CHIRAL PRINCIPAL SERIES CATEGORIES II: THE FACTORIZABLE WHITTAKER CATEGORY

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ABSTRACT. Ambitious proposals of Beilinson-Drinfeld-Gaitsgory suggest that local geometric Langlands should extend "factorizably" over the moduli of finite sets of points in a curve, which is known as Ran space. This framework gives an incarnation of the adèles in geometric Langlands. Combined with algebro-geometric methods, factorization methods are quite useful in both global and local geometric Langlands.

This paper is a part of their proposal. We extend Arkhipov-Berzukavnikov's results in local geometric Langlands to the factorization setting. To allow factorization, we use a version of Feigin-Frenkel's semi-infinite flag variety in place of the usual affine flag variety. Using this geometry, we give a new construction of the Arkhipov-Berzukavnikov functor (for some full subcategories anyway), and compare it to theirs.

CONTENTS

1. INTRODUCTION

1.1. This paper is a continuation of [Ras2]. The main result, Theorem 5.7.1, gives a Langlands dual description of a piece of the *factorizable* (unramified) principal series category. As such, it extends an equivalence of Arkhipov-Bezrukavnikov [AB] over the *Ran space*.

1.2. We assume the reader is familiar with the introduction to [Ras2]. In particular, we assume the reader has absorbed the motivation for some ideas from there already.

1.3. **Notation.** We fix k a field of characteristic 0 throughout this paper.

Let G be a split reductive group over a field k of characteristic 0. Let B be a Borel subgroup, let B^- be an opposed Borel, let $T = B \cap B^-$ be the corresponding Cartan, and let $N \subseteq B$ and $N^- \subseteq B^-$ be the unipotent radicals. We denote Lie algebras in the standard way. We denote the Langlands dual group by $\check{G}$, and let $\check{B}$ denote the corresponding Borel, and so on.

Let X be a smooth curve, let $x \in X$ be a k-point. Let K_x be Laurent series based at x, and let O_x be the subring of Taylor series. Let $G(K_x)$ denote the algebraic loop group and let $G(O_x)$ be the subgroup scheme of maps from the formal disc to G.

1.4. **D-modules in infinite type.** We need much from this theory. (Indeed, [Ras3] was written specifically to supplement §2 of the present paper.)

In [Ras3] (see also [Ber]), the theory of D-modules is extended to schemes and indschemes of infinite type. In fact, there are two such extensions, denoted $D^!(S)$ and $D^*(S)$ for an indscheme S. For *placid indschemes equipped with a dimension theory* (see [Ras3]), the two are naturally identified. In particular, this applies for $S = G(K_x)$. We refer to *loc. cit.* for more details here.

For S of ind-finite type, the two are also canonically identified with the traditional DG category of D-modules on an indscheme and so are denoted by $D(S)$.

1.5. **The principal series category.** The main player in this paper is the *(unramified) principal series category*, which is defined as $D^!(G(K_x))_{N(K_x)T(O_x)}$. Here the notation indicates coinvariants for $N(K_x)T(O_x) \subseteq G(K_x)$, as defined in [Ber]. This category is an analogue in geometric Langlands of the universal[1] unramified principal series representation of a p-adic reductive group. (We refer to [Cas] for the classical theory.)

We interpret this category as D-modules on the Feigin-Frenkel *semi-infinite flag variety* $\mathrm{Fl}_x^{\frac{\infty}{2}} = G(K_x)/N(K_x)T(O_x)$, and denote it by $D^!(\mathrm{Fl}_x^{\frac{\infty}{2}})$. Note that $\mathrm{Fl}_x^{\frac{\infty}{2}}$ does not make sense as an indscheme, since this is a quotient by a group indscheme. [2]

Each of the above makes sense *factorizably* over the Ran space. We refer to [BD2] and [Ras1] for what this means.

Remark 1.5.1. Our interest in this paper is specifically in the *factorizable* (or *chiral*) principal series category. The reason is that Ran space techniques are well-known to be essential for local-to-global techniques in geometric Langlands. In fact, the origin of this project was to prove Conjecture 6.6.2 of [Gai7], which can be done by combining the main theorem of this paper with Gaitsgory's (unpublished) factorizable derived Satake theorem.

Remark 1.5.2. The purpose of §2 is to give complete definitions of all the players in this paper, and to provide careful constructions of factorization structures. Therefore, in the introduction, we allow ourselves to be somewhat lax about this. In particular, we will often indicate a factorization category by its fiber at the point $x \in X$.

1.6. In this paper, we study the *Whittaker category* of $D^!(\mathrm{Fl}_x^{\frac{\infty}{2}})$. By definition, this means we take $N^-(K_x)$-invariants[3] against a non-degenerate character $\psi : N^-(K_x) \to \mathbb{G}_a$. We denote the resulting category by $\mathsf{Whit}_x^{\frac{\infty}{2}}$.

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[1]*Unramified principal series* means that one parabolically induces a representation from $T(K_x)/T(O_x)$. Then *universal* here means we take the regular representation.

[2]One can define quasi-maps versions of $\mathrm{Fl}_x^{\frac{\infty}{2}}$ in lieu of this (c.f. [FM]), but this defines a different category of D-modules that does not play the same fundamental role in local geometricLanglands.

[3]One can just as well take *coinvariants*: see [Ras5] Theorem 2.1.1. (Note that *loc. cit.* is not written factorizably, but the method immediately adapts.

Taking ψ to have conductor zero and incorporating some twists by half-forms, the assignment $x \mapsto \mathsf{Whit}_x^{\frac{\infty}{2}}$ extends to a *factorization category* over X. In particular, it extends over the Ran space.

1.7. The main conjecture of [Ras2] is about a Langlands dual spectral description of $x \mapsto \mathsf{Whit}_x^{\frac{\infty}{2}}$ as a factorization category. The main theorem of this paper, Theorem 5.7.1, proves this conjecture for a certain full subcategory of $\mathsf{Whit}_x^{\frac{\infty}{2}}$ (compatible with factorization), which is denoted $\mathsf{Whit}_{acc,x}^{\frac{\infty}{2}}$ in this paper. (The notation *acc* is an abbreviation for *accessible*, since this is the part of the category semi-infinite Whittaker category that we can understand by the methods of this paper.)

More precisely, we have a functor $\mathsf{Whit}_x^{sph} := \mathsf{Whit}(D(\mathrm{Gr}_{G,x})) \to \mathsf{Whit}_x^{\frac{\infty}{2}}$ given by pullback/pushforward along the correspondence $\mathrm{Gr}_{G,x} \leftarrow G(K_x)/B(O_x) \to \mathrm{Fl}_x^{\frac{\infty}{2}}$, and $\mathsf{Whit}_{acc,x}^{\frac{\infty}{2}}$ is defined to be generated under colimits by objects in the image of this functor and their translates under the natural action of the lattice of coweights $\check{\Lambda} = \mathrm{Gr}_{T,x} = T(K_x)/T(O_x)$ on $\mathrm{Fl}_x^{\frac{\infty}{2}}$ (acting "on the right").

1.8. As motivated in the introduction to [Ras2], the spectral side of our equivalence is expressed as (unital) factorization modules for the (derived) factorization algebra $\Upsilon_{\check{\mathfrak{n}}}$ (see [Ras1]). We refer to *loc. cit.* for a discussion of $\Upsilon_{\check{\mathfrak{n}}}$, and simply remind that it is a factorizable version of the homological Chevalley complex for $\check{\mathfrak{n}}$; the notation is taken from [BG2]. (See also §4.3 below.) We realize $\Upsilon_{\check{\mathfrak{n}}}$ as a factorization algebra in the factorization category of D-modules on the affine Grassmannian for T: this encodes the $\check{T}$ on $\check{\mathfrak{n}}$ and its Chevalley complex.

More precisely, the DG category $\Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,x}))$ of unital $\check{\Lambda}$-graded factorization modules for $\Upsilon_{\check{\mathfrak{n}}}$ based at $x \in X$ can be expressed explicitly:

$$\Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,x})) \simeq \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B}_{\check{T}}^\wedge) \supseteq \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B}).$$

Here the first equality is a special case of Theorem 8.22.1. The second inclusion follows from the fact that $\check{B}/\check{T}$ is an affine space, in particular contractible (c.f. Remark 5.7.2). (For $Z \subseteq Y$, $Y_Z^\wedge$ denotes the formal completion of Y along Z.)

The spectral side of our equivalence corresponds to the (factorizable version of) the full subcategory $\mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B})$ of $\Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,x}))$.

1.9. We remark that:

$$\mathsf{QCoh}(\check{\mathfrak{n}}/\check{B}) \simeq \mathsf{QCoh}\Big(\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x) \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)\Big) \qquad (1.9.1)$$

to make contact with the conjecture from [Ras2]. The subcategory $\mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B})$ is then the full subcategory of sheaves set-theoretically supported on $\mathrm{LocSys}_{\check{B}}(\mathcal{D}_x)$.

Remark 1.9.1. As was explained in the introduction to [Ras2], as we work factorizably instead of fixing $x \in X$, the factorization category of factorization modules for $\Upsilon_{\check{\mathfrak{n}}}$ should be understood in terms of the right hand side of (1.9.1).

There is room for confusion here, since any symmetric monoidal category (such as $\mathsf{QCoh}(\check{\mathfrak{n}}/\check{B})$) gives rise to a factorization category: c.f. [Ras2] §6. The resulting factorization category in this case is *not* compatible with $\Upsilon_{\check{\mathfrak{n}}}$, i.e., there is not a natural functor (unlike for local systems, c.f. *loc. cit.*).

1.10. **What are the difficulties in proving this theorem?** Langlands duality for reductive groups has a fundamentally combinatorial nature. However, $D^!(\mathrm{Fl}_x^{\frac{\infty}{2}})$ as defined above is not so amenable to combinatorics: there is no theory of middle extensions and so on. Moreover, since we take Whittaker equivariant D-modules, we are dealing with cosets for infinite-dimensional groups on both sides, so closures of orbits and such standard geometry are not well-behaved.

We discuss the strategy for circumventing these issues, and thereby the contents of this paper, in what follows. We separate the issues into two pieces: the construction of a comparison functor, and the proof that it is an equivalence.

1.11. **Construction of the functor.** Note that there is a canonical forgetful functor:

$$\Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,x})) \to D(\mathrm{Gr}_{T,x})\big(= \mathsf{Rep}(\check{T})\big).$$

The starting point for our construction is that there is also a functor:[4]

$$i_x^{\frac{\infty}{2},!} : D^!(\mathrm{Fl}_x^{\frac{\infty}{2}}) \to D(\mathrm{Gr}_{T,x})$$

geometrically given by !-restriction along $\mathrm{Gr}_{T,x} = B(K_x)/N(K_x)T(O_x) \hookrightarrow G(K_x)/N(K_x)T(O_x) = \mathrm{Fl}_x^{\frac{\infty}{2}}$. (More precisely, we consider !-restriction along $B(K_x) \hookrightarrow G(K_x)$ and apply $N(K_x)T(O_x)$-coinvariants.)

1.12. The main observation is that $i_x^{\frac{\infty}{2},!}$ factorizes, and that its source (and target) and *unital* factorization categories. The functor $i_x^{\frac{\infty}{2},!}$ even preserves unit objects. However, as we will discuss further, its restriction to $\mathsf{Whit}_x^{\frac{\infty}{2}} = \mathsf{Whit}(\mathrm{Fl}_x^{\frac{\infty}{2}}) \subseteq D^!(\mathrm{Fl}_x^{\frac{\infty}{2}})$ *does not* preserve unit objects, as the embedding $\mathsf{Whit}(\mathrm{Fl}_x^{\frac{\infty}{2}}) \hookrightarrow D^!(\mathrm{Fl}_x^{\frac{\infty}{2}})$ does not.

1.13. Here is a toy model for how we will proceed. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a lax unital (perhaps lax) monoidal functor between monoidal categories. Recall that this means that there are natural transformations:

$$\mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$$
$$F(\mathcal{F}) \otimes F(\mathcal{G}) \to F(\mathcal{F} \otimes \mathcal{G})$$

for $\mathbb{1}_-$ denoting the unit object and $\otimes$ denoting monoidal products. Of course, these natural transformations satisfy usual coherences.

In this case, F preserves algebras, and in particular, $F(\mathbb{1}_{\mathcal{C}})$ is a (unital) algebra in $\mathcal{D}$. Moreover, F upgrades to a lax monoidal functor $F^{enh} : \mathcal{C} \to F(\mathbb{1}_{\mathcal{C}})\text{--}\mathsf{bimod}(\mathcal{D})$ (where the target again consists of unital bimodules).

Clearly the functor F^{enh} has a "better chance" of being an equivalence than the original functor F.

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[4]In fact, we are omitting some cohomological shifts here, which appear to arise only incidentally.

To obtain the functor denoted by $i_x^{\frac{\infty}{2},!}$ in the paper, one should shift by $(2\check{\rho}, \check{\lambda})$ on the $\check{\lambda}$-degree piece. We refer to §2.25-2.26 for this material.

1.14.   As discussed in [Ras1], unital factorization categories are best understood asanalogous to lax monoidal categories, and factorization functors are (necessarily) analogous to lax monoidal functors.

A key part of the formalism from [Ras1] is that for a factorization functor $F : \mathcal{C} \to \mathcal{D}$ between unital factorization categories, F upgrades to a functor $F^{enh} : \mathcal{C} \to F(\text{unit}_{\mathcal{C}})\text{–mod}^{\text{fact}}_{un}(\mathcal{D})$ compatible with factorization.[5]

Then [Ras2] Theorem 4.6.1 allows us to compute that the unit object in $\mathsf{Whit}^{\frac{\infty}{2}}$ goes to $\Upsilon_{\check{\mathfrak{n}}}$ under the functor $i^{\frac{\infty}{2},!}$, and this identification is compatible with factorization structures. This calculation is the subject of §4; a somewhat heuristic version appeared in [Ras2] §1.33.

The upshot is that we obtain the desired functor:

$$i^{\frac{\infty}{2},!,enh}_x : \mathsf{Whit}^{\frac{\infty}{2}}_x \to \Upsilon_{\check{\mathfrak{n}}}\text{–mod}^{\text{fact}}_{un}(D(\mathrm{Gr}_{T,x})).$$

1.15. **A confusing point.** The calculation of where the unit object goes is done by a reduction to the finite-dimensional setting considered in [Ras2]. There a different *kind* of factorization algebra was denoted by $\Upsilon_{\check{\mathfrak{n}}}$; in this discussion, we denote the object from [Ras2] by $\Upsilon^{f.d.}_{\check{\mathfrak{n}}}$.

The "true" factorization algebra $\Upsilon_{\check{\mathfrak{n}}}$ is given by a compatible system of D-modules on $D(\mathrm{Gr}_{T,X^I})$ for all finite sets I; in particular, it defines an D-module on the Ran space version $\mathrm{Gr}_{T,\mathrm{Ran}_X}$ of the affine Grassmannian for T (and by pushforward, on Ran space itself).

The "not quite"[6] factorization algebra $\Upsilon^{f.d.}_{\check{\mathfrak{n}}}$ is given by a compatible system of D-modules on the scheme $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\text{eff}}$ of $\check{\Lambda}^{pos}$-valued divisors on X.

The two spaces are quite different: one can describe points of $\mathrm{Gr}_{T,\mathrm{Ran}_X}$ as a $\check{\Lambda}$-valued divisor D on X, plus a finite set of points $\{x_1, \ldots, x_n\} \subseteq X$, such that the divisor is supported on the union of these points. This allows a great deal of redundancy on the Ran space side: we allow points where the divisor is not supported as well.

Of course, $\Upsilon_{\check{\mathfrak{n}}}$ and $\Upsilon^{f.d.}_{\check{\mathfrak{n}}}$ satisfy a natural compatibility, formulated explicitly in §4. They can be roughly be regarded as "the same," which is why we allow ourselves the freedom to denote one by $\Upsilon_{\check{\mathfrak{n}}}$ in [Ras2] and another by $\Upsilon_{\check{\mathfrak{n}}}$ in this paper.

But there is a fundamental difference: $\Upsilon^{f.d.}_{\check{\mathfrak{n}}}$ consists of D-modules in the heart of the t-structure on $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\text{eff}}$, while $\Upsilon_{\check{\mathfrak{n}}}$ *does not* have any analogous property (as is easy to see explicitly using the chiral PBW theorem). This fact about t-structures is used in a crucial way in [Ras2] (following [BG2]) to connect the combinatorics of Langlands duality with this set of problems in geometric representation theory.

So even though we eventually only care about the Ran space version of $\Upsilon_{\check{\mathfrak{n}}}$, it is essential to consider its cousin $\Upsilon^{f.d.}_{\check{\mathfrak{n}}}$ to make the link with Langlands duality.

1.16. **Compatibility with Satake.** We further use [Ras2] Theorem 7.9.1 to express a compatibility with geometric Satake (as expressed through the *geometric Casselman-Shalika* theorem, see Theorem 4.14.1).

More precisely, *loc. cit.* readily implies that the composition:

$$\mathsf{Rep}(\check{G}) \overset{Casselman-Shalika}{\simeq} \mathsf{Whit}(D(\mathrm{Gr}_{G,x})) \to \mathsf{Whit}^{\frac{\infty}{2}} \xrightarrow{i^{\frac{\infty}{2},!,enh}_x} \Upsilon_{\check{\mathfrak{n}}}\text{–mod}^{\text{fact}}_{un}(D(\mathrm{Gr}_{T,x}))$$

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[5]Technically, the right hand side may only be a *lax* factorization category in the terminology of [Ras1]. In the example considered in this paper, this is not actually an issue because of the nice finiteness properties of $\Upsilon_{\check{\mathfrak{n}}}$. See [Ras2] Corollary 7.8.1.

[6]In [Ras2], we called this data a *graded* factorization algebra.

is computed *as a factorizable functor* by the functor forming the Chevalley complex of an $\check{G}$-representation with respect to the action of $\check{\mathfrak{n}}$.

This compatibility plays a key role in the proof of the main theorem. Note that it is fundamentally of higher categorical nature, and not only because (Ran space) $\Upsilon_{\check{\mathfrak{n}}}$ does not lie in the heart of a t-structure. But again, the fundamental issues arising here are dealt with using Theorem 7.9.1.

1.17. **Proof of the main theorem.** We now indicate the ideas that go into the proof of Theorem 5.7.1.

In §5, we use the theory of *ULA* objects (c.f. [Ras2] Appendix B to reduce to showing that the functor $i_x^{\frac{\infty}{2},!,enh}\big|_{\mathsf{Whit}^{\frac{\infty}{2}}_{acc,x}}$ is fully-faithful (with expected image) for our point $x \in X$. That is, we reduce from working over powers of the curve to working at a single point.[7],[8] (The nice ULAness properties here are fundamentally what make the subcategory $\mathsf{Whit}^{\frac{\infty}{2}}_{acc} \subseteq \mathsf{Whit}^{\frac{\infty}{2}}$ more accessible than the full category.)

1.18. We then prove the pointwise version by a comparison with Arkhipov-Bezrukavnikov theory [AB]. (It would be wonderful to find an independent proof.)

In §6-7, we identify $\mathsf{Whit}^{\frac{\infty}{2}}_{x}$ with *baby Whittaker* D-modules on the *affine* flag variety $\mathrm{Fl}^{\mathrm{aff}}_{G,x} := G(K_x)/I$ (for I the Iwahori subgroup). (It is essential here to work at a single point, since Iwahori does not behave well factorizably.) We highlight the key role played by these results: the category $D^!(\mathrm{Fl}^{\frac{\infty}{2}}_{x})$ is outside of the usual domain of geometric representation theory because of its infinite-dimensional nature, but we are able to import these techniques by our comparison with the ind-finite dimensional affine flag variety.

The main theorem of [AB] shows that this category of baby Whittaker D-modules on the affine flag variety is equivalent to $\mathsf{QCoh}(\check{\mathfrak{n}}/\check{B})$. Moreover, it follows from results in [FG2] that $\mathsf{Whit}^{\frac{\infty}{2}}_{acc,x}$ corresponds to the full subcategory of objects supported on $\mathbb{B}\check{B}$.

So it remains to show that our functor $i_x^{\frac{\infty}{2},!,enh}$ is equivalent to the identity functor:

$$\mathsf{QCoh}(\check{\mathfrak{n}}_0^{\wedge}/\check{B}) \overset{[AB]}{\simeq} \mathsf{Whit}^{\frac{\infty}{2}}_{acc,x} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_{T,x})) \supseteq \mathsf{QCoh}(\check{\mathfrak{n}}_0^{\wedge}/\check{B}).$$

1.19. To show this, we need to unwind the Arkhipov-Bezrukavnikov construction somewhat.

Recall that the main inputs for their construction are the natural $\mathsf{Rep}(\check{T})$-action and the $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$-action on $\mathsf{QCoh}(\check{\mathfrak{n}}/\check{B})$, coming from the structure maps $\mathbb{B}\check{T} \leftarrow \check{\mathfrak{n}}/\check{B} \to \check{\mathfrak{g}}/\check{G}$. (There is also a basic "Plücker" data relating the two, which plays a key role in their construction.)

1.20. We recall that $\mathsf{Rep}(\check{T})$-action on $\mathsf{QCoh}(\check{\mathfrak{n}}/\check{B})$ has to do with Mirkovic-Wakimoto sheaves on the affine flag variety. Under the equivalence with the semi-infinite flag variety, this just translates to the natural $\mathrm{Gr}_{T,x}$-action on $\mathrm{Fl}^{\frac{\infty}{2}}_{x}$. It is formal that $i_x^{\frac{\infty}{2},!,enh}$ is compatible with this action.

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[7]Admittedly, the necessity for such a reduction is not emphasized in the notation from the introduction. But this is a serious step.

[8]Preservation of ULA objects under a factorization functor is analogous to a lax monoidal functor being monoidal. So this reduction is like saying a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of monoidal categories if it is an equivalence of categories. This statement may sound stupid, but note that it fails if F is merely assumed to be lax monoidal.

1.21.   We treat the $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$-action using factorization methods. Namely, following a conjecture of Gaitsgory, we show that $\mathsf{QCoh}(\check{\mathfrak{g}}_0^{\wedge}/\check{G})$-actions can be fully-faithfully encoded in factorization module structures for the factorization category $\mathsf{Rep}(\check{G}) = \mathsf{QCoh}(\mathbb{B}\check{G}) = \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}))$.

This result is Theorem 9.13.1, and is a categorical version of the Beilinson-Drinfeld calculation of chiral modules over a commutative chiral algebra, c.f. [BD2] §3.6.18. Unsurprisingly, it relies on derived versions of their calculations: these appear as Theorems 8.13.1 and 8.22.1. The theory of topological associative algebras used in [BD2] §3.6 is not well-adapted to derived categories, so we need finiteness hypotheses in §8 that do not appear in [BD2] (and whose purpose is roughly to say some topological associative algebras are discretely topologized).

1.22.   The idea to encode the $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$ in factorization terms is based on the original Gaitsgory construction [Gai1] of central sheaves via nearby cycles, which is so essential to the Arkhipov-Bezrkuavnikov construction.

However, the relationship between the theory of factorization module categories and the Gaitsgory construction is not immediate. The relevant compatibility is established in §10.

1.23.   Given the reasonably conceptual ideas mentioned above, the remainder of the argument relies some explicit analysis related to the Arkhipov-Bezrukavnikov construction of Plücker data.

1.24. **Structure of this paper.** The purpose of §2 is to construct $\mathsf{Whit}^{\frac{\infty}{2}}$ as a unital factorization category. We include some detailed remarks on the structure of loop groups in the factorization setting here; the arguments are standard, but I'm not sure about references in this generality. This material relies essentially on ideas from [Ras3] and [Ras1], which are essentially extended appendices to this paper. The proof of a certain technical lemma necessary for the construction of units in factorizable Whittaker categories is postponed to §3.

In §4, we import the results from [Ras2] to the semi-infinite flag/Ran space framework introduced in §2. In particular, we construct the functor to factorization $\Upsilon_{\check{\mathfrak{n}}}$-modules in §4.12.

In §5, we define the accessible part subcategory of $\mathsf{Whit}^{\frac{\infty}{2}}$ referenced above and formulate the main theorem. We also analyze ULA objects here, and reduce the main theorem to verifying it over a single point.

In §6, we construct the equivalence $D^!(\mathrm{Fl}_x^{\frac{\infty}{2}}) \simeq D(\mathrm{Fl}_G^{\mathrm{aff}})$. In §7, we further show that Whittaker D-modules here coincide with the *baby Whittaker D-modules* considered in [AB] (where it is used as a trick to avoid considering infinite-dimensional orbits).

In §8, we calculate the DG categories of chiral modules over certain chiral algebras in "explicit" terms, following Beilinson-Drinfeld. (In particular, the main result here gives the explicit description of $\Upsilon_{\check{\mathfrak{n}}}-\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,x}))$ as $\mathsf{QCoh}(\check{\mathfrak{n}}_0^{\wedge}/\check{B}_{\check{T}})$.)

In §9, we show that $\mathsf{QCoh}(\check{\mathfrak{g}}_0^{\wedge}/\check{G})$-module structures can be recovered from a certain corresponding *factorization module category* structure. In §10, we relate this construction to the Gaitsgory nearby cycles construction [Gai1], which is one of the main inputs for [AB].

Finally, in §11, we prove the main result, Theorem 5.7.1.

1.25. **Conventions.** We follow [Ras2] in our use of (higher) categorical language and notation. We refer to *loc. cit.* for more details on this.

In §2-7, the difference between classical and derived algebraic geometry is not relevant because we deal with D-modules. However, beginning in §8, it is essential to use derived algebraic geometry. We sometimes use the phrases "scheme" or "stack" sloppily, meaning the corresponding *derived* notions, and use the phrase "classical scheme/stack" to emphasize that when we are working with the more traditional notions.

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2. Dramatis personae

2.1.   The goal for this section is to introduce the semi-infinite flag variety in the context of factorizable geometry, and its associated Whittaker D-modules.

A summary of what is achieved is given in §2.36, and may be motivating to read before the remainder of the section.

2.2. **A note on citations.** This section makes many references to the theory of chiral categories from [Ras1], and to the theory of D-modules in infinite type from [Ras3].

The references to [Ras1] primarily regard precise terminology and constructions, and can be ignored by the reader who is comfortable with the basic ideas. But note that we will make the following change for ease of notation: using the 1-affineness of X_{dR} [Gai6], we avoid the sheaves of categories language and work $D(X^I)$-module categories instead.

The references to [Ras3] are of two kinds. Some results are e.g. base change results of a general nature, and the reader who can take on faith that such things hold in the infinite type setting may skip past these results. The other kind of reference is to the constructions of various functors unique to the setting of D-modules in infinite type. Unfortunately, there is no easy escape here, and we only hope that having precise references will aid in learning the necessary material.

2.3.   We fix a smooth affine curve X.

2.4.   Let I be a finite set. Let Y be some fixed affine scheme.

We recall in §2.5-2.7 the definition of the *jet space* $Y(O)_{X^I}$ and the *algebraic loop space* (alias: *meromorphic jet space*) $Y(K)_{X^I}$.

2.5. **Jet spaces.** Let $n \in \mathbb{Z}^{\geqslant 0}$ be an integer.

For S an affine test scheme, we define the nth jet space $Y(O)_{X^I}^{(n)}$ to have S-points:

$$Y(O)_{X^I}^{(n)}(S) = \left\{ x = (x_i)_{i \in I} : S \to X^I \text{ and } \varkappa : \Gamma_x^{(n)} \to Y \right\} \tag{2.5.1}$$

where $\Gamma_x \subseteq X \times S$ is the sum of the graphs $\Gamma_{x_i} \subseteq X \times S$ of the maps x_i considered as relative divisors,[9] and $\Gamma_x^{(n)}$ is the nth infinitesimal neighborhood of Γ_x in $X \times S$. Note that $Y(O)_{X^I}^{(n)}$ is represented by a scheme of finite type over X^I.

As n varies, the spaces $Y(O)_{X^I}^{(n)}$ form an inverse system under affine structure maps. We let $Y(O)_{X^I}$ denote the projective limit.

The following is well-known: we include a proof for completeness.

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[9]In other words, take the composition map $S \to X^I \to \text{Div}^{|I|} X$, and note that $\text{Div}^{|I|} X$ is the Hilbert scheme of length $|I|$ subschemes of X.

Lemma 2.5.1. *Suppose Y a smooth scheme. Then for every pair $m, n \in \mathbb{Z}^{\geq 0}$, the scheme $Y(O)_{X^I}^{(n)}$ is smooth, and the structure maps:*

$$Y(O)_{X^I}^{(n+m)} \to Y(O)_{X^I}^{(n)}$$

are smooth, affine and surjective on geometric points.

Proof. We have already noted that the map is affine. The surjectivity follows by formal smoothness of Y.

Let S be an X^I-scheme that is affine, and let it be equipped with the structure map $x : S \to X^I$. A map $S \to Y(O)_{X^I}^{(n)}$ is equivalent to a map $\gamma : \Gamma_x^{(n)} \to Y$, so the cotangent complex $\Omega^1_{Y(O)_{X^I}^{(n)}/X^I}$ restricts to S as $\pi_* \gamma^*(\Omega^1_Y)$, where $\pi = \pi_n$ is the composition $\Gamma_x^{(n)} \hookrightarrow X \times S \to S$.

Because Y is smooth, Ω^1_Y is a vector bundle concentrated in a single cohomological degree. Therefore, the same is true for $\gamma^*(\Omega^1_Y)$. Because π is finite flat, $\pi_* \gamma^*(\Omega^1_Y)$ is also a vector bundle concentrated in exactly one degree. Therefore, we deduce smoothness of $Y(O)_{X^I}^{(n)}$ from the fact that the cotangent complex is a vector bundle.

It remains to show smoothness of the structure maps. We perform the relative tangent space computation. For $\gamma : \Gamma_x^{(n+m)} \to Y$, the relevant map is:

$$\pi_{n+m,*}(\gamma^*(T_Y)) \to \pi_{n,*}(\gamma^*(T_Y)|_{\Gamma_x^{(n)}})$$

where T_Y is the tangent complex (i.e., tangent sheaf) of Y. Since the maps π_i are affine, it suffices to show the surjectivity on $\Gamma_x^{(n+m)}$, before applying $\pi_{n+m,*}$. But this is obvious: we are dealing with a restriction map for vector bundles on an affine scheme. $\qquad \square$

2.6. Discs.

Let S be an affine test scheme and let $x = (x_i)_{i \in I} : S \to X^I$ be a map.

We define the *formal disc* $\widehat{\mathcal{D}}_x$ at x to be the formal completion of $X \times S$ along Γ_x. Note that $\widehat{\mathcal{D}}_x$ is an ind-affine indscheme.

We define the *adic disc* $\mathcal{D}_x \in \mathsf{AffSch}$ to be the value of the partially defined left adjoint of the functor $\mathsf{AffSch} \hookrightarrow \mathsf{PreStk}$ evaluated on $\widehat{\mathcal{D}}_x$. Note that ind-affineness of $\widehat{\mathcal{D}}_x$ implies that this functor is defined here: it is the spectrum of the limit of the corresponding commutative rings.

Observe that formation of $\widehat{\mathcal{D}}_x$ is étale local on X in the natural sense.

Note that $Y(O)_{X^I}$ is equivalently described as the moduli of maps $x : S \to X^I$ plus a map $\widehat{\mathcal{D}}_x \to Y$ or $\mathcal{D}_x \to Y$.

We define the *punctured disc* $\overset{o}{\mathcal{D}}_x \in \mathsf{Sch}$ at x as:

$$\overset{o}{\mathcal{D}}_x := \mathcal{D}_x \backslash \Gamma_x.$$

These constructions organize into the diagram:

$$\Gamma_x \longrightarrow \widehat{\mathcal{D}}_x \longrightarrow \mathcal{D}_x \longleftarrow \overset{o}{\mathcal{D}}_x$$
$$\downarrow$$
$$X \times S.$$

2.7. **Loop spaces.** Finally, we define:

$$Y(K)_{X^I}(S) = \left\{ x : S \to X^I \text{ and } \gamma : \overset{o}{\mathcal{D}}_x \to Y. \right\} \tag{2.7.1}$$

As in [KV] Proposition 3.5.2, $Y(K)_{X^I}$ is represented by an indscheme (of ind-infinite type), and formation of $Y(K)_{X^I}$ is étale local on X.

Remark 2.7.1. If Z is an affine X-scheme, then we have notions of "relative jets" and "relative meromorphic jets" that generalizes the constructions above when $Z = X \times Y$.

This is actually the level of generality we will be using in practice, but we find it convenient to write the material that follows in the product situation. See §2.11 and 2.16 for more discussion of this point.

Note that representability questions in the relative case reduce to the product case treated in [KV]: factor the map Z to X through its graph, and then the relative (resp. meromorphic) jets embed as a closed subscheme (resp. sub-indscheme) of the corresponding "absolute" jets.

2.8. **Factorization of the disc.** Let $\mathsf{Set}_{<\infty}$ denote the category of (possibly empty) finite sets under (possibly non-surjective) maps.

Let $f : I \to J$ be a map in $\mathsf{Set}_{<\infty}$, let S be an affine scheme and let $x = (x_j)_{j \in J} : S \to X^J$ be a map. Let $x' = (x'_i) = (x_{f(i)}) : S \to X^I$ be the map induced by f.

Note that $\Gamma_{x'}^{red}$ is a closed subscheme of Γ_x^{red}, giving a canonical map $\mathcal{D}_{x'} \to \mathcal{D}_x$. Therefore, we obtain an op-correspondence:

$$
\begin{array}{ccc}
 & \mathcal{D}_x \backslash \Gamma_{x'} & \\
 & \nearrow \quad \nwarrow & \\
\overset{o}{\mathcal{D}}_{x'} = \mathcal{D}_{x'} \backslash \Gamma_{x'} & \qquad\qquad & \mathcal{D}_x \backslash \Gamma_x = \overset{o}{\mathcal{D}}_x.
\end{array}
\tag{2.8.1}
$$

Remark 2.8.1. If f is surjective then the reduced schemes underlying Γ_x and $\Gamma_{x'}$ coincide. Therefore, in this case the right map in (2.8.1) is an isomorphism.

2.9. **Chiral categories.** Varying $I \in \mathsf{fSet}$, we obtain that the rules $I \mapsto Y(O)_{X^I}$ and $I \mapsto Y(K)_{X^I}$ factorize.

Applying [Ras3] Proposition 6.9.1, we obtain chiral categories on X_{dR}:

$$\left(I \mapsto D^!(Y(O)_{X^I}) \right) \text{ and} \left(I \mapsto D^!(Y(K)_{X^I}) \right).$$

Passing to the limit over I, we obtain the categories $D^!(Y(O)_{\mathrm{Ran}_X})$ and $D^!(Y(K)_{\mathrm{Ran}_X}) \in \mathsf{DGCat}_{cont}$.

