \mathcal{F})_i \to (\operatorname{gr}_{j+1} \mathcal{F})_{i-r}\right) \in \mathsf{Vect}^{\leq -n}. \tag{6.3.8}$$

Then we will show that for $i < -M - r$, F_i^{KK} is in $\mathsf{Vect}^{\leq -n}$.

As in the proof of (2), there exists a filtration $\widetilde{F}_\bullet$ on F_i^{KK} with associated graded terms computed by (6.3.6). It follows that for $i - rj < -M$, we have:

$$\widetilde{\operatorname{gr}}_j F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) \in \mathsf{Vect}^{\leq -n}.$$

On the other hand, if $i - rj \geq -M$, then our assumption $i < -M - r$ forces $0 > j + 1$, which as before, forces:

$$\widetilde{\operatorname{gr}}_j F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) = 0.$$

As:

$$\widetilde{F}_j F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) = 0$$

for $j < -1$, the above analysis implies that:

$$F_i^{\mathrm{KK}} \, \mathrm{C}_{\mathrm{dR}}(\mathcal{F} \overset{!}{\otimes} \exp) \in \mathsf{Vect}^{\leq -n}$$

which is what was to be shown.

□

Remark 6.3.1.2. The above method for verifying the convergence of a "Kazhdan-Kostant spectral sequence" are taken from the proof of [Ras4] Theorem 4.2.1. Indeed, the hypothesis and argument for (2) above are Step 6 from the proof of *loc. cit.* rendered into the present setting, and (3) is a natural variant.

6.4. **Proof of Theorem 6.1.2.1.**

6.4.1. We now turn to the proof of the theorem. We will first prove the theorem modulo a problem of spectral sequence convergence; the remainder of the section will verify this convergence.

6.4.2. It clearly suffices to show the result when $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{constr},\heartsuit}$. In this case, take a good filtration on $\mathcal{F}$ by applying Lemma 6.2.4.1.

Note that $\mathrm{Bun}_N^\Omega \to \mathrm{Bun}_G$ factors through $\mathrm{Bun}_N^\Omega / T$. Moreover, note that the map $\psi : \mathrm{Bun}_N^\Omega \to \mathbf{A}^1$ is $\mathbf{G}_m$-equivariant, using the action of $\mathbf{G}_m$ on the source via $2\check{\rho} : \mathbf{G}_m \to T$ and the $\mathbf{G}_m$-action which is the square of the homothety action on $\mathbf{A}^1$ (so $r = 2$ in our references to Lemma 6.3.1.1).

By (6.2.1) and Lemma 6.3.1.1, we see that $\operatorname{coeff}(\mathcal{F})[\dim \mathrm{Bun}_N^\Omega]$ has a canonical filtration $F_\bullet^{\mathrm{KK}}$ such that its associated graded is essentially computed by composing the correspondences:

$$\begin{tikzcd}[column sep=small]
 & T^* \mathrm{Bun}_G \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_N^\Omega \arrow[dl] \arrow[dr] & & \mathrm{Bun}_N^\Omega \arrow[dl, "d\psi"'] \arrow[dr] & \\
T^* \mathrm{Bun}_G & & T^* \mathrm{Bun}_N^\Omega & & \mathrm{Spec}(k).
\end{tikzcd}$$

Here *essentially* means the following: we are supposed to apply upper-! along the first leftward arrow and upper-$*$ along the second leftward arrow; however, for the first arrow, upper-! and upper-$*$ differ by tensoring with a graded line bundle, so up to this discrepancy, we can compose the correspondences well by base-change.

The composed correspondence is:

$$\begin{tikzcd} & \mathrm{Kos}_G^{\mathrm{glob}} \arrow[dl, "\sigma"'] \arrow[dr] & \\ T^* \mathrm{Bun}_G & & \mathrm{Spec}(k) \end{tikzcd}$$

where $\mathrm{Kos}_G^{\mathrm{glob}}$ is the global Kostant section; indeed, this is essentially the definition of the Kostant section. Note that $\mathcal{N}ilp \times_{\mathrm{Bun}_G} \mathrm{Kos}_G^{\mathrm{glob}} = \mathrm{Spec}(k)$ (as derived stacks), mapping to $\mathcal{N}ilp$ via $f^{\mathrm{glob}} \in \mathcal{N}ilp^{\mathrm{Kos}}$.

Now suppose $\mathcal{G} \in \mathsf{Coh}(\mathcal{N}ilp)^{\heartsuit}$. As $\mathcal{N}ilp^{\mathrm{Kos}}$ is smooth (all of $\mathcal{N}\mathring{i}lp^{\mathrm{reg}}$ is) and connected, the Euler characteristic of the (derived) fibers of $\mathcal{G}$ at points of $\mathcal{N}\mathring{i}lp^{\mathrm{Kos}}$ are constant. In particular, if $\iota : \mathcal{N}ilp \to T^* \mathrm{Bun}_G$ is the embedding, we see that the Euler characteristic of $\Gamma(\mathrm{Kos}_G^{\mathrm{glob}}, \sigma^* \iota_* \mathcal{G})$ is the rank of $\mathcal{G}$ at the generic point of $\mathcal{N}ilp^{\mathrm{Kos}}$. More generally, we deduce that for $\mathcal{H} \in \mathsf{Coh}(T^* \mathrm{Bun}_G)^{\heartsuit}$ set-theoretically supported on $\mathcal{N}ilp$, the Euler characteristic of $\Gamma(\mathrm{Kos}_G^{\mathrm{glob}}, \sigma^* \mathcal{H})$ is the multiplicity of $\mathcal{H}$ at the generic point of $\mathcal{N}\mathring{i}lp^{\mathrm{Kos}}$.

As we have a *good* filtration on $\mathcal{F}$, we see that $\mathrm{gr}_\bullet \mathcal{F}$ is set-theoretically supported on $\mathcal{N}ilp \supseteq \mathrm{SS}(\mathcal{F})$. Applying the definition of characteristic cycle, we now obtain:

$$\chi(\Gamma(\mathrm{Kos}_G^{\mathrm{glob}}, \sigma^* \mathrm{gr}_\bullet \mathcal{F})) = c_{\mathrm{Kos},\mathcal{F}}.$$

Reincorporating the twist by the graded line bundle discussed above, we see that:

$$\chi(\mathrm{gr}_\bullet^{\mathrm{KK}} \mathrm{coeff}(\mathcal{F})) = \epsilon \cdot c_{\mathrm{Kos},\mathcal{F}}.$$

It remains to see that the same equation holds for $\mathrm{coeff}(\mathcal{F})$ itself. For this, it suffices to see that its filtration $F_\bullet^{\mathrm{KK}}$ is bounded from below. We check this below using a convergence criterion from Lemma 6.3.1.1.

Remark 6.4.2.1. If we knew we could choose a good filtration on $\mathcal{F}$ such that $\mathrm{gr}_\bullet \mathcal{F}$ was a successive extension of sheaves supported on $\mathcal{N}ilp$ and flat at f^{glob}, we would obtain a similarly easy proof of Theorem 8.0.0.1. Unfortunately, we do not know a way to do this.[38]

6.4.3. As just stated, we now turn to the *spectral sequence convergence*. We will verify that the conditions of Lemma 6.3.1.1 (3) are verified for the filtered D-module $\psi_{*,\mathrm{dR}}(\mathfrak{p}^! \mathcal{F})$ on $\mathbf{A}^1$.

The first assumption is obvious: the original filtration on $\mathcal{F}$ is bounded from below (in the natural sense), so the same is true after applying D-module operations.

The third assumption follows e.g. from our work in §6.4.2: we saw there that $\mathrm{gr}_\bullet^{\mathrm{KK}}(\mathrm{coeff}(\mathcal{F}))$ is finite dimensional, and in particular, $\mathrm{gr}_i^{\mathrm{KK}} \mathrm{coeff}(\mathcal{F})$ is zero for all but finitely many i.

It remains to verify the second condition. We do so below.

[38]We tried to use the filtration of Kashiwara-Kawai from [KK] for this purpose, but we were not successful.

6.4.4. We now analyze the second condition from Lemma 6.3.1.1 (3), that is, we check the suitable boundedness of the secondary grading on:

$$\Gamma\big(\mathbf{A}^1, 0^*(\mathrm{gr}_\bullet\, \psi_{*,\mathrm{dR}}(\mathfrak{p}^!\mathcal{F}))\big). \tag{6.4.1}$$

Analysis exactly as in §6.4.2 shows that we can understand (6.4.1) via the following geometry.

We let $\mathrm{Kos}_G^{\mathrm{deg}} = \mathfrak{b}/N$ be the *degenerate Kostant slice* and let $\mathrm{Kos}_G^{\mathrm{deg,glob}}$ denote its global avatar:

$$\mathrm{Kos}_G^{\mathrm{deg,glob}} = \mathcal{M}aps(X, (\mathfrak{b}/N)/\mathbf{G}_m) \underset{\mathcal{M}aps(X,\mathbf{BG}_m)}{\times} \{\Omega_X^1\}.$$

where $\mathbf{G}_m$ acts by homotheties (i.e., this is the Ω_X^1-twisted version of $\mathcal{M}aps(X, \mathfrak{b}/N)$). Then (6.4.1) is computed (up to similarly tensoring by line bundles) by pulling back $\mathrm{gr}_\bullet(\mathcal{F})$ to $\mathrm{Kos}_G^{\mathrm{deg,glob}}$ and taking global sections.

In these terms, the secondary grading corresponds to the action of $\mathbf{G}_m$ on $\mathrm{Kos}_G^{\mathrm{deg}}$ (and hence $\mathrm{Kos}_G^{\mathrm{deg,glob}}$) via $2\check{\rho} : \mathbf{G}_m \to T$, as was considered above, noting that the pullback of $\mathrm{gr}_\bullet(\mathcal{F})$ to $\mathrm{Kos}_G^{\mathrm{deg,glob}}$ is $\mathbf{G}_m$-equivariant for this action.

The idea is that the boundedness follows because this action is contracting. We fill in the details in what follows.

6.4.5. To treat the notion of *positively graded* quasi-coherent sheaves on stacks, we digress to give some general axiomatics.

Consider $\mathbf{A}^1$ as a monoid under multiplication, so $\mathsf{QCoh}(\mathbf{A}^1)$ inherits a convolution monoidal structure. Let $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$ be a $\mathsf{QCoh}(\mathbf{A}^1)$-module category in what follows. For example, one can imagine $\mathcal{C} = A\text{–}\mathsf{mod}$ for a $\mathbf{Z}^{\geq 0}$-graded algebra A.

We have a full monoidal subcategory $\mathsf{QCoh}(\mathbf{G}_m) \subseteq \mathsf{QCoh}(\mathbf{A}^1)$, so $\mathsf{QCoh}(\mathbf{G}_m)$ acts on $\mathcal{C}$ as well, and we may form $\mathcal{C}^{\mathbf{G}_m,w}$. By functoriality, $\mathsf{QCoh}(\mathbf{A}^1/\mathbf{G}_m) = \mathsf{QCoh}(\mathbf{A}^1)^{\mathbf{G}_m,w}$ acts on $\mathcal{C}^{\mathbf{G}_m,w}$.

An object $\mathcal{F} \in \mathsf{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ amounts to a collection of objects $\mathcal{F}_n \in \mathsf{Vect}$ for $n \in \mathbf{Z}$ with connecting maps $\iota : \mathcal{F}_n \to \mathcal{F}_{n+1}$ defined for all n (i.e., to a filtered vector space – this is the Rees construction). The monoidal structure on this category sends $(\mathcal{F}_\bullet, \iota_\mathcal{F}), (\mathcal{G}_\bullet, \iota_\mathcal{G})$ to the object that in degree n is $\mathcal{F}_n \otimes \mathcal{G}_n$ with obvious connecting maps $\iota_\mathcal{F} \otimes \iota_\mathcal{G}$.

Consider the object $\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(n)$ which is 0 in degrees $< -n$, k in degrees $\geq -n$, and connecting maps are identities except where forced to be zero. There are evident maps $\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(n) \to \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(n+1)$ and $\mathrm{colim}\, \mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(n) = \mathcal{O}_{\mathbf{G}_m/\mathbf{G}_m}$ is the unit for the monoidal structure. Moreover, each $\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(n)$ is idempotent in the monoidal structure, and the map to the unit defines on $\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(n)$ the structure of idempotent coalgebra in $\mathsf{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$.

We let $\sigma_{\geq -n} : \mathcal{C}^{\mathbf{G}_m,w} \to \mathcal{C}^{\mathbf{G}_m,w}$ be the functor of acting by $\mathcal{O}_{\mathbf{A}^1/\mathbf{G}_m}(n)$. By the above, there is a natural transformation $\sigma_{\geq -n} \to \mathrm{id}$, and the essential image of $\sigma_{\geq -n}$ is the subcategory of objects $\mathcal{F} \in \mathcal{C}^{\mathbf{G}_m,w}$ for which the map

$$\sigma_{\geq -n}\mathcal{F} \to \mathcal{F}$$

is an isomorphism. We let $\mathcal{C}^{\mathbf{G}_m,w,\deg\geq -n}$ denote this essential image; the functor $\sigma_{\geq -n}$ provides a continuous right adjoint to the embedding $\mathcal{C}^{\mathbf{G}_m,w,\deg\geq -n} \to \mathcal{C}^{\mathbf{G}_m,w}$. We consider objects of $\mathcal{C}^{\mathbf{G}_m,w,\deg\geq -n}$ as having *graded degrees* $\geq -n$.

In the example $\mathcal{C} = A\text{–}\mathsf{mod}$, note that $A\text{–}\mathsf{mod}^{\mathbf{G}_m,w}$ consists of graded A-modules, $A\text{–}\mathsf{mod}^{\mathbf{G}_m,w,\geq -n}$ consists of modules in degrees $\geq -n$, and $\sigma_{\geq -n}$ takes a graded A-module $M = \oplus_{i\in\mathbf{Z}} M_i$ and forms $\sigma_{\geq -n}M = \oplus_{i\geq -n} M_i$ with its natural graded A-module structure.

We note explicitly that for each $\mathcal{F} \in \mathcal{C}^{\mathbf{G}_m,w}$, we have:

$$\mathcal{F} = \operatorname*{colim}_n \sigma_{\geq -n}\mathcal{F}. \tag{6.4.2}$$

It follows that if $\mathcal{F}$ is compact, then it necessarily has bounded below degrees, i.e., $\mathcal{F} \in \mathcal{C}^{\mathbf{G}_m,w,\geq -n}$ for some n.

We also note that any functor $F : \mathcal{C} \to \mathcal{D}$ of $\mathsf{QCoh}(\mathbf{A}^1)$-module categories induces $F^{\mathbf{G}_m,w} : \mathcal{C}^{\mathbf{G}_m,w} \to \mathcal{D}^{\mathbf{G}_m,w}$, which obviously commutes with functors $\sigma_{\geq -n}$ and therefore preserves objects of graded degrees $\geq -n$.

6.4.6. We will prove the following general result.

Lemma 6.4.6.1. *Suppose the monoid $\mathbf{A}^1$ acts on $\mathcal{Y}$, which is a quasi-compact algebraic stack locally almost of finite type with affine diagonal (over our field k of characteristic 0).*

Let $\mathcal{G} \in \mathsf{QCoh}(\mathcal{Y})^{\mathbf{G}_m,w}$ be a $\mathbf{G}_m$-equivariant complex such that $\tau^{\geq -n}\mathcal{G} \in \mathsf{Coh}(\mathcal{Y})$ is coherent for every n.

Then $\Gamma(\mathcal{Y},\mathcal{G}) \in \mathsf{Rep}(\mathbf{G}_m)$ has the property that $\tau^{\geq -n}\Gamma(\mathcal{Y},\mathcal{G})$ has bounded below degrees. I.e., if the grading is denoted $\Gamma(\mathcal{Y},\mathcal{G}) = \oplus_i \Gamma(\mathcal{Y},\mathcal{G})_i$, then $\tau^{\geq -n}\Gamma(\mathcal{Y},\mathcal{G})_i = 0$ for $i \ll 0$ (with bound depending on n).

Proof. Because $\Gamma(\mathcal{Y},-)$ has bounded amplitude (because of our assumptions on $\mathcal{Y}$ and k), there exists an integer $M \gg 0$ such that:

$$\tau^{\geq -n}\Gamma(\mathcal{Y},\mathcal{G}) = \tau^{\geq -n}\Gamma(\mathcal{Y},\tau^{\geq -n-M}\mathcal{G}).$$

Therefore, it suffices to show that:

$$\Gamma(\mathcal{Y},\tau^{\geq -n-M}\mathcal{G}) \in \mathsf{Rep}(\mathbf{G}_m)$$

has bounded below degrees for every n. Replacing $\mathcal{G}$ by $\tau^{\geq -n-M}\mathcal{G}$, we see that we can assume $\mathcal{G}$ is coherent to start.

Now note that $\mathsf{QCoh}(\mathbf{A}^1)$ acts on $\mathsf{QCoh}(\mathcal{Y})$ and $\mathsf{IndCoh}(\mathcal{Y})$, and the functor $\Psi : \mathsf{IndCoh}(\mathcal{Y}) \to \mathsf{QCoh}(\mathcal{Y})$ is a morphism of $\mathsf{QCoh}(\mathbf{A}^1)$-module categories. Therefore, we may apply the discussion from §6.4.5 freely.

As $\mathcal{G}$ is a compact object in $\mathsf{IndCoh}(\mathcal{Y})^{\mathbf{G}_m,w}$, we see that it lies in $\mathsf{IndCoh}(\mathcal{Y})^{\mathbf{G}_m,w,\deg\geq -N}$ for $N \gg 0$. Therefore, the same is true of $\Gamma(\mathcal{Y},\mathcal{G}) \in \mathsf{Rep}(\mathbf{G}_m)$, proving our claim.