Notation 2.9.1. We use the notation $D^!(Y(O))^{\mathrm{fact}}, D^!(Y(K))^{\mathrm{fact}} \in \mathsf{FactCat}(X_{dR})$ to denote the corresponding factorization categories.
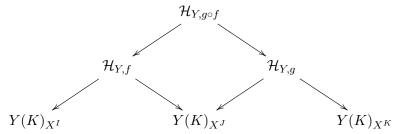
2.10. **Unital structures.** Suppose Y is an affine scheme of finite type.

Let $f : I \to J$ be a map in $\mathsf{Set}_{<\infty}$. Using the notation of §2.8, let $\mathcal{H}_{Y,f}$ denote the moduli of maps $x : S \to X^J$ plus a map $(\mathcal{D}_x \backslash \Gamma_{x'}) \to Y$, defined formally as in (2.7.1).

Applying (2.8.1), we obtain a correspondence:

$$\begin{array}{ccc} & \mathcal{H}_{Y,f} & \\ \alpha_{Y,f}\swarrow & & \searrow\beta_{Y,f} \\ Y(K)_{X^I} & & Y(K)_{X^J}. \end{array}$$

(2.10.1)

For f the identity, this correspondence is the identity correspondence. For $f : I \to J$ and $g : J \to K$, we obtain a canonical diagram:

$$\begin{array}{ccccc} & & \mathcal{H}_{Y,g\circ f} & & \\ & \swarrow & & \searrow & \\ & \mathcal{H}_{Y,f} & & \mathcal{H}_{Y,g} & \\ \swarrow & & \searrow\quad\swarrow & & \searrow \\ Y(K)_{X^I} & & Y(K)_{X^J} & & Y(K)_{X^K} \end{array}$$

where the middle diamond is Cartesian.

In other words, we obtain a functor $\mathsf{Set}_{<\infty} \to \mathsf{IndSch}_{corr}$ sending I to $Y(K)_{X^I}$. This functor is compatible with factorization in the natural sense.

Moreover, for f as above, one sees that the map:

$$\beta_{Y,f} : \mathcal{H}_{Y,f} \to Y(K)_{X^J}$$

is finitely presented. Therefore, by [Ras3] §6.3, we obtain that:

$$I \mapsto D^!\Big(Y(K)_{X^I}\Big)$$

defines a unital factorization category on X_{dR}:

$$D^!(Y(K))^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR})$$

refining our earlier non-unital factorization category.

Warning 2.10.1. We are abusing notation here in also letting $D^!(Y(K))^{\mathrm{fact}}$ denote the unital version of the factorization category. We will use this abuse throughout.

Remark 2.10.2. For a morphism $f : I \to J \in \mathsf{Set}_{<\infty}$, the corresponding map $D^!(Y(K)_{X^I}) \to D^!(Y(K)_{X^J})$ is the computed by the functor $\beta_{Y,f,*,!-dR}\alpha^!_{Y,f}$. We recall that the functor $\beta_{Y,f,*,!-dR}$ of !-dR $*$-pushforward is defined for any finitely presented morphism and is the functor of [Ras3] §6.3.

Remark 2.10.3. The unit object in $D^!(Y(K)_{\mathrm{Ran}_X})$ is obtained by !-dR $*$-pushforward of $\omega_{Y(O)_{\mathrm{Ran}_X}}$. Here, the symbol $\omega_{Y(O)_{\mathrm{Ran}_X}}$ refers to the compatible system of objects $(I \mapsto \omega_{Y(O)_{X^I}})$ and the term "!-dR $*$-pushforward" refers to the appropriate compatible system of such functors.

Remark 2.10.4. For a morphism $Y_1 \to Y_2$ of schemes of finite type, we obtain canonical maps $Y_1(K)_{X^I} \to Y_2(K)_{X^I}$. These maps are obviously compatible with the correspondences above and therefore define a canonical strictly unital morphism:

$$D^!(Y_2(K))^{\mathrm{fact}} \to D^!(Y_1(K))^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR})$$

computed as !-pullback over each X^I.

Notation 2.10.5. For I and J two finite sets, we will sometimes use the notation $\mathcal{H}_{Y,I,J}$ in place of $\mathcal{H}_{Y,f}$ with f the tautological embedding $I \hookrightarrow I \coprod J$.

2.11. Forms of algebraic groups.

We will be working with group schemes $\mathcal{G}$ over X that are *forms* of affine algebraic groups. See §2.16 to see the examples we will use.

We will say that two group schemes over X are *forms* of each other if they are isomorphic as group schemes étale[10] locally on X.

Therefore, being a form of an affine algebraic group means that the group scheme $\mathcal{G}$ is a smooth, affine group scheme that is a form of $\mathcal{G}_0 \times X$ for $\mathcal{G}_0$ an affine algebraic group. In this case, we abbreviate the situation in saying that $\mathcal{G}$ is a form of $\mathcal{G}_0$.

For the remainder of this section, we fix $\mathcal{G}$ an affine group scheme over X of the type above.

Example 2.11.1. Every reductive group scheme over X is a form of the associated split reductive group.

2.12.   In applying the Beauville-Laszlo principle [BL],[11] it is convenient to have the following well-known technical result. We include a proof for completeness.

Lemma 2.12.1. *Let $x : S \to X^I$ be a map from an affine scheme S. Let $\mathcal{G}$ be a form of an algebraic group over X. Then the restriction map:*

$$\{\mathcal{G}\text{–bundles on } \mathcal{D}_x\} \to \{\mathcal{G}\text{–bundles on } \widehat{\mathcal{D}}_x\}$$

is an equivalence of groupoids.

Proof. First, we claim that $\mathcal{O}_\mathcal{G}$, considered as a representation of $\mathcal{G}$ over X, is a union of subrepresentations that are finite rank vector bundles on X. Indeed, it is always true that comodules for an A-coalgebra B are a union of A-finitely generated submodules, and because X is a smooth curve, submodules of $\mathcal{O}_\mathcal{G}$ (which is flat) are necessarily flat.

Pulling $\mathcal{G}$ back to $\mathcal{D}_x$, we see that there are again "enough" vector bundle representations. Therefore, using the Tannakian formalism, we reduce to treating the case $\mathcal{G} = GL_{r,X}$.

Let $S = \mathrm{Spec}(A)$, and let A_n denote the commutative algebra of functions on the (affine) scheme $\Gamma_x^{(n)}$ (so $A_0 = A$). Let $B = \lim_n A_n$, so $\mathrm{Spec}(B) = \mathcal{D}_x$. Let $I_n \subseteq B$ denote the kernel of the (surjective) map $B \to A_n$.

Let $\mathcal{E}$ be a finitely generated projective B-module of rank r. Because $\mathcal{E}$ is a direct summand of a finite rank free B-module, $\mathcal{E} \xrightarrow{\simeq} \lim_n \mathcal{E}/I_n$. This proves fully-faithfulness.

It remains to show essential surjectivity. Here we need to show that the limit $\mathcal{E} := \lim_n \mathcal{E}_n$ of a compatible system $\{\mathcal{E}_n\}$ of rank r projective A_n-modules is a finitely generated projective B-module.

We can write $\mathcal{E}_0 \oplus \mathcal{E}_0' \xrightarrow{\simeq} A_0^{\oplus(r+s)}$ for $\mathcal{E}_0'$ a rank s vector bundle on $\mathrm{Spec}(A)$.

Therefore, by formal smoothness of $GL_{r+s}/GL_r \times GL_s$, we can lift the compatible system $\{\mathcal{E}_n\}$ to a compatible system $(\mathcal{E}_n, \mathcal{E}_n', \mathcal{E}_n \oplus \mathcal{E}_n' \xrightarrow{\simeq} A_n^{\oplus(r+s)})$ such that the $n = 0$ case is given by our earlier choice. But this obviously realizes $\mathcal{E}$ itself as a direct summand of a finite free module.

$\square$

---

[10]A warning: There is a risk that taking étale forms means that e.g. the associated affine Grassmannian will be an ind-algebraic space, not an indscheme, which is somewhat problematic since [Ras3] is written for indschemes. However, we note that 1) the forms we will take are *Zariski* locally trivial (c.f. §2.16), removing the problem for us in practice, and 2) the material in *loc. cit.* extends to the setting of algebraic spaces using [Ryd] and an appropriate generalization of the relevant material of [GR2]. For these reasons, we will ignore the issue in what follows and deal with D-modules on our indschemes without further mention.

[11]Which is necessarily about $\mathcal{D}$ — not $\widehat{\mathcal{D}}$ — since it involves the punctured disc.

In particular, we obtain the following corollary from formal smoothness of the map $X \to X/\mathcal{G}$.

Corollary 2.12.2. *In the notation above, a $\mathcal{G}$-bundle on $\mathcal{D}_x$ is trivial if and only if its restriction to S is.*

2.13. **The affine Grassmannian.** We will specialize the above material to the case of (relative) jets into $\mathcal{G}$, considered as in Remark 2.7.1.

Fix a finite set I.

In this case, $\mathcal{G}(O)_{X^I}$ is a group scheme over X^I Moreover, since each $\mathcal{G}(O)_{X^I}^{(n)}$ is a smooth group scheme over X^I, $\mathcal{G}(O)_{X^I}$ satisfies the hypotheses of [Ras3] Example 4.2.3 as a group scheme over X^I.

We also have the Beilinson-Drinfeld affine Grassmannian $\mathrm{Gr}_{\mathcal{G},X^I}$ with the usual $\mathcal{G}(K)_{X^I}$-equivariant (relative to the left action on the source) map $\pi_{\mathcal{G},X^I} : \mathcal{G}(K)_{X^I} \to \mathrm{Gr}_{\mathcal{G},X^I}$.

We recall that $\mathrm{Gr}_{\mathcal{G},X^I}$ parametrizes points $(x_i)_{i \in I}$ of X, a $\mathcal{G}$-bundle $\mathcal{P}_{\mathcal{G}}$ on X, and a trivialization τ of $\mathcal{P}_{\mathcal{G}}$ defined on $X \backslash \{x_i\}_{i \in I}$. This is understood in families in the usual way.

We have the following well-known result (proved by reduction[12] to $G = GL_n$):

Lemma 2.13.1. *The space $\mathrm{Gr}_{\mathcal{G},X^I}$ is an ind-algebraic space of ind-finite type. If $\mathcal{G}$ is reductive, then $\mathrm{Gr}_{\mathcal{G},X^I}$ is ind-proper over X^I. If $\mathcal{G}$ is Zariski-locally constant,[13] then then $\mathrm{Gr}_{\mathcal{G},X^I}$ is an indscheme of ind-finite type.*

We deduce:

Proposition 2.13.2. *The map $\pi_{\mathcal{G},X^I} : \mathcal{G}(K)_{X^I} \to \mathrm{Gr}_{\mathcal{G},X^I}$ realizes $\mathcal{G}(K)_{X^I}$ as an étale-locally trivial $\mathcal{G}(O)_{X^I}$-torsor over $\mathrm{Gr}_{\mathcal{G},X^I}$.[14]*

Proof. We follow [BD1] Theorem 4.5.1, where this is proved over a point.

After Zariski localization, we can assume that X admits an étale map to $\mathbb{A}^1$, and after étale localization, that $\mathcal{G}$ is constant (in particular, pulled back from $\mathbb{A}^1$), and therefore we reduce to the case $X = \mathbb{A}^1$. We abuse notation in also denoting by $\mathcal{G}$ the corresponding affine algebraic group.

We embed $X = \mathbb{A}^1$ into its compactification $\mathbb{P}^1$ with ∞ denoting the point complementary to $\mathbb{A}^1$.

In this case we will show that $\mathcal{G}(K)_{X^I} \to \mathrm{Gr}_{\mathcal{G},X^I}$ admits a section Zariski-locally on the target. Because $\mathcal{G}(K)_{X^I}$ acts transitively on geometric points of $\mathrm{Gr}_{\mathcal{G},X^I}$, it suffices to show that there is a Zariski neighborhood of the unit $X^I \subseteq \mathrm{Gr}_{\mathcal{G},X^I}$ that admits a section.

Form the fiber product:

$$\mathcal{U} := \mathrm{Gr}_{\mathcal{G},X^I} \underset{\mathrm{Bun}_{\mathcal{G}}(\mathbb{P}^1)}{\times} \mathbb{B}\mathcal{G}$$

where $\mathbb{B}\mathcal{G} \to \mathrm{Bun}_{\mathcal{G}}(\mathbb{P}^1)$ is the map defined by the trivial bundle. Note that $\mathbb{B}\mathcal{G} \to \mathrm{Bun}_{\mathcal{G}}(\mathbb{P}^1)$ is an open embedding (specifically because we deal with $\mathbb{P}^1$) and therefore the map $\mathcal{U} \to \mathrm{Gr}_{\mathcal{G},X^I}$ is an open embedding. Of course, the map $X^I \to \mathrm{Gr}_{\mathcal{G},X^I}$ factors through $\mathcal{U}$.

The composition:

$$\mathbb{B}\mathcal{G} \hookrightarrow \mathrm{Bun}_{\mathcal{G}}(\mathbb{P}^1) \xrightarrow{\mathrm{ev}_\infty} \mathbb{B}\mathcal{G}$$

---

[12]This reduction step is justified as in the proof of Lemma 2.12.1.

[13]I.e., Zariski-locally of the form $\mathcal{G}_0 \times X$ for $\mathcal{G}_0$ an affine algebraic group.

[14]In fact, Zariski-locally trivial if $\mathcal{G}$ is a Zariski form.

is the identity. Therefore, one obtains that $\mathcal{U}$ is the moduli of $(x_i)_{i \in I}$ in $X = \mathbb{A}^1$ and a map $\mathbb{P}^1 \backslash \{(x_i)_{i \in I}\} \to \mathcal{G}$ sending ∞ to $1 \in \mathcal{G}$. We obtain a map $\mathcal{U} \to \mathcal{G}(K)_{X^I}$ given by taking Laurent expansions, giving the desired section.

$\square$

Convention 2.13.3. For the ease of exposition, we systematically ignore the differences between schemes and algebraic spaces for the remainder of the section (since the forms we will use are Zariski-locally trivial).

The following now results from [Ras3] Example 6.8.4 and Lemma 2.5.1, since $\mathrm{Gr}_{\mathcal{G}, X^I}$ is an indscheme of ind-finite type.
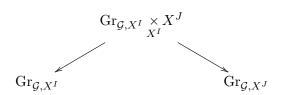
Corollary 2.13.4. $\mathcal{G}(K)_{X^I}$ *is a placid indscheme.*

We obtain the following from Construction 6.12.6 of [Ras3] §6.12.

Corollary 2.13.5. *The indscheme* $\mathcal{G}(K)_{X^I}$ *carries a canonical dimension theory* $\tau^{\mathcal{G}}$ *such that for any finite type subscheme* $T \subseteq \mathrm{Gr}_{\mathcal{G}, X^I}$ *we have:*

$$\tau^{\mathcal{G}}\big(\pi_{\mathcal{G}, X^I}^{-1}(T)\big) = \pi_{\mathcal{G}, X^I}^*(\dim_T).$$

2.14. Note that $I \mapsto \mathrm{Gr}_{\mathcal{G}, X^I}$ defines a *unital factorization indscheme*, i.e., for every $f : I \to J$ we have correspondences:

$$
\begin{array}{ccc}
& \mathrm{Gr}_{\mathcal{G}, X^I} \underset{X^I}{\times} X^J & \\
& \swarrow \qquad \searrow & \\
\mathrm{Gr}_{\mathcal{G}, X^I} & & \mathrm{Gr}_{\mathcal{G}, X^J}
\end{array}
$$

where the left map is obvious and the right map is given by sending:

$$\left((x_j)_{j \in J}, \mathcal{P}_{\mathcal{G}}, \tau \text{ a trivialization of } \mathcal{P}_{\mathcal{G}}|_{X \backslash \{x_{f(i)}\}_{i \in I}} \right) \in \mathrm{Gr}_{\mathcal{G}, X^J}$$
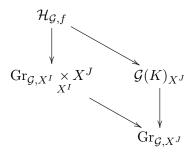
to the point:

$$\left((x_j)_{j \in J}, \mathcal{P}_{\mathcal{G}}, \tau|_{X \backslash \{x_j\}_{j \in J}} \right).$$

Here we note that $X \backslash \{x_j\}_{j \in J} \subseteq X \backslash \{x_{f(i)}\}_{i \in I}$, so that this restriction makes sense.

Therefore, $I \mapsto D(\mathrm{Gr}_{\mathcal{G}, X^I})$ defines a unital factorization category $D(\mathrm{Gr}_{\mathcal{G}})^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR})$.

Remark 2.14.1. The natural maps $\pi_{\mathcal{G}, X^I} : \mathcal{G}(K)_{X^I} \to \mathrm{Gr}_{\mathcal{G}, X^I}$ are compatible with the correspondences (2.10.1) for $\mathcal{G}(K)$. Moreover, for every $f : I \to J$, the square:

$$
\begin{array}{ccc}
& \mathcal{H}_{\mathcal{G}, f} & \\
& \downarrow \qquad \searrow & \\
\mathrm{Gr}_{\mathcal{G}, X^I} \underset{X^I}{\times} X^J & & \mathcal{G}(K)_{X^J} \\
& \searrow \qquad \downarrow & \\
& \mathrm{Gr}_{\mathcal{G}, X^J} &
\end{array}
$$

is Cartesian. Therefore, the functors $\pi^!_{\mathcal{G}, X^I}$ define a strictly unital factorization functor:

$$\pi^!_{\mathcal{G}} : D(\mathrm{Gr}_G) \to D^!(\mathcal{G}(K)). \tag{2.14.1}$$

Remark 2.14.2. Formation of the unital factorization indscheme $I \mapsto \mathrm{Gr}_{\mathcal{G}, X^I}$ is obviously functorial in $\mathcal{G}$: given a morphism $\mathcal{G}_1 \to \mathcal{G}_2$ we obtain morphisms $\mathrm{Gr}_{\mathcal{G}_1, X^I} \to \mathrm{Gr}_{\mathcal{G}_2, X^I}$ compatible with the unital factorization structures. Moreover, for every $I \to J$, the square:

$$
\begin{array}{ccc}
& \mathrm{Gr}_{\mathcal{G}_1, X^I} \underset{X^I}{\times} X^J & \\
& \swarrow \qquad \downarrow & \\
\mathrm{Gr}_{\mathcal{G}_1, X^I} & \mathrm{Gr}_{\mathcal{G}_1, X^I} \underset{X^I}{\times} X^J & \\
\downarrow & \swarrow & \\
& \mathrm{Gr}_{\mathcal{G}_2, X^I} &
\end{array}
$$

is (obviously) Cartesian.

Therefore, we obtain a strictly unital chiral functor:

$$D(\mathrm{Gr}_{\mathcal{G}_1}) \to D(\mathrm{Gr}_{\mathcal{G}_2})$$

given by de Rham pushforwards (which is well-behaved because all the indschemes present are ind-finite type).

2.15. Pure inner forms. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two smooth group schemes over X. Recall that they are said to be *pure inner forms* of each other if there is a specified *bitorsor* for these groups: a $\mathcal{G}_1$-torsor $\mathcal{P}$ on X with a commuting $\mathcal{G}_2$-action realizing $\mathcal{P}$ as a $\mathcal{G}_2$-torsor as well.

In this case, we have a canonical isomorphism of stacks:

$$X/\mathcal{G}_1 \xrightarrow{\simeq} X/\mathcal{G}_2.$$

For example, the map $X/\mathcal{G}_1 \to X/\mathcal{G}_2$ is defined by the $\mathcal{G}_2$-torsor $\mathcal{P}/\mathcal{G}_1$ on $X/\mathcal{G}_1$ (note that we can speak about $\mathcal{G}_2$-torsors on $X/\mathcal{G}_1$ because we have a canonical map $X/\mathcal{G}_1 \to X$).

In particular, if X is proper, we can identify the algebraic stacks:

$$\mathrm{Bun}_{\mathcal{G}_1} \xrightarrow{\simeq} \mathrm{Bun}_{\mathcal{G}_2}. \tag{2.15.1}$$

If $\mathcal{P}$ is a bitorsor for $\mathcal{G}_1$ and $\mathcal{G}_2$, observe $\mathcal{G}_2$ is the group scheme of $\mathcal{G}_1$-automorphisms of $\mathcal{P}$: this follows by considering the local case where $\mathcal{P}$ is trivialized as a $\mathcal{G}_1$-torsor. Therefore, given any group scheme $\mathcal{G}_1$ with a torsor $\mathcal{P}$ we canonically obtain a pure inner form $\mathcal{G}_2$ of $\mathcal{G}_1$ as the group scheme of automorphisms. Moreover, we see that pure inner forms of $\mathcal{G} = \mathcal{G}_1$ are classified by $\mathcal{G}_1$-torsors.

To summarize, for any $\mathcal{G}$ with torsor $\mathcal{P}$, we obtain a form $\mathcal{G}' := \mathrm{Aut}_{\mathcal{G}}(\mathcal{P})$.

2.16. Let $\mathcal{P}^{can}_T$ be the T-torsor $\check{\rho}(\Omega_X) := 2\check{\rho}(\Omega_X^{\frac{1}{2}})$ for $\Omega_X^{\frac{1}{2}}$ our choice of square root of the canonical bundle. For $\Gamma = B, B^-, G$, let $\mathcal{P}^{can}_\Gamma$ denote the induced Γ-torsor via the embedding $T \hookrightarrow \Gamma$.

Let G^{can}, B^{can}, and $B^{-,can}$ denote the corresponding pure inner forms of G, B and B^- respectively. Note that commutativity of T means that T^{can} is a constant group scheme.

Let $N^{-,can}$ denote the form of N^- obtained by twisting $\mathcal{P}^{can}_{B^-}$ by the adjoint action of B^- on N^-. Note that $N^{-,can}$ is *not* an inner form of N^-. We treat N^{can} similarly.

Example 2.16.1. Suppose that $G = GL_2$. Then G^{can} is the group scheme whose sections are matrices:

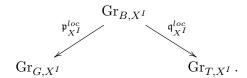$$\begin{pmatrix} f & \varphi \\ \omega & g \end{pmatrix}$$

with $f, g \in \mathcal{O}_X$, $\omega \in \Omega_X$, and $\varphi \in \Omega_X^{-1}$, and with determinant $fg - \varphi \otimes \omega \in \mathcal{O}_X$ everywhere non-zero.

Convention 2.16.2. To avoid including twists in the notation everywhere, we will write e.g. $G(K)_{X^I}$ for the relative jets into G^{can} (in the sense of Remark 2.7.1). The same goes for loop spaces, jet spaces, and affine Grassmannians for G and our other groups.

The truth is that these twists do not play a role at all until we discuss Whittaker invariants, and we could work just as well with any other twists of our groups until then (including the constant one). However, for reasons of notation, we choose to make the official policy that these twists are included at every step.

Remark 2.16.3. By (2.15.1), this twist gives rise to the same automorphic forms as the split form of G.

Notation 2.16.4. We will use the notation $\mathfrak{p}_{X^I}^{loc}$ and $\mathfrak{q}_{X^I}^{loc}$ for the maps:

$$
\begin{array}{ccc}
 & \mathrm{Gr}_{B,X^I} & \\
\mathfrak{p}_{X^I}^{loc} \swarrow & & \searrow \mathfrak{q}_{X^I}^{loc} \\
\mathrm{Gr}_{G,X^I} & & \mathrm{Gr}_{T,X^I} \,.
\end{array}
$$

(Here the notation *loc* indicates that these are "local" counterparts to the maps $\mathfrak{p} : \mathrm{Bun}_B \to \mathrm{Bun}_G$ and $\mathfrak{q} : \mathrm{Bun}_B \to \mathrm{Bun}_T$ from [BG1]).

By the above, $\mathfrak{p}_{*,dR}^{loc}$ and $\mathfrak{q}_{*,dR}^{loc}$ have canonical structures of (strictly) unital chiral functors.

2.17. **Group actions on categories.** It will be convenient to have the basic aspects of the theory of group action on categories available to us.

Remark 2.17.1. Because we need to work in a relative framework, it is not sufficient for us to appeal to [Ber].

Let S be a base scheme of finite type and let $\mathcal{H} \to S$ be a group indscheme over S that is placid as a mere indscheme.

By [Ras3] Proposition 6.9.1, the category $D^!(\mathcal{H})$ obtains the structure of coalgebra in the symmetric monoidal category $D(S)\text{–mod} \simeq \mathsf{ShvCat}_{/S_{dR}}$.

Definition 2.17.2. A category *(!–)acted on* by $\mathcal{H}$ (over S) is a left comodule for $D^!(\mathcal{H})$ in $\mathsf{ShvCat}_{/S_{dR}}$. We denote the corresponding category by $\mathcal{H}\text{–mod}$.

Example 2.17.3. If T is an indscheme over S with an action of $\mathcal{H}$, then by [Ras3] Proposition 6.9.1, $\mathcal{H}$ acts on $D^!(T)$.

Remark 2.17.4. The "Hopf algebra" structure on $\mathcal{H}$ implies that $\mathcal{H}\text{–mod}$ admits a symmetric monoidal structure with symmetric monoidal forgetful functor $\mathcal{H}\text{–mod} \to \mathsf{ShvCat}_{/S_{dR}}$. For $\mathsf{C}, \mathsf{D} \in \mathcal{H}\text{–mod}$, the coaction map on $\mathsf{C} \underset{D(S)}{\otimes} \mathsf{D}$ is induced in the obvious way from the coaction for C and D separately, and the !-restriction functor $D^!(\mathcal{H} \times_S \mathcal{H}) \to D^!(\mathcal{H})$ induced by the diagonal $\mathcal{H} \to \mathcal{H} \times_S \mathcal{H}$.

Remark 2.17.5. The forgetful functor $\mathcal{H}$–$\mathsf{mod} \to \mathsf{ShvCat}_{/S_{dR}}$ admits a right adjoint $\mathsf{C} \mapsto \mathsf{C} \otimes_{D(S)} D^!(\mathcal{H})$.

Moreover, we claim that $\mathcal{H}$–$\mathsf{mod} \to \mathsf{ShvCat}_{/S_{dR}}$ commutes with limits. Note that $D^!(\mathcal{H})$ is dualizable as an object of $\mathsf{ShvCat}_{/S_{dR}}$ by placidity. Therefore, tensoring in $\mathsf{ShvCat}_{/S_{dR}}$ with $D^!(\mathcal{H})$ commutes with limits, so the result is proved as [Lur] Corollary 4.2.3.5.

In particular, we see that every $\mathsf{C} \in \mathcal{H}$–$\mathsf{mod}$ admits a bar resolution:

$$\mathsf{C} = \lim_{\mathbf{\Delta}} \left(\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \rightrightarrows \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \underset{D(S)}{\otimes} D^!(\mathcal{H}) \Rrightarrow \dots \right)$$

Given C acted on by $\mathcal{H}$, we define the category $\mathsf{C}^{\mathcal{H}}$ of invariants C as the limit of the bar construction:

$$\mathsf{C}^{\mathcal{H}} := \lim_{[n] \in \mathbf{\Delta}} \left(\mathsf{C} \rightrightarrows D^!(\mathcal{H}) \underset{D(S)}{\otimes} \mathsf{C} \Rrightarrow \dots \right)$$

There is a tautological functor:

$$\mathrm{Oblv} : \mathsf{C}^{\mathcal{H}} \to \mathsf{C}.$$

Example 2.17.6. The category $D^!(\mathcal{H})$ acts on itself, and we have $D(S) \xrightarrow{\simeq} D^!(\mathcal{H})^{\mathcal{H}}$ by splitting the relevant cosimplicial object. Here the corresponding functor $D(S) \xrightarrow{\simeq} D^!(\mathcal{H})^{\mathcal{H}} \xrightarrow{\mathrm{Oblv}} D^!(\mathcal{H})$ is !-pullback.

Remark 2.17.7. Suppose that $\mathcal{H} = \cup_i \mathcal{H}_i$ is an ind-group scheme. Then for every C acted on by $\mathcal{H}$, we have:

$$\mathsf{C}^{\mathcal{H}} \xrightarrow{\simeq} \lim_i \mathsf{C}^{\mathcal{H}_i}.$$

Indeed, this follows by commuting limits with limits.

We recall that $D^!(\mathcal{H})$ is dualizable as a $D(S)$-module category with dual $D^*(\mathcal{H})$ because $\mathcal{H}$ is assumed placid. Under this duality, the coalgebra structure on $D^!(\mathcal{H})$ transfers to the canonical algebra structure on $D^*(\mathcal{H}) \in \mathsf{ShvCat}_{/S_{dR}}$ induced by the multiplication on $\mathcal{H}$.[15]

We therefore obtain:

Proposition 2.17.8. *Under the above hypotheses on $\mathcal{H}$, categories acted on by $\mathcal{H}$ are canonically equivalent to left $D^*(\mathcal{H})$-modules in $\mathsf{ShvCat}_{/S_{dR}}$.*

For C acted on by $\mathcal{H}$, we refer to the corresponding $D^*(\mathcal{H})$-action as *convolution*.

For the remainder of this discussion, we suppose that $\mathcal{H}$ is a group scheme over S, and moreover that $\mathcal{H}$ satisfies the hypotheses of [Ras3] Example 4.2.3, i.e., $\mathcal{H}$ is a filtered limit of smooth affine S-group schemes under smooth surjective homomorphisms.

By [Ras3] Proposition 4.11.1, the pullback $D(S) \to D^!(\mathcal{H})$ then admits a right adjoint in $\mathsf{ShvCat}_{/S_{dR}}$ of renormalized de Rham pushforward functor of [Ras3] §4.9.

We refer to [Lur] Theorem 6.2.4.2 and [Gai4] §4.4.7 for an introduction to the Beck-Chevalley formalism used below.

Proposition 2.17.9. *Under the above hypotheses on $\mathcal{H}$, the cosimplicial object defining $\mathsf{C}^{\mathcal{H}}$ satisfies the Beck-Chevalley conditions.*

---

[15]Here we are repeatedly using the canonical identification from [GR2] of $(f^!)^{\vee}$, the functor dual to $f^!$, with $f_{*,dR}$ for a morphism f of finite type schemes.

Corollary 2.17.10. *The functor* $\mathrm{Oblv} : \mathsf{C}^{\mathcal{H}} \to \mathsf{C}$ *admits a right adjoint* $\mathrm{Av}_{\mathcal{H},\mathsf{C},*} = \mathrm{Av}_{\mathcal{H},*} = \mathrm{Av}_*$
in $D(S)$. *In particular, formation of* Av_* *commutes with base-change of the (finite type) scheme* S.
Moreover, for a morphism $\mathsf{C} \to \mathsf{D}$ *of categories acted on by* $\mathcal{H}$, *the diagram:*

$$
\begin{array}{ccc}
\mathsf{C} & \longrightarrow & \mathsf{D} \\
\downarrow {\scriptstyle \mathrm{Av}_*} & & \downarrow {\scriptstyle \mathrm{Av}_*} \\
\mathsf{C}^{\mathcal{H}} & \longrightarrow & \mathsf{D}^{\mathcal{H}}
\end{array}
$$

commutes (i.e., the relevant natural transformation is a natural isomorphism). More precisely, Av_*
is given by convolution with $\omega_{\mathcal{H}}^{ren}$, *this object being defined by the dimension theory on* $\mathcal{H}$ *obtained*
by pullback from the standard dimension theory on S.

We we will use the following in the proof of Proposition 2.17.9.

Lemma 2.17.11. *For* C *acted on by* $\mathcal{H}$, *let*

$$
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \to \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H})
$$

be the endofunctor induced by the coaction map:

$$
\mathsf{C} \to \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H})
$$

and considering the right hand side as a $(D^!(\mathcal{H}), \overset{!}{\otimes})$-*module.*
Then this endofunctor is an equivalence.

Proof. Recall that $D^!(\mathcal{H})$ is dualizable as a $D(S)$-module category. Therefore, by Remark 2.17.5
we reduce to the case where $\mathsf{C} = \mathsf{D} \otimes_{D(S)} D^!(\mathcal{H})$ for $\mathsf{D} \in \mathsf{ShvCat}_{/S_{dR}}$. Here the result is obvious. $\square$

Proof of Proposition 2.17.9. For every integer n, the functor:

$$
\mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{n \text{ times}} \to \mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{(n+1) \text{ times}}
$$

coming from tensoring on the right with the pullback $D(S) \to D^!(\mathcal{H})$ admits a right adjoint, as
noted before. Moreover, we claim that for every morphism $[n] \to [m] \in \mathbf{\Delta}$, we need to show that
the following diagram commutes (i.e., the base-change map should be an equivalence):

$$
\begin{array}{ccc}
\mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{(n+1) \text{ times}} & \longrightarrow & \mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{n \text{ times}} \\
\downarrow & & \downarrow \\
\mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{(m+1) \text{ times}} & \longrightarrow & \mathsf{C} \underset{D(S)}{\otimes} \underbrace{D^!(\mathcal{H}) \underset{D(S)}{\otimes} \ldots \underset{D(S)}{\otimes} D^!(\mathcal{H})}_{m \text{ times}}
\end{array}
$$

where horizontal arrows are these left adjoints and vertical arrows are the structure maps, $[n+1] \to$
$[m + 1]$ being induced from $[n] \to [m]$ by adjoining a new infimum.

Rather than get bogged down in notation, we prove this instead for "representative" morphisms $[n] \to [m]$, the general argument being the same.

Namely, suppose that $n = 0$ and $m = 1$. If $0 \mapsto 1$, the commutativity is tautological. Therefore, suppose that $0 \mapsto 0$. Then the corresponding map $\mathsf{C} \to \mathsf{C} \otimes_{D(S)} D^!(\mathcal{H})$ is the coaction map, and we should prove that the diagram:

$$
\begin{array}{ccc}
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) & \longrightarrow & \mathsf{C} \\
\downarrow{\scriptstyle \mathrm{coact}\,\otimes\,\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{coact}} \\
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \underset{D(S)}{\otimes} D^!(\mathcal{H}) & \longrightarrow & \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H})
\end{array}
$$

commutes, where the horizontal arrows are given by taking renormalized de Rham cohomology in the last variable.

Intertwining the lower two terms using Lemma 2.17.11, we see that this diagram is isomorphic to:

$$
\begin{array}{ccc}
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) & \longrightarrow & \mathsf{C} \\
\downarrow & & \downarrow \\
\mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H}) \underset{D(S)}{\otimes} D^!(\mathcal{H}) & \longrightarrow & \mathsf{C} \underset{D(S)}{\otimes} D^!(\mathcal{H})
\end{array}
\tag{2.17.1}
$$

where now the two vertical arrows are induced by tensoring appropriately with the pullback $D(S) \to D^!(\mathcal{H})$.

To see that the diagram (2.17.1) commutes, it suffices to show that in the diagram:

$$
\begin{array}{ccc}
\mathcal{H} \underset{S}{\times} \mathcal{H} & \overset{p_1}{\longrightarrow} & \mathcal{H} \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi} \\
\mathcal{H} & \overset{\pi}{\longrightarrow} & S
\end{array}
$$

the natural transformation $p_2^! \pi_{*,ren} \to p_{1,*,ren} \pi^!$ arising from adjunction is an equivalence. To this end, we extend the diagram to:

$$
\begin{array}{ccccc}
\mathcal{H} \underset{S}{\times} \mathcal{H} & \longrightarrow & \mathcal{H} \times \mathcal{H} & \overset{p_1}{\longrightarrow} & \mathcal{H} \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{H}} \times \pi} & & \downarrow{\scriptstyle \pi} \\
\mathcal{H} & \overset{\Gamma_\pi}{\longrightarrow} & \mathcal{H} \times S & \overset{p_2}{\longrightarrow} & S
\end{array}
$$

where Γ_π indicates the graph of the map π. Now base-change is obvious for the right square, and for the left square it follows from [Ras3] Proposition 4.11.1.

$\square$

2.18. The unipotent case. Let S be a finite type base scheme again.

Definition 2.18.1. A *unipotent S-group scheme* is a smooth S-group scheme that has a central filtration by smooth S-group schemes with subquotients forms (in the sense of 2.11) of $\mathbb{G}_a \times S$.

A *prounipotent group S-scheme* is a group S-scheme that is a projective limit of unipotent S-group schemes under smooth surjective group homomorphisms.

A *unipotent group indscheme* over S is a group indscheme over S that is a union of closed subgroup schemes each of which is prounipotent.

Example 2.18.2. Any form $\mathcal{H}$ of a unipotent group $\mathcal{H}_0$ over $\mathrm{Spec}(k)$ is unipotent: indeed, this follows from comparing the lower central series of $\mathcal{H}$ with that of $\mathcal{H}_0$. The group scheme $N(O)_{X^I}$ is prounipotent. For any form $\mathcal{G}$ of an algebraic group, $\mathrm{Ker}\left(\mathcal{G}(O)_{X^I} \to \mathcal{G}\right)$ is prounipotent. The group indscheme $N(K)_{X^I}$ is a unipotent group indscheme over X^I.

Remark 2.18.3. Obviously unipotent group indschemes are placid.

Let $\mathcal{H}$ be a unipotent group indscheme over S for the remainder of this section.
The key feature for our purposes is the following:

Proposition 2.18.4. *For every* C *acted on by* $\mathcal{H}$, *the functor:*

$$\mathrm{Oblv} : \mathsf{C}^{\mathcal{H}} \to \mathsf{C}$$

is fully-faithful in $\mathsf{ShvCat}_{/S_{dR}}$.

Proof. By Remark 2.17.7 and [Ras1] Corollary A.4.5, we reduce to proving this in the case when $\mathcal{H}$ is a prounipotent group scheme over S.

In this case, note that $D(S) \to D^!(\mathcal{H})$ is fully-faithful and admits a fully-faithful right adjoint in $\mathsf{ShvCat}_{/S_{dR}}$. Indeed, under the identification $D^! \simeq D^*$, $f^!$ identifies with $f^{*,dR}$ by [Ras3] Proposition 4.11.1, so the result follows from the contractibility of affine space.

Therefore, for any $\mathsf{D} \in \mathsf{ShvCat}_{/S_{dR}}$, the induced functor:

$$\mathsf{D} \to \mathsf{D} \underset{D(S)}{\otimes} D^!(\mathcal{H})$$

is fully-faithful.

By Lemma 2.17.11, we see that each morphism in the semicosimplicial diagram (underlying the cosimplicial diagram) defining $\mathsf{C}^{\mathcal{H}}$ is fully-faithful. By contractibility of the category of the semisimplex category (i.e., finite totally ordered sets under injections preserving the orders), we deduce the result from [Ras1] Corollary A.4.5.

$\square$

2.19. **Borel notation.** Let $N(K)T(O)_{X^I}$ denote the "connected component of the identity" in $B(K)$,[16] i.e., the group indscheme over X^I:

$$N(K)T(O)_{X^I} := B(K)_{X^I} \underset{T(K)_{X^I}}{\times} T(O)_{X^I}.$$

Remark 2.19.1. Note that $N(K)T(O)_{X^I}$ is an ind-group scheme: indeed, choose a coordinate t on X and then $N(K)T(O)_{X^I}$ is the union of the subgroups $\mathrm{Ad}_{-\check{\lambda}(t)}(B(O)_{X^I})$ for $\check{\lambda}$ a dominant coweight, and one readily checks that these subgroups do not depend on the choice of coordinate.

Remark 2.19.2. Varying the finite set I, one sees that $N(K)T(O)_{X^I}$ is another factorization group scheme. It has a unital structure under correspondences induced by that of $B(K)_{X^I}$.

---

[16]We remark that this is poor terminology scheme-theoretically: for example, $T(O)$ is not the connected component of the identity of $T(K)$ due to the existence of nilpotents.

2.20. **Semi-infinite flag variety.** In this section, we consider $G(K)_{X^I}$ acting on itself through the right action.

We define $D^!(\mathrm{Fl}^{\frac{\infty}{2}}_{X^I})$ as the $N(K)T(O)_{X^I}$-coinvariants category of $D^!(G(K)_{X^I})$.

We have a tautological functor:

$$\mathfrak{p}^{\frac{\infty}{2}}_{I,*,ren} : D^!(G(K)_{X^I}) \to D^!(\mathrm{Fl}^{\frac{\infty}{2}}_{X^I}).$$

These categories are compatible with restrictions between X^I as $I \in \mathsf{fSet}$ varies by [Ras3] Proposition 6.9.1 and by the base-change results of [Ras3] §6.3. Therefore, we obtain the category $D^!(\mathrm{Fl}^{\frac{\infty}{2}}_{\mathrm{Ran}_X})$, which arises as the global sections on an underlying sheaf of categories $D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}$ on $\mathrm{Ran}_{X_{dR}}$, equipped with the tautological functor:

$$\mathfrak{p}^{\frac{\infty}{2}}_{\mathrm{Ran}_X,*,ren} : D^!(G(K)_{\mathrm{Ran}_X}) \to D^!(\mathrm{Fl}^{\frac{\infty}{2}}_{\mathrm{Ran}_X}).$$

There is an evident structure of factorization category on $D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}$ (which we will upgrade to *unital* factorization category in what follows), equipped with the functor $\mathfrak{p}^{\frac{\infty}{2}}_{*,ren} : D^!(G(K))^{\mathrm{fact}} \to D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}$.

Remark 2.20.1. While the semi-infinite flag variety $\mathrm{Fl}^{\frac{\infty}{2}}_{X^I}$ does not exist as an indscheme, the notation follows the standard convention in the literature to pretend that it does. Then $\mathfrak{p}^{\frac{\infty}{2}}_I$ would be map $G(K)_{X^I} \to \mathrm{Fl}^{\frac{\infty}{2}}_{X^I}$.

Remark 2.20.2. The choice to work with coinvariants here instead of invariants is more natural for the purposes of §4.

2.21. **Intermediate Grassmannian.** We will need the following intermediate space between the semi-infinite flag variety $\mathrm{Fl}^{\frac{\infty}{2}}_{X^I}$ and Gr_{G,X^I}.

For each finite set I, let Gr_{G,B,X^I} be the *intermediate Grassmannian* parametrizing a point $x = (x_i)_{i \in I} \in X^I$, a G^{can}-bundle $\mathcal{P}$ on X with a trivialization on $X \backslash x = X \backslash \{x_i\}_{i \in I}$ and a reduction to B^{can} on $\widehat{\mathcal{D}}_x$ (this is understood in families in the usual manner).

Remark 2.21.1. For a closed point $x \in X$ with a trivialization of $\Omega^{1/2}_X|_{\mathcal{D}_x}$ (to eliminate the twist of §2.16), the fiber of $\mathrm{Gr}_{G,B,X}$ over a closed point $x \in X$ is the indscheme (of ind-infinite type) $G(K_x)/B(O_x)$.

We have obvious maps $\mathrm{Gr}_{G,B,X^I} \to \mathrm{Gr}_{G,X^I}$, and by Proposition 2.13.2, Gr_{G,B,X^I} is a placid indscheme. Clearly $I \mapsto \mathrm{Gr}_{G,B,X^I}$ factorizes.

Moreover, the unital structure (in the sense of correspondences) on $(I \mapsto G(K)_{X^I})$ defines a unital structure on $(I \mapsto \mathrm{Gr}_{G,B,X^I})$. For example, the unit map over X^I is given by the correspondence:

$$
\begin{array}{ccc}
 & G(O)_{X^I}/B(O)_{X^I} & \\
 & \swarrow \qquad \searrow & \\
X^I & & \mathrm{Gr}_{G,B,X^I} \, .
\end{array}
\tag{2.21.1}
$$

Therefore, the assignment:

$$I \mapsto D^!(\mathrm{Gr}_{G,B,X^I})$$

defines a unital factorization category $D^!(\mathrm{Gr}_{G,B})^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR})$ on X_{dR}.

2.22. We can more explicitly express the category $D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})$ by realizing it as a localization of $D^!(\mathrm{Gr}_{G,B,X^I})$ as follows.

We have a canonical functor:

$$\mathfrak{p}_{X^I,*,ren}^{\frac{\infty}{2},int} : D^!(\mathrm{Gr}_{G,B,X^I}) \to D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})$$

obtained by writing $D^!(\mathrm{Gr}_{G,B,X^I})$ as the $B(O)_{X^I}$-coinvariants of $D^!(G(K)_{X^I})$ via [Ras3] Proposition 6.7.1.

This is a functor of $D(X^I)$-module categories (i.e., sheaves of categories on X_{dR}^I), and we will show in §2.23 that it is a localization functor as such.

2.23. As in Remark 2.19.1, we can write $N(K)T(O)_{X^I}$ as a filtered union of subgroup schemes $\mathcal{K}_\alpha$ beginning with $B(O)_{X^I}$ and such that the subquotients are locally finite-dimensional affine spaces over X^I.

It follows tautologically that:

$$D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}}) \simeq \operatorname*{colim}_\alpha D^!(G(K)_{X^I})_{\mathcal{K}_\alpha}$$

with the coinvariant category on the right defined as the colimit of the appropriate bar construction.

By [Ras3] Proposition 6.7.1, we have a canonical identification:

$$D^!(G(K)_{X^I})_{B(O)_{X^I}} \simeq D^!(\mathrm{Gr}_{G,B,X^I})$$

with the equivalence induced by the functor of renormalized de Rham pushforward along $G(K)_{X^I} \to \mathrm{Gr}_{G,B,X^I}$.

We claim that for each of our distinguished subgroups $\mathcal{K}_\alpha$, the functor:

$$D^!(\mathrm{Gr}_{G,B,X^I}) \simeq D^!(G(K)_{X^I})_{B(O)_{X^I}} \to D^!(G(K)_{X^I})_{\mathcal{K}_\alpha} \qquad (2.23.1)$$

admits a fully-faithful left adjoint.

Indeed, there is a canonical indscheme (of ind-infinite type):

$$G(K)_{X^I}/\mathcal{K}_\alpha$$

so that $G(K)_{X^I} \to G(K)_{X^I}/\mathcal{K}_\alpha$ is a $\mathcal{K}_\alpha$-torsor (for $\mathcal{K}_\alpha = B(O)_{X^I}$, we obtain Gr_{G,B,X^I}).

By [Ras3] Proposition 6.7.1, we have:

$$D^!(G(K)_{X^I})_{\mathcal{K}_\alpha} \simeq D^!(G(K)_{X^I}/\mathcal{K}_\alpha)$$

so that the functor (2.23.1) corresponds to the renormalized pushforward:

$$D^!(\mathrm{Gr}_{G,B,X^I}) \to D^!(G(K)_{X^I}/\mathcal{K}_\alpha).$$

Then the existence of the left adjoint follows from [Ras3] Proposition 6.18.1: it is computed as the upper-! functor under this dictionary. Moreover, the fact that the fibers of our map are affine spaces implies the fully-faithfulness of this left adjoint.

Passing to the colimit over the groups $\mathcal{K}_\alpha$ and applying [Ras1] Proposition A.7.3, we obtain that the functor $\mathfrak{p}_{X^I,*,ren}^{\frac{\infty}{2},int}$ is a localization functor as desired.

Remark 2.23.1. Note that $D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})$ is *not* a localization of $D^!(G(K)_{X^I})$: the problem is that $B(O)$ admits the non-trivial reductive quotient T.

2.24. **Unitality of the semi-infinite flag variety.** For every finite set I, let $\mathcal{K}_I$ denote the kernel of the functor $\mathfrak{p}^{\frac{\infty}{2},int}_{X^I,*,ren}$.

For I and J two finite sets, let:

$$\mathcal{H}_{G,B,I,J} := \mathcal{H}_{G,B,f}$$

<div align="center">

$\alpha_{G,B}$ $\beta_{G,B}$ (2.24.1)

$X^I \times \mathrm{Gr}_{G,B,X^J}$ $\mathrm{Gr}_{G,B,X^{I\coprod J}}$

</div>

denote the associated unit correspondence, where $f : I \hookrightarrow I \coprod J$ is the tautological inclusion.

Lemma 2.24.1. *The unit functor* $\beta_{G,B,*,!-dR}\alpha^!_{G,B}$ *maps* $D(X^I) \otimes \mathcal{K}_J$ *to* $\mathcal{K}_{I\coprod J}$.

Proof. Suppose that $\mathcal{F} \in D(X^I) \otimes \mathcal{K}_J$. We need to show that:

$$\mathfrak{p}^{\frac{\infty}{2},int}_{X^{I\coprod J},*,ren}\beta_{G,B,*,!-dR}\alpha^!_{G,B}(\mathcal{F}) = 0.$$

Step 1. First, let us show that the left hand side is zero when restricted to $[X^I \coprod X^J]_{disj}$, the locus where the corresponding point in $\mathrm{Ran}_X \times \mathrm{Ran}_X$ lies in $[\mathrm{Ran}_X \times \mathrm{Ran}_X]_{disj}$.

Each of our functors is intertwined by this restriction to this open: indeed, this is obvious for $\beta_{G,B,*,!-dR}$ and $\alpha^!_{G,B}$, and for $\mathfrak{p}^{\frac{\infty}{2},int}_{X^{I\coprod J},*,ren}$ this follows by combining the analysis of §2.23 with [Ras3] Proposition 6.18.1.

Then the claim follows because our correspondence restricts to the obvious correspondence:

<div align="center">

$[G(O)_{X^I}/B(O)_{X^I} \times \mathrm{Gr}_{G,B,X^J}]$

$[X^I \times \mathrm{Gr}_{G,B,X^J}]_{disj}$ $[\mathrm{Gr}_{G,B,X^I} \times \mathrm{Gr}_{G,B,X^J}]_{disj}$

</div>

Here the notation $[-]_{disj}$ everywhere indicates that we restrict to $[X^I \times X^J]_{disj}$. Moreover, the map $\mathfrak{p}^{\frac{\infty}{2},int}_{X^{I\coprod J},*,ren}$ restricts to this locus as $\mathfrak{p}^{\frac{\infty}{2},int}_{X^I,*,ren} \otimes \mathfrak{p}^{\frac{\infty}{2},int}_{X^J,*,ren}$. From here, the claim is obvious.

Step 2. To complete the above analysis, we need the following digression.

Suppose that we are given $I = I_1 \coprod I_2$ and a map $\varepsilon : I_2 \to J$.

We associate to this datum a locally closed subscheme $Z \hookrightarrow X^I \times X^J$, defined as the locus of points $\big((x_i)_{i\in I}, (x_j)_{j\in J}\big)$ such that, for every $i \in I_1, j \in J$, we have $x_i \neq x_j$, and for every $i \in I_2$ we have $x_i = x_{\varepsilon(i)}$. (The scheme-theoretic meaning of $x_i \neq x_j$ for S-points is that the map $(x_i, x_j) : S \to X \times X$ factors through the complement to the diagonal).

For example, if $I_1 = I$, $I_2 = \varnothing$, then $Z = [X^I \times X^J]_{disj}$. In general, Z is isomorphic to $[X^{I_1} \times X^J]_{disj}$, and the map $Z \to X^I \times X^J$ factors as:

$$Z = [X^{I_1} \times X^J]_{disj} \hookrightarrow [X^{I_1} \times (X^{I_2 \coprod J})]_{disj} \hookrightarrow X^I \times X^J \qquad (2.24.2)$$

where the first map is the diagonal embedding defined by the surjection $I_2 \coprod J \overset{\varepsilon \times \mathrm{id}_J}{\twoheadrightarrow} J$.

Note that as the data $(I = I_1 \coprod I_2, \varepsilon : I_2 \to J)$ vary, the associated locally closed subschemes cover $X^I \times X^J$. Indeed, given a geometric point $\big((x_i)_{i\in I}, (x_j)_{j\in J}\big) \in X^I \times X^J$, let I_1 be the set of i such that $x_i \neq x_j$ for all $j \in J$, let I_2 be its complement, and define $\varepsilon : I_2 \to J$ by choosing for each $i \in I_2$ some $j \in J$ such that $x_i = x_j$.

We remark that this construction does not form a partition: there is some redundancy.

Step 3. Let $I = I_1 \coprod I_2$, $\varepsilon : I_2 \to J$ and Z be as above.

Using factorization and the composition (2.24.2), we see that the restriction of (2.24.1) to Z is isomorphic to:

$$G(O)_{X^{I_1}}/B(O)_{X^{I_1}} \times X^{I_2} \times \mathrm{Gr}_{G,B,X^J}$$

$$X^I \times \mathrm{Gr}_{G,B,X^J} \qquad\qquad\qquad \mathrm{Gr}_{G,B,X^{I_1}} \times X^{I_2} \times \mathrm{Gr}_{G,B,X^J} .$$

The same argument as in Step 1 implies that our functors are intertwined by !-restriction to Z in the obvious way. Therefore, we see that $\mathfrak{p}_{X^I \sqcup J, *, ren}^{\frac{\infty}{2}, int} \beta_{G,B,*,!-dR} \alpha_{G,B}^!(\mathcal{F})$ has vanishing !-restriction to the locus:

$$\mathrm{Gr}_{G,B,X^I \sqcup J} \underset{X^I \sqcup J}{\times} Z.$$

But this suffices, since varying our choice of $I = I_1 \coprod I_2$ and $\varepsilon : I_2 \to J$ we obtain a cover of $X^I \times X^J$ by locally closed subschemes.

$\square$

Therefore, varying I and J, we see that $D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}$ has a canonical structure of unital sheaf of categories. We will denote the corresponding object of $\mathsf{ShvCat}_{/\mathrm{Ran}_{X_{dR}}^{un}}$ by the same notation.

Lemma 2.24.2. *Let $f : I \to J$ be a surjection of finite sets. Then the functor:*

$$\mathcal{K}_I \underset{D(X^I)}{\otimes} D(X^J) \to \mathcal{K}_J$$

induced by !-restriction is an equivalence.

Proof. Let $\mathcal{K}_{X^I, \alpha} \subseteq N(K)T(O)_{X^I}$ be a subgroup scheme as in §2.23 (there denoted $\mathcal{K}_\alpha$, where there was only one finite set at play). Let $\mathcal{K}_{X^J, \alpha}$ denote the restriction of $\mathcal{K}_{X^I, \alpha}$ along the closed embedding:

$$\mathrm{Gr}_{G,B,X^J} = \mathrm{Gr}_{G,B,X^I} \underset{X^I}{\times} X^J \hookrightarrow \mathrm{Gr}_{G,B,X^I} . \qquad\qquad (2.24.3)$$

Note that $\mathcal{K}_{X^J, \alpha} \subseteq N(K)T(O)_{X^J}$ is a subgroup scheme of the same type as considered in §2.23.

Define $\mathcal{K}_{I, \alpha}$ and $\mathcal{K}_{J, \alpha}$ respectively as the kernels of the renormalized pushforward functors:

$$D^!(\mathrm{Gr}_{G,B,X^I}) \to D^!(G(K)_{X^I}/\mathcal{K}_{X^I, \alpha})$$
$$\text{resp. } D^!(\mathrm{Gr}_{G,B,X^J}) \to D^!(G(K)_{X^J}/\mathcal{K}_{X^J, \alpha}).$$

Because these pushforward functors admit fully-faithful left adjoints, the corresponding functors:

$$\mathcal{K}_{X^I, \alpha} \hookrightarrow D^!(\mathrm{Gr}_{G,B,X^I})$$
$$\mathcal{K}_{X^J, \alpha} \hookrightarrow D^!(\mathrm{Gr}_{G,B,X^J})$$

do as well. Moreover, they are $D(X^I)$-equivariant. Applying this to I, we see that the functor:

$$\mathcal{K}_{I,\alpha} \underset{D(X^I)}{\otimes} D(X^J) \to D^!(\mathrm{Gr}_{G,B,X^I}) \underset{D(X^I)}{\otimes} D(X^J)$$

is fully-faithful as well. By [Ras3] Proposition 6.9.1, the functor:

$$D^!(\mathrm{Gr}_{G,B,X^I}) \underset{D(X^I)}{\otimes} D(X^J) \to D^!(\mathrm{Gr}_{G,B,X^J})$$

is an equivalence, so we see that:

$$\mathcal{K}_{I,\alpha} \underset{D(X^I)}{\otimes} D(X^J) \to \mathcal{K}_{J,\alpha} \tag{2.24.4}$$

is fully-faithful.

Now observe that (2.24.3) is a finitely presented closed embedding (having been obtained by base-change from $X^J \hookrightarrow X^I$), and therefore the !-restriction functor admits a fully-faithful left adjoint of !-dR $*$-pushforward. This left adjoint is a morphism of $D(X^I)$-module categories by [Ras3] Remark 3.9.5. Moreover, by [Ras3] Proposition 4.12.1, we see that this !-dR $*$-pushforward functor coincides with renormalized pushforward up to cohomological shift, and therefore it maps $\mathcal{K}_{J,\alpha}$ to $\mathcal{K}_{I,\alpha}$.

Therefore, we see that (2.24.4) is essentially surjective and therefore an equivalence.

The proof of [Ras1] Proposition A.7.3 shows that the colimit $\mathrm{colim}_\alpha \mathcal{K}_{I,\alpha}$ considered as a subcategory of $D^!(\mathrm{Gr}_{G,B,X^I})$ coincides with $\mathcal{K}_I$; comparing with the same expression for $\mathcal{K}_J$, we obtain the result.

$\square$

Therefore, we see that the conditions of [Ras1] §6.5 are satisfied, so that $D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}$ obtains a canonical structure of unital chiral category. As such, it is equipped with the canonical strictly unital functor:

$$\mathfrak{p}^{\frac{\infty}{2},int,\mathrm{fact}}_{*,ren} : D^!(\mathrm{Gr}_{G,B})^{\mathrm{fact}} \to D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR}).$$

2.25. Semi-infinite restriction functor (preliminary). Fix a finite set I. Let $\widetilde{i}_{X^I} : \mathrm{Gr}_{B,X^I} \to \mathrm{Gr}_{G,B,X^I}$ denote the canonical map induced by the embedding $B \hookrightarrow G$. As in Remark 2.10.4, these maps give a canonical strictly unital chiral functor:

$$\widetilde{i}^! : D^!(\mathrm{Gr}_{G,B})^{\mathrm{fact}} \to D(\mathrm{Gr}_B)^{\mathrm{fact}}.$$

Proposition 2.25.1. *There is a unique unital chiral functor:*

$$\widetilde{i}^{\frac{\infty}{2},!,\mathrm{fact}} : D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}} \to D(\mathrm{Gr}_T)^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR}). \tag{2.25.1}$$

with an isomorphism:

$$\widetilde{i}^{\frac{\infty}{2},!} \circ \mathfrak{p}^{\frac{\infty}{2},int}_{*,ren} \simeq \mathfrak{q}^{loc}_{*,dR} \circ \widetilde{i}^! : D^!(\mathrm{Gr}_{G,B})^{\mathrm{fact}} \to D(\mathrm{Gr}_T)^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR}).$$

The unital functor $\widetilde{i}^{\frac{\infty}{2},!}$ is strictly unital.

Proof. By construction of the factorization structure on $D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}$, it suffices to show that for every finite set I, the kernel of the functor

$$\mathfrak{p}^{\frac{\infty}{2},int}_{X^I,*,ren} : D^!(\mathrm{Gr}_{G,B,X^I}) \to D^!(\mathrm{Fl}^{\frac{\infty}{2}}_{X^I})$$

is annihilated by the functor $\mathfrak{q}^{loc}_{X^I,*,dR} \circ \widetilde{i}^!_{X^I}$. Here $i_{X^I} : \mathrm{Gr}_{B,X^I} \to \mathrm{Gr}_{G,B,X^I}$ is the obvious map.

Let $\mathcal{K}_\alpha$ be a subgroup scheme of $N(K)T(O)_{X^I}$ as in §2.23. It suffices to show that $\widetilde{i}^!_{X^I}$ maps the kernel of the functor (2.23.1) into the kernel of the pushforward functor $\mathfrak{q}^{loc}_{X^I,*,dR}$ for the map $\mathfrak{q}^{loc}_{X^I} : \mathrm{Gr}_{B,X^I} \to \mathrm{Gr}_{T,X^I}$.

As in *loc. cit.*, (2.23.1) may be realized as the renormalized pushforward along the placid morphism:

$$\mathrm{Gr}_{G,B,X^I} \to G(K)_{X^I}/\mathcal{K}_\alpha.$$

Therefore, the result follows by the base-change property of [Ras3] Proposition 6.18.1, as applied to the (Cartesian) square in the diagram:

$$
\begin{array}{ccc}
\mathrm{Gr}_{B,X^I} & \longrightarrow B(K)_{X^I}/\mathcal{K}_\alpha \longrightarrow & \mathrm{Gr}_{T,X^I} \\
\downarrow{\scriptstyle i_{X^I}} & \downarrow & \\
\mathrm{Gr}_{G,B,X^I} & \longrightarrow G(K)_{X^I}/\mathcal{K}_\alpha. &
\end{array}
$$

$\square$

Remark 2.25.2. As in Remark 2.20.1, the notation $\widetilde{i}^{\frac{\infty}{2}}$ refers to the would-be embedding:

$$\mathrm{Gr}_T = B(K)/N(K)T(O) \hookrightarrow \mathrm{Fl}^{\frac{\infty}{2}}.$$

2.26. **Semi-infinite restriction functor (redux).** We will actually use a variant of the above functor, using a small correction by some cohomological shifts.

Note that Gr_{T,X^I} has a canonical degree map $\deg : \mathrm{Gr}_{T,X^I} \to \check{\Lambda}$ where $\check{\Lambda}$ is considered as a discrete indscheme (i.e., an infinite union of points). Indeed, thinking of Gr_{T,X^I} as the data of a point $(x_i)_{i \in I}$ and a $\check{\Lambda}$-valued divisor on X supported on (the formal completion of) the union of the points (x_i), we take the degree of the corresponding divisor.

This map is compatible with factorization in the sense that for $I = I_1 \coprod I_2$, the restriction of deg to:

$$\mathrm{Gr}_{T,X^I} \underset{X^I}{\times} [X^{I_1} \times X^{I_2}]_{disj} \simeq \left(\mathrm{Gr}_{T,X^{I_1}} \times \mathrm{Gr}_{T,X^{I_2}} \right) \underset{X^I}{\times} [X^{I_1} \times X^{I_2}]_{disj}$$

coincides with the restriction of the map:

$$\mathrm{Gr}_{T,X^{I_1}} \times \mathrm{Gr}_{T,X^{I_2}} \xrightarrow{\deg \times \deg} \check{\Lambda} \times \check{\Lambda} \xrightarrow{\mathrm{sum}} \check{\Lambda}.$$

It now follows that $D(\mathrm{Gr}_T)^{\mathrm{fact}}$ carries a factorizable automorphism, which on each Gr_{T,X^I} is the functor:

$$\mathcal{F} \mapsto \mathcal{F}[(-2\rho, \deg)].$$

I.e., on the preimage $\deg^{-1}(\check{\lambda})$, we take a cohomological shift by $(-2\rho, \check{\lambda})$.

Finally, we define functors:

$$i^{\frac{\infty}{2},!}_{X^I} : D(\mathrm{Fl}^{\frac{\infty}{2}}_{X^I}) \to D(\mathrm{Gr}_T)$$
$$i^{\frac{\infty}{2},!}_{X^I}(\mathcal{F}) := \widetilde{i}^{\frac{\infty}{2},!}_{X^I}[(-2\rho, \deg)]$$

which together define a factorizable functor.

We have similar factorizable functor $i^! : D(\mathrm{Gr}_{G,B}^{int})^{\mathrm{fact}} \to D(\mathrm{Gr}_B)^{\mathrm{fact}}$, where we apply similar cohomological shifts on connected components of the Gr_{B,X^I} in the exact same way at for Gr_{T,X^I}.

We have the following variant of Proposition 2.25.1.

Proposition 2.26.1. *There is a unique unital chiral functor:*

$$i^{\frac{\infty}{2},!,\mathrm{fact}} : D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}} \to D(\mathrm{Gr}_T)^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR}). \tag{2.26.1}$$

with an isomorphism:

$$i^{\frac{\infty}{2},!} \circ \mathfrak{p}_{*,ren}^{\frac{\infty}{2},int} \simeq \mathfrak{q}_{*,dR}^{loc} \circ i^! : D^!(\mathrm{Gr}_{G,B})^{\mathrm{fact}} \to D(\mathrm{Gr}_T)^{\mathrm{fact}} \in \mathsf{FactCat}_{un}(X_{dR}).$$

The unital functor $i^{\frac{\infty}{2},!}$ is strictly unital.

2.27. **Whittaker conditions.** The remainder of this section is devoted to imposing the Whittaker condition on $D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}$, and especially to establishing its structure as a unital factorization category.

2.28. **Whittaker character.** Observe that we have a canonical homomorphism:

$$N^-(K)_{X^I} \to (N^-/[N^-,N^-])(K)_{X^I} = \prod_{i \in \mathcal{I}_G} \mathrm{Tot.\,Sp.}(\Omega_X^1)(K)_{X^I} \overset{\prod_{i \in \mathcal{I}_G} \mathrm{Res}}{\longrightarrow} \prod_{i \in \mathcal{I}_G} \mathbb{G}_a \overset{\mathrm{sum}}{\to} \mathbb{G}_a$$

where $\mathrm{Tot.\,Sp.}(\Omega_X^1)$ indicates the total space of the bundle Ω_X^1, $\mathrm{Tot.\,Sp.}(\Omega_X^1)(K)_{X^I}$ denotes the corresponding meromorphic jet space, and Res denotes the residue map.

We then let $\psi_{X^I} \in D^!(N^-(K)_{X^I})$ denote the induced character D-module on $N^-(K)_{X^I}$ given by !-pulling back the character D-module $\psi \in D(\mathbb{G}_a)$. Note that ψ_{X^I} canonically descends to an object:

$$\widetilde{\psi}_{X^I} \in D(\mathrm{Gr}_{N^-,X^I}).$$

Let $D(X^I)^\psi$ denote the category $D(X^I)$ considered as a category acted on by $N^-(K)_{X^I}$) via the character D-module ψ^{loc}. Let $D(X^I)^{-\psi}$ denote the same, but with the character D-module ψ_{X^I} replaced by its pullback under the inversion map on $N^-(K)_{X^I}$.

2.29. For any category C acted on by $N^-(K)_{X^I}$, we let $\mathsf{Whit}_{X^I}(\mathsf{C}) = \mathsf{Whit}(\mathsf{C})$ denote the (!–) *Whittaker category*:

$$(\mathsf{C} \underset{D(X^I)}{\otimes} D(X^I)^{-\psi})^{N^-(K)_{X^I}}.$$

By unipotence, the functor:

$$\mathsf{Whit}(\mathsf{C}) \to \mathsf{C}$$

is locally fully-faithful.

Example 2.29.1. We have $\widetilde{\psi}_{X^I} \in \mathsf{Whit}(\mathrm{Gr}_{N^-,X^I})$. In fact, the functor $D(X^I) \to \mathsf{Whit}(\mathrm{Gr}_{N^-,X^I})$ given by tensoring with $\widetilde{\psi}_{X^I}$ is an equivalence.

Remark 2.29.2. The category constructed above is sometimes called the !-Whittaker category. It plays the role of Whittaker invariants. There is a dual construction of Whittaker coinvariants sometimes called the *-Whittaker category.

For further discussion of these points, see [Gai3] and [Ber].

2.30.   For each finite set I, define $\mathsf{Whit}^{abs}_{X^I}$ the *absolute Whittaker category over* X^I as $\mathsf{Whit}_{X^I}\big(D^!(G(K)_{X^I})\big)$. Varying I, we obtain a factorization category:

$$I \mapsto \mathsf{Whit}^{abs}_{X^I} := \mathsf{Whit}_{X^I}\big(D^!(G(K)_{X^I})\big)$$

Similarly, we obtain the chiral categories:

$$I \mapsto \mathsf{Whit}^{sph}_{X^I} := \mathsf{Whit}_{X^I}\big(D^!(\mathrm{Gr}_{G,X^I})\big)$$
$$I \mapsto \mathsf{Whit}^{int}_{X^I} := \mathsf{Whit}_{X^I}\big(D^!(\mathrm{Gr}_{G,B,X^I})\big).$$

2.31. **Unital structures on Whittaker categories.** We now describe the construction of unital factorization category structures on Whittaker categories.

Our key technical tool for this is the following lemma.

Lemma 2.31.1. *Let Z be one of the factorization spaces $G(K)$, Gr_G, or $\mathrm{Gr}_{G,B}$. Then for each pair I, J of finite sets, we have:*

(1) *The unit functor:*

$$D(X^I) \otimes D^!(Z_{X^J}) \to D^!(Z_{X^{I \sqcup J}})$$

admits a $D(X^I) \otimes D(X^J)$-linear right adjoint.
(2) *This right adjoint:*

$$D^!(Z_{X^{I \sqcup J}}) \to D(X^I) \otimes D^!(Z_{X^J})$$

preserves the Whittaker subcategories.
(3) *The induced functor:*

$$\mathsf{Whit}(D^!(Z_{X^{I \sqcup J}})) \to D(X^I) \otimes \mathsf{Whit}(D^!(Z_{X^J}))$$

admits a $D(X^I) \otimes D(X^J)$-linear left adjoint.

We will prove (1) and (2) in §2.32-2.33. The proof of (3) requires the introduction of some new ideas that are orthogonal to our current purposes, so we will delay this part of the argument to §3.