□

Now take $\mathcal{G}_0 = \operatorname{gr}_\bullet \mathcal{F} \in \mathsf{Coh}(T^*\operatorname{Bun}_G)$ (i.e., we forget the grading). Take $\mathcal{Y} = \operatorname{Kos}_G^{\text{deg,glob}}$ and let $\mathcal{G}$ be the pullback of $\mathcal{G}_0$, which carries a $\mathbf{G}_m$-equivariance structure ($\operatorname{Kos}_G^{\text{deg,glob}} \to T^*\operatorname{Bun}_G$ is $\mathbf{G}_m$-equivariant for the trivial action on $T^*\operatorname{Bun}_G$). As the pullback of a coherent sheaf, we have

$\tau^{\geq -n}\mathcal{G} \in \mathsf{Coh}(\mathrm{Kos}_G^{\mathrm{deg,glob}})$. Therefore, the requisite boundedness of degrees follows from Lemma 6.4.6.1 and the fact that the $2\check{\rho}$-action of $\mathbf{G}_m$ on $\mathrm{Kos}_G^{\mathrm{deg}}$ extends to an action of $\mathbf{A}^1$.[39]

Remark 6.4.6.2. Lemma 6.4.6.1 can be proved more directly when $\mathcal{Y}$ has the property that $\Gamma(\mathcal{Y}, -)$ is conservative, as in our example. Indeed, then $\tau^{\geq -n}\mathcal{G} = \tau^{\geq -n}\mathcal{P}$ for some perfect complex $\mathcal{P}$, and all perfect complexes are Karoubi-generated from objects $\mathcal{O}_{\mathcal{Y}}(m)$, and these obviously satisfy the conclusion. This is to say: one can avoid the discussion from §6.4.5 in this case.

7. Exactness of tempered Hecke functors

In this section, we establish exactness for Hecke functors acting on the *tempered* automorphic category. This material is a sort of digression: Theorem 7.1.0.1 does not mention Whittaker coefficients (although they are used in the proof). The results of this section are independent of the rest of the paper up to this point.

7.1. **Statement of the main result.** Fix a point $x \in X$. Recall from [FR] that we have the category $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$. There is a natural quotient functor:

$$p : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$$

with a fully faithful left adjoint p^L. The main theorem of [FR] asserts that this data is actually independent of the point $x \in X$, although we will not need this until the discussion in §7.7.

We consider the action of $\mathsf{Rep}(\check{G})$ on $D(\mathrm{Bun}_G)$ via Hecke functors at $x \in X$. For $V \in \mathsf{Rep}(\check{G})$, we let $\mathcal{S}_{V,x} \star -$ denote the corresponding endofunctor of $D(\mathrm{Bun}_G)$.

The goal for this section is to prove:

Theorem 7.1.0.1. *(1) There is a unique t-structure on $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ such that p is t-exact.*
(2) The action of $\mathsf{Rep}(\check{G})^{\heartsuit} \subseteq \mathsf{Rep}(\check{G})$ on $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ is by t-exact functors.

The results in this section adapt to etale sheaves in positive characteristic (conditional on derived geometric Satake in that context).

As we discuss in §7.7, the above result strengthens the main results of [Gai1]. In fact, our proof is dramatically simpler and has clear conceptual meaning;[40] it turns out the assertion is something purely local, and a simple application of known results about derived Satake and the spherical Whittaker category.

7.1.1. The argument is purely local. Therefore, we largely take $\mathcal{C}$ to be a $G(K)$-category throughout this section (for loops being based at x); the application will be when $\mathcal{C} = D^!(\mathrm{Bun}_G^{\mathrm{lvl},x})$, where $\mathrm{Bun}_G^{\mathrm{lvl},x}$ is the moduli scheme of G-bundles with complete level structure at x.

We remind that in this setting, one can speak of a t-structure on $\mathcal{C}$ being *strongly compatible* with the $G(K)$-action; see [Ras1] §10.

[39] Let us clarify one sign issue for the careful reader. We normalize our conventions so that if $\mathbf{A}^1$ acts on an affine scheme $\mathrm{Spec}(A)$, then A is considered to have $\mathbf{G}_m$-degrees ≥ 0, not ≤ 0 (although the latter might in some sense be the more natural, if less aesthetic, convention). This convention was implicit in Lemma 6.3.1.1, where t had degree n rather than $-n$, so we have used this convention consistently.

[40] By contrast, the construction of $\widetilde{D}(\mathrm{Bun}_{GL_n})$ from [Gai1] is ad hoc, and by its nature cannot generalize to other reductive groups. Our construction produces a different category with the same nice features, and which does generalize.

7.2. **Averaging from the spherical category.** Suppose $\mathcal{C}$ is a category with a $G(K)$-action. We use the notation of §4.3.

Suppose that $\mathcal{C}$ is equipped with a t-structure strongly compatible with the $G(K)$-action. Recall that for each group subscheme $\mathcal{K} \subseteq G(K)$, $\mathcal{C}^{\mathcal{K}}$ admits a canonical t-structure; it is characterized by the fact that $\mathrm{Oblv} : \mathcal{C}^{\mathcal{K}} \to \mathcal{C}$ is t-exact. The same applies in the presence of an additive character $\psi : \mathcal{K} \to \mathbf{G}_a$ and twisted invariants $\mathcal{C}^{\mathcal{K},\psi}$.

Lemma 7.2.0.1. *If $\mathcal{C} \in G(K)\text{–}\mathsf{mod}$ is equipped with a t-structure strongly compatible with the $G(K)$-action, then the !-averaging functor $\mathrm{Av}_!^{\psi}[(2\check{\rho},\rho)] : \mathcal{C}^{G(O)} \to \mathsf{Whit}^{\leq 1}(\mathcal{C})$ is t-exact.*

Proof. The proof is standard: we review it here.

The functor $\mathrm{Av}_*^{I_1^- \to I} : \mathcal{C}^{I_1^-} \to \mathcal{C}^I$ is an equivalence, and in particular, admits a left adjoint, which we denote $\mathrm{Av}_!^{I \to I_1^-}$. On general grounds (cf. [Ras4] Appendix B), there is a canonical natural transformation:

$$\mathrm{Av}_!^{I \to I_1^-}[-2\dim(I_1^- \cdot I/I)] \to \mathrm{Av}_*^{I \to I_1^-}.$$

We remark that the displayed dimension is $2(\check{\rho},\rho) + |\Delta^+|$, where Δ^+ is the set of positive roots for G.

As in [FG2] Lemma 15.1.2, for $\mathcal{F} \in \mathcal{C}^{G(O)}$, the cokernel of the natural map:

$$\mathrm{Av}_!^{I \to I_1^-}(\mathcal{F})[-2\dim(I_1^- \cdot I/I)] \to \mathrm{Av}_*^{I \to I_1^-}(\mathcal{F})$$

is partially integrable (in the sense of *loc. cit.*); in particular, the above map induces an isomorphism on applying $\mathrm{Av}_*^{\mathring{I}_1,\psi}$ (cf. [FG2] Proposition 14.2.1).

Now recall from [BBM] (see also [Ras2] Appendix A) that we have $\mathrm{Av}_!^{\mathring{I}_1,\psi} = \mathrm{Av}_*^{\mathring{I}_1,\psi}[2\dim N]$.

By Lemma 4.3.0.1, the functor in question is a composition:

$$\mathcal{C}^{G(O)} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^I \xrightarrow{\mathrm{Av}_*^{I \to I_1^-}} \mathcal{C}^{I_1^-} \xrightarrow{\mathrm{Av}_!^{\mathring{I}_1,\psi}} \mathsf{Whit}^{\leq 1}(\mathcal{C}).$$

Applying [BBM] as above, this may instead be written as:

$$\mathcal{C}^{G(O)} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^I \xrightarrow{\mathrm{Av}_*^{I \to I_1^-}} \mathcal{C}^{I_1^-} \xrightarrow{\mathrm{Av}_*^{\mathring{I}_1,\psi}[2\dim N]} \mathsf{Whit}^{\leq 1}(\mathcal{C}).$$

By [Ras4] Lemmas B.2.2-3 and the above, this functor has amplitude $\leq \dim(I_1^- \cdot I/I) + \dim(\mathring{I}_1 I_1^-/I_1^-) - 2\dim N = 2(\check{\rho},\rho)$.

On the other hand, by the above, we can also rewrite this functor as the composition:

$$\mathcal{C}^{G(O)} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^I \xrightarrow{\mathrm{Av}_!^{I \to I_1^-}[-2\dim(I_1^- \cdot I/I)]} \mathcal{C}^{I_1^-} \xrightarrow{\mathrm{Av}_!^{\mathring{I}_1,\psi}} \mathsf{Whit}^{\leq 1}(\mathcal{C}).$$

Because $\mathrm{Av}_!^{I \to I_1^-}$ is inverse, hence right adjoint, to $\mathrm{Av}_*^{I_1^- \to I}$, which has amplitude $\leq \dim(II_1^-/I_1^-) = 2(\check{\rho},\rho) + \dim N$ (by [Ras4] Lemma B.2.2), this $\mathrm{Av}_!$ functor has amplitude $\geq -2(\check{\rho},\rho) - \dim N$.[41] Applying [Ras4] Lemma B.2.3 to $\mathrm{Av}_!^{\mathring{I}_1,\psi}$, we see that the above functor has amplitude $\geq 2(\check{\rho},\rho) + \dim N - \dim N = 2(\check{\rho},\rho)$.

[41] This would be obvious from [Ras4] Lemma B.2.3, but that result requires the two subgroups in question to mutually lie in a compact open subgroup, where [Ras4] Lemma B.2.2 does not.

Combined with the above, we find that $\mathrm{Av}_!^{\psi} : \mathcal{C}^{G(O)} \to \mathsf{Whit}^{\leq 1}(\mathcal{C})$ has amplitude exactly $2(\check{\rho}, \rho)$.

□

Remark 7.2.0.2. As in [Ras4] Appendix B, a $G(K)$-category with strongly compatible t-structure has an induced t-structure on $\mathsf{Whit}(\mathcal{C})$. It is likely the case that $\iota_{1,\infty,!}[2(\check{\rho}, \rho)] : \mathsf{Whit}^{\leq 1}(\mathcal{C}) \hookrightarrow \mathsf{Whit}(\mathcal{C})$ is t-exact; this was shown in *loc. cit.* for $\mathcal{C} = \widehat{\mathfrak{g}}_{\kappa}\text{–}\mathsf{mod}$, as was observed there also for $\mathcal{C}$ being D-modules on a reasonable indscheme (equipped with a dimension theory, to obtain a t-structure). In this case, the above result would simply say that $\mathrm{Av}_!^{\psi} : \mathcal{C}^{G(O)} \to \mathsf{Whit}(\mathcal{C})$ is t-exact.

In other words, the use of baby Whittaker rather than full Whittaker in the above (and what follows) simply reflects our ignorance regarding this point.

7.3. **Construction of the t-structure.** Suppose again that $\mathcal{C}$ is a $G(K)$-category with a strongly compatible t-structure.

In this case, we may form $\mathcal{C}^{G(O)}$ and its tempered quotient $\mathcal{C}^{G(O),x-\mathrm{temp}}$. We let $p : \mathcal{C}^{G(O)} \to \mathcal{C}^{G(O),x-\mathrm{temp}}$ denote the canonical projection. We remind that p admits a fully faithful left adjoint p^L.

Proposition 7.3.0.1. *In the above setting, there is a unique t-structure on $\mathcal{C}^{G(O),x-\mathrm{temp}}$ such that the projection $p : \mathcal{C}^{G(O)} \to \mathcal{C}^{G(O),x-\mathrm{temp}}$ is t-exact.*

Proof. By [Ras5] Lemma 10.2.1,[42] it suffices to show that $\mathrm{Ker}(p) = \mathcal{C}^{G(O),x-\mathrm{anti-temp}}$ is closed under truncations, and that the resulting abelian category $\mathcal{C}^{G(O),x-\mathrm{anti-temp},\heartsuit}$ is closed under subobjects (cf. [Ras5] Remark 10.2.2).

By Lemma 4.2.0.1, we have:

$$\mathcal{C}^{G(O),x-\mathrm{anti-temp}} = \mathrm{Ker}\left(\mathrm{Av}_!^{\psi} : \mathcal{C}^{G(O)} \to \mathsf{Whit}^{\leq 1}(\mathcal{C})\right).$$

By Lemma 7.2.0.1, $\mathrm{Av}_!^{\psi}$ is t-exact up to shift; it follows immediately that its kernel is closed under truncations, and the heart of the kernel is closed under taking subobjects.

□

7.4. **More on Whittaker functors.** We continue to assume $\mathcal{C}$ is acted on by $G(K)$.

Fix a dominant coweight $\check{\lambda}$. In this case, we can perform two constructions.

- Let $\psi^{\check{\lambda}} : \mathrm{Ad}_{-(\check{\rho}+\check{\lambda})(t)} \overset{\circ}{I} \to \mathbf{G}_a$ be the composition:

$$\mathrm{Ad}_{-(\check{\rho}+\check{\lambda})(t)} \overset{\circ}{I} \xrightarrow{\mathrm{Ad}_{\check{\lambda}(t)}} \overset{\circ}{I}_1 \xrightarrow{\psi} \mathbf{G}_a.$$

  (I.e., take a character of conductor $\check{\lambda}$ for $N(K)$ and apply the corresponding baby Whittaker construction.)
- Take a representation $V^{\check{\lambda}} \in \mathsf{Rep}(\check{G})^{\heartsuit}$.

We consider $\mathcal{C}^{G(O)}$ as acted on by $\mathsf{Rep}(\check{G})$ via Satake.

[42] The cited lemma uses an adjunction in which the quotient admits a right adjoint, not a left adjoint. However, the proof in *loc. cit.* works for arbitrary DG categories, not necessarily cocomplete ones (although it is written in that context). Therefore, we may safely pass to opposite categories to deduce the claim (or observe that the argument in *loc. cit.* immediately applies in the present context).

Proposition 7.4.0.1. *In the above setting, there is a canonical commutative diagram:*

$$\begin{tikzcd} \mathcal{C}^{G(O)} \arrow[r, "V^{\check{\lambda}}\star -"] \arrow[d, equal] & \mathcal{C}^{G(O)} \arrow[r, "\mathrm{Av}_!^{\mathring{I}_1,\psi}"] & \mathcal{C}^{\mathring{I}_1,\psi} = \mathsf{Whit}^{\leq 1}(\mathcal{C}) \arrow[d, equal] \\ \mathcal{C}^{G(O)} \arrow[r, "\mathrm{Av}_*^{\psi^{\check{\lambda}}}{[(\check{\lambda},2\rho)]}"] & \mathcal{C}^{\mathrm{Ad}_{-(\check{\rho}+\check{\lambda})(t)}\mathring{I},\psi^{\check{\lambda}}} \arrow[r, "\check{\lambda}(t)\cdot -", "\simeq"'] & \mathcal{C}^{\mathring{I}_1,\psi} = \mathsf{Whit}^{\leq 1}(\mathcal{C}) \end{tikzcd} \tag{7.4.1}$$

Proof. This result is an easy application of the Casselman-Shalika formula. Specifically, we will see both sides are given by convolving with the same sheaf.

For a coweight $\check{\mu}$, let $\jmath^{\check{\mu}} : \mathring{I}_1 \cdot (-\check{\mu})(t)G(O)/G(O) \to \mathrm{Gr}_G$ be the locally closed embedding of the $\mathring{I}_1$-orbit through $(-\check{\mu})(t) \in \mathrm{Gr}_G$. For μ dominant, let $\psi^{\check{\mu},!}_{\mathring{I}_1}(\exp)$ denote the character sheaf on this orbit, normalized to lie in the same cohomological degree as the dualizing sheaf.

The top line in (7.4.1) is then given by convolution with

$$\jmath^0_!(\psi^{0,!}_{\mathring{I}_1}(\exp)) \star \mathcal{S}_{V^{\check{\lambda}}} \in \mathsf{Whit}^{\leq 1}(D(\mathrm{Gr}_G)).$$

for $\mathcal{S}_{V^{\check{\lambda}}}$ the spherical sheaf corresponding to $V^{\check{\lambda}}$.

The bottom line in (7.4.1) is given by convolution with:

$$\jmath^{\check{\lambda}}_*(\psi^{\check{\lambda},!}_{\mathring{I}_1}(\exp))[-2\dim\big((\mathrm{Ad}_{(-\check{\lambda}-\check{\rho})(t)}\,\mathring{I})\cdot G(O)/G(O)\big) + (\check{\lambda},2\rho)].$$

Here the first summand in the shift appears because we should use constant sheaves (rather than dualizing sheaves) for $*$-averaging, and the second summand appears simply because it is in (7.4.1). We observe that:

$$\dim\big((\mathrm{Ad}_{(-\check{\lambda}-\check{\rho})(t)}\,\mathring{I})\cdot G(O)/G(O)\big) = (\check{\lambda}+\check{\rho},2\rho)$$

so the above may be rewritten as:

$$\jmath^{\check{\lambda}}_*(\psi^{\check{\lambda},!}_{\mathring{I}_1}(\exp))[-(\check{\lambda}+2\check{\rho},2\rho)].$$

Finally, by the form of the geometric Casselman-Shalika formula given in [ABBGM] Theorem 2.2.2 and Corollary 2.2.3, we have:

$$\jmath^0_!(\psi^{0,!}_{\mathring{I}_1}(\exp)) \star \mathcal{S}_{V^{\check{\lambda}}}[-(\check{\rho},2\rho)] \simeq \jmath^{\check{\lambda}}_*(\psi^{\check{\lambda},!}_{\mathring{I}_1}(\exp))[-(\check{\lambda}+\check{\rho},2\rho)].$$

This yields the claim. □

7.5. **A generalization.** We remind that by construction, the quotient $\mathcal{C}^{G(O),x-\mathrm{temp}} = \mathcal{C}^{G(O)}/\mathcal{C}^{G(O),x-\mathrm{anti-temp}}$ inherits a (unique) $\mathsf{Rep}(\check{G})$-action for which the projection $p : \mathcal{C}^{G(O)} \to \mathcal{C}^{G(O),x-\mathrm{temp}}$ is $\mathsf{Rep}(\check{G})$-linear.