Corollary 2.31.2. *The factorization category $\mathsf{Whit}^{abs,\mathrm{fact}}$ admits a unique structure of unital factorization category such that $\mathsf{Whit}^{abs,\mathrm{fact}} \to D^!(G(K))^{\mathrm{fact}}$ upgrades to a unital chiral functor.*

For I and J two finite sets, the corresponding unit functor:

$$D(X^I) \otimes \mathsf{Whit}^{abs}_{X^J} \to \mathsf{Whit}^{abs}_{X^{I \sqcup J}}$$

is the left adjoint of Lemma 2.31.1 (3).

The same results hold with $G(K)$ replaced by Gr_G (resp. $\mathrm{Gr}_{G,B}$) and $\mathsf{Whit}^{abs,\mathrm{fact}}$ replaced by $\mathsf{Whit}^{sph,\mathrm{fact}}$ (resp. Whit^{int}).

Remark 2.31.3. We emphasize that in Corollary 2.31.2, e.g. the inclusion functor $\mathsf{Whit}^{abs,\mathrm{fact}} \to D^!(G(K))^{\mathrm{fact}}$ is lax unital, not strictly unital.

Proof that Lemma 2.31.1 implies Corollary 2.31.2. Lemma 2.31.1 exactly implies that the hypotheses of [Ras1] Proposition 6.4.2 are satisfied, and therefore *loc. cit.* implies the result.

$\square$

2.32. Let $\mathcal{G}$ be as in §2.11 and fix finite sets I and J.

We claim that the corresponding unit map:

$$D(X^I) \otimes D^!(\mathcal{G}(K)_{X^J}) \to D^!(\mathcal{G}(K)_{X^{I \amalg J}})$$

admits a continuous right adjoint, and we claim that this functor is a morphism of $D(X^I \times X^J)$-module categories.

Indeed, form the correspondence, using Notation 2.10.5:

$$
\begin{array}{ccc}
 & \mathcal{H}_{\mathcal{G},I,J} & \\
{\scriptstyle \alpha = \alpha_{\mathcal{G}}} \swarrow & & \searrow {\scriptstyle \beta_{\mathcal{G}} = \beta} \\
X^I \times \mathcal{G}(K)_{X^J} & & \mathcal{G}(K)_{X^{I \amalg J}}
\end{array}
\tag{2.32.1}
$$

with $f : I \hookrightarrow I \amalg J$ the tautological embedding. Then the unit map is computed as $\beta_{*,!-dR} \circ \alpha^!$.

Note that $\mathcal{H}_{\mathcal{G},I,J}$ is placid because $\mathcal{H}_{\mathcal{G},I,J} \to \mathcal{G}(K)_{X^{I \amalg J}}$ is a finitely presented closed embedding. We record for future use the observation that $\mathcal{H}_{\mathcal{G},I,J}$ therefore inherits a dimension theory from [Ras3] §6.13.

We immediately see from [Ras3] §6.5 that $\beta_{*,!-dR}$ has right adjoint $\beta^!$.

Lemma 2.32.1. *The map:*

$$\alpha : \mathcal{H}_{\mathcal{G},I,J} \to X^I \times \mathcal{G}(K)_{X^J}$$

is a placid morphism.[17]

Proof. We will prove this by an explicit construction.

Let $n, m \geq -1$ be two fixed integers. Define the indscheme $\mathcal{H}_{\mathcal{G},I,J}^{n,m}$ parametrizing:

$$
\left\{
\begin{array}{c}
x_I = (x_i)_{i \in I} \in X^I,\ x_J = (x_j)_{j \in J} \in X^J,\ \mathcal{P}_{\mathcal{G}} \text{ a } \mathcal{G}\text{-bundle on } X, \\
\tau \text{ a trivialization of } \mathcal{P}_{\mathcal{G}}|_{X \setminus \{x_j\}_{j \in J}}, \\
\sigma \text{ a trivialization of } \mathcal{P}_{\mathcal{G}} \text{ on } \Gamma_{x_I}^{(n)} \cup \Gamma_{x_J}^{(m)}.
\end{array}
\right\}
$$

Here, we use the natural convention that $\Gamma_x^{(-1)} = \varnothing$ for any $x : S \to X^K$. We emphasize that the symbol $\cup$ here indicates sum of effective divisors.

As in Lemma 2.5.1, as n and m vary, we obtain a projective system under maps that are affine smooth covers. Since for $n = m = -1$, we obtain $X^I \times \mathrm{Gr}_{\mathcal{G},X^J}$, we see that the $\mathcal{H}_{\mathcal{G},I,J}^{n,m}$ actually are indschemes.

By Lemma 2.12.1, we have:

$$\lim_{n,m} \mathcal{H}_{\mathcal{G},I,J}^{n,m} = \mathcal{H}_{\mathcal{G},I,J}$$

$$\lim_m \mathcal{H}_{\mathcal{G},I,J}^{-1,m} = X^I \times \mathcal{G}(K)_{X^J}.$$

Therefore, taking for $\mathcal{I}$ the filtered category $\mathbb{Z}^{\geq -1} \times \mathbb{Z}^{\geq -1}$ (with $\mathbb{Z}^{\geq -1}$ considered as a category by its ordering), we see that the map α can be written as obtained from the compatible affine smooth covering maps:

---

[17]This subsection requires the most subtle use of the notion of placid morphism, so we recall that the notion of *placid morphism* is introduced in *loc. cit.* §4.10 and §6.17, and is something like a pro-smooth morphism. The key point is [Ras3] Proposition 6.18.1, which roughly says that placid morphisms behave like smooth morphisms in this setting, and the implicit dimension shifts in the infinite-dimensional D-module theory make $\alpha^!$ behave like $\alpha^{*,dR}$.

$$\lim_{n,m} \mathcal{H}_{\mathcal{G},I,J}^{n,m} \to \lim_{m} \mathcal{H}_{\mathcal{G},I,J}^{-1,m}$$

giving the result.

$\square$

One easily shows that the dimension theories on $\mathcal{H}_{\mathcal{G},I,J}$ coming from α and β respectively coincide. Therefore, by [Ras3] Proposition 6.18.1, $\alpha^!$ admits the right adjoint $\alpha_{*,ren}$.

We record the following feature of $\alpha_{*,ren}$ for future use.

Lemma 2.32.2. *Suppose that $\mathcal{G}$ is a form of a unipotent algebraic group. Then the functor $\alpha^!$ is fully-faithful, i.e., the counit for the adjunction $(\alpha^!, \alpha_{*,ren})$ is an equivalence.*

Proof. We use the same notation as in Lemma 2.32.1.

Unipotence implies that the pullback functors for each of the maps:

$$\mathcal{H}_{\mathcal{G},I,J}^{n,m} \to \mathcal{H}_{\mathcal{G},I,J}^{n',m'}$$

are fully-faithful, since the fibers are fibrations with affine space fibers.

The argument easily follows from here — we form the commutative square:

$$
\begin{array}{ccc}
\mathcal{H}_{\mathcal{G}} & \xrightarrow{\ \alpha\ } & X^I \times \mathcal{G}(K)_{X^J} \\
\downarrow & & \downarrow \\
\mathcal{H}_{\mathcal{G},I,J}^{n,m} & \longrightarrow & \mathcal{H}_{\mathcal{G},I,J}^{-1,m}.
\end{array}
$$

and note that, by definition, it suffices to check that the counit is an equivalence after pushing forward to $\mathcal{H}_{\mathcal{G},I,J}^{-1,m}$ for every m. Moreover, we can check this after applying the counit to objects pulled back from $\mathcal{H}_{\mathcal{G},I,J}^{-1,m}$ (by smoothness of these structure maps). From here the claim is obvious.
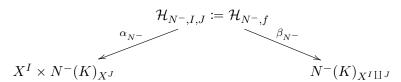
$\square$

Variant 2.32.3. We use the notation of (2.24.1) for the unit correspondence for Gr_{G,B,X^J}. Note that in general we have:

$$\mathcal{H}_{G,B,I,J} = \mathcal{H}_{G,I,J}/B(O)_{X^{I \sqcup J}}.$$

As above, the unit functor $\beta_{G,B,*,!-dR} \circ \alpha_{G,B}^!$ admits the right adjoint $\alpha_{G,B,*,ren} \circ \beta_{G,B}^!$.

We also note that the corresponding statement for Gr_G is true and vacuous.

2.33.   In the setting of §2.32 with $\mathcal{G}$ our twisted form of G, we claim that the functor $\alpha_{G,*,ren}\beta_G^!$ preserves the corresponding Whittaker equivariant subcategories on each side.

In the diagram:

$$
\begin{array}{ccc}
 & \mathcal{H}_{N^-,I,J} := \mathcal{H}_{N^-,f} & \\
{}_{\alpha_{N^-}}\swarrow & & \searrow{}^{\beta_{N^-}} \\
X^I \times N^-(K)_{X^J} & & N^-(K)_{X^{I \sqcup J}}
\end{array}
$$

the two corresponding character D-modules on $\mathcal{H}_{N^-,I,J}$ obtained by pullback from α or β obviously coincide.

Therefore, we can make sense of the Whittaker category of $D^!(\mathcal{H}_{G,I,J})$. Moreover, $\beta^!_G$ obviously preserve Whittaker categories. Therefore, it suffices to show that $\alpha_{*,ren}$ preserves these Whittaker equivariant categories.

We begin by showing that $\alpha_{G,*,ren}$ maps the $N^-(O)_{X^I\sqcup J}$-equivariant category of $D^!(\mathcal{H}_{N^-,I,J})$ to the $N^-(O)_{X^J}$-equivariant (i.e., $X^I \times N^-(O)_{X^J}$-equivariant) category of $D^!(X^I \times G(K)_{X^J})$.

We have the diagram:

$$
\begin{array}{ccc}
N^-(O)_{X^{I\sqcup J}} \underset{X^I\times X^J}{\times} \mathcal{H}_{G,I,J} & \xrightarrow{\quad\text{act}\quad} & \mathcal{H}_{G,I,J} \\
\downarrow{\alpha'_G} & & \downarrow{\alpha_G} \\
X^I \times N^-(O)_{X^J} \underset{X^I\times X^J}{\times} X^I \times G(K)_{X^J} & \xrightarrow{\quad\text{act}\quad} & X^I \times G(K)_{X^J}.
\end{array}
\qquad (2.33.1)
$$

Noting that the horizontal maps are placid, we claim:

Lemma 2.33.1. *The base-change map:*

$$
\text{act}^! \, \alpha_{G,*,ren} \to \alpha'_{G,*,ren} \, \text{act}^!
$$

is an equivalence.

Proof. The diagram (2.33.1) is isomorphic in the usual way to:

$$
\begin{array}{ccc}
N^-(O)_{X^{I\sqcup J}} \underset{X^I\times X^J}{\times} \mathcal{H}_{G,I,J} & \xrightarrow{\quad p_2\quad} & \mathcal{H}_{G,I,J} \\
\downarrow{\alpha'_G} & & \downarrow{\alpha_G} \\
X^I \times N^-(O)_{X^J} \underset{X^I\times X^J}{\times} X^I \times G(K)_{X^J} & \xrightarrow{\quad p_2\quad} & X^I \times G(K)_{X^J}.
\end{array}
$$

Therefore, it suffices to see that the base-change map is an isomorphism for this diagram.

We enlarge this diagram to:

$$
\begin{array}{ccccc}
N^-(O)_{X^{I\sqcup J}} \underset{X^I\times X^J}{\times} \mathcal{H}_{G,I,J} & \xrightarrow{\Delta} & N^-(O)_{X^{I\sqcup J}} \times \mathcal{H}_{G,I,J} & \xrightarrow{p_2} & \mathcal{H}_{G,I,J} \\
\downarrow{\alpha'} & & \downarrow{\alpha_{N^-}\times\alpha_G} & & \downarrow{\alpha_G} \\
X^I \times N^-(O)_{X^J} \underset{X^I\times X^J}{\times} X^I \times G(K)_{X^J} & \xrightarrow{\Delta} & X^I \times N^-(O)_{X^J} \times X^I \times G(K)_{X^J} & \xrightarrow{p_2} & X^I \times G(K)_{X^J}.
\end{array}
$$

where we have abused notation in several ways, not least of all that α_{N^-} denotes the restriction of α_{N^-} to $N^-(O)_{X^{I\sqcup J}}$. It suffices to show the base-change property for each of these squares separately.

For the left square above, note that this square is Cartesian, and that the maps Δ are finitely presented because $X^I \times X^J$ is finite type. Therefore, [Ras3] Proposition 6.18.1 implies the base-change property.

For the right square, the result follows immediately from Lemma 2.32.2.

□

From the lemma and Lemma 2.32.2, it is obvious that $\alpha_{G,*,ren}$ maps the $N^-(O)_{X^I \sqcup J}$-equivariant category of $D^!(\mathcal{H}_{N^-,I,J})$ to the $N^-(O)_{X^J}$-equivariant (i.e., $X^I \times N^-(O)_{X^J}$-equivariant) category of $D^!(X^I \times G(K)_{X^J})$.

The same argument as above applies verbatim to larger congruence subgroups with (or just as well, without) the twist by the Whittaker character (which restricts to $N^-(O)_{X^I \sqcup J}$ as the trivial character). Exhausting $N^-(K)_{X^I \sqcup J}$ by these compact open subgroups, we obtain the result.

Variant 2.33.2. As in Variant 2.32.3, the right adjoints to the unit functors for $\mathrm{Gr}_{G,B}$ and Gr_G also preserve the Whittaker subcategories.

2.34.   As was mentioned in §2.31, we now postpone the proof of the third condition from *loc. cit.* to §3, assuming it (and therefore Corollary 2.31.2) for the remainder of this section.

2.35.   Let I be a finite set. Define $\mathsf{Whit}^{\frac{\infty}{2}}_{X^I} \in \mathsf{ShvCat}_{/X^I_{dR}}$ as the $N(K)T(O)_{X^I}$-coinvariants of $\mathsf{Whit}^{abs}_{X^I}$. Varying I, we obtain a factorization category $\mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}} \in \mathsf{FactCat}(X_{dR})$.[18]

The lemmas of §2.24 apply verbatim, and therefore $\mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}}$ inherits a unital factorization category structure. The tautological functor:

$$\mathfrak{p}^{\frac{\infty}{2},int}_{*,ren} : \mathsf{Whit}^{int} \to \mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}}$$

is again strictly unital.

Moreover, we have an obvious lax unital chiral functor:

$$\mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}} \to D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}. \tag{2.35.1}$$

2.36.   The results of this section may be summarized as follows:

We have a diagram:

$$\mathrm{Gr}_G \longleftarrow G(K) \longrightarrow \mathrm{Gr}^{int}_{G,B} \longrightarrow \mathrm{Fl}^{\frac{\infty}{2}}$$

where subscripts have been removed and the right map is a fiction in the style of Remark 2.20.1. This induces a diagram:

$$
\begin{array}{ccccccc}
\mathsf{Whit}^{sph,\mathrm{fact}} & \longrightarrow & \mathsf{Whit}^{abs,\mathrm{fact}} & \longrightarrow & \mathsf{Whit}^{int} & \longrightarrow & \mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D(\mathrm{Gr}_G) & \longrightarrow & D^!(G(K))^{\mathrm{fact}} & \longrightarrow & D^!(\mathrm{Gr}^{int}_{G,B}) & \longrightarrow & D^!(\mathrm{Fl}^{\frac{\infty}{2}})^{\mathrm{fact}}
\end{array}
$$

of unital chiral categories. Here all functors are (lax) unital chiral functors defined appropriately as !-pullback or renormalized pushforward, and the the two horizontal lines consist of strictly unital chiral functors.

3. Fusion with the Whittaker sheaf (a technical point)

3.1.   This purpose of this section is to the complete the proof of Lemma 2.31.1 by proving (3) of *loc. cit.* The proof of the proposition is given by combining a *fusion* construction with some well-known facts about Drinfeld's compactification of Gr_{N^-}.

---

[18]It is natural to ask if formation of these coinvariants commute with the formation of the Whittaker invariants. In fact, this is the case: it follows from Theorem 2.1.1 of [Ras5], or rather, its (straightforward) generalization to the factorization setting: this result identifies Whittaker invariants and coinvariants canonically, and coinvariants commute with coinvariants. (For $G = GL_n$, one can use [Ber], or again, its factorizable generalization, instead.)

3.2.   Before proceeding, we begin with a somewhat informal description of the method in the case when I and J are singleton sets, and say for definiteness that $Z = G(K)$. We will use e.g. the notation:

$$G(K) \times G(K) \rightsquigarrow G(K)$$

for the space $G(K)_{X^2}$, where this should be read as describing a factorization space that is $G(K_x) \times G(K_y)$ away from the diagonal specializing to $G(K_x)$ over the diagonal.

Suppose that $\mathcal{F} \in \mathsf{Whit}_X^{abs} := \mathsf{Whit}(G(K)_X)$. We are supposed to show e.g. that we can !-average the induced object:

$$\delta_{G(O)_X} \boxtimes \mathcal{F} \rightsquigarrow \mathcal{F}$$

with respect to the Whittaker character (here $\delta_{G(O)_X}$ is the δ D-module on meromorphic jets supported on regular jets).[19]

We construct a space:

$$\mathrm{Gr}_{N^-} \times G(K) \rightsquigarrow G(K)$$

encoding the action of $N^-(K)$ on $G(K)$. Moreover, we show that given $\mathcal{F} \in \mathsf{Whit}(G(K)_X)$, we can form an object:

$$\psi_X \boxtimes \mathcal{F} \rightsquigarrow \mathcal{F} \tag{3.2.1}$$

encoding the Whittaker equivariance of $\mathcal{F}$. These constructions we refer to as *fusion*.

We moreover have a space:

$$\mathrm{Gr}_G \times G(K) \rightsquigarrow G(K)$$

encoding the action of $G(K)$ on itself. Moreover, the $*$-extension of (3.2.1) to this locus coincides with the !-extension. Indeed, it suffices to see this over the closure of $\big((\mathrm{Gr}_{N^-} \times G(K) \rightsquigarrow G(K)\big)$, and here it follows from the usual considerations of the Whittaker character of $N^-(K)$.

We then show that the pullback to $\big(G(K) \times G(K) \rightsquigarrow G(K)\big)$ of this D-module computes the desired left adjoint.

3.3.   We begin by studying the semi-infinite orbits of Gr_G in the factorization setting. Fix a finite set I and $\check{\lambda} = (\check{\lambda}_i)$ a collection of coweights for G defined for each $i \in I$.

Observe that there is a canonical section:

$$X^I \to \mathrm{Gr}_{T, X^I}$$

associated to $\check{\lambda}$. Indeed, it suffices to define a relative Cartier divisor valued in $\check{\Lambda}$ on the relative curve $X \times X^I \to X^I$, and we take $\sum_i \check{\lambda}_i \cdot [x_i]$, where $x_{i_0} : X^I \to X \times X^I$ is the section defined by:

$$(x_i)_{i \in I} \mapsto \Big(x_{i_0}, (x_i)_{i \in I}\Big)$$

and $[x_i]$ is the associated effective Cartier divisor.

Note that every geometric point of Gr_{T, X^I} is in the image of one of these sections for appropriate choice of $\check{\lambda}$.

---

[19]We note that the required task appears completely obvious in the given notation, due to the holonomicity of $\delta_{G(O)_X}$. However, this ignores the important "interaction" occurring over the diagonal, preventing such a naive argument from going through.

3.4.   We define $\mathrm{Gr}_{B,X^I}^{\check{\lambda}}$ as the fiber product:

$$\mathrm{Gr}_{B,X^I}^{\check{\lambda}} := \mathrm{Gr}_{B,X^I} \underset{\mathrm{Gr}_{T,X^I}}{\times} X^I$$

where the map $X^I \to \mathrm{Gr}_{T,X^I}$ is the section defined by $\check{\lambda}$.

Example 3.4.1. Suppose that $I = \{1, 2\}$. Then the fiber of Gr_{B,X^2} over $(x, y) \in X^2$ is $\mathrm{Gr}_{B,x}^{\check{\lambda}_1} \times \mathrm{Gr}_{B,y}^{\check{\lambda}_2}$ for $x \neq y$, and is $\mathrm{Gr}_{B,x}^{\check{\lambda}_1 + \check{\lambda}_2}$ for $x = y$.

3.5.   We give a variant of $\mathrm{Gr}_B^{\check{\lambda}}$ with $\overline{\mathrm{Gr}}_B$ replacing Gr_B.

First, note that we can define $\overline{\mathrm{Gr}}_{B,X^I}$ to parametrize points $x = (x_i)_{i \in I}$ in X^I, a G-bundle on X with a Drinfeld reduction to B, and a trivialization of this data away from $\{x_i\}_{i \in I}$, incorporating twists by $\mathcal{P}_T^{can}$ in the obvious way.

Remark 3.5.1. One easily finds that $\mathrm{Gr}_{B,X^I} \to \overline{\mathrm{Gr}}_{B,X^I}$ is a Zariski open embedding (in particular, schematic).

It is easy to see that the morphism:

$$\overline{\mathrm{Gr}}_{B,X^I} \to \mathrm{Gr}_{G,X^I} \underset{X^I}{\times} \mathrm{Gr}_{T,X^I}$$

is an ind-closed embedding, and in particular, that $\overline{\mathrm{Gr}}_{B,X^I}$ is an ind-proper indscheme.

We then define $\overline{\mathrm{Gr}}_{B,X^I}^{\check{\lambda}}$ using the map $\overline{\mathrm{Gr}}_{B,X^I} \to \mathrm{Gr}_{T,X^I}$, as with $\mathrm{Gr}_{B,X^I}^{\check{\lambda}}$. Note that $\overline{\mathrm{Gr}}_{B,X^I}^{\check{\lambda}} \to \mathrm{Gr}_{G,X^I}$ is an ind-closed embedding.

In the special case $\check{\lambda} = 0$ (i.e., each $\check{\lambda}_i = 0$), we use the notation $\overline{\mathrm{Gr}}_{N,X^I}$ for $\overline{\mathrm{Gr}}_{B,X^I}^0$.

3.6.   We have similarly spaces $\mathrm{Gr}_{B^-,X^I}^{\check{\lambda}}$, $\overline{\mathrm{Gr}}_{B^-,X^I}^{\check{\lambda}}$, and $\overline{\mathrm{Gr}}_{N^-,X^I}$ defined again as fiber products with the section $X^I \to \mathrm{Gr}_{T,X^I}$ defined by $\check{\lambda}$, via the natural map e.g. $\mathrm{Gr}_{B^-,X^I} \to \mathrm{Gr}_{T,X^I}$.

Observe that $N^-(K)_{X^I}$ acts on $\mathrm{Gr}_{B^-,X^I}^{\check{\lambda}}$ and $\overline{\mathrm{Gr}}_{B^-,X^I}^{\check{\lambda}}$ for each $\check{\lambda}$.

By the usual conductor considerations, one finds:

$$\mathsf{Whit}(D(\mathrm{Gr}_{B^-,X^I}^{\check{\lambda}})) = 0$$

when $-\check{\lambda}$ is not a dominant coweight.

Let $\jmath_{N^-,X^I}$ denote the open embedding $\mathrm{Gr}_{N^-,X^I} \hookrightarrow \overline{\mathrm{Gr}}_{N^-,X^I}$. As in Example 2.29.1, we have:

$$\jmath_{N^-,X^I,*,dR}(\widetilde{\psi}_{X^I}) \in \mathsf{Whit}(D(\overline{\mathrm{Gr}}_{N^-,X^I}))$$

and the above remarks imply that the induced functor:

$$D(X^I) \to \mathsf{Whit}(D(\overline{\mathrm{Gr}}_{N^-,X^I})) \tag{3.6.1}$$

given by tensoring with this object is an equivalence.

Variant 3.6.1. The above considerations also apply to describe the Whittaker coinvariants of $D(\overline{\mathrm{Gr}}_{N^-,X^I})$. Here one finds that the functor:

$$D(\overline{\mathrm{Gr}}_{N^-,X^I}) \to D(X^I)$$

given by !-restriction to Gr_{N^-,X^I} followed by twisting by the character $\widetilde{\psi}_{X^I}$ and then applying de Rham pushforward to X^I is an equivalence after applying Whittaker coinvariants. Indeed, this again follows by analysis of strata.

3.7. From actions to fusion. Fix $\mathcal{G}$ over X a form of an affine algebraic group and I and J two finite sets. Suppose that Z is an indscheme over X^J with an action of $\mathcal{G}(K)_{X^J}$.

Under certain hypotheses, we will construct a new indscheme $\mathrm{Fus}_{I,J}^{\mathcal{G}}(Z)$ that lives over $X^{I \sqcup J}$, and that over the disjoint locus of the base is isomorphic to the restriction of $\mathrm{Gr}_{\mathcal{G},X^I} \times Z$. The construction is inspired by [Gai1].

Recall the space $\mathcal{H}_{\mathcal{G},I,J}$ from §2.10 (see Notation 2.10.5 in particular). We have a morphisms:

$$
\begin{array}{ccc}
 & \mathcal{H}_{\mathcal{G},I,J} & \\
 & \swarrow \qquad \searrow & \\
\mathcal{G}(K)_{X^{I \sqcup J}} & & X^I \times \mathcal{G}(K)_{X^J}
\end{array}
\qquad (3.7.1)
$$

between placid group indschemes over $X^{I \sqcup J}$. In particular, $\mathcal{H}_{\mathcal{G},I,J}$ acts on $X^I \times Z$, using the action of $\mathcal{G}(K)_{X^J}$ on Z and the right leg of (3.7.1). We consider $\mathcal{H}_{\mathcal{G},I,J}$ acting on the right on $\mathcal{G}(K)_{X^{I \sqcup J}}$ via the left leg of (3.7.1). We obtain the diagonal action of $\mathcal{H}_{\mathcal{G},I,J}$ on:

$$
\mathcal{G}(K)_{X^{I \sqcup J}} \underset{X^{I \sqcup J}}{\times} X^I \times Z. \qquad (3.7.2)
$$

Definition 3.7.1. We say that the action of $\mathcal{G}(K)_{X^J}$ on Z is *fusive* if the quotient of (3.7.2) by the action of $\mathcal{H}_{\mathcal{G},I,J}$ exists as an indscheme for each I.

When the action is fusive, we let $\mathrm{Fus}_{I,J}^{\mathcal{G}}(Z)$ denote the corresponding quotient; see Remark 3.7.5 for a description of what the resulting space looks like.

Note that there is a canonical action of $\mathcal{G}(K)_{X^{I \sqcup J}}$ on $\mathrm{Fus}_{I,J}^{\mathcal{G}}(Z)$ arising from the action of $\mathcal{G}(K)_{X^{I \sqcup J}}$ on (3.7.2) through its action of the left on the first factor of *loc. cit.*

Example 3.7.2. Suppose that $Z = \mathrm{Gr}_{\mathcal{G},X^J}$, equipped with the usual action. This action is fusive: one easily finds that the desired quotient is $\mathrm{Gr}_{\mathcal{G},X^{I \sqcup J}}$, where the structure map:

$$
\mathcal{G}(K)_{X^{I \sqcup J}} \underset{X^{I \sqcup J}}{\times} (X^I \times \mathrm{Gr}_{\mathcal{G},X^J}) \to \mathrm{Gr}_{\mathcal{G},X^{I \sqcup J}}
$$

is defined by the action of $\mathcal{G}(K)_{X^{I \sqcup J}}$ on $\mathrm{Gr}_{\mathcal{G},X^{I \sqcup J}}$ and the unit map $X^I \coprod \mathrm{Gr}_{\mathcal{G},X^J} \to \mathrm{Gr}_{\mathcal{G},X^{I \sqcup J}}$.

Counterexample 3.7.3. The trivial action of $\mathcal{G}$ (i.e., its action as a group scheme over X on X itself) is *not* fusive.

Example 3.7.4. Suppose that $Z = \mathcal{G}(K)_{X^J}$, equipped with the left action. This action is again fusive: in this case, the desired quotient $\mathrm{Fus}_{I,J}^{\mathcal{G}}(\mathcal{G}(K)_{X^J})$ is the moduli of points $\left((x_i)_{i \in I}, (x_j)_{j \in J}\right) \in X^{I \sqcup J}$, a $\mathcal{G}$-bundle $\mathcal{P}_{\mathcal{G}}$ on X trivialized away from the points $\left((x_i)_{i \in I}, (x_j)_{j \in J}\right)$, and with an additional trivialization on the formal neighborhood of the points $(x_j)_{j \in J}$. One shows that this moduli is a placid indscheme in the usual way, using the increasing infinitesimal neighborhoods of the points x_j.

We have an obvious map $X^I \times \mathcal{G}(K)_{X^J} \to \mathrm{Fus}_{I,J}^{\mathcal{G}}(\mathcal{G}(K)_{X^J})$, realizing the latter as the locus where the $\mathcal{G}$-bundle $\mathcal{P}_{\mathcal{G}}$ is instead trivialized on the complement to the points $(x_j)_{j \in J}$. There is also an obvious action of $\mathcal{G}(K)_{X^{I \sqcup J}}$ on $\mathrm{Fus}_{I,J}^{\mathcal{G}}(\mathcal{G}(K)_{X^J})$, essentially coming from the action of jets on the affine Grassmannian. Therefore, as in Example 3.7.2, we obtain the structure map:

$$\mathcal{G}(K)_{X^I \coprod J} \underset{X^I \coprod J}{\times} (X^I \times \mathcal{G}(K)_{X^J}) \to \mathrm{Fus}^{\mathcal{G}}_{I,J}(\mathcal{G}(K)_{X^J})$$

by combining these two observations.

Remark 3.7.5. It is instructive to analyze the space $\mathrm{Fus}^{\mathcal{G}}_{I,J}(Z)$ in the combinatorially simplest case, in which $I = J = *$. In this case, away from the diagonal of X^2, we have $\mathcal{H}_{\mathcal{G},I,J} \simeq \mathcal{G}(O)_X \times \mathcal{G}(K)_X$, while over the diagonal it is isomorphic to $\mathcal{G}(K)_X$. Therefore, we have:

$$\mathrm{Fus}^{\mathcal{G}}_{*,*}(Z)|_{X^2 \backslash \Delta} \simeq \mathcal{G}(K)_X \times \mathcal{G}(K)_X \overset{\mathcal{G}(O)_X \times \mathcal{G}(K)_X}{\underset{X^2}{\times}} Z|_{X^2 \backslash \Delta} \overset{p_1 \times \mathrm{act}}{\simeq} \mathrm{Gr}_{G,X} \times Z|_{X^2 \backslash \Delta}$$

$$\mathrm{Fus}^{\mathcal{G}}_{*,*}(Z)|_{\Delta} \simeq \mathcal{G}(K)_X \overset{\mathcal{G}(K)_X}{\underset{X}{\times}} Z \overset{\mathrm{act}}{\simeq} Z.$$

Here the superscript of a group over a Cartesian product indicates that we take the quotient by the appropriate diagonal action.

3.8. **Fusion of sheaves.** Suppose in the setting of §3.7 that $\mathcal{G}(K)_{X^J}$ acts fusively on $Z \to X^J$. Suppose moreover that $\mathcal{F}$ is a $\mathcal{G}(K)_{X^J}$-equivariant D-module on Z, i.e., $\mathcal{F}$ is an object of the equivariant category:

$$D^!(Z)^{\mathcal{G}(K)_{X^J}}.$$

We obtain a new D-module:

$$\mathrm{Fus}^{\mathcal{G}}_{I,J}(\mathcal{F}) \in D^!(\mathrm{Fus}^{\mathcal{G}}_{I,J}(Z))^{\mathcal{G}(K)_{X^I \coprod J}} \tag{3.8.1}$$

by the following construction:

Note that:

$$\omega_{X^I} \boxtimes \mathcal{F} \in D^!(X^I \times Z) \tag{3.8.2}$$

is $X^I \times \mathcal{G}(K)_{X^J}$-equivariant (i.e., equipped with an equivariant structure), and therefore equivariant for $\mathcal{H}_{\mathcal{G},I,J}$ acting through the right leg of (3.7.1). Pulling back (3.8.2) along the map:

$$\mathcal{G}(K)_{X^I \coprod J} \underset{X^I \coprod J}{\times} (X^I \times Z) \to X^I \times Z$$

we obtain a D-module equivariant for the diagonal action of $\mathcal{H}_{\mathcal{G},I,J}$ considered in §3.7, and for the left action of $\mathcal{G}(K)_{X^I \coprod J}$ on the first factor of this space.

Descending to $\mathrm{Fus}^{\mathcal{G}}_{I,J}(Z)$ via the first of these equivariance observations, and appealing to the second, we obtain (3.8.1) as desired.

Example 3.8.1. In the setting of Remark 3.7.5, the D-module $\mathrm{Fus}^{\mathcal{G}}_{I,J}(\mathcal{F})$ is isomorphic to $\omega_{\mathrm{Gr}_{\mathcal{G},X}} \boxtimes \mathcal{F}$ away from the diagonal, and isomorphic to $\mathcal{F}$ over the diagonal.

Variant 3.8.2. Given $\widetilde{\mathcal{F}} \in D(X^I) \otimes D^!(Z)^{\mathcal{G}(K)_{X^J}}$, we claim that we can generalize the above construction to produce:

$$\mathrm{Fus}^{\mathcal{G}}_{I,J}(\widetilde{\mathcal{F}}) \in D^!(\mathrm{Fus}^{\mathcal{G}}_{I,J}(Z))^{\mathcal{G}(K)_{X^I \coprod J}}.$$

in such a way in the case $\widetilde{\mathcal{F}} = \omega_{X^I} \boxtimes \mathcal{F}$, we recover our earlier construction of $\mathrm{Fus}^{\mathcal{G}}_{I,J}(\mathcal{F})$.

Indeed, we simply replace $\omega_{X^I} \boxtimes \mathcal{F}$ in (3.8.2) by $\widetilde{\mathcal{F}}$.

Observe that this new construction is $D(X^I) \otimes D(X^J)$-linear.

Remark 3.8.3. We can reformulate this construction in the following way. The map:

$$X^I \times Z \to \mathrm{Fus}^{\mathcal{G}}_{I,J}(Z)$$

induces a restriction functor:

$$D^!(\mathrm{Fus}^{\mathcal{G}}_{I,J}(Z))^{\mathcal{G}(K)_{X^{I \sqcup J}}} \to D^!(X^I \times Z)^{\mathcal{H}_{\mathcal{G},I,J}}$$

that is an equivalence (c.f. [Ras3] Proposition 6.7.1) with inverse Fus.

Remark 3.8.4. The above construction can be performed more generally on any sheaf of categories on X^J_{dR} acted on by $\mathcal{G}(K)_{X^J}$.

3.9. **Compactification.** Suppose now that $\mathcal{G}$ is our preferred form of our reductive group G and that $Z \to X^J$ is acted on fusively by G.

We have a canonical map:

$$\mathrm{Fus}^{N^-}_{I,J}(Z) \hookrightarrow \mathrm{Fus}^{G}_{I,J}(Z).$$

We will presently use Drinfeld's method to construct $\overline{\mathrm{Fus}}^{N^-}_{I,J}(Z)$, a "compactification" of this map.

Example 3.9.1. We begin by explicitly treating the case of $Z = \mathrm{Gr}_{G,X^J}$ from Example 3.7.2.

In this case, we define $\overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})$ as the moduli of $\big((x_i)_{i \in I}, (x_j)_{j \in J}\big) \in X^{I \sqcup J}$, a G-bundle $\mathcal{P}$ on X with a polar Drinfeld reduction to N^- (in the $\mathcal{P}^{can}_T$-twisted sense), the poles being at the points x_j, and a trivialization of this datum on $X \backslash \{x_i, x_j\}_{i \in I, j \in J}$. Here a *polar Drinfeld reduction* of the specified type means that we give a Drinfeld reduction defined on the complement to the union of the graphs of the points x_j.

Remark 3.9.2. As in Remark 3.7.5, it is instructive to see what happens when $I = J = *$. In this case, one easily finds:

$$\mathrm{Fus}^{N^-}_{*,*}(\mathrm{Gr}_{G,X})|_{X^2 \backslash \Delta} \simeq \mathrm{Gr}_{N^-,X} \times \mathrm{Gr}_{G,X}|_{X^2 \backslash \Delta}$$

$$\mathrm{Fus}^{N^-}_{*,*}(\mathrm{Gr}_{G,X})|_{\Delta} \simeq \mathrm{Gr}_{G,X}$$

It is easy to see that the tautological map $\overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}) \to \mathrm{Gr}_{G,X^{I \sqcup J}}$ is an ind-closed embedding, and the natural map:

$$\mathrm{Fus}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}) \to \overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})$$

is an ind-open embedding.

Remark 3.9.3. Recall from [FGV] that for X a proper curve, the moduli space of a point of $x = (x_j) \in X^J$ and G-bundle on X with a polar Drinfeld reduction to N^- defined away from the points x_j is an ind-algebraic stack $\overline{\mathrm{Bun}}^{pol}_{N^-,X^J}$ locally of finite type (proof: bound the order of the poles allowed). Then $\overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})$ may be computed as the fiber product:

$$\begin{array}{ccc}
\overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}) & \longrightarrow & \mathrm{Gr}_{G,X^{I \sqcup J}} \\
\downarrow & & \downarrow \\
X^I \times \mathrm{Bun}^{pol}_{N^-,X^J} & \longrightarrow & \mathrm{Bun}^{pol}_{N^-,X^{I \sqcup J}} \,.
\end{array}$$

Before giving $\overline{\mathrm{Fus}}^{N^-}$ in the general case, we need to observe the existence of a certain group action.

Construction 3.9.4. Recall from §2.13 that $\pi_{G,X^I \sqcup J}$ denotes the structure map $G(K)_{X^I \sqcup J} \to \mathrm{Gr}_{G,X^I \sqcup J}$. We will construct an action of $\mathcal{H}_{G,I,J}$ on $\pi^{-1}_{G,X^I \sqcup J}(\overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}))$ (the action is on the right, so to speak).

Indeed, we have:

$$\pi^{-1}_{G,X^I \sqcup J}(\overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})) = \left\{ \begin{array}{l} x = \big((x_i)_{i \in I}, (x_j)_{j \in J}\big) \in X^{I \sqcup J}, \\ \text{a } G\text{-bundle } \mathcal{P}_G \text{ on } X \text{ with a} \\ \quad \mathcal{P}^{can}_T\text{-twisted Drinfeld reduction to } N^- \text{ on } X \backslash \{x_j\}, \\ \text{a trivialization of this datum on } X \backslash \{x_i, x_j\}_{i \in I, j \in J}, \\ \text{and a trivialization of } \mathcal{P}_G \text{ on } \widehat{\mathcal{D}}_x. \end{array} \right\}$$

and Beauville-Laszlo allows us to rewrite this as:

$$\left\{ \begin{array}{l} x = \big((x_i)_{i \in I}, (x_j)_{j \in J}\big) \in X^{I \sqcup J}, \\ \text{a } \mathcal{P}^{can}_T\text{-twisted map } \delta : \mathcal{D}_x \backslash \big(\cup_{j \in J} \Gamma_{x_j} \big) \to \overline{G/N}, \\ \text{and a lift of } \delta|_{\overset{o}{\mathcal{D}_x}} \text{ to a map } \overset{o}{\mathcal{D}}_x \to G. \end{array} \right\}.$$

The action of:

$$\mathcal{H}_{G,I,J} = \{ x = \big((x_i)_{i \in I}, (x_j)_{j \in J}\big) \in X^{I \sqcup J}, \mathcal{D}_x \backslash \big(\cup_{j \in J} \Gamma_{x_j} \big) \to G \}$$

on this space is now clear: it arises from the G-equivariant map $G \to \overline{G/N}/T$.

Construction 3.9.5. We are now equipped to define $\overline{\mathrm{Fus}}^{N^-}_{I,J}(Z)$.

We take it to be the quotient of:

$$\pi^{-1}_{G,X^I \sqcup J}(\overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})) \underset{X^{I \sqcup J}}{\times} X^I \times Z. \tag{3.9.1}$$

by the diagonal action of $\mathcal{H}_{G,I,J}$. Note that $N^-(K)_{X^I \sqcup J}$ acts $\overline{\mathrm{Fus}}^{N^-}_{I,J}(Z)$ through its left action on $\pi^{-1}_{G,X^I \sqcup J}(\overline{\mathrm{Fus}}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J}))$.

Remark 3.9.6. The quotient of:

$$\pi^{-1}_{G,X^I \sqcup J}(\mathrm{Fus}^{N^-}_{I,J}(\mathrm{Gr}_{G,X^J})) \underset{X^{I \sqcup J}}{\times} X^I \times Z$$

by $\mathcal{H}_{G,I,J}$ is obviously isomorphic to the quotient of:

$$N^-(K)_{X^I \sqcup J} \underset{X^{I \sqcup J}}{\times} X^I \times Z$$

by $\mathcal{H}_{N^-,I,J}$.

Lemma 3.9.7. *The restriction functor:*

$$\mathsf{Whit}_{X^I \sqcup J}(\overline{\mathrm{Fus}}^{N^-}_{I,J}(Z)) \to \mathsf{Whit}_{X^I \sqcup J}(\mathrm{Fus}^{N^-}_{I,J}(Z)) \tag{3.9.2}$$

is an equivalence.

Proof. Note that the map:

$$\pi_{G,X^I \sqcup J}^{-1}(\mathrm{Fus}_{I,J}^{N^-}(\mathrm{Gr}_{G,X^J})) \underset{X^I \sqcup J}{\times} X^I \times Z \hookrightarrow \pi_{G,X^I \sqcup J}^{-1}(\overline{\mathrm{Fus}}_{I,J}^{N^-}(\mathrm{Gr}_{G,X^J})) \underset{X^I \sqcup J}{\times} X^I \times Z$$

is an open embedding of ind-finite type.

Therefore, the functor (3.9.2) admits a right adjoint in $\mathsf{ShvCat}_{/X_{dR}^{I \sqcup J}}$ given by $(*, dR)$-extension. It suffices to check that the unit of the adjunction is an equivalence, and we can check this after restriction using a covering of $X^I \times X^J$ as in the proof of Lemma 2.24.1. Now the result follows because (3.6.1) is an equivalence.

$\square$

3.10.   Suppose that Z is an indscheme over X^J acted on fusively by $G(K)_{X^J}$, and let $\widetilde{\mathfrak{F}}$ be an object of $D(X^I) \otimes \mathsf{Whit}(D^!(Z))$. Twisting and untwisting by the character ψ and applying Variant 3.8.2, we form $\widetilde{\mathrm{Fus}}_{I,J}^{N^-}(\widetilde{\mathfrak{F}}) \in \mathsf{Whit}_{X^I \sqcup J}(D^!(\mathrm{Fus}_{I,J}^{\mathcal{G}}(Z)))$. By Lemma 3.9.7, this object canonically lifts to an object:

$$\overline{\mathrm{Fus}}_{I,J}^{N^-}(\widetilde{\mathfrak{F}}) \in \mathsf{Whit}_{X^I \sqcup J}(\overline{\mathrm{Fus}}_{I,J}^{N^-}(Z)).$$

Moreover, the assignment $\widetilde{\mathfrak{F}} \mapsto \overline{\mathrm{Fus}}_{I,J}^{N^-}(\widetilde{\mathfrak{F}})$ is obviously $D(X^I) \otimes D(X^J)$-linear.

We claim that the functor:

$$\mathsf{Whit}_{X^I \sqcup J}(\overline{\mathrm{Fus}}_{I,J}^{N^-}(Z)) \to D(X^I) \otimes \mathsf{Whit}_{X^J}(Z) \tag{3.10.1}$$

induced by restriction along the map:

$$X^I \times Z \to \overline{\mathrm{Fus}}_{I,J}^{N^-}(Z)$$

is an equivalence, with inverse provided by $\overline{\mathrm{Fus}}_{I,J}^{N^-}$. Indeed, this follows by combining Remark 3.8.3 with Lemma 3.9.7, and the observation that the functor:

$$D^!(X^I \times Z)^{\mathcal{H}_{N^-,I,J},\psi} \to D^!(X^I) \otimes \mathsf{Whit}(D^!(Z))$$

is an equivalence, where the superscript ψ indicates that we take invariants twisted with respect to the character of $N^-(K)_{X^I \sqcup J}$. We note that the last observation is trivial: the functor is fully-faithful since both are subcategories of $D^!(X^I \times Z)$, and is then an equivalence since $\mathcal{H}_{N^-}$ acts on $X^I \times Z$ through $X^I \times N^-(K)_{X^J}$.

3.11.   We now obtain that the !-restriction functor:

$$\mathsf{Whit}(D^!(\mathrm{Fus}_{I,J}^{G}(Z))) \to D(X^I) \otimes \mathsf{Whit}_{X^J}(D^!(Z))$$

admits a left adjoint. Indeed, from the equivalence (3.10.1), we need to show that the functor:

$$\mathsf{Whit}(D^!(\mathrm{Fus}_{I,J}^{G}(Z))) \to \mathsf{Whit}(D^!(\overline{\mathrm{Fus}}_{I,J}^{N^-}(Z)))$$

admits a left adjoint. But the map $\overline{\mathrm{Fus}}_{I,J}^{N^-}(Z) \hookrightarrow \mathrm{Fus}_{I,J}^{G}(Z)$ is a finitely presented closed embedding, so the functor of !-dR *-pushforward provides the desired left adjoint.

3.12.   We now establish the third point of Lemma 2.31.1. First, we specialize to the case $Z = G(K)_{X^I}$.

Recall that e.g. $\mathsf{Whit}_{X^I}^{abs}$ denotes the category of Whittaker D-modules on $G(K)_{X^I}$.

We have the Cartesian diagram:

$$
\begin{array}{ccc}
\mathcal{H}_{G,I,J} & \xrightarrow{\ \beta_G\ } & G(K)_{X^{I \sqcup J}} \\
\ \downarrow{\alpha_G} & & \downarrow \\
X^I \times G(K)_{X^J} & \longrightarrow & \mathrm{Fus}_{I,J}^G(G(K)_{X^J}).
\end{array}
\tag{3.12.1}
$$

We are supposed to show that the functor:

$$
\alpha_{G,*,ren}\beta_G^! : \mathsf{Whit}_{X^{I \sqcup J}}^{abs} \to D(X^I) \otimes \mathsf{Whit}_{X^J}^{abs}
$$

admits a left adjoint.

As in Lemma 2.32.1, the right and left vertical maps in (3.12.1) are placid. Therefore, by [Ras3] Proposition 6.18.1 we may compute $\alpha_{G,*,ren}\beta_G^!$ by base-change. Then the existence of the left adjoint follows from placidity of the right vertical map, [Ras3] Proposition 6.18.1, and §3.11.

The other cases for Z work similarly, since in each case the corresponding indscheme over $X^{I \sqcup J}$ maps placidly to $\mathrm{Fus}_{I,J}^G(Z_{X^J})$.

4. Semi-infinite restriction and Zastava spaces

4.1.   In this section, we use the results of [Ras2] to relate the functor $i^{\frac{\infty}{2},!}$ with Langlands duality.

The main results of this section are Theorems 4.4.1 and 4.15.1. We remark from the onset that these results are essentially reformulations of Theorems 4.6.1 and 7.9.1 from [Ras2].

4.2.   Because $i^{\frac{\infty}{2},!}$ is a (lax) unital functor of unital factorization categories, the formalism of chiral categories [Ras1] provides $i^{\frac{\infty}{2},!}(\mathrm{unit}_{\mathsf{Whit}^{\frac{\infty}{2}}}) \in D(\mathrm{Gr}_T)^{\mathrm{fact}}$ with a canonical structure of unital factorization algebra.

The first goal for this section to to compute this unital factorization algebra in Langlands dual terms.

4.3. **Construction of $\Upsilon_{\check{\mathfrak{n}}}$.** By commutativity of T, $D(\mathrm{Gr}_T)^{\mathrm{fact}}$ is a *commutative* unital factorization category. In fact, it is canonically identified (in the obvious way) with the commutative unital factorization category $\mathsf{Rep}(\check{T})^{\mathrm{fact}}$. associated $\mathsf{Rep}(\check{T})$.

We can view $\check{\mathfrak{n}}$ as a Lie algebra object in the symmetric monoidal category $\mathsf{Rep}(\check{T})$, or equivalently, as a $\check{\Lambda}$-graded Lie algebra.

We obtain a Lie-$*$ algebra $\check{\mathfrak{n}}_X$ in $D(\mathrm{Gr}_T)^{\mathrm{fact}}$. Define $\Upsilon_{\check{\mathfrak{n}}} \in \mathsf{Alg}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_T)^{\mathrm{fact}})$ as its chiral enveloping algebra:

$$
\Upsilon_{\check{\mathfrak{n}}} := U^{ch}(\check{\mathfrak{n}}_X) \in \mathsf{Alg}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_T)^{\mathrm{fact}}).
$$

4.4.   The first main result of this section is the following.

Theorem 4.4.1. *There is a canonical isomorphism:*

$$
i^{\frac{\infty}{2},!}(\mathrm{unit}_{\mathsf{Whit}^{\frac{\infty}{2}}}) \simeq \Upsilon_{\check{\mathfrak{n}}} \in \mathsf{Alg}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_T)^{\mathrm{fact}}).
$$

The proof of this theorem is given in §4.11 below after some preliminary constructions.

Remark 4.4.2. The word *canonical* is here (as always, perhaps) a bit ambiguous. Ultimately, this isomorphism is characterized via (the proof of) Proposition-Construction 4.10.1, and the isomorphism of Theorem 4.6.1 from [Ras2], which is (readily) characterized uniquely.

4.5. **The most confusing part.** As mentioned, Theorem 4.4.1 is essentially a reformulation of a result from [Ras2]. However, *loc. cit.* only dealt with configuration spaces of divisors on curves, whereas now we are treating Ran space.

We presently compare these two settings in §4.5-4.10.

4.6.   Recall the space $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ from [Ras2] §2.11: by definition, it parametrizes $\check{\Lambda}^{pos}$-valued divisors on X. Note that $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ is a union of components $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}$ indexed by $\check{\lambda} \in \check{\Lambda}^{pos}$, with each of these components a product over $\mathcal{I}_G$ of appropriate symmetric powers of the curve.

For the Ran space version: for each finite set I, define

$$\mathrm{Div}_{\mathrm{eff},X^I}^{\check{\Lambda}^{pos}} \subseteq \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times X^I$$

as the incidence locus, i.e., the locus of points where the divisor is supported on the given I-tuple of points (in the sense that its restriction to the complement of these points is zero).

These spaces have obvious structure maps between them for surjections $I \twoheadrightarrow J$, and we obtain $\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}$ by passing to the colimit.

4.7. **Ranification.** Note that $\mathrm{Div}_{\mathrm{eff},X^I}^{\check{\Lambda}^{pos}}$ embeds as a closed subscheme of Gr_{T,X^I}.

Define the *Ranification* functor $\mathcal{L}_{X^I}^{\mathrm{Ran}} : D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}) \to D(\mathrm{Gr}_{T,X^I})$ as via pullback pushforward along the diagram:
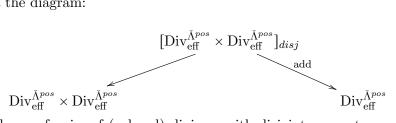
$$
\begin{array}{ccc}
\mathrm{Div}_{\mathrm{eff},X^I}^{\check{\Lambda}^{pos}} & \longrightarrow & \mathrm{Gr}_{T,X^I} \\
\downarrow & & \\
\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}. & &
\end{array}
$$

The functors $\mathcal{L}_{X^I}^{\mathrm{Ran}}$ are compatible under surjections $I \twoheadrightarrow J$, and therefore we obtain:

$$\mathcal{L}^{\mathrm{Ran}} : D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}) \to D(\mathrm{Gr}_{T,\mathrm{Ran}_X}).$$

Below, we will study the basic stability properties of the Ranification functor.

4.8.   Recall that there is a *chiral tensor product* $\overset{ch}{\otimes}$ on $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$. It is defined by !-pullback and ∗-pushforward via the diagram:

$$
\begin{array}{ccc}
 & [\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}]_{disj} & \\
 & \swarrow \qquad\qquad \searrow{\scriptstyle add} & \\
\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} & & \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}
\end{array}
$$

where this is the locus of pairs of (colored) divisors with disjoint supports mapping by inclusion and by addition of divisors.

We also have a ⋆-tensor product, which is just computed as $\mathcal{F} \overset{\star}{\otimes} \mathcal{G} := \mathrm{add}_{*,dR}(\mathcal{F} \boxtimes \mathcal{G})$.

This allows us to imitate the usual algebra of factorization algebras on these configuration spaces instead of on Ran space.

In particular, we can speak about factorization algebras, Lie-$*$ algebras, and chiral envelopes in $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$.

Remark 4.8.1. We emphasize the following point. Just as usual Lie-$*$ algebras are supported on $X \subseteq \mathrm{Ran}_X$, the analogue of Lie-$*$ algebras on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ is supported on the locus of divisors supported on a single point, i.e., the union:

$$\coprod_{0 \neq \check{\lambda} \in \check{\Lambda}^{pos}} X \hookrightarrow \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$$

corresponding to the maps $X \overset{x \mapsto \check{\lambda} \cdot x}{\hookrightarrow} \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$. We emphasize that $\check{\lambda} = 0$ is not allowed (since that would correspond to divisors supported at no points). In particular, in the degenerate case where $G = T$ there are no non-zero Lie-$*$ algebras on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ (since $\check{\Lambda}^{pos} = 0$ and $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} = \mathrm{Spec}(k)$).

4.9.   The basic compatibility between $\star$-tensor products is given by the following.

Lemma 4.9.1. *The functor $\mathcal{L}^{\mathrm{Ran}}$ is canonically lax symmetric monoidal with respect to the $\star$-tensor structures on the source and target.*

Proof. This follows from the pseudo-properness[20] of the canonical map:
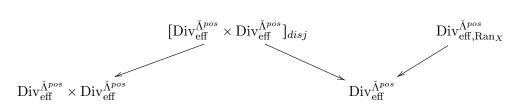
$$\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}} \to \left(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \right) \underset{\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}}{\times} \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\mathrm{Ran}_X}$$

where we use the addition map $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \to \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ to form the fiber product on the right. $\square$

Similarly, we have the following.

Lemma 4.9.2. *The functor $\mathcal{L}^{\mathrm{Ran}}$ is canonically colax symmetric monoidal with respect to the $\overset{ch}{\otimes}$-tensor structures.*

Proof. Up to pushforward to $\mathrm{Gr}_{T,\mathrm{Ran}_X}$, we compute $\mathcal{L}^{\mathrm{Ran}}(- \overset{ch}{\otimes} -)$ using the diagram:

$$[\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}]_{disj} \qquad\qquad\qquad \mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}}$$

$$\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \qquad\qquad\qquad \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$$

Similarly, we compute $\mathcal{L}^{\mathrm{Ran}}(-) \overset{ch}{\otimes} \mathcal{L}^{\mathrm{Ran}}(-)$ using the diagram:

$$[\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}}]_{disj}$$

$$\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \qquad\qquad\qquad\qquad \mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}}$$

Now the result follows from the fact that the canonical map:

---

[20]Recall that this condition ensures the existence of a left adjoint to the !-pullback functor.

$$[\mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}}]_{disj} \to [\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}]_{disj} \underset{\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}}{\times} \mathrm{Div}_{\mathrm{eff},\mathrm{Ran}_X}^{\check{\Lambda}^{pos}}$$

is étale (in particular, schematic) and quasi-compact: c.f. [Ras1] Lemma 6.18.1.

<div align="right">□</div>

Finally, we have the following compatibility between Lemmas 4.9.1 and 4.9.2. Recall that (in either the divisor or the Ran setting) there are maps:

$$- \overset{\star}{\otimes} - \to - \overset{ch}{\otimes} - \tag{4.9.1}$$

plus higher versions encoding homotopy associativity and commutativity of these tensor products.

Lemma 4.9.3. *For every* $\mathcal{F}, \mathcal{G} \in D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$*, the diagram:*

$$
\begin{array}{ccc}
\mathcal{L}^{\mathrm{Ran}}(\mathcal{F}) \overset{\star}{\otimes} \mathcal{L}^{\mathrm{Ran}}(\mathcal{G}) & \xrightarrow{Lem.4.9.1} & \mathcal{L}^{\mathrm{Ran}}(\mathcal{F} \overset{\star}{\otimes} \mathcal{G}) \\
\downarrow{\scriptstyle(4.9.1)} & & \downarrow{\scriptstyle(4.9.1)} \\
\mathcal{L}^{\mathrm{Ran}}(\mathcal{F}) \overset{ch}{\otimes} \mathcal{L}^{\mathrm{Ran}}(\mathcal{G}) & \xleftarrow{Lem.4.9.2} & \mathcal{L}^{\mathrm{Ran}}(\mathcal{F} \overset{ch}{\otimes} \mathcal{G})
\end{array}
$$

commutes. More precisely, the higher version of this statement encoding compatibility with symmetric monoidal structures holds.

Proof. This is a direct verification from the constructions.

<div align="right">□</div>

4.10. **Comparison of chiral enveloping algebras.** Let $L \in D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$ be a Lie-$*$ algebra. By Lemma 4.9.1, $\mathcal{L}^{\mathrm{Ran}}(L)$ is Lie algebra object of $D(\mathrm{Gr}_{T,\mathrm{Ran}_X})$ with respect to its $\overset{\star}{\otimes}$-tensor structure.[21] It follows formally that for $\iota : \mathrm{Gr}_{T,X} \to \mathrm{Gr}_{T,\mathrm{Ran}_X}$ the canonical embedding, $\iota_{*,dR}\iota^!(\mathcal{L}^{\mathrm{Ran}}(L))$ is a Lie-$*$ algebra on $D(\mathrm{Gr}_{T,\mathrm{Ran}_X})$.

In what follows, we let $U^{ch,\mathrm{fact}}(L) \in D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$ denote the factorization algebra associated with the chiral enveloping algebra of the Lie-$*$ algebra L, and similarly on Ran space. Recall that $U^{ch,\mathrm{fact}}(L)$ is the homological Chevalley complex of L (with respect to the $\overset{\star}{\otimes}$-tensor structure).

Proposition-Construction 4.10.1. *There is a canonical isomorphism of factorization algebras between* $U^{ch,\mathrm{fact}}(\iota_{*,dR}\iota^!\mathcal{L}^{\mathrm{Ran}}(L))$ *and* $\mathcal{L}^{\mathrm{Ran}}(U^{ch,\mathrm{fact}}(L))$.

Proof. First, we construct a map comparing the two.

We have a canonical map:

$$U^{ch,\mathrm{fact}}(\mathcal{L}^{\mathrm{Ran}}(L)) \to \mathcal{L}^{\mathrm{Ran}}(U^{ch,\mathrm{fact}}(L)) \in D(\mathrm{Gr}_{T,\mathrm{Ran}_X}). \tag{4.10.1}$$

Indeed, this is evident from lax symmetric monoidality of $\mathcal{L}^{\mathrm{Ran}}$ since $U^{ch,\mathrm{fact}}$ is computed as a homological Chevalley complex, i.e., up to a cohomological shift it computes the abelianization of these Lie algebra objects.

Moreover, observe that both sides of (4.10.1) are cocommutative coalgebras with respect to $\overset{ch}{\otimes}$: for the left hand side this follows from the canonical map (4.9.1), while for the right hand side it follows from Lemma 4.9.2. By Lemma 4.9.3 (and by unwinding the construction of the

---

[21]We emphasize that this object is a *generalized* Lie-$*$ algebra: a true Lie-$*$ algebra is supported on X.

cocommutative coalgebra structure on the homological Chevalley complex), the map (4.10.1) is a morphism of such cocommutative coalgebras.

Now using the canonical morphism $\iota_{*,dR}\iota^!\mathcal{L}^{\mathrm{Ran}}(L) \to \mathcal{L}^{\mathrm{Ran}}(L)$ of Lie algebra objects of $(D(\mathrm{Gr}_{T,\mathrm{Ran}_X}), \overset{\star}{\otimes})$, we obtain the desired morphism:

$$U^{ch,\mathrm{fact}}(\iota_{*,dR}\iota^!\mathcal{L}^{\mathrm{Ran}}(L)) \to \mathcal{L}^{\mathrm{Ran}}(U^{ch,\mathrm{fact}}(L)) \in D(\mathrm{Gr}_{T,\mathrm{Ran}_X}). \qquad (4.10.2)$$

We now deduce that (4.10.2) is an isomorphism by using the chiral PBW theorem.

By construction, (4.10.2) is compatible with chiral PBW filtrations. Therefore, it suffices to check that the map is an isomorphism on the associated graded. For simplicity, we observe further that both sides of (4.10.2) factorize, and therefore we can check this over $X \subseteq \mathrm{Ran}_X$.

The associated graded of the left hand side is then:

$$\mathrm{Sym}(\iota^!\mathcal{L}^{\mathrm{Ran}}(L)[1])$$

while the associated graded of the right hand side is:

$$\iota^!(\mathcal{L}^{\mathrm{Ran}}(\mathrm{Sym}(L[1])))$$

where Sym is computed using the usual $\overset{!}{\otimes}$ structures. Since $\mathcal{L}^{\mathrm{Ran}}$ is obviously symmetric monoidal[22] with respect to $\overset{!}{\otimes}$.

This immediately implies that the associated graded terms are isomorphic. Moreover, by construction, the map (4.10.2) is our given isomorphism on the first associated graded term. Finally, since the associated graded of the PBW filtration is naturally a commutative algebra, and a symmetric algebra at that, this suffices to see the claim.

$\square$

4.11.   We now apply Proposition 4.10.1 to deduce the desired result.

Proof of Theorem 4.4.1. Fix a finite set I.

By §2.36, we have strictly unital chiral functors:

$$\mathsf{Whit}^{sph,\mathrm{fact}} \to \mathsf{Whit}^{int,\mathrm{fact}} \xrightarrow{\mathfrak{p}^{\frac{\infty}{2},int}_{*,ren}} \mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}}.$$

Therefore, we need to compute the result of applying (renormalized) $*$-pushforwards and $!$-pullbacks of the Whittaker sheaf on Gr_{N^-} along the diagram:
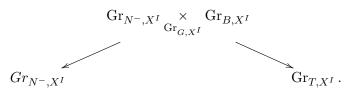
$$\mathrm{Gr}_{N^-,X^I} \to \mathrm{Gr}_{G,X^I} \leftarrow \mathrm{Gr}^{int}_{G,B,X^I} \xrightarrow{\mathfrak{p}^{\frac{\infty}{2},int}_{X^I}} \mathrm{Fl}^{\frac{\infty}{2}}_{X^I} \xleftarrow{i^{\frac{\infty}{2}}_{X^I}} \mathrm{Gr}_{T,X^I}.$$

Of course, the latter two maps are not meaningful, since $\mathrm{Fl}^{\frac{\infty}{2}}_{X^I}$ itself is not meaningful, but we do know the corresponding categories of D-modules and the corresponding functors.

Naively ignoring this same non-existence of $\mathrm{Fl}^{\frac{\infty}{2}}_{X^I}$, we have a commutative diagram:

---

[22]Really, it is non-unitally symmetric monoidal, but this is only because we choose to further pushforward to $\mathrm{Gr}_{T,\mathrm{Ran}_X}$ instead of contenting ourselves with $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff},\mathrm{Ran}_X}$.

$$\begin{array}{ccccccc}
\mathrm{Gr}_{N^-,X^I} \underset{\mathrm{Gr}_{G,X^I}}{\times} \mathrm{Gr}_{B,X^I} & \longrightarrow & \mathrm{Gr}_{B,X^I} & \!\!=\!\!\!=\!\!\!= & \mathrm{Gr}_{B,X^I} & \longrightarrow & \mathrm{Gr}_{T,X^I} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow i^{\frac{\infty}{2}}_{X^I} \\
\mathrm{Gr}_{N^-,X^I} & \longrightarrow & \mathrm{Gr}_{G,X^I} & \longleftarrow & \mathrm{Gr}^{int}_{G,B,X^I} & \longrightarrow & \mathrm{Fl}^{\frac{\infty}{2}}_{X^I}
\end{array}$$

with left and right squares Cartesian. Heuristically, this means we simply need to compute !-pullback and *-pushforward of the Whittaker sheaf on Gr_{N^-,X^I} along the correspondence:

$$\mathrm{Gr}_{N^-,X^I} \underset{\mathrm{Gr}_{G,X^I}}{\times} \mathrm{Gr}_{B,X^I}$$

$$Gr_{N^-,X^I} \qquad\qquad\qquad\qquad Gr_{T,X^I} \, .$$

Recalling that $\mathrm{Fl}^{\frac{\infty}{2}}_{X^I}$ does not actually exist, this manipulation is actually justified by Proposition 2.26.1. Note that these base-change computations are obviously compatible with factorization.

Next, recall the (open) Zastava space $\overset{o}{\mathcal{Z}} = \coprod_{\check\lambda \in \check\Lambda^{pos}} \overset{o}{\mathcal{Z}}{}^{\check\lambda}$, c.f. [Ras2] §2. Recall also that the map $\mathrm{Gr}_{N^-,X^I} \underset{\mathrm{Gr}_{G,X^I}}{\times} \mathrm{Gr}_{B,X^I} \to \mathrm{Gr}_{T,X^I}$ factors through $\mathrm{Div}^{\check\Lambda^{pos}}_{\mathrm{eff},X^I}$. We have a tautological Cartesian diagram:

$$\begin{array}{ccc}
\mathrm{Gr}_{N^-,X^I} \underset{\mathrm{Gr}_{G,X^I}}{\times} \mathrm{Gr}_{B,X^I} & \longrightarrow & \overset{o}{\mathcal{Z}}{}^{\check\Lambda^{pos}} \\
\downarrow & & \downarrow \\
\mathrm{Div}^{\check\Lambda^{pos}}_{\mathrm{eff},X^I} & \longrightarrow & \mathrm{Div}^{\check\Lambda^{pos}}_{\mathrm{eff}} \, .
\end{array}$$

Moreover, this diagram is compatible with factorization.

We now obtain the claim by base-change from Theorem 4.6.1 of [Ras2]. Note that the cohomological shifts of §2.26 occur because of the appearance of the IC sheaf of $\overset{o}{\mathcal{Z}}$ in the statement of Theorem 4.6.1 of [Ras2], and the fact that $\overset{o}{\mathcal{Z}}{}^{\check\lambda}$ is smooth of dimension $(2\rho, \check\lambda)$.

$\square$

4.12. **Construction of the functor.** A priori, the assignment:

$$I \mapsto \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_{T,X^I}))$$

defines a lax unital factorization category (c.f. [Ras1] §6.12 and §8.14). However, as in [Ras2] Corollary 7.8.1, this lax unital structure is actually strict, i.e., the above assignment is an honest unital factorization category.

From [Ras1] Proposition 8.14.1 and Theorem 4.4.1, we now obtain a functor:

$$i^{\frac{\infty}{2},!,enh} : \mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_T)^{\mathrm{fact}})$$

of unital factorization categories.

4.13. **Compatibility with Casselman-Shalika.** Below, we formulate Theorem 4.15.1, which is a kind of compatibility between $i^{\frac{\infty}{2},!,enh}$ and the factorizable (non-derived) geometric Satake theorem.

4.14. The following appears as Theorem 6.36.1 of [Ras2].

Theorem 4.14.1 (Geometric Casselman-Shalika theorem). *There is a canonical equivalence of unital factorization categories:*

$$\mathrm{CS} : \mathsf{Whit}^{sph,\mathrm{fact}} \xrightarrow{\simeq} \mathsf{Rep}(\check{G})^{\mathrm{fact}}.$$

We recall that this equivalence is constructed using the geometric Satake theorem.

4.15. Define the *geometric* Chevalley functor:

$$\mathrm{Chev}_{\check{\mathfrak{n}}}^{\mathrm{geom}} : \mathsf{Whit}^{sph,\mathrm{fact}} \to \Upsilon_{\check{\mathfrak{n}}}\mathrm{-mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_T)^{\mathrm{fact}}) \in \mathsf{FactCat}_{un}(X_{dR})$$

as the composition:

$$\mathsf{Whit}^{sph,\mathrm{fact}} \to \mathsf{Whit}^{int,\mathrm{fact}} \to \mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}} \xrightarrow{i^{\frac{\infty}{2},!,enh}} \Upsilon_{\check{\mathfrak{n}}}\mathrm{-mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_T)^{\mathrm{fact}}).$$

Define the *spectral* Chevalley functor:

$$\mathrm{Chev}_{\check{\mathfrak{n}}}^{\mathrm{spec}} : \mathsf{Rep}(\check{G})^{\mathrm{fact}} \to \Upsilon_{\check{\mathfrak{n}}}\mathrm{-mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_T)^{\mathrm{fact}}) \in \mathsf{FactCat}_{un}(X_{dR})$$

as the composition:

$$\mathsf{Rep}(\check{G})^{\mathrm{fact}} \xrightarrow{\mathrm{restriction}} \mathsf{Rep}(\check{B})^{\mathrm{fact}} \simeq \check{\mathfrak{n}}_X\mathrm{-mod} \xrightarrow{\mathrm{Ind}^{ch}} \Upsilon_{\check{\mathfrak{n}}}\mathrm{-mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_T)^{\mathrm{fact}})$$

where $\check{\mathfrak{n}}_X\mathrm{-mod}$ is the factorization category of Lie-$*$ modules for $\check{\mathfrak{n}}_X$ in $D(\mathrm{Gr}_T)^{\mathrm{fact}}$, and where Ind^{ch} is the chiral induction functor for Lie-$*$ modules.

We now have the following compatibility:

Theorem 4.15.1. *The following diagram of unital factorization categories canonically commutes:*

This is deduced from [Ras2] Theorem 7.9.1 by the same method by which we deduced Theorem 4.4.1 from [Ras2] Theorem 4.6.1.