We now prove:

Theorem 7.5.0.1. *Suppose $G(K)$ acts on $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$, and that $\mathcal{C}$ is equipped with a t-structure that is strongly compatible with this action.*

Then for every $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$, the functor:

$$V \star - : \mathcal{C}^{G(O),x-\mathrm{temp}} \to \mathcal{C}^{G(O),x-\mathrm{temp}}$$

is t-exact with respect to the t-structure from Proposition 7.3.0.1.

Proof. We treat right and left exactness separately.

Step 1. First, we show that $V \star - : \mathcal{C}^{G(O),x-\mathrm{temp}} \to \mathcal{C}^{G(O),x-\mathrm{temp}}$ is right t-exact.

It suffices to prove this result for irreducible representations. Therefore, we take $V = V^{\check{\lambda}}$.

Suppose $\mathcal{F} \in \mathcal{C}^{G(O),\leq 0}$ is connective. It suffices to show that $p(V^{\check{\lambda}} \star \mathcal{F}) \in \mathcal{C}^{G(O),x-\mathrm{temp},\leq 0}$.

By Lemma 7.2.0.1 and Lemma 4.2.0.1, we have a (necessarily unique) commutative diagram:

$$\begin{tikzcd} \mathcal{C}^{G(O)} \arrow[d, "p"] \arrow[dr, "\mathrm{Av}_!^{\psi}[(2\check{\rho},\rho)]"] & \\ \mathcal{C}^{G(O),x-\mathrm{temp}} \arrow[r, dotted] & \mathsf{Whit}^{\leq 1}(\mathcal{C}) \end{tikzcd}$$

in which (crucially!) the bottom arrow is t-exact and conservative.

Therefore, it suffices to show that:

$$\mathrm{Av}_!^{\psi}(V^{\check{\lambda}} \star \mathcal{F})[(2\check{\rho},\rho)] \in \mathsf{Whit}^{\leq 1}(\mathcal{C})^{\leq 0}.$$

By Proposition 7.4.0.1, we can rewrite this term as:

$$\mathrm{Av}_*^{\psi^{\check{\lambda}}}(\mathcal{F})[(\check{\lambda}+\check{\rho},2\rho)].$$

By in this form, the desired estimate follows the usual estimates for the amplitude of Av_* functors: see [Ras4] Lemma B.2.2.

Step 2. We now prove left t-exactness. It suffices to prove this for finite dimensional V. Then the functor $V\star - : \mathcal{C}^{G(O),x-\mathrm{temp}} \to \mathcal{C}^{G(O),x-\mathrm{temp}}$ is right adjoint to $V^{\vee}\star - : \mathcal{C}^{G(O),x-\mathrm{temp}} \to \mathcal{C}^{G(O),x-\mathrm{temp}}$. The latter is right t-exact by the above, so the former must be left t-exact as desired.

□

Finally, Theorem 7.1.0.1 follows by taking $\mathcal{C} = D^!(\mathrm{Bun}_G^{\mathrm{lvl},x})$ in Theorem 7.5.0.1.

7.6. **Variant for nilpotent sheaves.** Note that the spherical Hecke action $\mathcal{H}_x^{\mathrm{sph}} = D(\mathrm{Gr}_G)^{G(O)} \curvearrowright D(\mathrm{Bun}_G)$ preserves $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$. Therefore, we may form:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{x-\mathrm{temp}} := \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \underset{\mathsf{IndCoh}(\Omega_0\check{\mathfrak{g}}/\check{G})}{\otimes} \mathsf{QCoh}(\Omega_0\check{\mathfrak{g}}/\check{G}).$$

as for $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$. The same applies for $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$.

By functoriality, we then have a commutative diagram:

$$\begin{tikzcd} \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{anti-temp}} \arrow[r] \arrow[d] & D(\mathrm{Bun}_G)^{\mathrm{anti-temp}} \arrow[d] \\ \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \arrow[r] \arrow[d, "p"] & D(\mathrm{Bun}_G) \arrow[d, "p"] \\ \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[r] & D(\mathrm{Bun}_G)^{\mathrm{temp}}. \end{tikzcd}$$

The horizontal functors are fully faithful e.g. because $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$ admits a $\mathcal{H}_x^{\mathrm{sph}}$-linear right adjoint.

We see from Theorem 7.1.0.1 that $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$ is closed under truncations and subobjects (since this is true for $D(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$ and $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ separately). Therefore, $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{temp}}$ inherits a t-structure for which the projection $p : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ is t-exact. Then the bottom horizontal arrow above is t-exact (and conservative, being fully faithful) for this t-structure.

From the diagram above and Theorem 7.1.0.1, we see that Hecke functors are t-exact on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{temp}}$ with respect to the above t-structure.

7.7. **Relation to Gaitsgory's work for GL_n.** In this section, we briefly indicate how the above results can be used to better understand the main results of [Gai1].

7.7.1. *A variant with moving points.* Recall from [FR] that $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ is canonically independent of the point $x \in X$. We therefore use the notation $D(\mathrm{Bun}_G)^{\mathrm{temp}}$ instead.

Let $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ be given. Recall that there is a Hecke functor:

$$H_{V,X} : D(\mathrm{Bun}_G) \to D(X \times \mathrm{Bun}_G) = D(X) \otimes D(\mathrm{Bun}_G)$$

whose !-fibers at points $x \in X$ give the usual Hecke functors at points.

It is easy to see that the functor $H_{V,X}$ induces a functor:

$$H^{\mathrm{temp}}_{V,X} : D(\mathrm{Bun}_G)^{\mathrm{temp}} \to D(X) \otimes D(\mathrm{Bun}_G)^{\mathrm{temp}}.$$

We have the following generalization of Theorem 7.1.0.1:

Theorem 7.7.1.1. *The functor*

$$H^{\mathrm{temp}}_{V,X}[-1] : D(\mathrm{Bun}_G)^{\mathrm{temp}} \to D(X) \otimes D(\mathrm{Bun}_G)^{\mathrm{temp}}$$

is t-exact.

This follows by performing the proof of Theorem 7.1.0.1 over X, and applying [FR].

7.7.2. We now observe that Theorem 7.7.1.1 yields a quotient of $D(\mathrm{Bun}_G)$ with the properties described in [Gai1] §2.12. Namely, Hecke functors are t-exact, and $D_{\mathrm{cusp}}(\mathrm{Bun}_G) \subseteq D(\mathrm{Bun}_G)$ is right orthogonal to $D(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$ by [Ber3].

The construction of such a quotient is the main technical input in [Gai1]; see the discussion of *loc. cit.* §2.13.

Our argument is valid for general reductive groups G. Moreover, the degree restrictions in *loc. cit.* are not necessary here. Finally, we observe that the quotient we consider here has evident conceptual meaning in geometric Langlands, which was not the case for the quotient considered in [Gai1].

Finally, we remark that even for GL_n, our methods are much simpler than those in [Gai1].

However, to obtain a result for ℓ-adic sheaves, one needs some additional input. First, one needs derived Satake for ℓ-adic sheaves (which has been announced by Arinkin-Berzukavnikov). More seriously, one would need the independence of[43] $\mathsf{Shv}(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ of the point $x \in X$; [FR] shows this only for $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ in the ℓ-adic context.

We also refer back to Remark 1.5.2.1 for more context on our result.

[43]Here, unlike in the rest of the paper, Shv denotes ℓ-adic sheaves, not regular holonomic D-modules.

8. Whittaker coefficients of nilpotent sheaves

In this section, we establish favorable properties of Whittaker coefficients for sheaves with nilpotent singular support.

For our later applications, the main result of this section is:

Theorem 8.0.0.1. *The functor* $\mathrm{coeff}[\dim \mathrm{Bun}_G] : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Vect}$ *is t-exact.*

Using [AGKRRV2], we deduce this result from a theorem of Kevin Lin.

8.1. **Around Lin's theorem.** We begin this section by describing Lin's result and deducing some immediate consequences of it.

8.1.1. *Formulation.* Below, we let[44] $\underline{\mathsf{e}}_{\mathrm{Bun}_G} \in D(\mathrm{Bun}_G)$ denote the constant sheaf, i.e., $\underline{\mathsf{e}}_{\mathrm{Bun}_G} = \omega_{\mathrm{Bun}_G}[-2\dim \mathrm{Bun}_G]$.

We let $\Delta = \Delta_{\mathrm{Bun}_G}$ denote the diagonal map $\mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$, and we let $\pi_{\mathrm{Bun}_N^\Omega} : \mathrm{Bun}_N^\Omega \to \mathrm{Spec}(k)$ denote the projection.

Theorem 8.1.1.1 ([Lin])**.** *There is a canonical isomorphism:*

$$(\mathrm{coeff} \otimes \mathrm{id})(\Delta_! \underline{\mathsf{e}}_{\mathrm{Bun}_G}) := (\pi_{\mathrm{Bun}_N^\Omega} \times \mathrm{id})_{*,\mathrm{dR}}\Big((\mathfrak{p} \times \mathrm{id})^! (\Delta_! \underline{\mathsf{e}}_{\mathrm{Bun}_G}) \overset{!}{\otimes} p_1^! \psi^! (\exp)\Big)[-\dim \mathrm{Bun}_N^\Omega] \simeq \mathcal{P}oinc_! [-2\dim \mathrm{Bun}_G] \in D(\mathrm{Bun}_G).$$

Remark 8.1.1.2. We highlight that this theorem is a particular isomorphism between two explicit sheaves on Bun_G. It is in the spirit of many[45] results on quasi-maps spaces in geometric Langlands. The proof uses geometry of the Vinberg degeneration (via [Che]) and Zastava spaces. Specifically, the argument *(i)* constructs a map, *(ii)* shows that the map is an isomorphism at the cuspidal level (using that the pseudo-identity and identity coincide there, cf. [Gai7]), and *(iii)* checks that the map is an isomorphism after applying constant term functors, using Zastava geometry (and other tools) to study the results.

8.1.2. *Derivation from geometric Langlands conjectures.* We now (heuristically) show how Theorem 8.1.1.1 is predicted by standard compatibilities from geometric Langlands. We first work up to shifts,[46]

Specifically, we recall from [Gai7] §Conjecture 0.2.3 that:

$$\Delta_! \underline{\mathsf{e}}_{\mathrm{Bun}_G} \in D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G)$$

up to shifts is supposed to correspond to:

$$(\Psi_{\mathcal{N}ilp} \otimes \Psi_{\mathcal{N}ilp}) \Delta_*^{\mathsf{IndCoh}} (\omega_{\mathrm{LS}_{\check{G}}})$$

[44]The notation is a bit funny; we follow [AGKRRV1] in letting e be opaque notation for the field k, thought of as *the field of coefficients for our sheaf theory.*

We find this notation $\underline{\mathsf{e}}_{-}$ a bit more geometrically communicative for constant sheaves than $\underline{k}_{-}$... although it gets tricky for a point.

[45]Cf. [BG2], [BFGM], [Che], [Sch], [SW], among others.

[46]Unfortunately, the compatibility between both Eisenstein series and strange duality with Langlands duality are often only stated up to shifts (and tensoring with line bundles). It is not our purpose here to correct that issue in the literature here, which unfortunately leaves the ambiguity in shifts at the end.

Our understanding is that the forthcoming work of Ben-Zvi–Sakellaridis–Venkatesh will systematically clarify such issues, including the precise compatibility between both Eisenstein series and miraculous duality with geometric Langlands.

under geometric Langlands. Here we abuse notation in letting $\mathcal{N}ilp \subseteq T^*[-1](\mathrm{LS}_{\check{G}})$ denote the spectral global nilpotent cone, as in [AG]. We also let $\Psi_{\mathcal{N}ilp} : \mathsf{IndCoh}(\mathrm{LS}_{\check{G}}) \to \mathsf{IndCoh}_{\mathcal{N}ilp}(\mathrm{LS}_{\check{G}})$ denote the natural projection.

On the other hand, recall that the functor:

$$\mathrm{coeff} : D(\mathrm{Bun}_G) \to \mathsf{Vect}$$

is supposed to correspond to:

$$\Gamma^{\mathsf{IndCoh}}(\mathrm{LS}_{\check{G}}, -) : \mathsf{IndCoh}_{\mathcal{N}ilp}(\mathrm{LS}_{\check{G}}) \to \mathsf{Vect}.$$

Combining these two assertions, we see that:

$$(\mathrm{coeff} \otimes \mathrm{id})(\Delta_! \underline{\mathsf{e}}_{\mathrm{Bun}_G}) \in D(\mathrm{Bun}_G) \tag{8.1.1}$$

should correspond (up to shifts) to:

$$(\Gamma^{\mathsf{IndCoh}}(\mathrm{LS}_{\check{G}}, -) \otimes \mathrm{id})((\Psi_{\mathcal{N}ilp} \otimes \Psi_{\mathcal{N}ilp})\Delta_*^{\mathsf{IndCoh}}(\omega_{\mathrm{LS}_{\check{G}}})) = \omega_{\mathrm{LS}_{\check{G}}} \in \mathsf{IndCoh}_{\mathcal{N}ilp}(\mathrm{LS}_{\check{G}}).$$

We now observe that using the symplectic structure on $\mathrm{LS}_{\check{G}}$, we have:

$$\omega_{\mathrm{LS}_{\check{G}}} \simeq \mathcal{O}_{\mathrm{LS}_{\check{G}}}[\mathrm{v.}\dim(\mathrm{LS}_{\check{G}})]$$

where $\mathrm{v.}\dim(\mathrm{LS}_{\check{G}})$ is the Euler characteristic of the cotangent complex of $\mathrm{LS}_{\check{G}}$, which is $2\dim \mathrm{Bun}_G$.

Now observe that coeff is corepresented by $\mathcal{P}oinc_!$ while $\Gamma^{\mathsf{IndCoh}}(\mathrm{LS}_{\check{G}}, -)$ is corepresented by $\mathcal{O}_{\mathrm{LS}_{\check{G}}}$, so these two objects must correspond to each other under geometric Langlands.

Combining these observations, we find that (8.1.1) and $\mathcal{P}oinc_!$ both correspond to $\omega_{\mathrm{LS}_{\check{G}}}$ (up to shifts) under geometric Langlands, so we expect the two to be isomorphic (up to shifts): this is the assertion of Lin's theorem.

Remark 8.1.2.1. There are various ways to recover the precise shift. First, it is built into the proof of Lin's theorem, specifically, the construction of the comparison map in Theorem 8.1.1.1; we prefer not to describe the comparison map here and leave it to Lin's forthcoming work.

However, assuming Theorem 8.1.1.1 *up to shifts*, the precise value is also forced by known results. Specifically, let $\mathrm{Eis}_{!*} \in D(\mathrm{Bun}_G)$ be the $*$-pushforward of the IC sheaf on Drinfeld's compactification $\overline{\mathrm{Bun}}_B$. According to [BHKT] Appendix B (11.18), $\mathrm{coeff}_!(\mathrm{Eis}_{!*}) = k[\dim \mathrm{Bun}_G]$.[47] As $\mathrm{Eis}_{!*}$ is Verdier self-dual, we see that $\mathrm{coeff}(\mathrm{Eis}_{!*}) = k[-\dim \mathrm{Bun}_G]$. Finally, it is standard to see that $\mathrm{Eis}_{!*}$ has nilpotent singular support.

On the other hand, a version of Lin's theorem with an additional shift (beyond the stated one) by $N \in \mathbf{Z}$ would ultimately yield a version of our Theorem 8.2.1.1 with the same shift by N appearing. The only one consistent with the above calculation with compactified Eisenstein series is $N = 0$.

[47] The reader who glances at [BHKT] Appendix B (11.18) will find additional shifts. The shift by $-\dim \mathrm{Bun}_N^{\Omega}$ in *loc. cit.* does not appear for us simply because we defined $\mathrm{coeff}_!$ with a shift by $\dim \mathrm{Bun}_N^{\Omega}$ built in.

There is also a shift by $-\dim \mathrm{Bun}_T$ in *loc. cit.* We observe that in the notation of *loc. cit.*, we should take $\mathcal{F} = \mathrm{IC}_{\mathrm{Bun}_T} \in \mathsf{Shv}(\mathrm{Bun}_T)^{\heartsuit}$ to recover our specific example. Moreover, in *loc. cit.* (11.18), the $\lambda = 0$ term involves a $*$-fiber of $\mathcal{F}$ at $\check{\rho}(\Omega_X) \in \mathrm{Bun}_T$; as $\mathrm{IC}_{\mathrm{Bun}_T} = \mathsf{e}_{\mathrm{Bun}_T}[\dim \mathrm{Bun}_T]$, this yields an additional shift by $\dim \mathrm{Bun}_T$ cancelling the one appearing in the equation, and ultimately leading the precise value stated here.

8.1.3. *Relation to miraculous duality.* Recall the Drinfeld-Gaitsgory *miraculous duality* functor from [Gai7]:[48]

$$\mathrm{Mir} : D(\mathrm{Bun}_G)^\vee \to D(\mathrm{Bun}_G).$$

We remind that [Gai7] Theorem 0.1.6 asserts that this functor is an equivalence.

Corollary 8.1.3.1. $\mathrm{Mir}(\mathrm{coeff}) = \mathcal{P}oinc_![-2\dim \mathrm{Bun}_G]$.

This result is a formal consequence of 8.1.1.1. We review the relevant ideas here.

First, we record the following obvious result.

Lemma 8.1.3.2. *Let $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$ be given, and let $\lambda : \mathcal{C} \to \mathsf{Vect}$ be a functor. Then:*

$$(\lambda \otimes \mathrm{id}) : \mathcal{C} \otimes \mathcal{C}^\vee \to \mathsf{Vect} \otimes \mathcal{C}^\vee = \mathcal{C}^\vee$$

maps the unit[49] *$u_{\mathcal{C}}$ to λ.*

Proof of Corollary 8.1.3.1. The defining property of miraculous duality is that the functor:

$$D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G)^\vee \xrightarrow{\mathrm{id} \otimes \mathrm{Mir}} D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

sends $u_{D(\mathrm{Bun}_G)}$ to $\Delta_! \underline{\mathrm{e}}_{\mathrm{Bun}_G}$. (We remark that each arrow above is an equivalence.)

We now have the commutative diagram:

$$\begin{tikzcd}
D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G)^\vee \arrow[r, "\mathrm{id} \otimes \mathrm{Mir}"] \arrow[d, "\mathrm{coeff} \otimes \mathrm{id}"] & D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \arrow[r] & D(\mathrm{Bun}_G \times \mathrm{Bun}_G) \arrow[d, "\mathrm{coeff} \otimes \mathrm{id}"'] \\
D(\mathrm{Bun}_G)^\vee \arrow[rr, "\mathrm{Mir}"] & & D(\mathrm{Bun}_G)
\end{tikzcd}$$

We calculate the image of $u_{D(\mathrm{Bun}_G)}$ in two ways. By the above, if we traverse the upper leg of the diagram, we obtain $\Delta_! \underline{\mathrm{e}}_{\mathrm{Bun}_G}$, so Theorem 8.1.1.1 implies we obtain $\mathcal{P}oinc_![-2\dim \mathrm{Bun}_G]$ after applying the right arrow. On the other hand, if we apply the left arrow, Lemma 8.1.3.2 implies we obtain $\mathrm{coeff} \in D(\mathrm{Bun}_G)^\vee$, which maps to $\mathrm{Mir}(\mathrm{coeff})$ on applying the bottom arrow.

□

8.2. **Whittaker coefficients of nilpotent sheaves.**

8.2.1. *Comparison of coefficient functors.* We now prove the following assertion:

Theorem 8.2.1.1. *There is a canonical isomorphism of functors:*

$$\mathrm{coeff} \simeq \mathrm{coeff}_![-2\dim \mathrm{Bun}_G] : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Vect}.$$

Before proving this theorem, we record the following result:

Lemma 8.2.1.2 ([AGKRRV2], Corollary 4.3.7)**.** *Let $\lambda \in \mathsf{Shv}(\mathrm{Bun}_G)^\vee$ be given.*[50] *Then we have a canonical isomorphism:*

$$\lambda|_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)} \simeq C_{c,\mathrm{dR}}(\mathrm{Bun}_G, \mathrm{Mir}(\lambda) \overset{*}{\otimes} -)$$

[48] In [Gai7], this functor is denoted $\text{Ps-id}_{\mathrm{Bun}_G,!}$. Our notation is taken instead from [AGKRRV2].

[49] I.e., the object corresponding to $\mathrm{id}_{\mathcal{C}}$ under $\mathcal{C} \otimes \mathcal{C}^\vee \simeq \mathsf{End}_{\mathsf{DGCat}_{\mathrm{cont}}}(\mathcal{C})$.

[50] We can work with D-modules just as well as sheaves here; but in the non-holonomic case, one would need to remark that $\mathrm{Mir}(\lambda) \overset{*}{\otimes} -$ may take values in the pro-category (although $C_{c,\mathrm{dR}}(\mathrm{Bun}_G, -)$ will then map the result into $\mathsf{Vect} \subseteq \mathsf{ProVect}$).

of functors:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Vect}.$$

Proof of Theorem 8.2.1.1. For $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$, Lemma 8.2.1.2 yields:

$$\mathrm{coeff}(\mathcal{F}) = C_c(\mathrm{Bun}_G, \mathcal{F} \overset{*}{\otimes} \mathrm{Mir}(\mathrm{coeff})).$$

Applying Corollary 8.1.3.1, the right hand side is:

$$C_c(\mathrm{Bun}_G, \mathcal{F} \overset{*}{\otimes} \mathcal{P}oinc_![-2\dim \mathrm{Bun}_G]).$$

Applying the formula (5.4.1) and base-change, the right hand side is $\mathrm{coeff}_!(\mathcal{F})[-2\dim \mathrm{Bun}_G]$, yielding the claim.

□

8.2.2. We deduce:

Corollary 8.2.2.1. *For $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ locally compact, we have:*

$$\mathrm{coeff}(\mathbf{D}^{\mathrm{Verdier}}\mathcal{F}) = \mathrm{coeff}(\mathcal{F})^\vee[-2\dim \mathrm{Bun}_G].$$

That is, $\mathrm{coeff}[\dim \mathrm{Bun}_G]$ *commutes with Verdier duality on* $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$*.*

Indeed, this follows from Lemma 5.4.2.1 and Theorem 8.2.1.1.

8.2.3. *Variant with conductor.* Now fix D a $\check{\Lambda}^+$-valued divisor on X.

We have the following generalization of Theorem 8.2.1.1.

Corollary 8.2.3.1. *There is a canonical isomorphism of functors:*

$$\mathrm{coeff}_D \simeq \mathrm{coeff}_{D,!}[-2\dim \mathrm{Bun}_G] : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Vect}.$$

Proof. This is immediate from Theorem 8.2.1.1, Theorem 5.3.0.1, and Corollary 5.4.3.1.

□

As with Corollary 8.2.2.1, we have:

Corollary 8.2.3.2. *For D as above,* $\mathrm{coeff}_D[\dim \mathrm{Bun}_G]$ *commutes with Verdier duality on* $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$*.*

8.3. **Exactness.** We now prove Theorem 8.0.0.1. In fact, we prove the following generalization.

Theorem 8.3.0.1. *For every D a $\check{\Lambda}^+$-valued divisor on X, the functor:*

$$\mathrm{coeff}_D[\dim \mathrm{Bun}_G] : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Vect}$$

is t-exact.

Proof.

Step 1. First, we prove this result in the *sufficiently large* case, cf. §5.1.7.

We begin by naively estimating the amplitude of coeff_D on $D(\mathrm{Bun}_G)$. Decomposing into steps, we observe:

- The functor $\mathfrak{p}_D^!$ has amplitude $\leq \dim \mathrm{Bun}_G - \dim \mathrm{Bun}_N^{\Omega(-D)}$.
- Tensoring with the character is designed to be t-exact.

- By Lemma 5.1.7.2, for D sufficiently large, $\mathrm{C}_{\mathrm{dR}}(\mathrm{Bun}_N^{\Omega(-D)}, -)$ is right t-exact.
- We recall that there is a shift by $\dim \mathrm{Bun}_N^{\Omega(-D)}$ in the definition of coeff_D.

Combining these observations, we see that coeff_D has cohomological amplitude $\leq \dim \mathrm{Bun}_G$.

On the other hand, the same reasoning shows that[51] $\mathrm{coeff}_{D,!} : \mathsf{Shv}(\mathrm{Bun}_G) \to \mathsf{Vect}$ has amplitude $\geq -\dim \mathrm{Bun}_G$.

Now the exactness follows from Corollary 8.2.3.1.

Step 2. We now prove the result for $D = 0$.

First, we note that $\mathcal{P}oinc_! \in D(\mathrm{Bun}_G)$ lies in the left orthogonal to $D(\mathrm{Bun}_G)^{\text{anti-temp}}$; indeed, this follows immediately from derived Satake and the definition of temperedness. It follows that the functor $\mathrm{coeff} = \underline{\mathrm{Hom}}(\mathcal{P}oinc_!, -)$ factors through the projection to $D(\mathrm{Bun}_G)^{\mathrm{temp}}$, i.e., we have:

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[d, "p"] \arrow[dr, "\mathrm{coeff}"] & \\ D(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[r, dotted, "\widetilde{\mathrm{coeff}}"] & \mathsf{Vect}. \end{tikzcd}$$

The same applies in the presence of a divisor.

Let $x \in X$ be a point and let $\check{\lambda}$ be a coweight with $\check{\lambda} \cdot x$ sufficiently large (e.g., $\check{\lambda} = 2n\check{\rho}$ for $n \gg 0$). The Hecke action of $\mathsf{Rep}(\check{G})$ on $D(\mathrm{Bun}_G)$ below is considered at the point $x \in X$.

Clearly the trivial representation is a summand of $V^{\check{\lambda}} \otimes V^{-w_0(\check{\lambda})}$. Therefore, coeff is a summand of $\mathrm{coeff}((V^{\check{\lambda}} \otimes V^{-w_0(\check{\lambda})}) \star -)$. Therefore, it suffices to show that the latter functor is t-exact. For $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\leq 0}$, we have:

$$\mathrm{coeff}\left((V^{\check{\lambda}} \otimes V^{-w_0(\check{\lambda})}) \star \mathcal{F}\right) = \mathrm{coeff}(V^{\check{\lambda}} \star (V^{-w_0(\check{\lambda})} \star \mathcal{F})) \overset{\text{Thm. 5.3.0.1}}{=}$$
$$\mathrm{coeff}_{\check{\lambda}\cdot x}(V^{-w_0(\check{\lambda})} \star \mathcal{F}) = \widetilde{\mathrm{coeff}}_{\check{\lambda}\cdot x}(p(V^{-w_0(\check{\lambda})} \star \mathcal{F})) = \widetilde{\mathrm{coeff}}_{\check{\lambda}\cdot x}(V^{-w_0(\check{\lambda})} \star p(\mathcal{F})).$$

Now by Theorem 7.1.0.1 (and §7.6), $p(\mathcal{F}) \in \mathsf{Shv}(\mathrm{Bun}_G)^{\mathrm{temp},\leq 0}$, so $V^{-w_0(\check{\lambda})} \star p(\mathcal{F}) \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{temp},\leq 0}$ by Theorem 7.1.0.1. It follows from Step 1 that $\widetilde{\mathrm{coeff}}_{\check{\lambda}\cdot x}(V^{-w_0(\check{\lambda})} \star p(\mathcal{F}))$ is in degrees $\leq \dim \mathrm{Bun}_G$, giving right exactness of $\widetilde{\mathrm{coeff}}_{\check{\lambda}\cdot x}(V^{-w_0(\check{\lambda}\cdot x)} \star -)[\dim \mathrm{Bun}_G]$, so (by the above), right t-exactness of $\mathrm{coeff}[\dim \mathrm{Bun}_G]$ as well.

The same logic applies for left t-exactness, giving the claim.

Step 3. We now deduce the claim for general D. In fact, this is obvious from the $D = 0$ case, given Theorem 7.1.0.1 (in the form of §7.6, and using [FR] to allow divisors with support at multiple points) and Theorem 5.3.0.1.

□

[51] One can work just as well with $D_{\mathrm{hol}}(\mathrm{Bun}_G)$ here.

Part 3. Conservativeness of the Whittaker functor

9. Regular nilpotent singular support and Hecke functors

9.1. **Statement of the result.** This section is dedicated to the proof of the following result.

Theorem 9.1.0.1. *Suppose $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ has the property that $\mathrm{SS}(\mathcal{F}) \cap \mathcal{N}\overset{\circ}{i}lp \neq \emptyset$, i.e., $\mathrm{SS}(\mathcal{F}) \not\subseteq \mathcal{N}ilp_{\mathrm{irreg}}$.*

Then there exists D a $\check{\Lambda}^+$-valued divisor on X such that $\mathcal{N}ilp^{\mathrm{Kos}} \subseteq \mathrm{SS}(\mathcal{V}^D \star \mathcal{F})$.

Combined with Theorem 4.1.0.1, we obtain:

Corollary 9.1.0.2. *Suppose $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$. Then either:*

(1) $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$, or:
(2) There exists D a $\check{\Lambda}^+$-valued divisor on X such that $\mathcal{N}ilp^{\mathrm{Kos}} \subseteq \mathrm{SS}(\mathcal{V}^D \star \mathcal{F})$.

9.2. **A local result.** We begin with a purely local result concerning Hecke modifications and affine Springer fibers.

9.2.1. We work around an implicit point $x \in X(k)$ with coordinate t. Below, it is convenient for indexing purposes to consider Gr_G as the quotient $G(O)\backslash G(K)$, i.e., we quotient on the *left*. There is a residual $G(K)$-action on the right. We let $\mathrm{Gr}_G^{\check{\mu}}$ denote the $G(O)$-orbit through $\check{\mu}(t) \in \mathrm{Gr}_G$, and we let $\overline{\mathrm{Gr}}_G^{\check{\mu}}$ denote its closure.

Let $\xi \in \mathfrak{g}((t))$ be given. We define the *affine Springer fiber* Spr^ξ as:

$$\mathrm{Spr}^\xi := G(O)\backslash\{g \in G(K) \mid \mathrm{Ad}_g(\xi) \in \mathfrak{g}[[t]]\} \subseteq \mathrm{Gr}_G .$$

9.2.2. Below, we fix a k-point $\varphi \in \mathcal{N}(O)$, i.e., a nilpotent element $\varphi \in \mathfrak{g}[[t]]$. We suppose that φ is *generically regular*, i.e., the induced element of $\mathfrak{g}((t))$ is regular.

As in §2.5.5, there is a canonical *discrepancy* $\mathrm{disc}(\varphi) \in \check{\Lambda}^+_{G^{\mathrm{ad}}}$ attached to this element. Specifically, the perspective of *loc. cit.* attaches a $\check{\Lambda}^+_{G^{\mathrm{ad}}}$-valued divisor on the formal disc to φ, but we can think of this simply as an element of $\check{\Lambda}^+_{G^{\mathrm{ad}}}$ via its degree.

Specifically, if $\check{\mu} \in \check{\Lambda}^+_{G^{\mathrm{ad}}}$ is a coweight, saying φ has discrepancy $\check{\mu}$ means that it can be $G(O)$-conjugated into:

$$\mathrm{Ad}_{T(O)\check{\mu}(t)}(e) + [\mathfrak{n}, \mathfrak{n}][[t]] \subseteq \mathfrak{g}[[t]] \tag{9.2.1}$$

for $e \in \oplus_{i \in \mathcal{I}_G} \mathfrak{n}_{\alpha_i} \subseteq \mathfrak{n}$ a regular nilpotent element.

Remark 9.2.2.1. For $\check{\mu} \in \check{\Lambda}^+_{G^{\mathrm{ad}}}$, the construction of §2.5.4 yields a locally closed scheme $\mathcal{N}(O)_{\check{\mu}} \subseteq \mathcal{N}(O)$ parametrizing generically regular $\varphi \in \mathcal{N}(O)$ with discrepancy $\check{\mu}$. Specifically, we take:

$$\mathcal{N}(O)_{\check{\mu}} := \widetilde{\mathcal{N}}(O) \underset{\prod_{i \in \mathcal{I}_G} (\mathbf{A}^1/\mathbf{G}_m)(O)}{\times} \mathrm{Spec}(k)$$

where $\mathrm{Spec}(k) \to \prod_{i \in \mathcal{I}_G} (\mathbf{A}^1/\mathbf{G}_m)(O)$ corresponds to $\check{\mu}$ (i.e., it is the point $(t^{(\check{\mu}, \alpha_i)})_{i \in \mathcal{I}_G}$).

9.2.3. For $\xi \in \mathfrak{g}((t))$ regular nilpotent and $\check{\mu} \in \check{\Lambda}^+_{G^{\mathrm{ad}}}$, let $\mathrm{Spr}^{\xi}_{\check{\mu}} \subseteq \mathrm{Spr}^{\xi}$ denote the locally closed subscheme:

$$\mathrm{Spr}^{\xi}_{\check{\mu}} \coloneqq G(O)\backslash\{g \in G(K) \mid \mathrm{Ad}_g(\xi) \in \mathfrak{g}[[t]], \mathrm{disc}(\mathrm{Ad}_g(\xi)) = \check{\mu}\} \subseteq \mathrm{Gr}_G .$$

In other words, we take:

$$\mathcal{N}(O)_{\check{\mu}}/G(O) \underset{\mathfrak{g}((t))/G(K)}{\times} \{\xi\}.$$

9.2.4. *Main local result.* We now have:

Proposition 9.2.4.1. *Let $\varphi \in \mathring{\mathcal{N}}(O)$ be an (*everywhere*) regular nilpotent element of $\mathfrak{g}[[t]]$.*

Suppose $\check{\lambda} \in \check{\Lambda}^+$ is a dominant coweight, and let $\overline{\check{\lambda}} \in \check{\Lambda}^+_{G^{\mathrm{ad}}}$ denote the induced coweight for G^{ad}.

Then:

$$(\overline{\mathrm{Gr}}^{\check{\lambda}}_G \cap \mathrm{Spr}^{\varphi}_{\overline{\check{\lambda}}})^{\mathrm{red}} = \mathrm{Spec}(k) \in \mathrm{Gr}_G .$$

That is, the displayed intersection is the point $\check{\lambda}(t)$, at least at the reduced level.

Proof. Below, we let $\pi : G(K) \to G(O)\backslash G(K) = \mathrm{Gr}_G$ denote the projection.

The assertion clearly depends only on the $G(O)$-orbit of φ. Therefore, we can assume $\varphi = e \in \oplus_{i\in\mathcal{I}_G}\mathfrak{n}_{\alpha_i} \subseteq \mathfrak{n} \subseteq \mathfrak{n}[[t]]$ (with each projection $e_i \in \mathfrak{n}_{\alpha_i}$ of e necessarily non-zero, of course).

Suppose $g \in G(K)$ is a k-point with $\pi(g) \in \overline{\mathrm{Gr}}^{\check{\lambda}}_G$ with $\mathrm{Ad}_g(e) \in \mathfrak{g}[[t]]$ having discrepancy $\overline{\check{\lambda}}$. We will show $g \in G(O)\check{\lambda}(t)$. Clearly this would suffice.

By (9.2.1), we can find $\gamma \in G(O)$ and $h \in N(K)T(O)\check{\lambda}(t)$ such that:

$$\mathrm{Ad}_{\gamma}\,\mathrm{Ad}_g(e) = \mathrm{Ad}_h(e).$$

The assertion about g depends only depends on its left $G(O)$-coset; therefore, we may replace g by γg to instead write:

$$\mathrm{Ad}_g(e) = \mathrm{Ad}_h(e).$$

By assumption, we can write $h = n\tau\check{\lambda}(t)$ for $n \in N(K)$ and $\tau \in T(O)$.