5. FORMULATION OF THE MAIN THEOREM AND LOCAL ACYCLICITY IN $\mathsf{Whit}^{\frac{\infty}{2}}$

5.1. The full category $\mathsf{Whit}^{\frac{\infty}{2}}_{X^I}$ is technically quite difficult to work with: applying co/invariants on both sides with respect to an infinite-dimensional group is technically challenging.

In this section, we define the *accessible* subcategory $\mathsf{Whit}^{\frac{\infty}{2}}_{X^I,acc}$ of $\mathsf{Whit}^{\frac{\infty}{2}}_{X^I}$, whose origins are of more finite-dimensional origins.

We then formulate the main theorem of this paper, Theorem 5.7.1, which is a Langlands duality duality theorem for $\mathsf{Whit}^{\frac{\infty}{2}}_{X^I,acc}$.

Finally, we will show in Theorem 5.10.1 that the categories $\mathsf{Whit}^{\frac{\infty}{2}}_{X^I,acc}$ have some remarkable technical properties which make them easier to work with: namely, we show that they are *ULA* over X^I in the sense of [Ras2] Appendix B. This result will ultimately allow us to prove Theorem 5.7.1, which is about factorizable categories, to its corresponding version over a point.

5.2.   We need to recall two standard ideas before proceeding: the notion of monodromic object, and the fact that $G(O)$-monodomic objects can be !-averaged against the Whittaker character.

5.3. Review of monodromicity. We need the following brief interlude.

Suppose that $\mathcal{G}$ is a group scheme acting on a category $\mathcal{C} \in \mathsf{DGCat}_{cont}$. Recall that the *monodromic* subcategory $\mathcal{C}^{\mathcal{G}-\mathrm{mon}} \subseteq \mathcal{C}$ is the cocomplete subcategory generated under colimits by the image of the functor $\mathrm{Oblv} : \mathcal{C}^{\mathcal{G}} \to \mathcal{C}$.

Since Oblv admits the continuous right adjoint Av_*, the embedding $\mathcal{C}^{\mathcal{G}-\mathrm{mon}} \hookrightarrow \mathcal{C}$ admits a continuous right adjoint as well. Moreover, we deduce that:

$$\mathcal{C}^{\mathcal{G}-\mathrm{mon}} = D^*(\mathcal{G})^{\mathcal{G}-\mathrm{mon}} \underset{D^*(\mathcal{G})}{\otimes} \mathcal{C}. \tag{5.3.1}$$

Note here that $D^*(\mathcal{G})^{\mathcal{G}-\mathrm{mon}}$ is the full subcategory of $D^*(\mathcal{G})$ generated by $k_{\mathcal{G}} = \omega_{\mathcal{G}}^{ren}$.

Finally, note that the above makes sense if $\mathcal{G}$ is a placid group scheme over X^I (or any other base). Then as usual, $\mathcal{C}$ is a priori assumed to have a $D(X^I)$-module category structure. It follows from the above that the adjoint functors $\mathcal{C}^{\mathcal{G}-\mathrm{mon}} \rightleftarrows \mathcal{C}$ are $D(X^I)$-linear. In particular, it follows that the embedding $\mathcal{C}^{\mathcal{G}-\mathrm{mon}} \hookrightarrow \mathcal{C}$ preserves objects ULA[23] over X^I.

5.4. !-averaging. We have the following toy model for constructing Whittaker equivariant objects. For the moment, we fix a point in our curve and let e.g. $G(K)$ denote the loop group based at that point.

Proposition 5.4.1. *Let $\mathcal{C}$ be a category acted on by $G(K)$ (i.e., a $D^*(G(K))$-module category).*
   The left adjoint $\mathrm{Av}_!^{\psi}$ to the forgetful functor $\mathsf{Whit}(\mathcal{C}) \to \mathcal{C}$ is defined on the subcategory $\mathcal{C}^{G(O)} \subseteq \mathcal{C}$.

Proof. By definition, it suffices to show that the functor $\mathsf{Whit}(\mathcal{C}) \to \mathcal{C}^{G(O)}$ given by forgetting and then $*$-averaging admits a left adjoint.

We have a functor $\mathcal{C}^{G(O)} \to \mathsf{Whit}(\mathcal{C})$ defined by convolution with the unit object in $\mathsf{Whit}(D(\mathrm{Gr}_G))$, i.e., the cleanly extended Whittaker sheaf on Gr_{N^-}. It is then straightforward to construct the unit and counit maps for the adjunction by working directly with the corresponding D-modules on Gr_G: we remark that ind-properness of Gr_G plays a crucial role in this verification.
□

It follows formally that if $\mathcal{F} \in \mathcal{C}^{G(O)-\mathrm{mon}}$ is compact, then $\mathrm{Av}_!^{\psi}(\mathcal{F})$ is compact as an object of $\mathsf{Whit}(\mathcal{C})$.

Remark 5.4.2. The same construction works over X_{dR}^I. That is, if $\mathcal{C} \in D(X^I)-\mathsf{mod}$ is equipped with a $D^*(G(K)_{X^I})$-module structure, then the corresponding functor $\mathrm{Av}_!^{\psi} : \mathcal{C}^{G(O)_{X^I}} \to \mathsf{Whit}(\mathcal{C})$ is defined and $D(X^I)$-linear, and it sends objects $\mathcal{F} \in \mathcal{C}^{G(O)_{X^I}-\mathrm{mon}}$ ULA (over X^I) to a ULA object of $\mathsf{Whit}(\mathcal{C})$.

5.5. Definition of $\mathsf{Whit}_{X^I,acc}^{\frac{\infty}{2}}$. Define $\mathsf{Whit}_{X^I,acc}^{\frac{\infty}{2}}$ to be generated under colimits by objects in the image of the functor:

$$\mathrm{Av}_!^{\psi} : D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})^{G(O)_{X^I}-\mathrm{mon}} \to \mathsf{Whit}_{X^I}^{\frac{\infty}{2}}.$$

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[23]Recall that ULA means that the corresponding object of $\mathcal{C} \otimes_{D(X^I)} \mathsf{QCoh}(X^I)$ is compact.

5.6.   It is immediate to verify that:

$$I \mapsto \mathsf{Whit}^{\frac{\infty}{2}}_{X^I,acc}$$

is a full factorization subcategory of $I \mapsto \mathsf{Whit}^{\frac{\infty}{2}}_{X^I}$.

Moreover, the construction of the unit maps for $\mathsf{Whit}^{\frac{\infty}{2}}$ implies that the unit maps:

$$D(X^I) \otimes \mathsf{Whit}^{\frac{\infty}{2}}_{X^J} \to \mathsf{Whit}^{\frac{\infty}{2}}_{X^{I \amalg J}}$$

map $D(X^I) \otimes \mathsf{Whit}^{\frac{\infty}{2}}_{X^J,acc}$ to $\mathsf{Whit}^{\frac{\infty}{2}}_{X^{I \amalg J},acc}$. Indeed, this functor is $D(X^{I \amalg J})$-linear, so it suffices to see that it sends objects of the form $\omega_{X^I} \boxtimes \mathcal{F}$ with $\mathcal{F} \in \mathsf{Whit}^{\frac{\infty}{2}}_{X^J,acc}$ to objects in $\mathsf{Whit}^{\frac{\infty}{2}}_{X^{I \amalg J},acc}$. We can then work over strata of $X^{I \amalg J}$ and apply factorization, and then the claim follows from the fact that the unit object of $\mathsf{Whit}^{\frac{\infty}{2}}_{X^K}$ lies in $\mathsf{Whit}^{\frac{\infty}{2}}_{X^K,acc}$ for every finite set K.

In this way, we obtain the full unital factorization subcategory $\mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}}_{acc}$ of $\mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}}$.

5.7. **Formulation of the main theorem.** We now give the main theorem of this paper.

Theorem 5.7.1. *The functor $i^{\frac{\infty}{2},!,enh}$ is a fully-faithful functor of unital factorization categories:*

$$\mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}}_{acc} \to \Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_T)^{\mathrm{fact}}).$$

Remark 5.7.2. As follows from Theorem 8.22.1 and Koszul duality, the fiber of the righthand side at a point is equivalent to $\mathsf{QCoh}(\check{\mathfrak{n}}^\wedge_0/\check{B}^\wedge_{\check{T}})$. Here $\check{B}^\wedge_{\check{T}}$ is the formal completion of $\check{B}$ along $\check{T}$. On the other hand, Arkhipov-Berzukavnikov (combined with the comparisons of Theorems 6.2.1 and 7.3.1) implies that the left hand side is equivalent to $\mathsf{QCoh}(\check{\mathfrak{n}}^\wedge_0/\check{B})$.

To understand the difference, let us consider instead the simpler problem of comparing $\mathbb{B}\check{N}^\wedge_e$ with $\mathbb{B}\check{N}$. Quasi-coherent sheaves on the former is $\check{\mathfrak{n}}$–mod, while the latter gives $\mathsf{Rep}(\check{N})$. By unipotence, $\mathsf{Rep}(\check{N})$ is a full subcategory of $\check{\mathfrak{n}}$–mod. A similar argument implies the fully-faithfulness of the functor above. (Moreover, one can e.g. use this and factorization to obtain a complete, explicit description of the image of the functor $i^{\frac{\infty}{2},!,enh}$.)

Equivalently, this result means that for each finite set I, the functor $\mathsf{Whit}^{\frac{\infty}{2}}_{acc,X^I} \to \Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_{T,X^I}))$ is fully-faithful.

The remainder of this text is devoted to proving this result.

5.8. **ULAness.** We now collect one technical result which we will need in the sequel. The reader may prefer to skip this material for now, and return to it as needed.

5.9.   At this point, we assume the reader is familiar with the contents of [Ras2] Appendix B regarding ULA objects and ULA categories.

5.10.   The main technical result we will need is the following one.

Theorem 5.10.1. *For every finite set I, $\mathsf{Whit}^{\frac{\infty}{2}}_{acc,X^I}$ is ULA as a $D(X^I)$-module category, with the unit object $\mathrm{unit}_{\mathsf{Whit}^{\frac{\infty}{2}}_{X^I}}$ being a ULA.*

5.11.   The following result will be key to verifying Theorem 5.10.1.

Proposition 5.11.1. *The unit object* $\text{unit}_{D^!(\text{Fl}_{X^I}^{\frac{\infty}{2}})} \in D^!(\text{Fl}_{X^I}^{\frac{\infty}{2}})^{G(O)_{X^I}-\text{mon}}$ *is ULA.*

The proof of Proposition 5.11.1 will be given in §5.12-5.14 below.

Remark 5.11.2. By Remark 5.4.2, Proposition 5.11.1 implies that the unit object of $\mathsf{Whit}_{X^I}^{\frac{\infty}{2}}$ is ULA.

5.12.   Let $\check{\lambda} = (\check{\lambda}_i)_{i \in I} \in \check{\Lambda}^I$ be a vector of coweights. This defines an open $U_{X^I}^{\check{\lambda}} \subseteq G(K)_{X^I}$ as the inverse image of the complement to $\overline{\text{Gr}}_{B,X^I}^{\check{\lambda}} \subseteq \text{Gr}_{G,X^I}$. Note that $N(K)_{X^I}$ acts on the left on $U_{X^I}^{\check{\lambda}}$.

Lemma 5.12.1. *The map:*

$$\operatorname*{colim}_{\check{\lambda} \in \check{\Lambda}^I} D^!(U_{X^I}^{\check{\lambda}}) \to D^!(G(K))_{X^I} \in D(X^I)-\mathsf{mod}$$

is an equivalence. Here all functors are de Rham pushforward functors (along quasi-compact open embeddings), and we are letting $\check{\lambda}$ limit to $-\infty$.

Proof. Since $D^!(G(K)_{X^I})$ is the colimit of $D^!$ of its reasonable subschemes (under pushforwards), it suffices to verify this upon intersecting with the inverse image of a finite type subscheme in the Grassmannian. But such a subscheme is contained in a cofinal set of $U_{X^I}^{\check{\lambda}}$, giving the claim. $\square$

By Corollary 6.5.3, Proposition 5.11.1 follows from the next result.

Lemma 5.12.2. *For every $\check{\lambda} \in (-\check{\Lambda}^{pos})^I$, the projection of $\omega_{G(O)_{X^I}} \overset{pushforward}{\in} D^!(U_{X^I}^{\check{\lambda}})$ to $D^!(U_{X^I}^{\check{\lambda}})_{N(K)T(O)_{X^I}}$ is ULA.*

We will prove this lemma below.

5.13.   We now have the following result for *invariant* categories.

Lemma 5.13.1. *The partially-defined left adjoint $\text{Av}_!^{\frac{\infty}{2}}$ to the forgetful functor $D^!(U_{X^I}^{\check{\lambda}})^{N(K)T(O)} \to D^!(U_{X^I})$ is defined on the full subcategory $D^!(U_{X^I})^{B(O)_{X^I} \underset{X^I}{\times} G(O)_{X^I}-\text{mon}}$.*

We emphasize again that the $N(K)T(O)_{X^I}$-action and the $G(O)_{X^I}$-action occur on different sides; the notation indicates that we are taking objects that are monodromic for the $B(O)_{X^I}$-action on the left and the $G(O)_{X^I}$-action on the right.

Proof of Lemma 5.13.1. We will prove this using Braden's theorem, c.f. [Bra] and [DG].

Let $V_{X^I}^{\check{\lambda}} \subseteq \text{Gr}_{G,X^I}$ be the complement of $\overline{\text{Gr}}_{B,X^I}^{\check{\lambda}}$, so $U_{X^I}^{\check{\lambda}} \to V_{X^I}^{\check{\lambda}}$ is $G(O)_{X^I}$-torsor. Since this map is placid and therefore pullback is a left adjoint. Since the pullback $D(V_{X^I}^{\check{\lambda}})^{B(O)_{X^I}-\text{mon}} \to D(U_{X^I}^{\check{\lambda}})$ generates $D(U_{X^I}^{\check{\lambda}})^{B(O)_{X^I} \underset{X^I}{\times} G(O)_{X^I}-\text{mon}}$ under colimits, it suffices to see that we can !-average objects of $D(V_{X^I}^{\check{\lambda}})^{B(O)_{X^I}-\text{mon}}$.

Note that $D(V_{X^I}^{\check{\lambda}})^{B(O)_{X^I}-\text{mon}}$ is generated under colimits by objects supported on the union of finitely many $N(K)_{X^I}$-orbits (equivalently: $N(K)T(O)_{X^I}$-orbits). Therefore, it suffices to show that such objects can be !-averaged.

We claim now that it suffices to see that the operation of taking $*$-restriction to an orbit $\text{Gr}_{B,X^I}^{\check{\mu}} \subseteq V_{X^I}^{\check{\lambda}}$ and then !-pushforward to X^I is defined (i.e., the left adjoint to the relevant functor $D(X^I) \to D(V_{X^I}^{\check{\lambda}})$).

Indeed, this immediately implies that $*$-restriction to an orbit:

$$D(V_{X^I}^{\check\lambda})^{N(K)T(O)_{X^I}-\mathrm{mon}} \to D(\mathrm{Gr}_B^{\check\mu})^{N(K)T(O)_{X^I}-\mathrm{mon}} \simeq D(X^I)$$

is defined, which by a Cousin argument implies that !-extension from such an orbit is also defined.[24] We then see that the operation of $*$-restricting to an orbit, !-averaging, and then !-extending from the orbit is defined, so another Cousin argument does the trick.

That this functor is defined is now immediate from Braden's theorem and the usual realization of semi-infinite orbits as the attracting locus for an appropriate $\mathbb{G}_m$-action: c.f. [MV] §3. We remark that there are no issues in applying Braden's theorem to an indscheme of ind-finite type here, since Gr_{G,X^I} is a union of finite type $B(O)_{X^I}$-stable subschemes. Similarly, the generalization to the factorization setting here is immediate (and standard).

$\square$

5.14.   We have the following application of the above result.

Corollary 5.14.1. *The category* $D^!(U_{X^I}^{\check\lambda})_{N(K)T(O)_{X^I}}^{G(O)_{X^I}-\mathrm{mon}}$ *is dualizable as a* $D(X^I)$-*module category, and is canonically*[25] *dual to* $D^!(U_{X^I}^{\check\lambda})^{G(O)_{X^I}-\mathrm{mon},N(K)T(O)_{X^I}}$.

Remark 5.14.2. A similar statement holds for $D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})^{G(O)_{X^I}-\mathrm{mon}}$, as can be shown using the results here. Perhaps it is reasonable to expect it to hold for $D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})$ itself.

Proof of Corollary 5.14.1. First, note that $D^!(U_{X^I}^{\check\lambda})^{G(O)_{X^I}-\mathrm{mon}}$ itself is dualizable over X^I. Indeed, this category is a $D(X^I)$-linear retract of $D^!(G(K)_{X^I})$ (since monodromic categories always are, and since we have adjoint push and pull functors from this open) which is dualizable over X^I.

By a standard argument (c.f. [Gai5]), it suffices to show that for every $\mathcal{D} \in D(X^I)$–mod, the functor:

$$D^!(U_{X^I}^{\check\lambda})^{G(O)_{X^I}-\mathrm{mon},N(K)T(O)_{X^I}} \underset{D(X^I)}{\otimes} \mathcal{D} \to \big(D^!(U_{X^I}^{\check\lambda})^{G(O)_{X^I}-\mathrm{mon}} \underset{D(X^I)}{\otimes} \mathcal{D}\big)^{N(K)T(O)_{X^I}}$$

is an equivalence.

We see that the right hand side is a full subcategory of:

$$\big(D^!(U_{X^I}^{\check\lambda})^{G(O)_{X^I}-\mathrm{mon}} \underset{D(X^I)}{\otimes} \mathcal{D}\big)^{B(O)_{X^I}}.$$

We claim that the left hand side is also. Indeed, the functor:

$$D^!(U_{X^I}^{\check\lambda})^{G(O)_{X^I}-\mathrm{mon},N(K)T(O)_{X^I}} \to D^!(U_{X^I}^{\check\lambda})^{G(O)_{X^I}-\mathrm{mon},B(O)_{X^I}}$$

admits a $D(X^I)$-linear left adjoint by Lemma 5.13.1, and therefore remains fully-faithful after tensoring with any $D(X^I)$-module category. Moreover, because $B(O)_{X^I}$-invariants coincide with coinvariants, we can pull the invariants out of the tensor product here, giving the claim.

$\square$

We now deduce Lemma 5.12.2, completing the proof of Proposition 5.11.1.

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[24]Note that this conclusion would not be valid if we had not removed the closure of an orbit: it is important for this point that there should be only finitely many orbits in the closure.

[25]Note that everywhere here we are using the dimension theory on $U_{X^I}^{\check\lambda}$ to identify $D^!$ with D^*.

Proof of Lemma 5.12.2. The embedding $D^!(U_{X^I}^{\check{\lambda}})^{G(O)_{X^I}-\mathrm{mon}} \to D^!(U_{X^I}^{\check{\lambda}})$ admits a $D(X^I)$-linear right adjoint, and therefore it suffices to see that the "unit" object in this subcategory is ULA.

This unit object is obtained by $*$-pushforward from the object of $D^!(N(K)_{X^I})_{N(K)_{X^I}} = D(X^I)$ corresponding to ω_{X^I}. This object is obviously ULA over X^I.

Recall the following paradigm. Let $G : \mathcal{C} \to \mathcal{D} \in \mathsf{DGCat}_{cont}$ be a functor of dualizable categories. Then for $\mathcal{F} \in \mathcal{C}$ compact, $G(\mathcal{F})$ is compact if and only the left adjoint to $G^\vee : \mathcal{D}^\vee \to \mathcal{C}^\vee$ is defined on $\mathbb{D}(\mathcal{F}) \in \mathcal{C}^\vee$ (where $\mathbb{D}(\mathcal{F})$ is the object defined by the functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -) \in \mathsf{Hom}(\mathcal{C}, \mathsf{Vect}) = \mathcal{C}^\vee$). By rigiditiy of $\mathsf{QCoh}(X^I)$, one immediately obtains a similar statement for $D(X^I)$-module categories, where *compact* is replaced by *ULA* and functors are $D(X^I)$-linear throughout.

Therefore, by Corollary 5.14.1, to see that the unit in $D^!(U_{X^I}^{\check{\lambda}})^{G(O)_{X^I}-\mathrm{mon}}$ is ULA, it suffices to see that the left adjoint to the !-restriction functor $D^!(U_{X^I}^{\check{\lambda}})^{N(K)T(O)_{X^I}} \to D^!(N(K)T(O)_{X^I})^{N(K)T(O)_{X^I}}$ is defined and $D(X^I)$-linear. But we have seen this already in the course of the proof of Lemma 5.13.1.

$\square$

5.15.   We now have the following result, which tautologically implies what remains of Theorem 5.10.1:

Proposition 5.15.1. $D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})^{G(O)_{X^I}-\mathrm{mon}}$ *is ULA as a $D(X^I)$-module category.*

Proof. For $\check{\lambda} = (\check{\lambda}_i)_{i \in I} \in \check{\Lambda}^I$, let $\mathrm{unit}_{D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})}^{\check{\lambda}} \in D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})^{G(O)_{X^I}-\mathrm{mon}}$ denote object obtained by translating the unit object via the Gr_{T,X^I} action on $\mathrm{Fl}_{X^I}^{\frac{\infty}{2}}$. By Proposition 5.11.1, $\mathrm{unit}_{D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})}^{\check{\lambda}}$ is ULA. Therefore, it suffices to show that these objects generate $D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})$ as a $D(X^I)$-module category.

By Lemma 5.12.1, it suffices to show that the objects $\mathrm{unit}_{D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})}^{\check{\lambda}}$ generate $D^!(U_{X^I}^{\check{\mu}})^{G(O)_{X^I}}$ for every vector of coweights $\check{\mu}$ whenever $\check{\lambda}$ ranges over those vectors of coweights for which $\mathrm{unit}_{D^!(\mathrm{Fl}_{X^I}^{\frac{\infty}{2}})}^{\check{\lambda}}$ lies in $D^!(U_{X^I}^{\check{\mu}})^{G(O)_{X^I}} \subseteq D^!(G(K)_{X^I})^{G(O)_{X^I}}$.

But this is a statement about the union of strata in Gr_{G,X^I}, and therefore usual (ind-)finite type geometry allows us to apply a Cousin argument to reduce to the case of a single stratum, and here the result is clear.

$\square$

6. Iwahori vs. semi-infinite flags

6.1.   Let $x \in X$ be a closed point. Let $\mathsf{Whit}_x^{\frac{\infty}{2}}$ be the fiber of $\mathsf{Whit}_X^{\frac{\infty}{2}}$ at this point, i.e., $N(K_x)T(O_x)$-coinvariants of the Whittaker invariants of $D^!(G(K_x))$.

Let $I \subseteq G(O)$ be the Iwahori subgroup, i.e., the inverse image of B under the projection $G(O) \to G$. Let $\overset{o}{I}$ denote its prounipotent radical, i.e., the inverse image of N. We let $\overset{o}{I^-}$ be defined similarly but using N^- instead.

The purpose of §6-7 is to show that $\mathsf{Whit}_x^{\frac{\infty}{2}}$ coincides with the category $D(\mathrm{Fl}_{G,x}^{\mathrm{aff}})^{\overset{o}{I^-},\psi_{\overset{o}{I^-}}}$ considered in [AB].

There are two comparisons to be made: in the present section, we treat the $N(K_x)T(O_x)$ side, and in §7, we treat the Whittaker side.

6.2.   The main result of this section is the following.

Theorem 6.2.1. *Let $\mathcal{C}$ be a category acted on by $G(K_x)$.[26] Then the functor:*

$$\mathcal{C}_I \xrightarrow{\text{Nm}} \mathcal{C}_{B(O_x)} \to \mathcal{C}_{N(K_x)T(O_x)}$$

is an equivalence. Here Nm *is the* norm *map, which by definition corresponds to* Oblv *under the equivalences $\mathcal{C}_I \simeq \mathcal{C}^I$ and $\mathcal{C}_{B(O_x)} \simeq \mathcal{C}^{B(O_x)}$.*

Remark 6.2.2. Note that this result is borrowed from the theory of reductive p-adic groups: c.f. [Cas] Proposition 2.4.

Corollary 6.2.3. *For $\mathcal{C}$ as above, the functor $\mathcal{C}^{N(K_x)T(O_x)} \xrightarrow{\text{Oblv}} \mathcal{C}^{B(O_x)} \xrightarrow{\text{Av}_*} \mathcal{C}^I$ is an equivalence.*

Proof that Theorem 6.2.1 implies Corollary 6.2.3. We have:

$$\underline{\text{Hom}}_{D^*(G(K_x))-\text{mod}}(D^*(G(K_x))_{N(K_x)T(O_x)}, \mathcal{C}) \simeq \mathcal{C}^{N(K_x)T(O_x)}$$

and similarly for Iwahori invariants. Therefore, we deduce the result from Theorem 6.2.1 applied to the regular representation.

$\square$

6.3.   For every $\check{\lambda} \in \check{\Lambda}$, we use the notation:

$$I^{\check{\lambda}} := \text{Ad}_{-\check{\lambda}(t)}(I) \subseteq G(K_x)$$

$$B(O_x)^{\check{\lambda}} := \text{Ad}_{-\check{\lambda}(t)}(B(O_x)) \subseteq G(K_x)$$

where $t \in K_x$ is a uniformizer.

Remark 6.3.1. The normalization with $-\check{\lambda}(t)$ is so we can work with $\check{\lambda} \in \check{\Lambda}^+$ instead of $-\check{\Lambda}^+$.

6.4.   The key fact we will use is the following one.

Lemma 6.4.1. *For $\mathcal{C}$ acted on by $G(K_x)$ and $\check{\lambda}, \check{\eta}$ coweights, the functor:*

$$\text{Av}_*^{I^{\check{\mu}}} : \mathcal{C}^{I^{\check{\lambda}}} \to \mathcal{C}^{I^{\check{\mu}}}$$

(given by forgetting to $I^{\check{\lambda}} \cap I^{\check{\mu}}$ and then averaging) is an equivalence.

Proof. Up to translations, this follows from the invertibility of Mirkovic-Wakimoto sheaves in the Iwahori-Hecke algebra (see [AB] Lemma 8).

$\square$

Remark 6.4.2. We denote the inverse functor by $\text{Av}_!^{I^{\check{\lambda}}}$, since it is evidently given by (forgetting down to $I^{\check{\lambda}} \cap I^{\check{\mu}}$ and then) applying such a !-averaging.

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[26]I.e., a $D^!(G(K_x))$-comodule category in DGCat_{cont}, or equivalently, a $D^*(G(K_x))$-module category.

6.5. Before preceding, we record a technical general lemma we will need. The reader may prefer to skip this section and refer back to it as necessary.

Suppose that $\mathfrak{I}$ is a filtered category, and suppose we are given diagrams:

$$i \mapsto \mathcal{C}_i \in \mathsf{DGCat}_{cont}$$
$$i \mapsto \mathcal{D}_i \in \mathsf{DGCat}_{cont}.$$

Let $\mathcal{C}$ (resp. $\mathcal{D}$) denote the colimit category in DGCat_{cont}. For $\alpha : i \to j \in \mathfrak{I}$, let ψ_α (resp. φ_α) denote the structure functor $\mathcal{C}_i \to \mathcal{C}_j$ (resp. $\mathcal{D}_i \to \mathcal{D}_j$). We let $\psi_i : \mathcal{C}_i \to \mathcal{C}$ and $\varphi_i : \mathcal{D}_i \to \mathcal{D}$ denote the structure functors.

Suppose we are given compatible functors $F_i : \mathcal{C}_i \to \mathcal{D}_i$, and suppose that each functor F_i admits a continuous right adjoint G_i. We do not assume that the functors G_i are compatible with the structure maps (though they are automatically *lax* compatible).

Let F denote the induced functor $F : \mathcal{C} \to \mathcal{D}$.

Construction 6.5.1. For every i, define the continuous functor "$G\varphi_i$" $: \mathcal{D}_i \to \mathcal{C}$ by the formula:[27]

$$\text{``}G\varphi_i\text{''} := \operatorname*{colim}_{\alpha : i \to j} \psi_j G_j \varphi_\alpha.$$

For $\beta : k \to i$, observe that we have:

$$\text{``}G\varphi_i\text{''} \circ \varphi_\beta = \operatorname*{colim}_{\alpha : i \to j} \psi_j G_j \varphi_\alpha \varphi_\beta = \operatorname*{colim}_{\gamma : k \to j} \psi_j G_j \varphi_\gamma = \text{``}G\varphi_k\text{''}$$

where we use filteredness to deduce the second equality. There, we have a functor $G : \mathcal{D} \to \mathcal{C}$ characterized by the identities $G\varphi_i \simeq \text{``}G\varphi_i\text{''}$.

Lemma 6.5.2. *The functor G is the right adjoint to the functor F.*

Proof. We construct the unit and counit of the adjunction explicitly.

Let i be a fixed index. We have:

$$FG\varphi_i = \operatorname*{colim}_{\alpha : i \to j} F\psi_j G_j \varphi_\alpha = \operatorname*{colim}_{\alpha : i \to j} \varphi_j F_j G_j \varphi_\alpha \to \operatorname*{colim}_{\alpha : i \to j} \varphi_j \varphi_\alpha = \varphi_i.$$

These functors are compatible as we vary i, and therefore define a natural transformation:

$$FG \to \operatorname{id}_{\mathcal{D}}.$$

Fixing i again, we similarly obtain:

$$\psi_i = \operatorname*{colim}_{\alpha : i \to j} \psi_j \psi_\alpha \to \operatorname*{colim}_{\alpha : i \to j} \psi_j G_j F_j \psi_\alpha = \operatorname*{colim}_{\alpha : i \to j} \psi_j G_j \varphi_\alpha F_i = G\varphi_i F_i = GF\psi_i$$

and then by passing to the limit, we obtain the natural transformation:

$$\operatorname{id}_{\mathcal{C}} \to GF.$$

One easily finds that these natural transformations define the counit and unit of an adjunction. $\square$

---

[27]Note that for maps $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ of indices, we have the map $\psi_j G_j \varphi_\alpha = \psi_k \psi_\beta G_j \varphi_\alpha \to \psi_k G_k \varphi_{\beta\alpha} = \psi_k G_k \varphi_\beta \circ \varphi_\alpha$ given by the base-change map $\psi_\beta G_j \to G_k \varphi_\beta$, meaning that the arrows go in the correct direction in our colimit diagram.

Corollary 6.5.3. *Suppose that $\mathfrak{I}$ is a filtered as above and $i \mapsto \mathcal{D}_i \in \mathsf{DGCat}_{cont}$ is a diagram with structure maps denoted by φ as above.*

Suppose i_0 is a fixed index in $\mathfrak{I}$ and we are given $\mathcal{F}_{i_0} \in \mathcal{D}_{i_0}$ such that, for every $\alpha : i_0 \to j$, the functor $\mathcal{D}_{i_0} \to \mathcal{D}_j$ sends $\mathcal{F}_{i_0}$ to a compact object $\varphi_\alpha(\mathcal{F}_{i_0})$ in $\mathcal{D}_j$.

Then $\varphi_{i_0}(\mathcal{F})$ is compact in $\mathcal{D} = \operatorname{colim}_i \mathcal{D}_i$. Moreover, for every $\alpha : i_0 \to j$, the resulting continuous functor:

$$\mathcal{D}_j \to \mathcal{D} \xrightarrow{\operatorname{Hom}_{\mathcal{D}}(\varphi_{i_0}(\mathcal{F}),-)} \mathsf{Vect}$$

is computed explicitly by the formula:

$$\mathcal{G} \mapsto \operatorname*{colim}_{\beta : j \to k} \underline{\operatorname{Hom}}_{\mathcal{D}_k}(\varphi_{\beta\alpha}(\mathcal{F}_{i_0}), \varphi_\beta(\mathcal{G})).$$

Proof. First, replacing $\mathfrak{I}$ by $\mathfrak{I}_{i_0/}$ by filteredness, we may assume i_0 is initial in $\mathfrak{I}$. Then for any $j \in \mathfrak{I}$, let $\mathcal{F}_j \in \mathcal{D}_j$ obtained from functoriality from $\mathcal{F}_{i_0}$ using the structure functor $\mathcal{D}_{i_0} \to \mathcal{D}_j$. Let $\mathcal{F} \in \mathcal{D}$ denote the object $\varphi_{i_0}(\mathcal{F}_{i_0})$.

Then we apply Lemma 6.5.2 with $\mathcal{C}_j = \mathsf{Vect}$ for every j, with the compatible functors $\mathsf{Vect} \to \mathcal{D}_j$ given by $k \mapsto \varphi_{\alpha_j}(\mathcal{F}_{i_0})$. Note that the corresponding functor $\mathsf{Vect} \to \mathcal{D}$ sends the trivial vector space k to $\mathcal{F}$.

The lemma applies because each of these functors admits the continuous right adjoint $\underline{\operatorname{Hom}}_{\mathcal{D}_j}(\mathcal{F}_j, -)$.

Then Lemma 6.5.2 ensures that the functor $\mathsf{Vect} \to \mathcal{D}$, $k \mapsto \mathcal{F}$, admits a continuous right adjoint $\underline{\operatorname{Hom}}_{\mathcal{D}}(\mathcal{F}, -)$, and therefore $\mathcal{F}$ is compact. Then the explicit formula for the right adjoint given in Lemma 6.5.2 translates to the stated formula for $\underline{\operatorname{Hom}}_{\mathcal{D}}(\mathcal{F}, -)$.

$\square$

6.6.   We now give the proof of Theorem 6.2.1.

Proof of Theorem 6.2.1. For every $\check{\lambda} \in \check{\Lambda}$, let $\mathfrak{p}^{\check{\lambda}}$ denote the projection functor $\mathcal{C}^{B(O_x)^{\check{\lambda}}} \simeq \mathcal{C}_{B(O_x)^{\check{\lambda}}} \to \mathcal{C}_{N(K_x)T(O_x)}$. For $\check{\lambda} = 0$, we use the notation $\mathfrak{p}$ instead.

Step 1. First, we show that $\mathcal{C}^I \to \mathcal{C}_{N(K_x)T(O_x)}$ generates the target under colimits.

Certainly $\mathcal{C}_{N(K_x)T(O_x)}$ is generated under colimits by the image of the functor $\mathfrak{p}$.