Observe that gh^{-1} centralizes e; by standard facts about regular nilpotent elements, this implies $gh^{-1} \in N(K)Z_G(K)$ (we remind that $Z_G \subseteq G$ is the center of G). This implies:

$$g = gh^{-1} \cdot h \in N(K)T(O)Z_G(K)\check{\lambda}(t) = \check{\lambda}(t)N(K)T(O)Z_G(K).$$

Therefore:

$$\pi(g) \in \coprod_{\check{\zeta}\in\check{\Lambda}_{Z(G)^\circ}} (\check{\lambda} + \check{\zeta})(t)N(K) \subseteq \mathrm{Gr}_G .$$

(By standard convention, we have omitted a π before $(\check{\lambda} + \check{\zeta})(t)$.)

It is well-known (cf. [MV]) that for $\check{\eta} \in \check{\Lambda}^+$, we have:

$$\overline{\mathrm{Gr}}^{\check{\lambda}}_G \cap \check{\eta}(t)N(K) \neq \emptyset \Leftrightarrow 0 \leq \check{\eta} \leq \check{\lambda}.$$

It follows that:

$$\pi(g) \in \overline{\mathrm{Gr}}^{\check{\lambda}}_G \cap \check{\lambda}(t)N(K).$$

In addition, it is well-known (cf. [MV]) that this intersection is the single point $\check{\lambda}(t)$.

Therefore, we see that $\pi(g) = \check{\lambda}(t)$, as desired.

□

9.3. **Proof of Theorem 9.1.0.1.**

9.3.1. *Bounding singular support from below.* Let $f : \mathcal{H} \to \mathcal{Y}$ be a map of algebraic stacks with $\mathcal{Y}$ smooth and let $\Lambda \subseteq T^*\mathcal{H}$ be a closed conical substack. An *isolated pair for* Λ is a point $(x, \xi) \in \mathcal{H} \times_{\mathcal{Y}} T^*\mathcal{Y}$ with $x \in \mathcal{H}$ (a field-valued point) and $\xi \in T^*_{f(x)}\mathcal{Y}$ such that:

- $df(\xi) \in \Lambda|_x$.
- The intersection $\mathcal{H} \times_{\mathcal{Y}} T^*\mathcal{Y} \cap df^{-1}(\Lambda)$ is zero-dimensional at (x, ξ).

We have:

Theorem 9.3.1.1 ([AGKRRV1] Theorem 20.1.3)**.** *Suppose* $\mathcal{F} \in \mathsf{Shv}(\mathcal{H})$ *with* (x, ξ) *an isolated pair for* $\mathrm{SS}(\mathcal{F})$. *Then* $(f(x), \xi) \in \mathrm{SS}(f_{*,\mathrm{dR}}(\mathcal{F}))$.

Remark 9.3.1.2. In the étale setting, one needs additional hypotheses; cf. [Sai] and [AGKRRV1] Remark 20.1.5.

9.3.2. *Proof.* Choose a coweight $\check{\lambda} \in \check{\Lambda}$ such that:[52]

$$\mathring{\mathcal{N}ilp}^{-w_0(\check{\lambda})+(2-2g)\check{\rho}} \subseteq \mathrm{SS}(\mathcal{F}).$$

We may do this as we assumed $\mathrm{SS}(\mathcal{F}) \subseteq \mathcal{N}ilp$ and $\mathring{\mathcal{N}ilp} \cap \mathrm{SS}(\mathcal{F}) \neq \emptyset$, so some irreducible component of $\mathring{\mathcal{N}ilp}$ must lie in $\mathrm{SS}(\mathcal{F})$.

The relevance (cf. §2.5.4) of $-w_0(\check{\lambda}) + (2 - 2g)\check{\rho}$ exactly means that $\check{\lambda} \in \check{\Lambda}^+$ (equivalently: $-w_0(\check{\lambda}) \in \check{\Lambda}^+$).

Let $x \in X$ be a k-point and let $\check{\lambda}$ be a dominant coweight. Let $\mathring{\mathcal{S}}^{\check{\lambda}} \in D(\mathrm{Gr}_{G,x})$ denote the $*$-extension of the dualizing sheaf on the $\check{\lambda}$-orbit. By geometric Satake, this object has a finite filtration in the derived category with associated graded objects being usual spherical sheaves (up to shifts). Therefore, by the standard interaction of singular support in exact triangles, it suffices to show that $\mathrm{SS}(\mathring{\mathcal{S}}^{\check{\lambda}} \star \mathcal{F}) \cap \mathcal{N}ilp^{\mathrm{Kos}} \neq \emptyset$.

Indeed, let $\mathrm{Bun}_G \xleftarrow{p_1} \mathcal{H}^{\check{\lambda}}_x \xrightarrow{p_2} \mathrm{Bun}_G$ be the corresponding Hecke correspondence associated to x and the coweight $\check{\lambda}$, normalized so the p_2 has fibers that are twisted forms of $\mathrm{Gr}^{\check{\lambda}}_G$. We obtain a correspondence:

$$\begin{tikzcd}
& T^*\mathrm{Bun}_G \underset{\mathrm{Bun}_G}{\times} \mathcal{H}^{\check{\lambda}}_x \arrow[dl] \arrow[dr] & & T^*\mathrm{Bun}_G \underset{\mathrm{Bun}_G}{\times} \mathcal{H}^{\check{\lambda}}_x \arrow[dl] \arrow[dr] & \\
T^*\mathrm{Bun}_G & & T^*\mathcal{H}^{\check{\lambda}}_x & & T^*\mathrm{Bun}_G
\end{tikzcd}$$

[52]The funny indexing is chosen this way for later convenience.

where the first correspondence relates to p_1 and the second relates to p_2. As in [AGKRRV1] §20, the composition of these two correspondences is:

$$\begin{tikzcd} & \{((\mathcal{P}_{G,1},\varphi_1),(\mathcal{P}_{G,2},\varphi_2),\tau)\} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ T^*\operatorname{Bun}_G & & T^*\operatorname{Bun}_G \end{tikzcd}$$

where the top line indicates that $(\mathcal{P}_{G,i},\varphi_i) \in \operatorname{Higgs}_G$ $(i = 1, 2)$ and τ is an isomorphism of these Higgs bundles away from x so that the underlying isomorphism $\mathcal{P}_{G,1}|_{X\setminus x} \simeq \mathcal{P}_{G,2}|_{X\setminus x}$ has relative position $\check{\lambda}$.

The singular support of $p_1^!(\mathcal{F})$ is computed by the usual naive estimate because p_1 is a smooth. Applying Theorem 9.3.1.1, we find that it suffices to find points $(\mathcal{P}_{G,1},\varphi_1) \in \operatorname{SS}(\mathcal{F}) \subseteq T^*\operatorname{Bun}_G$ and a point $(\mathcal{P}_{G,2},\varphi_2) \in T^*\operatorname{Bun}_G$ such that:

$$((\mathcal{P}_{G,1},\varphi_1),(\mathcal{P}_{G,2},\varphi_2),\tau) \in (\alpha\times\beta)^{-1}\big(\operatorname{SS}(\mathcal{F}) \times (\mathcal{P}_{G,2},\varphi_2)\big)$$

and the right hand side should be zero-dimensional at this point; indeed, in this case, we necessarily have $(\mathcal{P}_{G,2},\varphi_2) \in \operatorname{SS}(f_{*,\mathrm{dR}}\mathcal{F})$ (by Theorem 9.3.1.1).

Let $\mathcal{P}_{G,1}$ be the G-bundle induced from the T-bundle $\check{\rho}(\Omega^1_X)(-\check{\lambda}\cdot x)$.[53] There is a canonical nilpotent Higgs field f_D^{glob} on $\mathcal{P}_{G,1}$, generalizing the $D = 0$ case from §2.5.6.

Note that the point $(\mathcal{P}_{G,1}, f_D^{\mathrm{glob}})$ lies in $\overset{\circ}{\mathcal{N}ilp}^{-w_0(\check{\lambda})+(2-2g)\check{\rho}}$, so lies in $\operatorname{SS}(\mathcal{F})$.

We take $(\mathcal{P}_{G,2},\varphi_2)$ to be the base-point of $\overset{\circ}{\mathcal{N}ilp}^{\mathrm{Kos}}$; i.e., we apply the above construction with D replaced by 0. We have an evident choice of Hecke modification τ.

By Proposition 9.2.4.1, we have:

$$(\alpha\times\beta)^{-1}\big(\overset{\circ}{\mathcal{N}ilp}^{-w_0(\check{\lambda})+(2-2g)\check{\rho}} \times (\mathcal{P}_{G,2},\varphi_2)\big) \tag{9.3.1}$$

is the single point constructed above.

In more detail: Hecke modifications of $(\mathcal{P}_{G,2},\varphi_2)$ are determined by their restrictions to the formal neighborhood of x. Moreover, any point $(\widetilde{\mathcal{P}}_{G,1},\widetilde{\varphi}_1,\tau)$ of (9.3.1) must have that the discrepancy divisor of $\widetilde{\varphi}_1$ is supported only at x, since $(\mathcal{P}_{G,2},\varphi_2)$ has vanishing discrepancy divisor and the two are isomorphism away from x. As we know the degree of $\widetilde{\mathcal{P}}_{G,1}$ (as it is a $\check{\lambda}$-modification of $\mathcal{P}_{G,2}$) and have assumed $(\widetilde{\mathcal{P}}_{G,1},\widetilde{\varphi}_1) \in \overset{\circ}{\mathcal{N}ilp}^{-w_0(\check{\lambda})+(2-2g)\check{\rho}}$, the discussion of Remark 2.5.5.1 determines the degree of the discrepancy divisor, whose here is found to be $\overline{-w_0(\check{\lambda})}$, i.e., the image of $-w_0(\check{\lambda})$ in $\check{\Lambda}_{G^{\mathrm{ad}}}$. We track signs: $\mathcal{P}_{G,2}$ is a modification of $\mathcal{P}_{G,1}$ of type $\check{\lambda}$, so $\mathcal{P}_{G,1}$ is a modification of $\mathcal{P}_{G,2}$ of type $-w_0(\check{\lambda})$. Choosing arbitrarily a trivialization of $\mathcal{P}_{G,2}$ on the formal disc of x, the proposition now applies and yields our claim.

10. Conservativeness of Whittaker coefficients

At this point, the proof of the main theorem is essentially just a matter of combining our previous results. Specifically, this is true in the nilpotent setting; we deduce the assertion for general D-modules using a straightforward application of the method of [AGKRRV1] §21.

[53]For the reader's convenience in verifying some formulae below, we note that if we twist by w_0, we see that this G-bundle is also induced from the T-bundle $(-\check{\rho})(\Omega^1_X)(-w_0(\check{\lambda})\cdot x)$.

10.1. **Conservativeness for nilpotent sheaves.**

10.1.1. Let $\mathcal{N}ilp^{\neq \mathrm{Kos}} \subseteq \mathcal{N}ilp$ be the union of all components of $\mathcal{N}ilp$ *besides* $\mathcal{N}ilp^{\mathrm{Kos}}$; this is a closed conical subset of $\mathcal{N}ilp$ because $\mathcal{N}ilp^{\mathrm{Kos}}$ is open in $\mathcal{N}ilp$.

We have:

Lemma 10.1.1.1. $\mathsf{Shv}_{\mathcal{N}ilp^{\neq \mathrm{Kos}}}(\mathrm{Bun}_G)$ *is exactly the kernel of the functor* $\mathrm{coeff} : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Vect}$.

Proof.

Step 1. First, we show $\mathsf{Shv}_{\mathcal{N}ilp^{\neq \mathrm{Kos}}}(\mathrm{Bun}_G) \subseteq \mathrm{Ker}(\mathrm{coeff})$.

Note that $\mathsf{Shv}_{\mathcal{N}ilp^{\neq \mathrm{Kos}}}(\mathrm{Bun}_G) \subseteq \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ is closed under truncations and subobjects by definition of singular support. Moreover, again by definition, this category is left complete for its t-structure; in particular, an object is zero if and only if all its cohomology groups are zero.

Therefore, by t-exactness (up to shift) of coeff (Theorem 8.0.0.1), it suffices to show $\mathrm{coeff}(\mathcal{F}) = 0$ for $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp^{\neq \mathrm{Kos}}}(\mathrm{Bun}_G)^{\heartsuit}$. Any such object is the union of its constructible subobjects, so we may assume $\mathcal{F}$ is constructible (by exactness again).

In this case, $\mathrm{coeff}(\mathcal{F}) \in \mathsf{Vect}^c$ and lies in a single cohomological degree. Therefore, it suffices to show that its Euler characteristic is zero. This follows from the assumption on $\mathcal{F}$ and from the index theorem, Theorem 6.1.2.1.

Step 2. Next, we show that if $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ with $\mathcal{N}ilp^{\mathrm{Kos}} \subseteq \mathrm{SS}(\mathcal{F})$, then $\mathrm{coeff}(\mathcal{F}) \neq 0$.

The logic is the essentially same as in the previous step. There exists some integer i and some construcible subobject $\mathcal{G} \subseteq H^i(\mathcal{F})$ such that $\mathcal{N}ilp^{\mathrm{Kos}} \subseteq \mathrm{SS}(\mathcal{G})$.

In this case, $\mathrm{coeff}(\mathcal{G})$ is concentrated in cohomological degree $\dim \mathrm{Bun}_G$. Moreover, it is non-zero by Theorem 6.1.2.1; here we remind that for objects of $\mathsf{Shv}^{\heartsuit}$, the characteristic cycle assigns *positive* integers to components of the singular support.

By exactness of coeff, we then have:

$$H^{\dim \mathrm{Bun}_G}(\mathrm{coeff}(\mathcal{G})) \hookrightarrow H^{\dim \mathrm{Bun}_G}(\mathrm{coeff}(H^i(\mathcal{F}))) = H^{\dim \mathrm{Bun}_G + i}\, \mathrm{coeff}(\mathcal{F}).$$

Clearly this implies $\mathrm{coeff}(\mathcal{F}) \neq 0$.

□

Corollary 10.1.1.2. *Suppose* $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ *has non-zero projection to* $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{temp}}$.

Then there is a $\check{\Lambda}^+$*-valued divisor* D *on* X *such that* $\mathrm{coeff}_D(\mathcal{F}) \neq 0$.

Proof. By assumption, $\mathcal{F} \notin \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{anti-temp}}$. Therefore, by Theorem 4.1.0.1, $\mathcal{F} \cap \mathring{\mathcal{N}ilp} \neq \emptyset$.

Therefore, by Theorem 9.1.0.1, there is a $\check{\Lambda}^+$-valued divisor D on X such that $\mathcal{N}ilp^{\mathrm{Kos}} \subseteq \mathrm{SS}(\mathcal{V}^D \star \mathcal{F})$. In this case, by Lemma 10.1.1.1 and Theorem 5.3.0.1, we obtain:

$$\mathrm{coeff}_D(\mathcal{F}) = \mathrm{coeff}(\mathcal{V}^D \star \mathcal{F}) \neq 0.$$

□

10.2. **Enhanced coefficient functors.** It is now convenient to introduce the following functor encoding all Whittaker coefficient functors simultaneously.

10.2.1. *Nilpotent setting.* Recall the prestack $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}$ from [AGKRRV1]. By *loc. cit.* Theorem 14.3.2, there is a canonical *spectral* action of $\mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ that is suitably compatible with Hecke functors.

Moreover, the category $\mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ is canonically self-dual by [AGKRRV1] Corollary 7.8.9. As in *loc. cit.*, we let:

$$\Gamma_! = \Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}, -) : \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \to \mathsf{Vect}$$

denote the functor dual to the structure sheaf $\mathcal{O}_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}} \in \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

10.2.2. On formal grounds, we obtain a canonical functor:

$$\mathrm{coeff}^{\mathrm{enh}} : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$$

fitting into a commutative diagram:

$$\begin{tikzcd}
\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \arrow[d, "\mathrm{coeff}^{\mathrm{enh}}"] \arrow[dr, "\mathrm{coeff}"] & \\
\mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \arrow[r, "\Gamma_!"] & \mathsf{Vect}.
\end{tikzcd} \tag{10.2.1}$$

Namely, the construction proceeds as follows. We have an action functor:

$$\mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \otimes \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G).$$

Dualizing the first tensor factor and applying its self-duality, we obtain a functor:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \otimes \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G).$$

Now compose this functor with $\mathrm{id} \otimes \mathrm{coeff}$. By construction, this functor has the desired property.

By construction, $\mathrm{coeff}^{\mathrm{enh}}$ factors through $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{temp}}$. We abuse notation in also denoting this functor by $\mathrm{coeff}^{\mathrm{enh}}$.

10.2.3. More generally, for $\mathcal{G} \in \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ and $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$, if we let $\star$ denote the action of the former category on the latter, we have:

$$\mathrm{coeff}(\mathcal{G} \star \mathcal{F}) = \Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}, \mathcal{G} \otimes \mathrm{coeff}^{\mathrm{enh}}(\mathcal{F})).$$

Recalling that Hecke functors (at points) factor through the action of $\mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ and applying Theorem 5.3.0.1, we see that $\mathrm{coeff}_D(\mathcal{F})$ can be algorithmically extracted from $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F})$.

In particular, we obtain:

Theorem 10.2.3.1. *The functor:*

$$\mathrm{coeff}^{\mathrm{enh}} : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)^{\mathrm{temp}} \to \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$$

is conservative.

Indeed, this is immediate from Corollary 10.1.1.2 and Lemma 10.1.1.1.

10.2.4. *Variant for D-modules.* Recall from [Gai2] (see also [Gai6] §4.3-4.5 and §11.1) that there is a canonical action of $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$ on $D(\mathrm{Bun}_G)$, similar to the above functors.

As $\mathrm{LS}_{\check{G}}$ is a QCA stack, $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$ is canonically self-dual; this time, the functor dual to $\mathcal{O}$ is simply usual global sections.