Note that:

$$\operatorname*{colim}_{\check{\lambda} \in \check{\Lambda}^+} \delta_{I^{\check{\lambda}} \cap I} \simeq \delta_{B(O_x)}.$$

Therefore, for $\mathcal{F} \in \mathcal{C}^{B(O_x)}$, we have:

$$\mathcal{F} \simeq \operatorname*{colim}_{\check{\lambda} \in \check{\Lambda}^+} \operatorname{Av}_*^{I^{\check{\lambda}} \cap I}(\mathcal{F})$$

and therefore $\mathcal{C}_{N(K_x)T(O_x)}$ is generated under colimits by the images of the functors $\mathcal{C}^{I^{\check{\lambda}} \cap I} \hookrightarrow \mathcal{C}^{B(O_x)} \xrightarrow{\mathfrak{p}} \mathcal{C}_{N(K_x)T(O_x)}$ as $\check{\lambda}$ ranges over $\check{\Lambda}^+$.

Now observe that for any $\mathcal{F} \in \mathcal{C}^{B(O_x)}$, we have:

$$\mathfrak{p}(\operatorname{Av}_*^{B(O_x)^{\check{\lambda}}}(\mathcal{F})) \xrightarrow{\simeq} \mathfrak{p}(\mathcal{F})$$

by definition of the coinvariants. For $\mathcal{F} \in \mathcal{C}^{I^{\check{\lambda}} \cap I}$, we then see that $\operatorname{Av}_*^{B(O_x)^{\check{\lambda}}}(\mathcal{F})$ is $I^{\check{\lambda}}$-equivariant, so that, by Lemma 6.4.1, we have:

$$\mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{F}) \xrightarrow{\simeq} \mathrm{Av}_*^{B(O_x)^{\check\lambda}} \mathrm{Av}_!^I \mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{F})$$

and therefore:

$$\mathfrak{p}(\mathrm{Av}_!^I \mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{F})) = \mathfrak{p}^{\check\lambda}(\mathrm{Av}_*^{B(O_x)^{\check\lambda}} \mathrm{Av}_!^I \mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{F})) \simeq \mathfrak{p}^{\check\lambda}(\mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{F})) = \mathfrak{p}(\mathcal{F}).$$

Therefore, since the former term is $\mathfrak{p}$ applied to an Iwahori-equivariant object, we obtain the claim.

Step 2. Next, suppose that $\mathcal{F} \in \mathcal{C}^I$ is compact.

From Lemma 6.4.1, we find that $\mathrm{Av}_*^{I^{\check\lambda}}(\mathcal{F})$ is compact in $\mathcal{C}^{I^{\check\lambda}}$ and therefore compact in $\mathcal{C}^{B(O_x)^{\check\lambda}}$. For $\check\lambda \in \check\Lambda^+$, we have $\mathrm{Av}_*^{I^{\check\lambda}}(\mathcal{F}) = \mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{F})$, so, we conclude that $\mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{F})$ is compact for every $\check\lambda \in \check\Lambda^+$.

Now observe that for any $\mathcal{G} \in \mathcal{C}^I$, the map:

$$\underline{\mathrm{Hom}}_{\mathcal{C}^I}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathcal{C}^{B(O_x)^{\check\lambda}}}(\mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{F}), \mathrm{Av}_*^{B(O_x)^{\check\lambda}}(\mathcal{G}))$$

is an isomorphism, since we can compute these averages as $\mathrm{Av}_*^{I^{\check\lambda}}$.

Therefore, Corollary 6.5.3 implies that:

$$\underline{\mathrm{Hom}}_{\mathcal{C}^I}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathcal{C}_{N(K_x)T(O_x)}}(\mathfrak{p}(\mathcal{F}), \mathfrak{p}(\mathcal{G}))$$

is an equivalence for every $\mathcal{G}$.

Step 3. Combining Steps 1 and 2, we obtain that our functor is an equivalence whenever $\mathcal{C}^I$ is compactly generated.

In particular, this applies to $\mathcal{C} = D^*(G(K_x))$, since $D^*(G(K_x))_I \simeq D(\mathrm{Fl}_{G,x}^{\mathrm{aff}})$ is compactly generated.

To treat the case of general $\mathcal{C}$, we use the same method as Corollary 6.2.3:

$$\mathcal{C}_I \simeq \mathcal{C} \underset{D^*(G(K_x))}{\otimes} D^*(G(K_x))_I \simeq \mathcal{C} \underset{D^*(G(K_x))}{\otimes} D^*(G(K_x))_{N(K_x)T(O_x)} \simeq \mathcal{C}_{N(K_x)T(O_x)}.$$

$\square$

7. Comparison of baby and big Whittaker categories

7.1. To complete the task set in §6.1, this section will compare the *baby Whittaker* category $D(\mathrm{Fl}_{G,x}^{\mathrm{aff}})^{\overset{o}{I^-},\psi_{o_{I^-}}}$ considered in [AB] to $\mathsf{Whit}(D(\mathrm{Fl}_{G,x}^{\mathrm{aff}}))$, which by Theorem 6.2.1 is equivalent to $\mathsf{Whit}(D^!(\mathrm{Fl}_x^{\frac{\infty}{2}}))$, the main category considered in this paper.

Our main result is Theorem 7.3.1, showing that these two categories are equivalent.

7.2. **Shifted Whittaker objects.** For $\mathcal{C}$ a category acted on by $G(K_x)$, we let $\mathsf{Whit}'(\mathcal{C})$ denote the *shifted Whittaker category*, where we use the character $N^-(K_x) \to \mathbb{G}_a$ given by the composition $N^-(K_x) \xrightarrow{\mathrm{Ad}_{\check\rho(t)}(-)} N^-(K_x) \to \mathbb{G}_a$ with the second map the standard (conductor zero) character. We use the notation ψ' for the corresponding character sheaf on $N^-(K_x)$.

Remark 7.2.1. For G with connected center, $\mathsf{Whit}'(\mathcal{C})$ is canonically equivalent to $\mathsf{Whit}(\mathcal{C})$ for any $\mathcal{C}$ acted on by $G(K)$: the equivalence is effected by the action of $\check\rho(t) \in G(K)$.

Note that the resulting character is non-trivial on elements of the form $\exp(f_i)$, instead of the usual $\exp(\frac{f_i}{t})$. The point of it is that there is a canonical "non-degenerate"[28] character $\psi_{\overset{o}{I^-}}$ of $\overset{o}{I^-}$ which coincides with the above character on the intersection. This character is the one used in [AB], and corresponding equivariant categories are called "baby Whittaker."

7.3. We have a functor $\mathsf{Whit}'(D(\mathrm{Fl}^{\mathrm{aff}}_{G,x})) \to D(\mathrm{Fl}^{\mathrm{aff}}_{G,x})^{\overset{o}{I^-},\psi_{\overset{o}{I^-}}}$ given by forgetting the Whittaker condition and then $*$-averaging against $\overset{o}{I^-},\psi_{\overset{o}{I^-}}$. We denote this functor by $\mathrm{Av}_*^{\overset{o}{I^-},\psi_{\overset{o}{I^-}}}$.

It is easy to see that this functor admits a left adjoint, since every object in the right hand side is (ind-)holonomic and because $\overset{o}{I^-} \cap N^-(K_x) \subseteq N^-(K_x)$ is a compact open subgroup: one applies [Ras3] Proposition 6.19.2. We denote this left adjoint by $\mathrm{Av}_!^{\mathsf{Whit}'}$.

Theorem 7.3.1. *The adjoint functors:*

$$D(\mathrm{Fl}^{\mathrm{aff}}_{G,x})^{\overset{o}{I^-},\psi_{\overset{o}{I^-}}} \xrightleftharpoons[\mathrm{Av}_*^{\overset{o}{I^-},\psi_{\overset{o}{I^-}}}]{\mathrm{Av}_!^{\mathsf{Whit}'}} \mathsf{Whit}'(D(\mathrm{Fl}^{\mathrm{aff}}_{G,x}))$$

are mutually inverse equivalences.

Remark 7.3.2. In [Ras5], we suggest a systematic framework for when baby Whittaker should coincide with the full Whittaker category: the corresponding Whittaker category, which conjecturally decomposes over $\mathrm{LocSys}_{\check{G}}(\overset{o}{\mathcal{D}}_x)$, should lie over the locus of local systems with regular singularities. We refer to *loc. cit.* for evidence for this conjecture and for higher slope generalizations.

7.4. Let $1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}}$ denote the canonical point of $\mathrm{Fl}_{G^{\mathrm{aff}}}$.

7.5. **Relevant orbits.** We begin by analyzing which orbits admit baby and shifted Whittaker sheaves on $\mathrm{Fl}^{\mathrm{aff}}_{G,x}$.

Let $W^{\mathrm{aff},\mathrm{ext}}$ denote the *extended affine Weyl group* $W \ltimes \check{\Lambda}$. Let W^{aff} be the *non-extended affine Weyl group* given as the semidirect product of W and the $\mathbb{Z}$-span of the coroots.

Remark 7.5.1. After a choice of Borel in G, one knows that W^{aff} picks up a canonical structure of Coxeter group, i.e., the corresponding simple reflections are determined. We use the Borel B^- in making these conventions. This choice reflects the fact that we are using $\overset{o}{I^-}$ and $N^-(K_x)$ for our characters. (But we continue to reference positive and dominant co/weights for G using B to define positivity).

We alert the reader that the same convention is implicitly used in [AB].

Remark 7.5.2. Recall that the length function on W^{aff} extends in a standard way to one on $W^{\mathrm{aff},\mathrm{ext}}$. This will be recalled explicitly in the proof of Proposition 7.5.9.

Notation 7.5.3. In the affine Weyl group, we use the notation $w\check{\lambda}$ to denote the product of the elements w and $\check{\lambda}$. This should not be confused with $w(\check{\lambda})$, the result of letting the Weyl group act on $\check{\Lambda}$.

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[28]We use scare quotes here because an expert in Kac-Moody representations would not call it non-degenerate: it vanishes on the affine root space spanned by $te_{\alpha_{max}}$.

The map $W^{\mathrm{aff,ext}} \to \mathrm{Fl}^{\mathrm{aff}}_{G,x}$ given by $\check{\lambda}w \mapsto \check{\lambda}(t)w 1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}}$ (we choose representatives in G for elements of the Weyl group) gives a set of points indexing both the $\overset{o}{I}{}^-$ orbits and the $N^-(K_x)$ orbits on $\mathrm{Fl}^{\mathrm{aff}}_{G,x}$.

Remark 7.5.4. The closure relations among the former are given by the Bruhat ordering on the extended affine Weyl group, while closure relations among the latter are given by the *semi-infinite Bruhat ordering*, c.f. [FFKM] §5. However, we will not explicitly need either of these facts in what follows.

For $g \in G(K_x)$ with $\overline{g}$ the induced point $g \cdot 1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}}$ in $\mathrm{Fl}^{\mathrm{aff}}_{G,x}$, note that the orbit $N^-(K_x)\overline{g}$ supports a shifted Whittaker sheaf[29] if and only if:

$$\mathfrak{n}^-(K_x) \cap \mathrm{Ad}_g(\mathrm{Lie}(I)) \subseteq \mathrm{Ker}(\psi'_{N^-(K_x)}) \tag{7.5.1}$$

and similarly, the orbit supports a baby Whittaker sheaf if and only if:

$$\mathrm{Lie}(\overset{o}{I}{}^-) \cap \mathrm{Ad}_g \mathrm{Lie}(I) \subseteq \mathrm{Ker}(\psi_{\overset{o}{I}{}^-}). \tag{7.5.2}$$

For our explicit orbit representatives, we easily find:

Proposition 7.5.5. *For $\check{\lambda}w \in W^{\mathrm{aff,ext}}$, the corresponding $N^-(K_x)$-orbit (resp. $\overset{o}{I}{}^-$-orbit) supports a Whittaker sheaf if and only if:*

$$\begin{cases} (\check{\lambda}, \alpha_i) \leqslant 0 & \text{if } w^{-1}(\alpha_i) > 0 \\ (\check{\lambda}, \alpha_i) < 0 & \text{if } w^{-1}(\alpha_i) < 0 \end{cases} \tag{7.5.3}$$

for every $i \in \mathcal{I}_G$.

Definition 7.5.6. We say that $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ (or the corresponding $N^-(K_x)$ or $\overset{o}{I}{}^-$ orbit) is *relevant* if (7.5.3) is satisfied.

Remark 7.5.7. As we will see in the proof of Proposition 7.5.9, the inequalities (7.5.3) force the generalization where we allow general positive roots α in place of the simple roots α_i.

Remark 7.5.8. If $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ is relevant, then $B(O_x) \cdot \check{\lambda}w = \check{\lambda}w \in \mathrm{Fl}^{\mathrm{aff}}_{G,x}$. It follows that:

$$\overset{o}{I}{}^- \cdot \check{\lambda}w \subseteq N^-(K_x) \cdot \check{\lambda}w.$$

To compare with [AB], we include the following computation, well-known and implicit in *loc. cit.*, but for which we are not sure of a good reference and therefore include for the reader's convenience. The reader may safely skip this material.

Proposition 7.5.9. $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ *is relevant if and only if $\check{\lambda}w$ is the unique element of minimal length in $W \cdot \check{\mu}$ for some $\check{\mu} \in \check{\Lambda}$.*

Proof. The existence of a unique minimal length element in this coset follows from the fact that W is a parabolic subgroup (in the sense of Coxeter groups) in the affine Weyl group W^{aff}.

Recall that we can compute the length of an element $\check{\lambda}w \in W^{\mathrm{aff,ext}}$ by the formula:[30]

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[29]I.e. Whit$'$ of the corresponding orbit is non-zero.

[30]This formula relies on the convention of Remark 7.5.1. One usually finds this formula written relative to the positive Borel, in which case the formula would have last term $|(\check{\lambda}, \alpha) - 1|$, but switching α with $-\alpha$ everywhere, we obviously recover the formula in its given form.

$$\ell(\check{\lambda}w) = \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)>0}} |(\check{\lambda},\alpha)| + \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)<0}} |(\check{\lambda},\alpha)+1|.$$

For $\check{\lambda} = w(\check{\mu})$, so that $\check{\lambda}w = w\check{\mu}$, we find:

$$\ell(w\check{\mu}) = \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)>0}} |(w(\check{\mu}),\alpha)| + \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)<0}} |(w(\check{\mu}),\alpha)+1| = \tag{7.5.4}$$
$$\sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)>0}} |(\check{\mu},w^{-1}(\alpha))| + \sum_{\substack{\alpha>0 \text{ a root} \\ w^{-1}(\alpha)<0}} |(\check{\mu},w^{-1}(\alpha))+1|.$$

Let $w_{\check{\mu}}$ be the minimal length element of W such that $w_{\check{\mu}}(-\check{\mu})$ lies in the dominant chamber: the uniqueness of a minimal length such element is again guaranteed by the fact that the appropriate stabilizer group is a parabolic subgroup of W.

We claim that $w_{\check{\mu}}$ is characterized in W by the identities:

$$\begin{cases} (w_{\check{\mu}}(\check{\mu}),\alpha) \leqslant 0 & \text{for } \alpha > 0 \text{ with } w_{\check{\mu}}^{-1}(\alpha) > 0 \\ (w_{\check{\mu}}(\check{\mu}),\alpha) < 0 & \text{for } \alpha > 0 \text{ with } w_{\check{\mu}}^{-1}(\alpha) < 0. \end{cases} \tag{7.5.5}$$

Indeed, we have $(w_{\check{\mu}}(\check{\mu}),\alpha) \leqslant 0$ for all $\alpha > 0$ by dominance of $-w_{\check{\mu}}(\check{\mu})$. Then recall that for $\alpha > 0$, $w^{-1}(\alpha) < 0$ is equivalent to $\ell(s_\alpha w) < \ell(w)$.[31] Therefore, if we had $w_{\check{\mu}}^{-1}(\alpha) < 0$ and $(w_{\check{\mu}}(\check{\mu}),\alpha) = 0$, this would force:

$$\ell(s_\alpha w_{\check{\mu}}) < \ell(w_{\check{\mu}})$$
$$s_\alpha w_{\check{\mu}}(\check{\mu}) = w_{\check{\mu}}(\check{\mu}) - (w_{\check{\mu}}(\check{\mu}),\alpha)\alpha = w_{\check{\mu}}(\check{\mu})$$

contradicting the minimality of $w_{\check{\mu}}$.

We see from this argument that it is enough to verify (7.5.5) in the case that α is a simple root.

Next, we claim that $w_{\check{\mu}}$ minimizes (7.5.4).

Indeed, let $w \in W$ other than $w_{\check{\mu}}$. Since we noted that $w_{\check{\mu}}$ is characterized by the identities (7.5.5) for α a simple root, we see that $w \neq w_{\check{\mu}}$ implies that either there exists a simple root α_i with $w^{-1}(\alpha_i) > 0$ and $w(\check{\mu},\alpha_i) > 0$, or else there exists α_i with $w^{-1}(\alpha_i) < 0$ and $(w(\check{\mu}),\alpha_i) \geqslant 0$.

In the former case, using the fact that s_i permutes the non-α_i positive roots, one finds:

$$\ell(s_i w\check{\mu}) - \ell(w\check{\mu}) = |(s_i w(\check{\mu}),\alpha_i)+1| - |(w(\check{\mu}),\alpha_i)| = |-(w(\check{\mu}),\alpha_i)+1| - |(w(\check{\mu}),\alpha_i)| = -1$$

and in the latter case, one similarly finds:

$$\ell(s_i w\check{\mu}) - \ell(w\check{\mu}) = |(s_i w(\check{\mu}),\alpha_i)| - |(w(\check{\mu}),\alpha_i)+1| = |(w(\check{\mu}),\alpha_i)| - |(w(\check{\mu}),\alpha_i)+1| = -1.$$

Either way, $\ell(s_i w\check{\mu}) < \ell(w\check{\mu})$, meaning that $w\check{\mu}$ was not of minimal length.

Finally, one immediately sees that in terms of $\check{\lambda} = w_{\check{\mu}}(\check{\mu})$, (7.5.5) exactly translates into (7.5.3), as desired (appealing to the fact that it is enough to verify (7.5.5) for simple roots.)

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[31]This fact is completely standard for α a simple root, but perhaps warrants a proof for general $\alpha > 0$ since e.g. it does not appear in [Hum] Chapter 1. We prove the claim by induction on $\ell(w)$, the case $\ell(w) = 0$ being obvious. Choose $i \in \mathcal{I}_G$ with $w(\alpha_i) < 0$; let s_i denote the corresponding simple reflection. If $w(\alpha_i) \neq -\alpha$, then $\ell(ws_i) < \ell(w)$ and $(ws_i)^{-1}(\alpha) < 0$, so by induction, $\ell(s_\alpha ws_i) < \ell(ws_i) = \ell(w) - 1$, but $\ell(s_\alpha ws_i) \geqslant \ell(s_\alpha w) - 1$, giving the claim in this case. Otherwise, $ws_i(\alpha_i) = \alpha$. Then $(ws_i)^{-1}s_\alpha ws_i = s_i$, since this this is a reflection swapping the sign of α_i. We obtain $s_\alpha w = ws_i$, but $w(\alpha_i) < 0$ implies that $\ell(s_\alpha w) = \ell(ws_i) < \ell(w)$.

$\square$

7.6. Minimal orbits.

We introduce two parallel pictures for $\overset{o}{I}{}^-$ and $N^-(K_x)$ orbits on $\mathrm{Fl}^{\mathrm{aff}}_{G,x}$.

We define the *minimal* $N^-(K_x)$-orbit (resp. $\overset{o}{I}{}^-$) orbit to be the orbit through $1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}}$.

We define $j_!^{\min,\mathsf{Whit}'} \in \mathsf{Whit}'(D(\mathrm{Fl}^{\mathrm{aff}}_{G,x}))$ and $j_!^{\min,\mathrm{baby}} \in D(\mathrm{Fl}^{\mathrm{aff}}_{G,x})^{\overset{o}{I}{}^-,\psi_{\overset{}{I}{}^-}}$ be the !-extensions of the relevant character sheaves supported on these orbits.[32]

7.7. Cleanness.

The main point in proving Theorem 7.3.1 are the following two *cleanness* results.

Remark 7.7.1. Suppose that $j : U \hookrightarrow Z$ is a locally closed embedding of schemes of finite type. Recall that $\mathcal{F} \in D(Z)$ is said to be *cleanly extended* from U if the maps $j_! j^!(\mathcal{F}) \to \mathcal{F} \to j_{*,dR} j^{*,dR}(\mathcal{F})$ are isomorphisms. This definition extends to the setting of ind-schemes of ind-finite type in the obvious way.

Proposition 7.7.2. *The object* $j_!^{\min,\mathrm{baby}}$ *is cleanly extended from the orbit* $\overset{o}{I}{}^- \cdot 1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}}$.

Proposition 7.7.3. *The object* $j_!^{\min,\mathsf{Whit}'}$ *is cleanly extended from the orbit* $N^-(K_x) \cdot 1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}}$.

Each of these results follows easily from the closure relations noted above, but we give complete proofs below.

Proof of Proposition 7.7.2. We have:

$$\overset{o}{I}{}^- \cdot 1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}} = N^- \cdot 1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}} \overset{\mathrm{open}}{\subseteq} G/B \overset{\mathrm{closed}}{\subseteq} \mathrm{Fl}^{\mathrm{aff}}_{G,x}.$$

On N^-, our sheaf is a non-degenerate character sheaf, and this obviously extends cleanly to G/B.

$\square$

Proof of Proposition 7.7.3. We use the techniques of §3 freely here.

Let $Z \subseteq \mathrm{Fl}^{\mathrm{aff}}_{G,x}$ be the pullback of $\overline{\mathrm{Gr}}_{N^-} \subseteq \mathrm{Gr}_G$. Then Z is ind-closed in $\mathrm{Fl}^{\mathrm{aff}}_{G,x}$ and contains the orbit $N^-(K_x) \cdot 1_{\mathrm{Fl}^{\mathrm{aff}}_{G,x}}$ as an ind-open subscheme.

Clearly the only $N^-(K_x)$-orbits in Z pass through points $\check{\lambda}w$ with $\check{\lambda} \in \check{\Lambda}^{pos}$.

We claim that the only such $\check{\lambda}w$ supporting a Whittaker sheaf is $\check{\lambda} = 0$, $w = 1$. Indeed, as in the proof of Proposition 7.5.9, the inequalities (7.5.3) force the same inequalities for a general positive root, not merely a simple root. Then we see $\check{\lambda} \in \check{\Lambda}^{pos}$ forces:

$$0 \leqslant (\check{\lambda}, \rho)(\check{\lambda}, \frac{1}{2}\sum_{\alpha > 0} \alpha) = \frac{1}{2}\sum_{\alpha > 0}(\check{\lambda}, \alpha) \leqslant 0$$

so we must have equality, forcing $\check{\lambda} = 0$, and then we further see from (7.5.3) that we must have $w = 1$ as well.

This now gives the cleanness result.

$\square$

---

[32]To see that $j_!^{\min,\mathsf{Whit}'}$ actually lies in the shifted Whittaker subcategory, exhaust $N^-(K_x)$ by compact open subgroups and exploit placidity of these subgroups.

Corollary 7.7.4. *The unit and counit maps:*

$$j_!^{\cdot\min,\text{baby}} \to \text{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}} \text{Av}_!^{\text{Whit}'}(j_!^{\cdot\min,\text{baby}})$$

$$\text{Av}_!^{\text{Whit}'} \text{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}(j_!^{\cdot\min,\text{Whit}'}) \to j_!^{\cdot\min,\text{Whit}'}$$

are isomorphisms.

Proof. By Remark 7.5.8, we obtain that:

$$\text{Av}_!^{\text{Whit}'}(j_!^{\cdot\min,\text{baby}}) \simeq j_!^{\cdot\min,\text{Whit}'}.$$

Note that Remark 7.5.8 implies that the only relevant $\overset{o}{I}{}^-$-orbit intersecting $N^-(K_x) \cdot 1_{\text{Fl}_{G,x}^{\text{aff}}}$ is $\overset{o}{I}{}^- \cdot 1_{\text{Fl}_{G,x}^{\text{aff}}}$.

Therefore, applying cleanness of the $j_!^{\cdot\min,\text{Whit}'}$, we obtain that $\text{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}(j_!^{\cdot\min,\text{Whit}'})$ is the $*$-extension of our character sheaf from $\overset{o}{I}{}^- \cdot 1_{\text{Fl}_{G,x}^{\text{aff}}}$. Moreover, applying cleanness of the latter, we obtain:

$$\text{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}(j_!^{\cdot\min,\text{Whit}'}) \simeq j_!^{\cdot\min,\text{Whit}'}$$

as desired.

$\square$

7.8. Compatibility with the affine Hecke algebra.

Both categories $D(\text{Fl}_{G,x}^{\text{aff}})^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ and $\text{Whit}'(D(\text{Fl}_{G,x}^{\text{aff}}))$ are acted on by the geometric affine Hecke algebra $H_{\text{aff}} := D(\text{Fl}_{G,x}^{\text{aff}})^I$ by the convolution action of H_{aff} on $D(\text{Fl}_{G,x}^{\text{aff}})$.

Moreover, the functor $\text{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ is given by a convolution, and therefore commutes with H_{aff}-actions.

One can further see that $\text{Av}_!^{\text{Whit}'}$ commutes with the H_{aff}-actions by exploiting the ind-properness of $\text{Fl}_{G,x}^{\text{aff}}$. Alternatively: we don't actually need this fact; we will only need that $\text{Av}_!^{\text{Whit}'}$ commutes with convolution with Mirkovic-Wakimoto sheaves, and this follows formally from their invertibility and the fact that $\text{Av}_*^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ commutes with such convolutions.

7.9. We now prove Theorem 7.3.1.

Proof of Theorem 7.3.1. The category $D(\text{Fl}_{G,x}^{\text{aff}})^{\overset{o}{I}^-,\psi_{\overset{o}{I}^-}}$ is compactly generated by objects !-extended from relevant orbits, and similarly for $\text{Whit}'(D(\text{Fl}_{G,x}^{\text{aff}}))$. For $\check{\lambda}w \in W^{\text{aff},\text{ext}}$ relevant, let $j_!^{\check{\lambda}w,\text{baby}}$ and $j_!^{\check{\lambda}w,\text{Whit}'}$ denote the corresponding objects.

As in [AB] Lemma 4, the object $j_!^{\check{\lambda}w,\text{baby}}$ is obtained from $j_!^{\min,\text{baby}}$ by convolving with an appropriate invertible object of H_{aff}.

Therefore, by Corollary 7.7.4 and §7.8, the unit map of the adjunction applied to $j_!^{\check{\lambda}w,\text{baby}}$ is an equivalence.

Moreover, we claim that:

$$\mathrm{Av}_!^{\mathsf{Whit}'}(j_!^{\check{\lambda}w,\mathrm{baby}}) \xrightarrow{\simeq} j_!^{\check{\lambda}w,\mathsf{Whit}'}.$$

Indeed, this is immediate from Remark 7.5.8. Therefore, $j_!^{\check{\lambda}w,\mathsf{Whit}'}$ is similarly obtained from $j_!^{\min,\mathrm{baby}}$ by convolving with the appropriate invertible object of H_{aff}. Therefore, as for the baby Whittaker category, we see that the counit for $j_!^{\check{\lambda}w,\mathsf{Whit}'}$ is an equivalence.

By compact generation, we now obtain the result.

$\square$

8. Modules over some factorization algebras

8.1. In [BD2], the *abelian* category of factorization modules for classical factorization algebras are expressed in more familiar linear algebra terms, e.g., modules over associative algebras. The goal for this section is to prove a technical result, Theorem 8.13.1, which is a derived version of the ideas from [BD2], and will play an important role in §9.

However, the techniques from [BD2] are truly specific to the abelian categorical framework, and we accordingly obtain results only for much more restrictive classes of factorization algebras.

Remark 8.1.1. In fact, one can show that the results analogous to those of [BD2] are false in general (at least in their naive formulations): as is now a standard pattern in homological algebra, it appears that the problem occurs "in cohomological degree $-\infty$." It remains an interesting problem how to properly understand the relationship between factorization modules in the DG setting and the punctured disc in less restrictive settings than that treated below.

8.2. We fix a smooth curve X in this section and a closed point $x \in X$.

Let $U \subseteq X$ be the complement to X, and let:

$$x \xhookrightarrow{i} X \xhookleftarrow{j} U$$

denote the embeddings.

8.3. **Modifications.** Let $I \mapsto \mathsf{C}_{X_{dR}^I} \in D(X^I)\text{–}\mathsf{mod}$ be a unital commutative factorization category on X_{dR}.

Let:

$$I \mapsto \mathcal{A}_{U^I} \in \mathsf{C}_{U^I} := \mathsf{C}_{X_{dR}^I} \underset{D(X^I)}{\otimes} D(U^I)$$

be a unital commutative factorization algebra in $\mathsf{C}|_{U_{dR}}$. We denote the datum of this factorization algebra by $\mathcal{A}$.

Definition 8.3.1. A *modification* of $\mathcal{A}$ is a unital commutative factorization algebra $\mathcal{A}' \in \mathsf{C}$ with an isomorphism of $\mathcal{A}'|_{U_{dR}} \simeq \mathcal{A}$ (as commutative factorization algebras).

8.4. We continue to let $\mathcal{A} \in \mathsf{ComAlg}_{un}^{\mathrm{fact}}(\mathsf{C}|_{U_{dR}})$ be a unital commutative factorization algebra.

Definition 8.4.1. We say that $\mathcal{A}$ *admits a universal modification* if the left adjoint to the restriction functor $\mathsf{ComAlg}_{un}^{\mathrm{fact}}(\mathsf{C}) \to \mathsf{ComAlg}_{un}^{\mathrm{fact}}(\mathsf{C}|_{U_{dR}})$ is defined on $\mathcal{A}$.

In this case, we let $\mathcal{A}^{univ} \in \mathsf{ComAlg}^{\mathrm{fact}}(\mathsf{C})$ denote the corresponding extension of $\mathcal{A}$.

Remark 8.4.2. One can see that the restriction functor above admits a *right* adjoint: this right adjoint sends $\mathcal{A}$ to the (unital, commutative) factorization algebra defined by the assignment:

$$I \mapsto j_{I,*,dR}(\mathcal{A}_{U^I}) \tag{8.4.1}$$

where $j_I : U^I \hookrightarrow X^I$ is the embedding.

Remark 8.4.3. Using the right adjoint (8.4.1), one easily sees that the tautological map $\mathcal{A} \to \mathcal{A}^{univ}|_{U_{dR}}$ is an isomorphism, meaning that $\mathcal{A}^{univ}$ is a modification of $\mathcal{A}$ and justifying the terminology. Of course, $\mathcal{A}^{univ}$ is initial among all modifications of $\mathcal{A}$.

Remark 8.4.4. Suppose that $\mathcal{A}$ admits a universal modification $\mathcal{A}^{\mathrm{univ}}$. Let C_x denote the fiber of C at $x \in X$. Following [BD1] §2.6 and [BD2] §2.4, we use the notation:

$$H_\nabla(\overset{o}{\mathcal{D}}_x, \mathcal{A}) := i^!(\mathcal{A}^{\mathrm{univ}}_{X_{dR}}) \in \mathsf{ComAlg}(\mathsf{C}_x)$$

for this fiber, and refer to this commutative algebra as the *local conformal blocks* of $\mathcal{A}$ along $\overset{o}{\mathcal{D}}_x$.

If $\mathcal{A}$ is a unital commutative factorization algebra on X_{dR} (not just U_{dR}), we use the same notation for the local conformal blocks of $\mathcal{A}|_{U_{dR}}$.

8.5.   For any unital commutative factorization algebra $\mathcal{A} \in \mathsf{ComAlg}_{un}^{\mathrm{fact}}(\mathsf{C})$, there is a canonical functor:

$$i^!(\mathcal{A}_{X_{dR}})\text{--}\mathsf{mod}(\mathsf{C}_x) \to \mathcal{A}\text{--}\mathsf{mod}_{un,x}^{\mathrm{fact}}(\mathsf{C}).$$

Since the right hand side depends only on $\mathcal{A}|_{U_{dR}}$, we can replace $\mathcal{A}$ by any modification.

In particular, we obtain a functor:

$$H_\nabla(\overset{o}{\mathcal{D}}_x, \mathcal{A})\text{--}\mathsf{mod}(\mathsf{C}_x) \to \mathcal{A}\text{--}\mathsf{mod}_{un,x}^{\mathrm{fact}}(\mathsf{C}). \tag{8.5.1}$$

8.6.   With these constructions in hand, we can now ask the motivating questions of this section:
- When do universal modifications exist?
- If a universal modification of $\mathcal{A}$ exists, is the functor (8.5.1) an equivalence?

Remark 8.6.1. In the non-derived setting of [BD2], it is shown that the (classical) pro-commutative algebra associated with the cofiltered diagram of modifications of a (classical) commutative factorization algebra governs the abelian category of modules for this algebra.

Our takeaway from this is that universal modifications do not typically exist, but that when they do, we can at least be optimistic that the answer to the second question will be affirmative.

8.7.   In what follows, we will give a setting in which universal modifications exist and the functor (8.5.1) is an equivalence.

For example, for the factorization category $I \mapsto D(X^I)$, we will see that *constant* commutative factorization algebra, i.e., those whose commutative D_X-algebra is of the form $A \otimes \omega_X$ for $A \in \mathsf{ComAlg}$, demonstrate such nice behavior.

8.8.   Recall from [Ras1] §7 (c.f. also [Ras2] §6) that for $\mathcal{C} \in \mathsf{ComAlg}(\mathsf{DGCat}_{cont})$, there is a canonical unital commutative factorization category:[33]

$$I \mapsto \mathcal{C}_{X_{dR}^I} \in D(X^I)\text{--}\mathsf{mod}.$$

We let $\mathcal{C}^{\mathrm{fact}}$ refer to the corresponding unital commutative factorization category.

Moreover, for every $A \in \mathsf{ComAlg}(\mathcal{C})$, there is an accompanying (commutative) unital factorization algebra:

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[33]In [Ras2], it was denoted $I \mapsto \mathcal{C}_{X^I}$ instead. Hopefully this will not cause too much confusion.

$$I \mapsto A^{\mathrm{fact}}_{X^I_{dR}} \in \mathcal{C}_{X^I_{dR}}.$$

We let $A^{\mathrm{fact}} \in \mathsf{ComAlg}^{\mathrm{fact}}_{un}(\mathcal{C})$ denote the corresponding factorization algebra. We refer to A^{fact} as a *constant* (commutative) factorization algebra in $\mathcal{C}^{\mathrm{fact}}$, and use this notation A^{fact} to indicate the existence of $A \in \mathsf{ComAlg}(\mathcal{C})$ giving rise to A.

Warning 8.8.1. For clarity, we note that for constant factorization algebras as above, only $A_{X_{dR}}$ is actually constant in any sense: for general $\mathcal{C}$, asking if $A_{X^2_{dR}}$ is constant does not make sense, and even for $\mathcal{C} = \mathsf{Vect}$, $A_{X^2_{dR}}$ is not typically constant.