Therefore, we obtain a functor:

$$\mathrm{coeff}^{\mathrm{enh}} : D(\mathrm{Bun}_G) \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$$

fitting into a commutative diagram:

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[d, "\mathrm{coeff}^{\mathrm{enh}}"'] \arrow[dr, "\mathrm{coeff}"] & \\ \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[r, "\Gamma"] & \mathsf{Vect}. \end{tikzcd}$$

10.2.5. We now state the natural compatibility between the above two constructions.

Let $\iota : \mathrm{LS}_{\check{G}}^{\mathrm{restr}} \to \mathrm{LS}_{\check{G}}$ denote the natural map. The symmetric monoidal functor:

$$\iota^* : \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

admits a left adjoint $\iota_?$ (denoted $\iota_!$ in [AGKRRV1] §7.1.3); this functor is naturally the dual to ι^* for the self-duality of both sides. In particular, $\Gamma(\mathrm{LS}_{\check{G}}, \iota_?(-)) = \Gamma_!$.

We obtain commutative diagrams:

$$\begin{tikzcd} \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}^{\mathrm{enh}}"] \arrow[d] & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \arrow[d, "\iota_?"] \\ D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}^{\mathrm{enh}}"] & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) \end{tikzcd}$$

and:

$$\begin{tikzcd} \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}^{\mathrm{enh}}"] & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \\ D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}^{\mathrm{enh}}"] \arrow[u] & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}). \arrow[u, "\iota^*"'] \end{tikzcd}$$

Here the left arrow in the second diagram is the right adjoint to the embedding, and the commutativity of this latter diagram follows from [AGKRRV1] Proposition 14.5.3.

10.3. **Conservativeness for general D-modules.** We now conclude the proof of our main result, Theorem 10.3.3.1 below.

10.3.1. *Field extensions.* We briefly digress to discuss field extensions.

Suppose k'/k is a (possibly transcendental) field extension. For $\mathcal{Y}$ over k, we let $\mathcal{Y}_{k'}$ denote the base-change of $\mathcal{Y}$ to k'.

For prestacks over k', we write $D_{/k'}(-)$ to denote D-modules considered relative to the field k'. We use $\mathsf{Shv}_{/k'}$ similarly: this means the ind-category version of regular holonomic objects of $D_{/k'}(-)$.

10.3.2. Let $x \in X(k)$ be fixed, and let $x' \in X'(k')$ be the induced point.

There is a natural equivalence:

$$\mathcal{H}_x^{\mathrm{sph}} \otimes \mathsf{Vect}_{k'} \simeq \mathcal{H}_{x'}^{\mathrm{sph}}.$$

Here the right hand side is taken to be defined with D-modules over k', as above. In other words, up to extending scalars, the spherical Hecke categories are the same. This identification is compatible in the natural sense with derived Satake.

It follows from the definitions that for $\mathcal{C} \in \mathcal{H}_x^{\mathrm{sph}}\text{–}\mathsf{mod}$ with $\mathcal{C}' := \mathcal{C} \otimes \mathsf{Vect}_{k'}$, we have commutative diagrams:

$$\begin{tikzcd}
\mathcal{C}^{x-\text{anti–temp}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}^{x-\text{temp}} \arrow[d] \\
\mathcal{C}'^{,x-\text{anti–temp}} \arrow[r] & \mathcal{C}' \arrow[r] & \mathcal{C}'^{,x-\text{temp}}.
\end{tikzcd}$$

There are similar functors if we work with the adjoints to the horizontal arrows. Moreover, the vertical arrows induce isomorphisms after tensoring with $\mathsf{Vect}_{k'}$.

10.3.3. *General case.* Finally, we show:

Theorem 10.3.3.1. *The functor*

$$\mathrm{coeff}^{\mathrm{enh}} : D(\mathrm{Bun}_G)^{\mathrm{temp}} \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$$

is conservative.

Suppose $\mathcal{F} \in D(\mathrm{Bun}_G)$ is given with $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}) \in \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$ vanishing. We need to show that $\mathcal{F}$ is anti-tempered, i.e., its image in $D(\mathrm{Bun}_G)^{\mathrm{temp}}$ is zero.

As in [AGKRRV1] Lemma 21.4.6, it suffices to show that for any field extension k'/k and any $\sigma \in \mathrm{LS}_{\check{G}}(k')$, the image of $\mathcal{F}$ in:

$$D(\mathrm{Bun}_G)^{\mathrm{temp}} \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{Vect}_{k'}$$

is zero.

The functor:

$$\mathsf{Shv}_{/k',\mathcal{N}ilp_{k'}}(\mathrm{Bun}_{G,k'}) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G},k'})}{\otimes} \mathsf{Vect}_{k'} \to D(\mathrm{Bun}_G) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{Vect}_{k'}$$

is an equivalence by [AGKRRV1] Proposition 13.5.3. The same applies for tempered variants by §10.3.2.

From §10.2.5, we have a commutative diagram:

$$\begin{tikzcd}
\mathsf{Shv}_{/k',\mathcal{N}ilp_{k'}}(\mathrm{Bun}_{G,k'})^{\mathrm{temp}} \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G},k'})}{\otimes} \mathsf{Vect}_{k'} \arrow[d, "\simeq"] \arrow[dr] & \\
D(\mathrm{Bun}_G)^{\mathrm{temp}} \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{Vect}_{k'} \arrow[r] & \mathsf{Vect}_{k'}.
\end{tikzcd} \tag{10.3.1}$$

Here the rightward arrows are induced by extension of scalars from the functors $\mathrm{coeff}^{\mathrm{enh}}$. The left arrow in this diagram is conservative by Theorem 10.2.3.1, observing that we have a commutative diagram:

$$\begin{tikzcd} \mathsf{Shv}_{/k',\mathcal{N}ilp_{k'}}(\mathrm{Bun}_{G,k'})^{\mathrm{temp}} \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G},k'})}{\otimes} \mathsf{Vect}_{k'} \arrow[r] \arrow[d] & \mathsf{Shv}_{/k',\mathcal{N}ilp_{k'}}(\mathrm{Bun}_{G,k'})^{\mathrm{temp}} \arrow[d, "\mathrm{coeff}^{\mathrm{enh}}"] \\ \mathsf{Vect}_{k'} \arrow[r] & \mathsf{QCoh}(\mathrm{LS}_{\check{G},k'}) \end{tikzcd}$$

with horizontal and right arrows conservative. Therefore, the right arrow in (10.3.1) is conservative, implying $\mathcal{F}$ maps to zero in this category, yielding the claim.

11. The structure of Hecke eigensheaves

In this section, we use our earlier results to deduce structural properties of Hecke eigensheaves. The main result is Theorem 11.1.4.1.

Throughout this section, to simplify the discussion a bit, we assume the ground field k to be algebraically closed (in addition to being of characteristic 0).

11.1. **Setup.**

11.1.1. *Notation for local systems.* Throughout this section, fix $\sigma \in \mathrm{LS}_{\check{G}}(k)$ an *irreducible* $\check{G}$-local system on X; i.e., σ does not admit a reduction to any parabolic $\check{P} \subsetneq \check{G}$.

11.1.2. Our discussion will be nicest when σ is *very irreducible* in the sense described below.

Let $\mathrm{Aut}(\sigma)$ denote the algebraic[54] group of automorphisms of σ as a local system. Note that a choice of point $x \in X(k)$ induces an embedding $\mathrm{Aut}(\sigma) \hookrightarrow \check{G}$.

Definition 11.1.2.1. We say that σ is *very irreducible* if the natural map $Z_{\check{G}} \to \mathrm{Aut}(\sigma)$ is an isomorphism; here $Z_{\check{G}}$ is the center of $\check{G}$.

Remark 11.1.2.2. Of course, for GL_n, an irreducible local system is very irreducible.

Remark 11.1.2.3. Very irreducible local systems form an open substack of $\mathrm{LS}_{\check{G}}$ that is non-empty when the genus of the curve is greater than 1. This follows from the existence of opers (without singularity) on X, which are always very irreducible, cf. [BD] §3.1.

Probably very irreducible local systems are dense in $\mathrm{LS}_{\check{G}}$ (for genus > 1), but we are not sure.

Remark 11.1.2.4. It is not hard to see that $\mathrm{Aut}(\sigma)/Z_{\check{G}}$ is zero-dimensional. Therefore, the gap between irreducible and very irreducible local systems concerns finite groups.

Example 11.1.2.5. For completeness, we provide an explicit (quite elementary) example of an irreducible local system that is not very irreducible. The group is PGL_2.

Suppose $k = \mathbf{C}$ and X has genus 2; we freely use Riemann-Hilbert. It is simple to see that the topological fundamental group $\pi_1^{\mathrm{top}}(X)$ surjects onto the symmetric group S_3. Indeed, the former has standard generators a_1, b_1, a_2, b_2 with defining relation $[a_1, b_1][a_2, b_2] = 1$, while the latter can be generated by elements r and s with r an element of order 3 and s a transposition with defining relation $srs = r^{-1}$. Then the map:

$$a_1 \mapsto r, \;\; b_1 \mapsto s, \;\; a_2 \mapsto r^2, \;\; b_2 \mapsto s$$

[54] In particular, $\mathrm{Aut}(\sigma)$ is something classical – we ignore the finer derived structures on $\mathrm{Aut}(\sigma)$.

defines our desired surjective homomorphism.

Then S_3 has a unique (up to isomorphism) irreducible 2-dimensional representation, which via the homomorphism above induces an irreducible GL_2-local system on X. The induced PGL_2-local system is also irreducible, but is easily seen not to be *very* irreducible.

11.1.3. Let $k_\sigma \in \mathsf{QCoh}(\mathrm{LS}_{\check{G}})^\heartsuit$ denote the structure sheaf of σ, i.e., the $*$-pushforward of $k \in \mathsf{Vect} \simeq \mathsf{QCoh}(\mathrm{Spec}(k))$ along the map:

$$\mathrm{Spec}(k) \xrightarrow{\sigma} \mathrm{LS}_{\check{G}}$$

11.1.4. *The main result.* The goal of this section is to outline a proof of the following result.

Theorem 11.1.4.1. *There exists $\mathcal{F}_\sigma \in D(\mathrm{Bun}_G)$ an eigensheaf for σ such that:*

- *$\mathcal{F}_\sigma$ is perverse up to shifts (i.e., locally compact, concentrated in cohomological degree* 0*, and with regular singularities).*
- *If σ is very irreducible, the restriction of $\mathcal{F}_\sigma$ to every connected component of* Bun_G *is an irreducible perverse sheaf.*

In addition, one has:

- *$\mathcal{F}_\sigma$ is cuspidal.*
- $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}_\sigma) \simeq k_\sigma[-\dim \mathrm{Bun}_G] \in \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$.

We emphasize that the argument combines work of many other authors, and some portion of what follows is simply a matter of proving a folklore result by stringing together results of other authors. In particular, the existence of a non-zero *complex* of sheaves $\mathcal{F}_\sigma$ (Theorem 11.2.1.1 below) is the culmination of work of many authors, notably Arinkin, Gaitsgory, Frenkel-Gaitsgory, Beilinson-Drinfeld, and not us.

We consider our contribution to be to the questions of perversity and irreducibility of $\mathcal{F}_\sigma$. This is a classical question: the above result proves [Fre] Conjecture 1.[55] We highlight that it has previously been unknown how to deduce any result of this type from the categorical properties of the geometric Langlands conjecture.

Remark 11.1.4.2. At some points, complete proofs of some key assertions are missing in the literature. Most glaringly: we need a generalization of [BG3]; *loc. cit.* is written for the Borel only, and we need the (folklore) generalization to a general parabolic subgroup.[56]

Remark 11.1.4.3. In the case of general irreducible σ, $\mathcal{F}_\sigma$ ought to be a semi-simple perverse sheaf with irreducible factors indexed by isomorphism classes irreducible representations $W_i \in \mathsf{Rep}(\mathrm{Aut}(\sigma))^\heartsuit$, with each simple factor $\mathcal{F}_{\sigma,W_i}$ appearing with multiplicity $\dim(W_i)$. We are unable to unconditonally prove this assertion at the moment, but our methods combined with the categorical geometric Langlands conjecture yield this conclusion, which seems not to have been previously contemplated.

Remark 11.1.4.4. The condition $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}_\sigma) \simeq k_\sigma[-\dim \mathrm{Bun}_G]$ is often referred to as the *Whittaker normalization* of a Hecke eigensheaf (at least, up to conventions regarding the shift). It plays a key role for us in what follows, cf. Theorem 11.2.1.2.

[55] Technically, *loc. cit.* ignores the discrepancy between irreducible and very irreducible local systems. For irreducible σ that are not very irreducible, the conjecture from *loc. cit.* is not reasonable; cf. Remark 11.1.4.3.

[56] We also remark that the literature has at other times appealed to this same folklore generalization. See e.g. [BHKT] Appendix A.

Remark 11.1.4.5. The assumption that k is algebraically closed is used in the reference to [Ari], and specifically to the existence of Whittaker normalized $\mathcal{F}_\sigma$; otherwise, we may a priori need to extend the ground field to obtain an oper structure (conjecturally, this should not be necessary).

11.2. **Formulation of intermediate results.** We now formulate a series of results from which we deduce Theorem 11.1.4.1.

11.2.1. *Existence of eigensheaves.* Crucially, we have:

Theorem 11.2.1.1 (Folklore)**.** *There exists an object $\mathcal{F}_\sigma \in D(\mathrm{Bun}_G)$ such that:*

- *$\mathcal{F}_\sigma$ is a Hecke eigensheaf with eigenvalue σ (see §11.3.1 for the definition).*
- $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}_\sigma) \simeq k_\sigma[-\dim \mathrm{Bun}_G] \in \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$.

Briefly: the proof is via the Kac-Moody localization method pioneered by Beilinson-Drinfeld, appealing to later developments due to Frenkel-Gaitsgory, independent ideas of Gaitsgory, and Arinkin. We review the relevant results in Appendix A.

In §11.3, we will show the following result.

Theorem 11.2.1.2. *Any object $\mathcal{F}_\sigma$ satisfying the conclusion of Theorem 11.2.1.1 also satisfies the conclusion of Theorem 11.1.4.1.*

Remark 11.2.1.3 (Application to [BD])*.* Suppose σ admits an oper structure (without singularities). In this case, σ is necessarily very irreducible (cf. Remark 11.1.2.3). We can take $\mathcal{F}_\sigma$ to be the D-module constructed in [BD] §5.1.1.;[57] that $\mathcal{F}_\sigma$ is Whittaker-normalized is a special case of the discussion in Appendix A. Therefore, Theorem 11.2.1.2 implies that the eigensheaves constructed in [BD] are perverse sheaves with irreducible restrictions to each connected component of Bun_G, answering in the affirmative a question of Beilinson-Drinfeld (see [BD] §5.2.7).

11.2.2. *Cuspidality.* We have the following result:

Theorem 11.2.2.1 (Braverman-Gaitsgory)**.** *Let $P \subseteq G$ be a parabolic subgroup with Levi quotient M. Let $\check{P} \subseteq \check{G}$ be the dual parabolic.*

Let $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})_{\check{P}} \subseteq \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$ denote the full subcategory of objects supported (set theoretically) on the image of the (proper) morphism $\mathrm{LS}_{\check{P}} \to \mathrm{LS}_{\check{G}}$.

Then the composition:

$$D(\mathrm{Bun}_M) \xrightarrow{\mathrm{Eis}_!} D(\mathrm{Bun}_G)$$

maps into the full subcategory:

$$D(\mathrm{Bun}_G) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{QCoh}(\mathrm{LS}_{\check{G}})_{\check{P}} \subseteq D(\mathrm{Bun}_G) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) = D(\mathrm{Bun}_G).$$

Proof. For $\check{P} = \check{B}$, this follows from the Hecke property of Eisenstein series shown in [BG3] Theorem 8.8 (see also *loc. cit.* Theorem 1.11).

In general, it is expected that the results of [BG3] generalize without major changes to parabolics. In particular, the assertion of the present theorem is asserted (and refined) by Gaitsgory in [Gai6] Proposition 11.1.3.

$\square$

[57] In [BD], our σ is denoted $\mathcal{F}$ and our $\mathcal{F}_\sigma$ is denoted $M_{\mathcal{F}}$.

Let $\mathrm{LS}_{\check{G}}^{\mathrm{irred}} \subseteq \mathrm{LS}_{\check{G}}$ denote the open parametrizing irreducible $\check{G}$-local systems; i.e., $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$ is the complement to the images of the maps $\mathrm{LS}_{\check{P}} \to \mathrm{LS}_{\check{G}}$ for $\check{P} \neq \check{G}$. Let j denote the relevant open embedding.

By adjunction, we have adjoint functors:

$$j^* : D(\mathrm{Bun}_G) \rightleftarrows D(\mathrm{Bun}_G) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}}) : j_*$$

with j_* being fully faithful.

Corollary 11.2.2.2. *For M a Levi besides G, the composition:*

$$D(\mathrm{Bun}_M) \xrightarrow{\mathrm{Eis}_!} D(\mathrm{Bun}_G) \xrightarrow{j^*} D(\mathrm{Bun}_G) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}})$$

is zero.

As cuspidal objects of Bun_G are exactly those objects in the right orthogonal to Eisenstein series along proper parabolics (cf. [DG2] §1.4), we obtain:

Corollary 11.2.2.3. *Any object of $D(\mathrm{Bun}_G)$ in the essential image of the functor:*

$$j_* : D(\mathrm{Bun}_G) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}}) \to D(\mathrm{Bun}_G)$$

is cuspidal.

Remark 11.2.2.4. The geometric Langlands conjectures predict that conversely, any cuspidal object of $D(\mathrm{Bun}_G)$ lies in the essential image of the above functor. This converse appears to be out of reach using present methods.

11.3. **Proof of Theorem 11.2.1.2.**

11.3.1. Let $D_\sigma(\mathrm{Bun}_G)$ denote the category:

$$D_\sigma(\mathrm{Bun}_G) := \mathrm{Hom}_{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})\text{-}\mathsf{mod}}(\mathsf{Vect}, D(\mathrm{Bun}_G)) \simeq D(\mathrm{Bun}_G) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathsf{Vect}$$

where $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$ acts on Vect via the symmetric monoidal functor of pullback along $\sigma : \mathrm{Spec}(k) \to \mathrm{LS}_{\check{G}}$ and we are using self-duality of Vect in the above identification.

By definition, a *Hecke eigensheaf* with eigenvalue σ is an object of this category $D_\sigma(\mathrm{Bun}_G)$.

Let $D_{\widehat{\sigma}}(\mathrm{Bun}_G) \subseteq D(\mathrm{Bun}_G)$ denote the full subcategory generated under colimits by the essential image of $D_\sigma(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$.

We now have:

Lemma 11.3.1.1 ([AGKRRV1]). *(1) Every object of $D_{\widehat{\sigma}}(\mathrm{Bun}_G)$ lies in $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ and has regular singularities.*

(2) The embedding $D_{\widehat{\sigma}}(\mathrm{Bun}_G) \hookrightarrow \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ extends to a decomposition:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) = D_{\widehat{\sigma}}(\mathrm{Bun}_G) \times \mathsf{Shv}_{\mathcal{N}ilp,\neq\sigma}(\mathrm{Bun}_G).$$

Proof. These are all structural results from [AGKRRV1].

That every object of $D_{\widehat{\sigma}}(\mathrm{Bun}_G)$ lies in $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ (and in particular, has regular singularities) is [AGKRRV1] Proposition 14.5.3 combined with *loc. cit.* Main Corollary 16.5.6.

The asserted product decomposition follows from [AGKRRV1] Corollary 14.3.5.

□

Remark 11.3.1.2. Only (2) above uses irreducibility of σ.

11.3.2. By Lemma 11.3.1.1 (2), the subcategory:

$$D_{\widehat{\sigma}}(\mathrm{Bun}_G) \subseteq \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$$

is closed under truncation functors for the natural t-structure on the right hand side; therefore, this subcategory inherits a canonical t-structure (uniquely characterized by t-exactness of the above embedding).

11.3.3. We now consider the functor:

$$\mathrm{coeff}^{\mathrm{enh}}_{\widehat{\sigma}} : D_{\widehat{\sigma}}(\mathrm{Bun}_G) \to \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$$

given as the restriction of $\mathrm{coeff}^{\mathrm{enh}}$ (in its $\mathsf{Shv}_{\mathcal{N}ilp}$ incarnation, cf. §10.2.2).

By the argument of §1.6.2, combining Theorem 8.0.0.1, Theorem 7.1.0.1, and [AGKRRV1] Theorem 1.4.5, we find that the functor:

$$\mathrm{coeff}^{\mathrm{enh}}_{\widehat{\sigma}}[\dim \mathrm{Bun}_G] : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$$

is t-exact. The same applies for $\mathrm{coeff}^{\mathrm{enh}}_{\widehat{\sigma}}$.

Moreover, by Corollary 11.2.2.3, any object of $D_{\widehat{\sigma}}(\mathrm{Bun}_G)$ is cuspidal, hence, by [Ber3], tempered. Therefore, Theorem 10.3.3.1 implies that $\mathrm{coeff}^{\mathrm{enh}}_{\widehat{\sigma}}$ is conservative, as well as t-exact (up to shift).

11.3.4. *Irreducible case.* We now clearly obtain the assertion of Theorem 11.2.1.2 in the case of (possibly not very) irreducible σ.

Namely, for $\mathcal{F}_\sigma \in D_\sigma(\mathrm{Bun}_G)$ as in the statement of the theorem. We abuse notation in also letting $\mathcal{F}_\sigma$ denote the corresponding object of $D_{\widehat{\sigma}}(\mathrm{Bun}_G)$.

We have already noted that any object of $D_{\widehat{\sigma}}(\mathrm{Bun}_G)$ has regular singularities and is cuspidal.

For $i \in \mathbf{Z}$, t-exactness of $\mathrm{coeff}^{\mathrm{enh}}$ yields:

$$\mathrm{coeff}^{\mathrm{enh}}(H^i(\mathcal{F}_\sigma)) = H^{i+\dim \mathrm{Bun}_G} \, \mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}_\sigma).$$

By assumption on $\mathcal{F}_\sigma$, the right hand side vanishes for $i \neq 0$. As $\mathrm{coeff}^{\mathrm{enh}}$ is conservative, this means $H^i(\mathcal{F}_\sigma) = 0$ for $i \neq 0$, so $\mathcal{F}_\sigma$ is ind-perverse.

We prove that $\mathcal{F}_\sigma$ is perverse in Remarks 11.3.5.2 and 11.3.6.1 below.

11.3.5. *Very irreducible case; G is semi-simple.* We now prove Theorem 11.2.1.2 for σ very irreducible. The argument is more direct for G semi-simple, so we impose this assumption at first.

Recall that the connected components of Bun_G are labeled by elements $c \in \pi_1^{\mathrm{alg}}(G)$. For $c \in \pi_1^{\mathrm{alg}}(G)$, let Bun_G^c denote the corresponding connected component and let $\mathcal{F}_\sigma = \oplus_{c \in \pi_1^{\mathrm{alg}}(G)} \mathcal{F}_\sigma^c$ denote the decomposition of $\mathcal{F}_\sigma$ by connected components (so $\mathcal{F}_\sigma^c$ is the restriction of $\mathcal{F}_\sigma$ to Bun_G^c).

It suffices to show:

Lemma 11.3.5.1. *(1) For each $c \in \pi_1^{\mathrm{alg}}(G)$, $\mathcal{F}_\sigma^c$ is non-zero.*

(2) The length of $\mathcal{F}_\sigma \in D(\mathrm{Bun}_G)^\heartsuit$ as a perverse[58] *sheaf is $\leq |\pi_1^{\mathrm{alg}}(G)|$.*

Indeed, this suffices as we then have (for ℓ denoting length):

$$|\pi_1^{\mathrm{alg}}(G)| \leq \sum_{c \in \pi_1^{\mathrm{alg}}(G)} \ell(\mathcal{F}_\sigma^c) = \ell(\mathcal{F}_\sigma) \leq |\pi_1^{\mathrm{alg}}(G)|$$

with the first inequality being (1) and the second being (2); this forces the inequalities to be equalities, which then forces each summand to be exactly 1, as desired.

Proof of Lemma 11.3.5.1. First, (1) is immediate from the eigensheaf property, since $\mathcal{F}_\sigma$ is non-zero.

Alternatively, we can see this from the Whittaker normalization. Calculating $\mathrm{coeff}_D(\mathcal{F}_\sigma)$ from $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}_\sigma) = k_\sigma[-\dim \mathrm{Bun}_G]$ (as in §10.2.3), we find that $\mathrm{coeff}_D(\mathcal{F}_\sigma)$ is non-zero for *every* D; however, by definition, we have:

$$\mathrm{coeff}_D(\mathcal{F}_\sigma) = \mathrm{coeff}_D(\mathcal{F}_\sigma^c)$$

where $c \in \pi_1^{\mathrm{alg}}(G)$ is the class represented by the cocharacter $(\check{\rho} \cdot \deg(\Omega_X^1)) - \deg(D)$. This clearly implies that $\mathcal{F}_\sigma^c$ must be non-zero for each c.

We now prove (2). Note that any subquotient of $\mathcal{F}_\sigma \in D(\mathrm{Bun}_G)$ also lies in $D_{\widehat{\sigma}}(\mathrm{Bun}_G)^\heartsuit$. As $\mathrm{coeff}_{\widehat{\sigma}}^{\mathrm{enh}}[\dim \mathrm{Bun}_G]$ is conservative and t-exact, it follows that:

$$\ell(\mathcal{F}_\sigma) \leq \ell(\mathrm{coeff}_{\widehat{\sigma}}^{\mathrm{enh}}(\mathcal{F}_\sigma)[\dim \mathrm{Bun}_G]) = \ell(k_\sigma). \tag{11.3.1}$$

But $k_\sigma \in \mathsf{QCoh}(\mathrm{LS}_{\check{G}})^\heartsuit$ is calculated as a pushforward along the composition:

$$\mathrm{Spec}(k) \to \mathbf{B}Z_{\check{G}} \hookrightarrow \mathrm{LS}_{\check{G}}$$

where the second map is a closed embedding by very irreducibility of σ. Therefore, the length of k_σ is the same as the length of the regular representation of $Z_{\check{G}}$. As $Z_{\check{G}}$ is finite abelian, we have:

$$\ell(k_\sigma) = |Z_{\check{G}}| = |\pi_1^{\mathrm{alg}}(G)|$$

as desired.

□

Remark 11.3.5.2. In the case of (not necessarily very) irreducible σ, the statement of Lemma 11.3.5.1 (2) should instead say $\ell(\mathcal{F}_\sigma) \leq \sum \dim(W_i)$ for notation as in Remark 11.1.4.3, i.e., the W_i are isomorphism classes of irreducible representations of $\mathrm{Aut}(\sigma)$, the automorphism group of σ. We note that the same argument as in Lemma 11.3.5.1 (2) yields this bound. As in Remark 11.1.2.4, $\mathrm{Aut}(\sigma)$ is a finite group (when G is semi-simple), so this upper bound is finite. We again reiterate: the categorical geometric Langlands conjecture predicts that this upper bound is an

[58]A priori, $\mathcal{F}$ is ind-perverse. The finiteness in this bound amounts to perversity.

equality; however, it is not clear how to a priori obtain the lower bound in this setting without the categorical conjecture.

In particular, this estimate implies that $\mathcal{F}_\sigma$ is perverse (not ind-perverse) under this relaxed hypothesis, resolving a leftover point from §11.3.5 (for G semi-simple).

11.3.6. *Very irreducible case; general reductive G.* In the previous section, it was important that $\mathcal{F}_\sigma$ ultimately had finite length, so that the inequality (11.3.1) was forced to be an equality. In turn, this corresponded to the geometric fact that $\mathrm{LS}_{\check{G}}$ is Deligne-Mumford in a neighborhood of σ. This will not be the case when $\pi_1^{\mathrm{alg}}(G)$ is infinite; we describe the remedy below.

Let Z_G° be the connected component of the identity in the center of G. Let $\check{\Lambda}_{Z_G^\circ} \subseteq \check{\Lambda}$ be the sublattice of coweights of the torus Z_G°. Note that the torus dual to Z_G° is $\check{G}^{\mathrm{ab}}$, the abelianization of $\check{G}$.

Fix $x \in X(k)$ a k-point. This choice yields[59] an action of the lattice $\check{\Lambda}_{Z_G^\circ}$ on Bun_G. It also yields a map $\mathrm{LS}_{\check{G}} \to \mathbf{B}\check{G}$ by restriction of a local system to $x \in X$, and then by composition, a map $\mathrm{LS}_{\check{G}} \to \mathbf{B}\check{G}^{\mathrm{ab}}$.

These are compatible in the following sense. The action of $\mathsf{QCoh}(\check{\Lambda}_{Z_G^\circ})$ (with its convolution monoidal structure) on $D(\mathrm{Bun}_G)$ is the same as the one obtained from:

$$\mathsf{QCoh}(\check{\Lambda}_{Z_G^\circ}) \simeq \mathsf{Rep}(\check{G}^{\mathrm{ab}}) = \mathsf{QCoh}(\mathbf{B}\check{G}^{\mathrm{ab}}) \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) \curvearrowright D(\mathrm{Bun}_G).$$

Indeed, this is an immediate consequence of the construction of the spectral action (via Satake) and a basic compatibility of geometric Satake.

Therefore, up to trivializing[60] the restriction of σ at x, we obtain a commutative diagram:

$$\begin{tikzcd}
D_\sigma(\mathrm{Bun}_G) \arrow[rr] \arrow[d] & & \mathsf{Vect} \arrow[d] \\
D_{\widehat{\sigma}}(\mathrm{Bun}_G/\check{\Lambda}_{Z_G^\circ}) = D_{\widehat{\sigma}}(\mathrm{Bun}_G)^{\check{\Lambda}_{Z_G^\circ}} \arrow[rr] \arrow[d] & & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}} \underset{\mathbf{B}\check{G}^{\mathrm{ab}}}{\times} \mathrm{Spec}(k)) \arrow[d] \\
D_{\widehat{\sigma}}(\mathrm{Bun}_G) \arrow[rr, "\mathrm{coeff}_{\widehat{\sigma}}^{\mathrm{enh}}"] & & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}).
\end{tikzcd}$$

Here the left arrows are forgetful functors, and the right arrows are the evident pushforwards.

Observe that $\mathrm{LS}_{\check{G}}^{\mathrm{restr}} \underset{\mathbf{B}\check{G}^{\mathrm{ab}}}{\times} \mathrm{Spec}(k)$ is Deligne-Mumford in an open neighborhood of the point σ. Therefore, the analysis of §11.3.5 goes through when considering the middle arrow, up to replacing $\pi_1^{\mathrm{alg}}(G)$ with the finite group $\pi_1^{\mathrm{alg}}(G/Z_G^\circ)$. Noting that for each $c \in \pi_1^{\mathrm{alg}}(G) = \pi_0(\mathrm{Bun}_G)$, the map $\mathrm{Bun}_G^c \to \mathrm{Bun}_G/\check{\Lambda}_{Z_G^\circ}$ is the embedding of a connected component, this clearly yields the result in the case of general G and very irreducible σ.

[59]Via the natural map $\check{\Lambda}_{Z_G^\circ} = (\mathrm{Gr}_{Z_G^\circ,x})^{\mathrm{red}} \to \mathrm{Bun}_{Z_G^\circ}$ and the evident action of $\mathrm{Bun}_{Z_G^\circ}$ on Bun_G.

[60]This has the effect of simplifying the notation below by removing certain twists involving this restriction. In other words, this trivialization is innocuous and chosen out of laziness.

Remark 11.3.6.1. Combining the above method with that of Remark 11.3.5.2 yields the perversity (i.e., local compactness) of $\mathcal{F}_\sigma$ for general G and irreducible σ; specifically, this shows that the restriction of $\mathcal{F}_\sigma$ to each connected component of Bun_G has finite length. This finally resolves the leftover point from §11.3.5.

A. Existence of Hecke eigensheaves via localization

In this appendix, we prove Theorem 11.2.1.1, the existence of Whittaker-normailzed Hecke eigensheaves (at least when k is algebraically closed). The construction of $\mathcal{F}_\sigma$ from *loc. cit.* is via the localization construction of Beilinson-Drinfeld developed in [BD], using refinements of the critical level Kac-Moody theory due to Frenkel-Gaitsgory and Arinkin's existence of oper structures on irreducible local systems.

At times, we use mild extensions of existing results that are not well recorded in the literature. In general, we point to [ABGRR] and [CF] for an introduction to the relevant circle of ideas.

We reiterate that we do not claim originality for the material here, which we generally consider to be folklore consequences of the work of others.

A.1. **Background on opers and localization.**

A.1.1. *Opers with singularities: local aspects.* Let $x \in X(k)$ be a marked point with coordinate t. We let $\mathcal{D}_x = \mathrm{Spec}(k[[t]])$ be a disc and let $\lambda \in \Lambda^+$ be a dominant weight. Recall that there is a scheme $\mathrm{Op}^\lambda_{\check{G},x}$ of *opers with singularity* λ (at x). These are defined in [FG2] §2.9, where they are denoted $\mathrm{Op}^{\lambda,\mathrm{reg}}_{\check{\mathfrak{g}}}$. We remind that these opers are $\check{G}$-local systems on the disc $\mathcal{D}_x$ (i.e., points of $\mathbf{B}\check{G}$) equipped with extra structure.

Remark A.1.1.1. We remind that any k-point of $\mathrm{Op}_{\check{G}}(\mathring{\mathcal{D}})$, the indscheme of opers on the punctured disc, underlying a local system on $\mathcal{D}_x$ (rather than $\mathring{\mathcal{D}}_x$) lies in $\mathrm{Op}^\lambda_{\check{G},x}$ for some λ.[61]

A.1.2. We use Kac-Moody notation as in [Ras5]. In particular, we let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–}\mathsf{mod}^{G(O)}$ be the Kazhdan-Lusztig category at critical level, which we also denote by $\mathsf{KL}_{\mathrm{crit},x}$. Let $\mathbb{V}^\lambda_{\mathrm{crit}} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–}\mathsf{mod}^{G(O),\heartsuit}$ be the Weyl module, i.e., the module $\mathrm{ind}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}_{\mathfrak{g}[[t]]}(V^\lambda)$ where V^λ is the G-representation corresponding to λ, acted on by $\mathfrak{g}[[t]]$ via the evaluation homomorphism $\mathfrak{g}[[t]] \to \mathfrak{g}$.

By [FG4] Theorem 1, there is a natural action of $\mathrm{Fun}(\mathrm{Op}^\lambda_{\check{G},x})$ by endomorphisms on $\mathbb{V}^\lambda_{\mathrm{crit}}$. Therefore, we obtain a functor:

$$\mathsf{QCoh}(\mathrm{Op}^\lambda_{\check{G},x}) \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–}\mathsf{mod}^{G(O)} \tag{A.1.1}$$

sending the structure sheaf in the left hand side to the Weyl module $\mathbb{V}^\lambda_{\mathrm{crit}}$.

Moreover, by [FG1] Theorem 1.10, the above is compatible with the $\mathsf{Rep}(\check{G})$ actions on both sides. Specifically, the map $\mathrm{Op}^\lambda_{\check{G},x} \to \mathbf{B}\check{G}$ induces a symmetric monoidal functor $\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\mathrm{Op}^\lambda_{\check{G},x})$, while $\mathsf{Rep}(\check{G})$ acts on the right hand side via geometric Satake. For us, [FG1] Theorem 1.10 amounts to the assertion that the above functor is naturally $\mathsf{Rep}(\check{G})$-linear. We refer to [Ras5] Corollary 7.10.1 for homotopical details regarding a similar situation.

Notation A.1.2.1. Later, we will wish to explicitly note the dependence on the point x. We will write $\mathbb{V}^{\lambda\cdot x}_{\mathrm{crit}} \in \mathsf{KL}_{\mathrm{crit},x}$ in this case.

[61]See for example the second equation of [FG3] §2.2, where this is stated explicitly. We remark that, as in *loc. cit.*, the assertion holds for k replaced by any reduced k-algebra, but not for non-reduced algebras.

A.1.3. Suppose $x \in X(k)$ is a marked point, which the reader should imagine was implicitly equipped with the coordinate t in the previous discussion.

There is a localization functor:

$$\mathrm{Loc}_x : \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–}\mathsf{mod}^{G(O)} \to D(\mathrm{Bun}_G)$$

where we implicitly choose a square root of the canonical bundle on Bun_G, as in [BD] §4, to identify $D_{\mathrm{crit}}(\mathrm{Bun}_G)$ with $D(\mathrm{Bun}_G)$.

This functor is equivariant for the action of the spherical Hecke category at x acting on both sides, and in particular, $\mathsf{Rep}(\check{G})$-linear for Hecke functors (again: at x).

We recall that $\mathrm{Loc}_x(\mathbb{V}^{\lambda}_{\mathrm{crit}})$ is the critically twisted D-module:

$$\mathrm{ind}_{\mathrm{crit}}(\mathcal{V}^{\lambda \cdot x}) \in D_{\mathrm{crit}}(\mathrm{Bun}_G) \simeq D(\mathrm{Bun}_G)$$

induced from induced from the vector bundle $\mathcal{V}^{\lambda \cdot x} := \mathrm{ev}_x^*(V^{\lambda})$ where where the map $\mathrm{ev}_x : \mathrm{Bun}_G \to \mathbf{B}G$ takes the fiber at x of a G-bundle on X.

A.1.4. By composition, we obtain a $\mathsf{Rep}(\check{G})$-linear functor:

$$\mathcal{L}\mathrm{oc}_x : \mathsf{QCoh}(\mathrm{Op}^{\lambda}_{\check{G},x}) \to D(\mathrm{Bun}_G)$$

sending the structure sheaf to $\mathrm{ind}_{\mathrm{crit}}(\mathcal{V}^{\lambda \cdot x})$.

A.1.5. *Ran space extension.* We now use a variant of the above with multiple points, working over Ran space.

Fix points $x_1, \ldots, x_n \in X$. We let $S := \{x_1, \ldots, x_n\} \subseteq X$. We also fix dominant weights $\lambda_1, \ldots, \lambda_n$.

Let $\mathrm{Ran}_{X,S}$ denote the marked Ran space, as in [Gai3]. As in [ABGRR] and [CF], there is a natural category $\mathsf{KL}_{\mathrm{crit},\mathrm{Ran}_X}$ over $\mathrm{Ran}_{X,\mathrm{dR}}$ with fiber $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–}\mathsf{mod}^{G(O)}$ at a marked point $x \in X(k)$. We can pull it back to $\mathrm{Ran}_{X,S,\mathrm{dR}}$ to obtain a similar category $\mathsf{KL}_{\mathrm{crit},\mathrm{Ran}_{X,S}}$.

The category $\mathsf{Rep}(\check{G})_{\mathrm{Ran}_X}$ acts by Hecke functors on $\mathsf{KL}_{\mathrm{crit},\mathrm{Ran}_X}$. We can pull back to $\mathrm{Ran}_{X,S,\mathrm{dR}}$ to obtain an action of $\mathsf{Rep}(\check{G})_{\mathrm{Ran}_{X,S}}$ on $\mathsf{KL}_{\mathrm{crit},\mathrm{Ran}_{X,S}}$.

We also have relative affine scheme $\mathrm{Op}^{\sum \lambda_i x_i}_{\check{G},\mathrm{Ran}_{X,S}} \to \mathrm{Ran}_{X,S,\mathrm{dR}}$ whose fibers parametrize a finite set $\Sigma \subseteq X$ containing our marked points S, plus a $\check{G}$-bundle with connection on the formal completion to X at Σ, which is equipped with an oper structure of type λ_i at each $x_i \in S$ and type 0 (i.e., regular) at each $y \in \Sigma \setminus S$.

A.1.6. We let:

$$\mathbb{V}^{\sum \lambda_i \cdot x_i}_{\mathrm{crit}} \in \mathsf{KL}_{\mathrm{crit},S} = \otimes_{i=1}^n \mathsf{KL}_{\mathrm{crit},x_i}$$

denote the object $\boxtimes_{i=1}^n \mathbb{V}^{\lambda \cdot x_i}_{\mathrm{crit}}$.

By unital chiral algebra techniques, there is an induced object:

$$\mathbb{V}^{\sum \lambda_i \cdot x_i}_{\mathrm{crit},\mathrm{unit}} \in \mathsf{KL}_{\mathrm{crit},\mathrm{Ran}_{X,S}}$$

obtained by inserting the vacuum representation at points away from S.

Using standard (non-derived!) chiral algebra techniques, we readily obtain an extension of [FG4] that yields a ($D(\mathrm{Ran}_{X,S})$-linear) functor (generalizing (A.1.1)):

$$\mathsf{QCoh}(\mathrm{Op}^{\sum\lambda_i x_i}_{\check{G},\mathrm{Ran}_{X,S}}) \to \mathsf{KL}_{\mathrm{crit},\mathrm{Ran}_{X,S}} \tag{A.1.2}$$

sending the structure sheaf on the left hand side to $\mathbb{V}^{\sum\lambda_i\cdot x_i}_{\mathrm{crit},\mathrm{unit}}$. Moreover, this functor is naturally $\mathsf{Rep}(\check{G})_{\mathrm{Ran}_{X,S}}$-linear.

A.1.7. We can then compose with the $\mathsf{Rep}(\check{G})_{\mathrm{Ran}_{X,S}}$-linear localization functor:

$$\mathrm{Loc}_{\mathrm{Ran}_{X,S}} : \mathsf{KL}_{\mathrm{crit},\mathrm{Ran}_{X,S}} \to D(\mathrm{Bun}_G \times \mathrm{Ran}_{X,S})$$

to obtain a $\mathsf{Rep}(\check{G})_{\mathrm{Ran}_{X,S}}$-linear functor:

$$\mathsf{QCoh}(\mathrm{Op}^{\sum\lambda_i x_i}_{\check{G},\mathrm{Ran}_{X,S}}) \to D(\mathrm{Bun}_G \times \mathrm{Ran}_{X,S}).$$

Taking cohomology along $\mathrm{Ran}_{X,S}$ then yields a[62] $\mathsf{Rep}(\check{G})_{\mathrm{Ran}_{X,S},\mathrm{indep}} = \mathrm{Ran}(\check{G})_{\mathrm{Ran}_X,\mathrm{indep}}$-linear functor:

$$\mathcal{L}\mathrm{oc}_{\mathrm{Ran}_{X,S}} : \mathsf{QCoh}(\mathrm{Op}^{\sum\lambda_i x_i}_{\check{G},\mathrm{Ran}_{X,S}})_{\mathrm{indep}} \to D(\mathrm{Bun}_G). \tag{A.1.3}$$

Remark A.1.7.1. We abuse notation in omitting the divisor $\sum\lambda_i x_i$ from the notation $\mathcal{L}\mathrm{oc}$.

A.1.8. *Globalization.* Now let $\mathrm{Op}^{\mathrm{glob},\sum\lambda_i x_i}_{\check{G}}$ denote the scheme of *global* opers with singularity, i.e., $\check{G}$-local systems on X with $\check{B}$-reductions as a $\check{G}$-bundle satisfying the usual oper condition away from $S = \{x_1,\dots,x_n\}$ and satisfying the version with singularity λ_i at x_i.

There is a natural symmetric monoidal functor:

$$\mathrm{Loc}_{\mathrm{Op}_{\check{G}}} : \mathsf{QCoh}(\mathrm{Op}^{\sum\lambda_i x_i}_{\check{G},\mathrm{Ran}_{X,S}})_{\mathrm{indep}} \to \mathsf{QCoh}(\mathrm{Op}^{\mathrm{glob},\sum\lambda_i x_i}_{\check{G}})$$

admitting a fully faithful right adjoint. In particular, $\mathrm{Loc}_{\mathrm{Op}_{\check{G}}}$ is a quotient functor.

It follows from the constructions that (A.1.3) factors as:

$$\begin{tikzcd}
\mathsf{QCoh}(\mathrm{Op}^{\sum\lambda_i x_i}_{\check{G},\mathrm{Ran}_{X,S}})_{\mathrm{indep}} \arrow[d, "\mathrm{Loc}_{\mathrm{Op}_{\check{G}}}"'] \arrow[dr, "\mathcal{L}\mathrm{oc}_{\mathrm{Ran}_{X,S}}"] & \\
\mathsf{QCoh}(\mathrm{Op}^{\mathrm{glob},\sum\lambda_i x_i}_{\check{G}}) \arrow[r, dotted, "\mathcal{L}\mathrm{oc}^{\mathrm{glob}}"] & D(\mathrm{Bun}_G).
\end{tikzcd}$$

The functor $\mathcal{L}\mathrm{oc}^{\mathrm{glob}}$ is a priori $\mathsf{Rep}(\check{G})_{\mathrm{Ran}_{X,S},\mathrm{indep}}$-linear; but as $\mathsf{Rep}(\check{G})_{\mathrm{Ran}_{X,S},\mathrm{indep}}$ acts through its quotient $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$, this functor is actually $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$-linear.

[62]Here and elsewhere, the subscript $(-)_{\mathrm{indep}}$ is taken to mean the *independent category*, see [Ber2] for detailed discussion.

A.1.9. Next, we claim that there is a commutative diagram:

$$\begin{tikzcd} \mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\mathrm{glob},\sum \lambda_i x_i}) \arrow[d, "\mathcal{L}\mathrm{oc}^{\mathrm{glob}}"'] \arrow[dr, bend left, "\pi_*"] & \\ D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}^{\mathrm{enh}}(-)"] & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) \end{tikzcd} \tag{A.1.4}$$

where π is the natural map $\mathrm{Op}_{\check{G}}^{\mathrm{glob},\sum \lambda_i x_i} \to \mathrm{LS}_{\check{G}}$.

Indeed, since these two functors $\mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\mathrm{glob},\sum \lambda_i x_i}) \to \mathsf{QCoh}(\mathrm{LS}_{\check{G}})$ are $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$-linear, it suffices to produce a commutative diagram:

$$\begin{tikzcd} \mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\mathrm{glob},\sum \lambda_i x_i}) \arrow[d, "\mathcal{L}\mathrm{oc}^{\mathrm{glob}}"'] \arrow[drr, bend left, "\Gamma(\mathrm{LS}_{\check{G}}{,}\pi_*(-))"] & & \\ D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}^{\mathrm{enh}}(-)"] & \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[r, "\Gamma(\mathrm{LS}_{\check{G}}{,}\pi_*(-))"] & \mathsf{Vect}. \end{tikzcd}$$

Equivalently, it suffices to produce a commutative diagram:

$$\begin{tikzcd} \mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\mathrm{glob},\sum \lambda_i x_i}) \arrow[d, "\mathcal{L}\mathrm{oc}^{\mathrm{glob}}"'] \arrow[dr, bend left, "\Gamma(\mathrm{Op}_{\check{G}}^{\mathrm{glob},\sum \lambda_i x_i}{,}-)"] & \\ D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}"] & \mathsf{Vect}. \end{tikzcd}$$

By construction of $\mathcal{L}\mathrm{oc}^{\mathrm{glob}}$ and [FG4] Theorem 2, it suffices to construct a commutative diagram:

$$\begin{tikzcd} \mathsf{KL}_{\mathrm{crit},\mathrm{Ran}_{X,S}} \arrow[r, "\Psi"] \arrow[d, "\mathrm{Loc}_{\mathrm{Ran}_{X,S}}"'] & D(\mathrm{Ran}_{X,S}) \arrow[d, "\mathrm{C}_{\mathrm{dR}}(\mathrm{Ran}_{X,S}{,}-)"] \\ D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}"] & \mathsf{Vect}. \end{tikzcd}$$

Here the functor Ψ displayed above is the quantum Drinfeld-Sokolov functor (at a point: this functor is BRST for $\mathfrak{n}((t))$ twisted by the standard character).

The necessary commutative diagram now follows from [Gai5] Corollary 6.4.4 and Proposition 7.3.2.[63] [64]

A.2. **Proof of Theorem 11.2.1.1.** Let us return to our fixed, irreducible local system $\sigma \in \mathrm{LS}_{\check{G}}(k)$.

First, we note a certain subtlety regarding opers. The "right" definition of opers is given in [Gai6] and involves fixing the induced $\check{T}$-bundle to be $\rho(\Omega_X)$. This convention differs from the one used in [BD] and [Ari]; we refer to the latter notion as *BD opers* to distinguish. When $\check{G}$ is adjoint, the two

[63]In fact, Theorem 5.1.5 from [Gai5] immediately yields a stronger statement than we are using here. However, it references a certain functor denoted in [Gai5] by D-S$^{\mathrm{KL}}$, and whose construction is not given there. There is folklore knowledge about how to construct this functor, so this point can be overcome; still, we prefer to circumvent it using the above simplification.

[64]Note that [Gai5] does not account for the shift $[-\dim \mathrm{Bun}_N^{\Omega}]$ in the definition of our functor coeff. This shift is explained in [CF] Theorem 4.0.5 (see also [CF] Example 4.0.4 and the appearance of $\mathrm{CT}_*^{\mathrm{shifted}}$ in §2.4).

notions coincide. When $\check{G}$ has connected center, any BD oper structure is easily seen to lift to an actual oper structure after restricting to an open.[65] In general, the latter assertion is not evident.[66]

With that said, let us first assume $\check{G}$ has connected center. Then by [Ari] Theorem A (and more precisely, its Corollary 1.1), there exists a dense open $U \subseteq X$ such that $\sigma|_U$ admits a BD oper structure (without defect on U). By assumption on $\check{G}$, up to further restricting U, σ therefore admits a "true" oper structure. By Remark A.1.1.1, this means that there exists a $\check{\Lambda}^+$-valued divisor $\sum \check{\lambda}_i x_i$ on X such that σ lifts to a k-point $\chi_\sigma \in \mathrm{Op}_{\check{G}}^{\mathrm{glob},\sum \lambda_i x_i}$.

Finally, take:

$$\mathcal{F}_\sigma := \mathcal{L}\mathrm{oc}^{\mathrm{glob}}(k_{\chi_\sigma})[-\dim \mathrm{Bun}_G]$$

for $k_{\chi_\sigma} \in \mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\mathrm{glob},\sum \lambda_i x_i})$ the skyscraper sheaf at the point χ_σ.

That $\mathcal{F}_\sigma$ is a Hecke eigensheaf follows from $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$-linearity of $\mathcal{L}\mathrm{oc}^{\mathrm{glob}}$. The Whittaker normalization:

$$\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}_\sigma) \simeq k_\sigma[-\dim \mathrm{Bun}_G]$$

follows from (A.1.4).

Finally, let us extend to general G. We have the central isogeny $G_1 := [G,G]^{\mathrm{sc}} \times Z(G)^\circ \to G$, where $[G,G]^{\mathrm{sc}}$ is the simply-connected cover of $[G,G]$ and $Z(G)^\circ$ is the connected component of the identity in the center $Z(G)$.

Note that σ induces a $\check{G}_1$-local system $\sigma_{\check{G}_1}$. By [GR1] Theorem 8.4.8, there is an equivalence:

$$\alpha : D(\mathrm{Bun}_{G_1}) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}_1})}{\otimes} \mathsf{QCoh}(\mathrm{LS}_{\check{G}}) \simeq D(\mathrm{Bun}_G)$$

of $\mathsf{QCoh}(\mathrm{LS}_{\check{G}})$-module categories. Under this equivalence, the induced functor:

$$D(\mathrm{Bun}_{G_1}) \to D(\mathrm{Bun}_G)$$

is given by !-pushforward, and therefore sends $\mathcal{P}oinc_{G_1,!}$ to $\mathcal{P}oinc_{G,!}$. Therefore, the equivalence α commutes with forming $\mathrm{coeff}^{\mathrm{enh}}$ (for G_1 vs. G).

Taking the fiber of α at the point σ, we obtain an equivalence:

$$D_{\sigma_{\check{G}_1}}(\mathrm{Bun}_{G_1}) \simeq D_\sigma(\mathrm{Bun}_G)$$

between eigensheaf categories, and this equivalence is compatible with taking coeff, so we see that the existence of a normalized eigensheaf for $\sigma_{\check{G}_1}$ implies the same for σ.

Remark A.2.0.1. The reader might object to the citation to [GR1], which builds on the results of this paper and proves much stronger results than we are considering presently. However, §8 from *loc. cit.* is self-contained and can be read independently from the rest of [GR1].

[65] Namely, the $\check{B}$-bundle $\mathcal{P}_{\check{B}}$ of our oper has induced $\check{T}^{\mathrm{ad}}$-bundle being $\rho(\Omega^1_X)$. The obstruction to lifting this isomorphism to $\check{T}$-bundles is a $Z_{\check{G}}$-bundle on X, and this is necessarily Zariski-locally trivial when $Z_{\check{G}}$ is a torus.

[66] The issue is about the fixed square root of Ω^1_X implicit in considering $\rho(\Omega_X)$.

E.g., if $\check{G} = SL_2$, a BD oper is a rank 2 bundle $\mathcal{E}$ with connection ∇, a flat isomorphism $\alpha : \Lambda^2(\mathcal{E},\nabla) \simeq (\mathcal{O}, d)$ and a line bundle $\mathcal{L} \subseteq \mathcal{E}$ such that ∇ maps $\mathcal{L}$ isomorphically onto $(\mathcal{E}/\mathcal{L}) \otimes \Omega^1_X$. Note that this data induces an isomorphism $\sigma : \mathcal{L}^{\otimes 2} \simeq \Omega^1_X$. A "true" oper is one where $\mathcal{L}$ is fixed to be the once and for all fixed square root of Ω^1_X.

Note that if we have a BD oper structure on $(\mathcal{E},\nabla)|_U$, there can be significant difficulties in comparing this to a true oper structure on U; the BD oper structure defines a square root of Ω^1_U that may not extend to a square root of Ω^1_X.

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