8.9.   We have the following result on the existence of universal modifications.

Proposition 8.9.1. *For $A^{\mathrm{fact}} \in \mathsf{ComAlg}^{\mathrm{fact}}_{un}(\mathcal{C}^{\mathrm{fact}})$ a constant factorization algebra, the universal modification of $A^{\mathrm{fact}}|_{U_{dR}}$ exists.*

The proof will be given in §8.12 after some preliminary constructions.

8.10.   We recall the following result, valid for any commutative factorization category:

Lemma 8.10.1. *The restriction functor:*

$$\mathsf{ComAlg}^{\mathrm{fact}}_{un}(\mathcal{C}^{\mathrm{fact}}) \to \mathsf{ComAlg}(\mathcal{C}_{X_{dR}})$$

is an equivalence.

Proof (sketch). This result is well-known in the classical setting and not difficult in the derived setting, so we do not give a proof here. However, justc to convince the reader that the result holds for general commutative factorization categories, we indicate how to recover $\mathcal{A}_{X^2_{dR}} \in \mathcal{C}_{X^2_{dR}}$ from the commutative algebra structure on $\mathcal{A}_{X_{dR}}$.

Let κ denote the functor:

$$\kappa : \mathcal{C}_{X_{dR}} \otimes \mathcal{C}_{X_{dR}} \to \mathcal{C}_{X^2_{dR}} \in D(X^2)\text{--}\mathsf{mod}$$

encoding the commutative factorization category structure on $\mathcal{C}^{\mathrm{fact}}$. Let $- \overset{!}{\otimes} - := \Delta^!(\kappa(- \boxtimes -))$ denote the induced binary operation on $\mathcal{C}_{X_{dR}}$.

Then $\mathcal{A}_{X^2_{dR}}$ is the pushout:

$$
\begin{array}{ccc}
\Delta_{*,dR}(\mathcal{A}_{X_{dR}} \overset{!}{\otimes} \mathcal{A}_{X_{dR}}) & \longrightarrow & \mathcal{A}_{X_{dR}} \boxtimes \mathcal{A}_{X_{dR}} \\
\downarrow & & \downarrow \\
\Delta_{*,dR}(\mathcal{A}_{X_{dR}}) & \longrightarrow & \mathcal{A}_{X^2_{dR}}.
\end{array}
$$

$\square$

8.11.   **A special case.** Suppose that $V \in \mathcal{C}$ is given. Let $A = \mathrm{Sym}(V)$. We will explicitly construct the universal modification of A^{fact} in this case.

First, note that $j_!(V \otimes \omega_U) \in \mathcal{C}_{X_{dR}} = \mathcal{C} \otimes D(X)$ is defined.

We then claim that $A^{\mathrm{fact},\mathrm{univ}}$ is the unital commutative factorization algebra associated with $\mathrm{Sym}(j_!(V \otimes \omega_U)) \in \mathsf{ComAlg}(\mathcal{C}_{X_{dR}})$ via Lemma 8.10.1.

Indeed, for any modification $\mathcal{A}' \to A^{\mathrm{fact}}$, we have the canonical map:

$$V \otimes \omega_U \to \mathcal{A}'|_{U_{dR}} \simeq A^{\text{fact}}|_{U_{dR}} = \text{Sym}(V \otimes \omega_U).$$

This gives the desired map:

$$\text{Sym}(j_!(V \otimes \omega_U)) \to \mathcal{A}' \in \text{ComAlg}(\mathcal{C}_{X_{dR}})$$

by adjunction.

8.12.   We now deduce the proposition.

Proof of Proposition 8.9.1. The class of unital commutative factorization algebras admitting universal modifications is closed under colimits, since the universal modification is a partially-defined left adjoint.

Moreover, the assignment $A \mapsto A^{\text{fact}}$ obviously commutes with sifted colimits. Since every $A \in \text{ComAlg}(\mathcal{C})$ is a geometric realization of commutative algebras of the form $\text{Sym}(V)$ for $V \in \mathcal{C}$ (by the Barr-Beck theorem, say), we obtain the result from §8.11.

□

8.13.   We now formulate the main result of this section.

Theorem 8.13.1. *Suppose that $\mathcal{C}$ is compactly generated and rigid symmetric monoidal in the sense of* [Gai5] *(i.e., every compact object admits a dual).*

Then for $A^{\text{fact}} \in \text{ComAlg}_{un}^{\text{fact}}(\mathcal{C}^{\text{fact}})$ a constant factorization algebra, the functor:

$$H_\nabla(\overset{o}{\mathcal{D}}_x, A^{\text{fact}})\text{–mod}(\mathcal{C}) \to A^{\text{fact}}\text{–mod}_{un,x}^{\text{fact}}(\mathcal{C}^{\text{fact}}).$$

of (8.5.1) *is an equivalence.*

The remainder of this section is devoted to the proof of this theorem. Its proof can be found in §8.23.

8.14. **Lie-$*$ algebras.**   We will deduce Theorem 8.13.1 from a similar result, Theorem 8.22.1, which works instead for factorization algebras associated with Lie-$*$ algebras.

Therefore, we hope that the reader will excuse the extended digression that follows.

8.15.   Suppose as above that C is a commutative factorization category on X_{dR}.

Let $L \in \mathsf{C}_{X_{dR}}$ be a Lie-$*$ algebra. Recall that we have the associated chiral Lie algebra $U^{ch}(L) \in \mathsf{C}_{X_{dR}}$: this is the (unital version of the) chiral envelope of L. Let $C_\bullet^{ch}(L) \in \text{Alg}^{\text{fact}}(\mathsf{C})$ denote theassociated factorization algebra.

Recall that we have the DG categories $L\text{–mod}_x$ of Lie-$*$ modules for L at x, and $U^{ch}(L)\text{–mod}_x$ of (unital) chiral modules for $U^{ch}(L)$ at x. These categories are related by adjoint functors:

$$L\text{–mod} \underset{\text{Oblv}}{\overset{\text{Ind}^{ch}}{\rightleftarrows}} U^{ch}(L)\text{–mod}_x \xrightarrow{\;\simeq\;} C_\bullet^{ch}(L)\text{–mod}_{un,x}^{\text{fact}}.$$

8.16.   For L as above, note that $j_{*,dR}j^!(L)$ admits a canonical Lie-$*$ structure.

Lemma 8.16.1. *Suppose that $L \xrightarrow{\simeq} j_{*,dR}j^!(L)$. Then Ind^{ch} and Oblv are mutually inverse equivalences.*

Proof. As in [BD2] §3.7.16-18, fully-faithfulness of Ind^{ch} follows from the chiral PBW theorem for modules (see [Ras1] §7.19 for a version for this derived context). Since Oblv is conservative, this gives the result.

□

8.17. Rigidity assumption. For the remainder of §8, we impose the assumption that $\mathcal{C}$ is compactly generated and rigid symmetric monoidal.

8.18. Nice Lie-$*$ algebras. Let $L \in \mathcal{C}_{X_{dR}}$ be a Lie-$*$ algebra. Below, we introduce a condition of *niceness* for L, which plays a similar role that the existence of universal modifications played in the commutative setting.

At first pass, the reader should take the following definition to simply mean that the $*$-fiber $L_x = i^{*,dR}(L) \in \mathcal{C}$ of L at x is defined.[34]

However, for technical reasons, we need the following strengthening of this condition.

Definition 8.18.1. L is *nice* (at $x \in X$) if for every $M \in \mathcal{C}_{X_{dR}}$, the left adjoint to the functor:

$$\mathcal{C} \otimes \mathcal{C}_{X_{dR}} = \Big(\mathcal{C} \otimes \mathsf{Vect}\Big) \otimes \mathcal{C}_{X_{dR}} \xrightarrow{(\mathrm{id}_{\mathcal{C}} \otimes i_{*,dR}) \otimes \mathrm{id}_{\mathcal{C}_{X_{dR}}}} \Big(\mathcal{C} \otimes D(X)\Big) \otimes \mathcal{C}_{X_{dR}} = \mathcal{C}_{X_{dR}} \otimes \mathcal{C}_{X_{dR}}$$

is defined on $L \boxtimes M \in \mathcal{C}_{X_{dR}} \otimes \mathcal{C}_{X_{dR}}$, and is computed by $L_x \boxtimes M$ (i.e., the relevant natural map from this restriction to $L_x \boxtimes M$ is an isomorphism).

Remark 8.18.2. We remark that the above definition makes sense for arbitrary objects of $\mathcal{C}_{X_{dR}}$, i.e., it does not make reference to the Lie-$*$ algebra structure on L.

8.19. Here is the first key feature of nice Lie-$*$ algebras.

Proposition-Construction 8.19.1. *For $L \in \mathcal{C}_{X_{dR}}$ a nice Lie-$*$ algebra, the $*$-fiber $L_x \in \mathcal{C}$ of L inherits a canonical structure of Lie algebra in $\mathcal{C}$.*

To construct this structure, we use the following lemma.

Lemma 8.19.2. *For nice L as above and for every integer n, the $*$-restriction of:*

$$L^{\boxtimes n} \in \mathcal{C}_{X_{dR}}^{\otimes n} = \mathcal{C}^{\otimes n} \otimes D(X^n)$$

to the point $(x, \ldots, x) \in X^n$ is defined, i.e., if the left adjoint to the pushforward functor:

$$\mathcal{C}^{\otimes n} = \mathcal{C}^{\otimes n} \otimes \mathsf{Vect} \xrightarrow{\mathrm{id} \otimes (i^{\times n})_{*,dR}} \mathcal{C}^{\otimes n} \otimes D(X^n) = \mathcal{C}_{X_{dR}}^{\otimes n}$$

is defined on this exterior product. Moreover, the resulting fiber is canonically isomorphic to $L_x \boxtimes \ldots \boxtimes L_x \in \mathcal{C}^{\otimes n}$.

Proof. Immediate from the definition of niceness. $\square$

Proof of Proposition-Construction 8.19.1. Let n be a given integer. Let κ_n denote the functor:

$$\kappa_n : \mathcal{C}_{X_{dR}} \otimes \ldots \otimes \mathcal{C}_{X_{dR}} \to \mathcal{C}_{X^n}$$

encoding the commutative factorization category structure. By rigidity of $\mathcal{C}$, κ_n admits a $D(X^n)$-linear right adjoint: indeed, this follows from [Ras2] Propositions B.7.1 and 6.16.1.

Suppose we are given an n-ary operation in the Lie operad. The Lie-$*$ structure on L provides an associated map:

$$\kappa_n(L \boxtimes \ldots \boxtimes L) \to \Delta_{*,dR}(L) \tag{8.19.1}$$

where $\Delta : X \to X^n$ is the diagonal map.

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[34]That is, that the left adjoint to the functor: $\mathcal{C} = \mathcal{C} \otimes \mathsf{Vect} \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes i_{*,dR}} \mathcal{C} \otimes D(X) = \mathcal{C}_{X_{dR}}$ is defined on L.

Let $i^n = i \circ \Delta$ denote the embedding of $(x, \ldots, x)$ into X^n. Since κ_n admits a $D(X^I)$-linear right adjoint, it commutes with forming $i^{n,*,dR}$.

Therefore, applying $i^{n,*,dR}$ to (8.19.1) and computing the left hand side via Lemma 8.19.2, we obtain a map:

$$L_x \otimes \ldots \otimes L_x \to L_x \in \mathcal{C}.$$

We emphasize that the tensor product appearing here is with respect to the symmetric monoidal structure on $\mathcal{C}$.

$\square$

8.20. Similarly, we obtain the following result.

Proposition 8.20.1. *For $L \in \mathcal{C}_{X_{dR}}$ a nice Lie-$*$ algebra, one has a canonical isomorphism:*

$$L\text{–}\mathsf{mod}_x \xrightarrow{\simeq} L_x\text{–}\mathsf{mod}(\mathcal{C})$$

where in the right hand side, L_x has been equipped with the Lie algebra structure constructed above.

Proof. The proof is as above: e.g., the action map of L on M is encoded by a map:

$$\kappa(L \boxtimes i_{*,dR}(M)) \to (\Delta \circ i)_{*,dR}(M) \in \mathcal{C}_{X_{dR}^2}.$$

As before, rigidity of $\mathcal{C}$ and niceness of L mean that this is equivalent to a map:

$$L_x \otimes M \to M \in \mathcal{C}.$$

$\square$

8.21. We now show that some particular Lie-$*$ algebras are nice.

Proposition 8.21.1. *Suppose that $\mathfrak{l}$ is a Lie algebra in $\mathcal{C}$. Then $L := \mathfrak{l} \boxtimes k_X \in \mathcal{C} \otimes D(X)$ and $j_{*,dR}j^!(L) = \mathfrak{l} \boxtimes j_{*,dR}(k_U)$ are nice Lie-$*$ algebras.*

Since the definition of niceness does not refer to the Lie-$*$ algebra structure of an object of $\mathcal{C}_{X_{dR}}$, this follows from rigidity and compact generation of $\mathcal{C}$ via the following lemma.

Lemma 8.21.2. *Suppose $G : \mathcal{D}_2 \to \mathcal{D}_1 \in \mathsf{DGCat}_{cont}$ is given. Suppose that $\mathcal{F} \in \mathcal{D}_1$ is compact such that the partially-defined left adjoint F to G is defined on $\mathcal{F}$.*

Then for every compactly generated category $\mathcal{E} \in \mathsf{DGCat}_{cont}$ and every $\mathcal{G} \in \mathcal{E}$, the partially-defined left adjoint to $G \otimes \mathrm{id}_{\mathcal{E}} : \mathcal{D}_2 \otimes \mathcal{E} \to \mathcal{D}_1 \otimes \mathcal{E}$ is defined on $\mathcal{F} \boxtimes \mathcal{G}$ and is computed by $F(\mathcal{F}) \boxtimes \mathcal{G}$.

Proof. We reduce at once to assuming $\mathcal{G}$ is compact.

The composite functor:

$$\mathcal{D}_2 \xrightarrow{G} \mathcal{D}_1 \xrightarrow{\mathrm{Hom}_{\mathcal{D}_1}(\mathcal{F},-)} \mathsf{Vect} \in \mathsf{DGCat}_{cont}$$

admits a left adjoint sending k to $F(\mathcal{F})$.

It follows that the composite functor:

$$\mathcal{D}_2 \otimes \mathcal{E} \xrightarrow{G \otimes \mathrm{id}_{\mathcal{E}}} \mathcal{D}_1 \otimes \mathcal{E} \xrightarrow{\mathrm{Hom}_{\mathcal{D}_1}(\mathcal{F},-) \otimes \mathrm{id}_{\mathcal{E}}} \mathcal{E} \xrightarrow{\mathrm{Hom}_{\mathcal{E}}(\mathcal{G},-)} \mathsf{Vect}$$

admits a left adjoint sending k to $F(\mathcal{F}) \boxtimes \mathcal{G}$, which is what we needed to show.

$\square$

8.22.   We now deduce the following result from Lemma 8.16.1 and the preceding.

Theorem 8.22.1. *For $\mathfrak{l} \in \mathsf{LieAlg}(\mathcal{C})$, there is a canonical isomorphism:*

$$i^{*,dR} j_{*,dR}(\mathfrak{l} \otimes k_U)\text{–}\mathsf{mod}(\mathcal{C}) \simeq C_\bullet(\mathfrak{l} \otimes k_X)\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}$$

8.23. **Back to the commutative case.** We now return to the commutative setting to prove Theorem 8.13.1.

Proof of Theorem 8.13.1. Recall that we are supposed to show that:

$$H_\nabla(\overset{o}{\mathcal{D}}_x, A^{\mathrm{fact}})\text{–}\mathsf{mod}(\mathcal{C}) \to A^{\mathrm{fact}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}(\mathcal{C}^{\mathrm{fact}}) \tag{8.23.1}$$

is an equivalence for $A \in \mathsf{ComAlg}(\mathcal{C})$.

It is easy to see that both sides commute with sifted colimits in the variable A. Therefore, we reduce to the case where $A = \mathrm{Sym}(V)$ for $V \in \mathcal{C}$.

Recall from §8.11 that in this case $A^{\mathrm{fact,univ}}$ is $\mathrm{Sym}(j_!(V \otimes \omega_U))$. Observe that this factorization algebra is the chiral envelope of the abelian Lie-$*$ algebra $j_!(V \otimes \omega_U)[-1]$ (the cohomological shift occurs because the passage from Lie-$*$ algebras to factorization algebras is given by a Chevalley complex construction).

By Theorem 8.22.1, we can therefore compute:

$$A^{\mathrm{fact}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}(\mathcal{C}^{\mathrm{fact}}) \simeq \mathrm{Sym}(i^{*,dR} j_{*,dR}(V \otimes \omega_U[-1]))\text{–}\mathsf{mod}(\mathcal{C}).$$

Note that there is a canonical isomorphism $i^{*,dR} j_{*,dR}(V \otimes \omega_U[-1]) \simeq i^! j_!(V \otimes \omega_U) \in \mathcal{C}$: this comes from applying $i^!$ to the boundary morphism in the triangle:

$$j_!(V \otimes \omega_U) \to j_{*,dR}(V \otimes \omega_U) \to i_{*,dR} i^{*,dR} j_{*,dR}(V \otimes \omega_U) \xrightarrow{+1}.$$

Therefore, we obtain an equivalence:

$$H_\nabla(\overset{o}{\mathcal{D}}_x, A^{\mathrm{fact}})\text{–}\mathsf{mod}(\mathcal{C}) = \mathrm{Sym}(i^! j_!(V \otimes \omega_U))\text{–}\mathsf{mod}(\mathcal{C}) \simeq A^{\mathrm{fact}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}(\mathcal{C}^{\mathrm{fact}}).$$

Tracing the above constructions, one readily finds that this equivalence is induced by the functor (8.23.1), giving the result.

$\square$

8.24. **Variant: coefficients in a module category.** We now discuss a more general set-up in which the above computations hold, in which we take modules with coefficients in a general $\mathcal{C}$-module category. (For §8.24, we can briefly relax the assumption that $\mathcal{C}$ is a rigid monoidal category).

Suppose that $\mathcal{M} \in \mathcal{C}\text{–}\mathsf{mod}$. A variant of the construction $I \mapsto \mathcal{C}_{X^I}$ defines a structure of unital factorization module category at x for $\mathcal{C}^{\mathrm{fact}}$ on $\mathcal{M}$.

More precisely, recall that a factorization module category is an assignment to each *pointed* finite set $* \in I$ of an object $\mathcal{M}_{X^{\overset{o}{I}}_{dR},x} \in D(X^{\overset{o}{I}} \times \{x\})\text{–}\mathsf{mod}$ where $\overset{o}{I} := I \backslash *$. For the above construction, one then has:

$$\mathcal{M}_{X^I_{dR},x} = \lim_{I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K} D(V(p)) \otimes \mathcal{C}^{\otimes \overset{o}{K}} \otimes \mathcal{M}$$

where the indexing category is a twisted arrow category as in [Ras2] §6, and where $V(p) \subseteq X^{\overset{o}{I}} \times \{x\}$ is the subset of points $(x_i)_{i \in I} \in X^I$ with $x_* = x$ and $x_i \neq x_j$ for $p(i) \neq p(j)$.

We record the following result for later use in §9; the reader may safely skip it for now.

Proposition 8.24.1.   *(1) If $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is dualizable, then the assignment:*

$$\mathcal{M} \mapsto \mathcal{M}_{X_{dR}^I, x}$$

commutes with limits.
    (2) Moreover, if $\mathcal{C}$ is rigid symmetric monoidal, then the assignment:

$$\mathcal{M} \mapsto \mathcal{M}_{X_{dR}^I, x}$$

commutes with colimits.

Proof. The first part is easy: the point is that dualizability of $\mathcal{C}$ implies that each term:

$$\mathcal{M} \mapsto D(V(p)) \otimes \mathcal{C}^{\overset{o}{\otimes} K} \otimes \mathcal{M}$$

commutes with limits.

For the second part: first, observe that the diagram defining $\mathcal{M}_{X_{dR}^I, x}$ can be understood as a diagram in $\mathcal{C}$–mod, letting $\mathcal{C}$ act on the $\mathcal{M}$ term in each expression.

Consider now the case that $\mathcal{M} = \mathcal{C}$. Then each term in the diagram defining $\mathcal{C}_{X_{dR}^I, x}$ admits a dual, and by [Ras2] Corollary 6.18.2, the colimit of the dual diagram is dualizable. It follows that formation of this limit commutes with tensor products over $\mathcal{C}$.

Applying this to the tensor product with $\mathcal{M}$, we obtain:

$$\mathcal{C}_{X_{dR}^I, x} \underset{\mathcal{C}}{\otimes} \mathcal{M} = \Big(\lim_{I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K} D(V(p)) \otimes \mathcal{C}^{\otimes K} \Big) \underset{\mathcal{C}}{\otimes} \mathcal{M} = \lim_{I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K} \big(D(V(p)) \otimes \mathcal{C}^{\otimes K} \underset{\mathcal{C}}{\otimes} \mathcal{M} \big) =: \mathcal{M}_{X_{dR}^I, x}.$$

But the left hand side obviously commutes with colimits in $\mathcal{M}$, giving the claim. $\qquad\square$

8.25.   Now in the setting of §8.5, we obtain a functor:

$$H_{\nabla}(\overset{o}{\mathcal{D}}_x, A^{\mathrm{fact}})\text{–}\mathsf{mod}(\mathcal{M}) \to A^{\mathrm{fact}}\text{–}\mathsf{mod}_{un,x}^{\mathrm{fact}}(\mathcal{M}) \tag{8.25.1}$$

generalizing (8.5.1). Here the right hand side should be understood as factorization modules in the factorization module category constructed above.

We have the following result, generalizing Theorem 8.13.1.

Theorem 8.25.1. *For $\mathcal{C}$ rigid and compactly generated, the functor* (8.25.1) *is an equivalence.*

The proof is exactly the same as for Theorem 8.13.1: one gives a version for Lie-$*$ algebras and deduces it formally from there.

9. FUSION: GENERALITIES

9.1.   Let Γ be an affine algebraic group and let $x \in X$ be a closed point.

Following ideas from [BD2] and [Gai1], Gaitsgory has suggested the following conjecture.

Conjecture. *There is a fully-faithful functor:*

$$\mathsf{ShvCat}_{/\operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)} \to \mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}})$$

from sheaves of categories over $\operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ to chiral module categories a x for the commutative factorization category $\mathsf{Rep}(\Gamma)^{\mathrm{fact}}$ associated with the symmetric monoidal DG category $\mathsf{Rep}(\Gamma)$.

Moreover, Gaitsgory suggests how the construction of this functor should go: this is essentially given in §9.7 below.

Remark 9.1.1. We refer to [Ras4] for further discussion of the notion of sheaf of categories over $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}})$ (especially for Γ reductive).

9.2. The purpose of this section is to prove the above conjecture in a formal neighborhood of the trivial local system in $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$: see Theorem 9.13.1 below.

9.3. **Regular local systems factorizably.** We now define $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I}$ a prestack X_{dR}^I encoding a factorizable version of the space of local systems of the disc at a point $y \in X$.[35]

Since local systems on the disc at a point is just $\mathbb{B}\Gamma$, we can define this space as the jet space into $\mathbb{B}\Gamma$ considered as a constant stack over X_{dR}.

More precisely, we make the following definitions. For convenience,[36] we work with prestacks (not with stacks), and let $\mathbb{B}^{naive}\Gamma$ denote the prestack $S \mapsto \mathbb{B}\big(\Gamma(S)\big)$ (as opposed to its sheafification for some topology). Define $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I}$ as the prestack:

$$S \mapsto \Big\{ y = (y_i) : S^{red} \to X^I, \Gamma_{y,dR} \underset{S_{dR}}{\times} S \to \mathbb{B}^{naive}G \Big\}$$

where as usual $\Gamma_y \subseteq X \times S^{red}$ is the union of the graphs of the maps y_i.

9.4. **Expressing** LocSys **via gauge forms.** We now express $\mathrm{LocSys}_\Gamma(\mathcal{D})$ as a quotient of regular 1-forms on the disc by the gauge action.

Let $\Omega_X^1(O)_{X_{dR}^I} \to X_{dR}^I$ be the affine D_{X^I}-scheme[37] of regular jets into the total space of the line bundle Ω_X^1. I.e., it is the prestack whose S-points (for $S \in \mathsf{AffSch}$) are:

$$S \mapsto \Big\{ y = (y_i)_{i \in I} : S^{red} \to X^I, \omega \in \Omega^\infty\big(\Gamma(\widehat{\mathcal{D}}_y, \Omega_X^1 \boxtimes \mathcal{O}_S|_{\widehat{\mathcal{D}}_y})\big) \Big\}$$

Here $\widehat{\mathcal{D}}_y$ is the formal completion in $X \times S$ (not in $X \times S^{red}$!) of the union of the Γ_{y_i}.

Remark 9.4.1. The reader should think of this space as the factorization version of the scheme of 1-forms on the formal disc.

Let $\mathrm{Lie}(\Gamma) \otimes \Omega_X^1(O)_{X_{dR}^I} \to X_{dR}^I$ denote the analogue where we take $\mathrm{Lie}(\Gamma)$-valued 1-forms.

Similarly, we have an affine group D_{X^I}-scheme $G(O)_{X_{dR}^I} \to X_{dR}^I$, whose pullback along $X^I \to X_{dR}^I$ recovers $G(O)_{X^I}$.

Each of these D_{X^I}-schemes is *classical*, and therefore we have can define the gauge action of $\Gamma(O)_{X_{dR}^I}$ on $\mathrm{Lie}(\Gamma) \otimes \Omega_X^1(O)_{X_{dR}^I}$ by the usual formula.

We have a canonical map:

$$\Big(\mathrm{Lie}(\Gamma) \otimes \Omega_X^1(O)_{X_{dR}^I} \Big) / G(O)_{X_{dR}^I} \to \mathrm{LocSys}_\Gamma(\widehat{\mathcal{D}})_{X_{dR}^I}$$

---

[35]The space defined below could just as well be called local systems on the formal disc, as opposed to the adic disc: we use the adic notation just for consistency when we move to punctured discs.

[36]This is a mild thing to do, because ultimately we only care about sheaves and not about the spaces themselves. For a more serious geometric study of spaces of local systems, of course one should sheafify for an appropriate topology. We would rather not bother here with the irrelevant (for us) question of which topology, and therefore choose none at all.

[37]Recall that these words mean the structure map to X_{dR}^I is (representable and) affine.

where the left hand side is the prestack (i.e., non-sheafified) quotient (formed relative to X_{dR}^I). Indeed, it is well-known that the classical prestack underlying the right hand side is the left hand side. This map is readily seen to be an isomorphism (so the right hand side is classical too), as one readily sees by computing cotangent complexes and noting that both sides are convergent prestacks (in the sense of derived algebraic geometry).

9.5. Local systems on the punctured disc.

Recall that we have fixed $x \in X$.

Let $\Omega^1_{o\atop \mathcal{D}_x}$ denote the (classical) indscheme of 1-forms on the punctured disc. I.e., this is the meromorphic jet space at x of the total space of the bundle $\Omega^1_{o\atop \mathcal{D}_x}$. Let $\mathrm{Lie}(\Gamma) \otimes \Omega^1_{o\atop \mathcal{D}_x}$ denote the version for $\mathrm{Lie}(\Gamma)$-valued 1-forms.

We again have a gauge action of $G(K_x)$ on $\mathrm{Lie}(\Gamma) \otimes \Omega^1_{o\atop \mathcal{D}_x}$. Let $\mathrm{LocSys}(\overset{o}{\mathcal{D}}_x)$ denote the prestack quotient of $\mathrm{Lie}(\Gamma) \otimes \Omega^1_{o\atop \mathcal{D}_x}$ by $G(K_x)$.

9.6. Factorization module structures.

Next, observe that $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ is naturally a *unital factorization module space* (we spell out in Remark 9.6.2 below what this means explicitly) for the factorization space over X_{dR}:

$$I \mapsto \mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I}.$$

Indeed, the corresponding structures on gauge forms and on the gauge group are obtained by restriction from the constructions of §2, and we can pass to the quotient to obtain the claim.

For every set I, let[38] $p_I : \mathcal{Z}_I \to X_{dR}^I \times \{x\}$ denote the corresponding space encoding the factorization module structure.

Remark 9.6.1. The space $\mathcal{Z}_I$ should be understood as parametrizing an I-tuple $y = (y_i)_{i \in I}$ of points on X (more precisely, on X_{dR}), and a local system on $(\mathcal{D}_y \cup \mathcal{D}_x) \backslash x$, i.e., the union of the discs at the points y_i with the disc at x punctured at the point x. The only technical issue is to say these words correctly, and gauge forms provide a convenient method.

Remark 9.6.2. Suppose we have a pair of finite sets I and J. Let $[X^I \times (X^J \times \{x\})]_{disj}$ denote the locus of points where $((y_i)_{i \in I}, (y_j)_{j \in J}, x)$ where $y_i \neq y_j$ for all $i \in I$, $j \in J$ and $y_i \neq x$ for all $i \in I$. We use the similar notation for de Rham spaces.

At first order, the factorization module structure provides an isomorphism:

$$\mathcal{Z}_{I \sqcup J} \underset{X_{dR}^{I \sqcup J} \times \{x\}}{\times} [X_{dR}^I \times (X_{dR}^J \times \{x\})]_{disj} \simeq$$

$$\left(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I} \times \mathcal{Z}_J \right) \underset{X_{dR}^I \times X_{dR}^J \times \{x\}}{\times} [X_{dR}^I \times (X_{dR}^J \times \{x\})]_{disj}$$

Example 9.6.3. For $I = \varnothing$, we have $\mathcal{Z}_\varnothing = \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$, and for $I = \{*\}$, we have $\mathcal{Z}_*$ is a degeneration along $X_{dR} \times \{x\}$:

$$\mathrm{LocSys}_\Gamma(\mathcal{D}_y) \times \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x) \rightsquigarrow \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x).$$

---

[38]The factor $\{x\}$ is included for two reasons: first, if x is not a k-point, we are effecting the appropriate (finite) extension of fields; moreover, it serves as a convenient placeholder for remembering how factorization module structures work.

Note that for every I, there is a canonical projection map:

$$q_I : \mathcal{Z}_I \to \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x).$$

9.7. **Construction of the functor.** We now construct a functor:

$$\mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}})} \to \mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\check{G})).$$

Essentially, this functor is:

$$p_{I,*}q_I^* : \mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}})} \to \mathsf{ShvCat}_{/X_{dR}^I \times \{x\}}.$$

To see that this defines a functor of the desired type, we need two technical results.

Recall from [Gai6] (see also [Ras1] Appendix A) that pushforwards for sheaves of categories satisfy arbitrary base-change. However, in order for the geometry above to linearize to factorization module structures, we still need to check that certain Künneth type results hold. That is, a priori, $I \mapsto \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I}) \in D(X^I)\text{–}\mathsf{mod} \simeq \mathsf{ShvCat}_{/X_{dR}^I}$ is only a lax factorization category, and we only have a lax factorization module structure.

Second, we need to check that the unital (a priori, lax) factorization category $I \mapsto \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I})$ identifies canonically with the unital factorization category $\mathsf{Rep}(\Gamma)^{\mathrm{fact}}$.

These results are presented in §9.8-9.9. Note that for the logical flow, we present them with the order reversed from the above.

9.8. $\mathrm{LocSys}_\Gamma(\mathcal{D})$ **and** $\mathsf{Rep}(\Gamma)$**.** We postpone the proof of the following result, which is a bit digressive, to Appendix A.

Lemma 9.8.1. *For every finite set I, the canonical functor:*

$$\mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I}) \to \mathsf{Rep}(\Gamma)_{X_{dR}^I} \in D(X^I)\text{–}\mathsf{mod}$$

is an equivalence.

Here $\mathsf{Rep}(\Gamma)_{X_{dR}^I}$ is the (X_{dR}^I-term) of the factorization category associated with $\mathsf{Rep}(\Gamma) \in \mathsf{ComAlg}(\mathsf{DGCat}_{cont})$, c.f. [Ras2] §6. This canonical functor comes from construction of $\mathsf{Rep}(\Gamma)_{X_{dR}^I}$ as a limit as in loc. cit.*, and from the tautological identification:*

$$\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}} \simeq X_{dR} \times \mathbb{B}^{naive}\Gamma.$$

9.9. **Künneth results.** Recall the problem as stated in §9.7: we need to check that certain pushforwards of certain sheaves of categories can be computed as tensor products.

For example, for I, J two finite sets, we need to check that the natural map:[39]

$$\mathsf{Rep}(\Gamma)_{X_{dR}^I} \otimes p_{J,*}q_J^*(-)|_{[X_{dR}^I \times (X_{dR}^J \times \{x\})]_{disj}} \to p_{I \coprod J,*}q_{I \coprod J}^*(-)|_{[X_{dR}^I \times (X_{dR}^J \times \{x\})]_{disj}}$$

is an equivalence, where this map arises from factorization and Lemma 9.8.1 (note that the variable entry $(-)$ here takes values from sheaves of categories over $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$).

This result follows from Lemma 9.9.2 below combined with the following result.

---

[39]Note that we are not distinguishing between thinking about $\mathsf{Rep}(\Gamma)_{X_{dR}^I}$ as a $D(X^I)$-module category and as a sheaf of categories on X_{dR}^I. Of course, this is not an outrageous sin because of the 1-affineness of X_{dR}^I.

Lemma 9.9.1. *Let $\mathcal{Y}$ be a 1-affine prestack. Then for any prestack $\mathcal{Z}$ and any sheaf of categories* C *on $\mathcal{Z}$, the canonical map:*

$$\mathsf{QCoh}(\mathcal{Y}) \otimes \Gamma(\mathcal{Z}, \mathsf{C}) \to \Gamma(\mathcal{Y} \times \mathcal{Z}, p_2^*(\mathsf{C}))$$

is an equivalence.

Proof. By base-change for pushforwards for sheaves of categories, we have:

$$\Gamma(\mathcal{Y} \times \mathcal{Z}, p_2^*(\mathsf{C})) = \Gamma(\mathcal{Y}, \pi_{\mathcal{Y}}^* \Gamma(\mathcal{Z}, \mathsf{C}))$$

for $\pi_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{Spec}(k)$ the structure map. But the right hand side is $\mathsf{QCoh}(\mathcal{Y}) \otimes \Gamma(\mathcal{Z}, \mathsf{C})$ by 1-affineness of $\mathcal{Y}$.

$\square$

In Appendix A, we will also prove the following result, which is needed to apply the above in our situation.

Lemma 9.9.2. *The prestacks* $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I}$ *and* $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}$ *are 1-affine.*

9.10. **Example: the unramified case.** We claim that the functor:

$$\mathsf{Rep}(\Gamma)\text{--mod} \simeq \mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\mathcal{D}_x)} \xrightarrow{\text{pushforward}} \mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)} \to \mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}})$$
$$(9.10.1)$$

identifies canonically with the functor of §8.24.

Indeed, as in Lemma 9.8.1, there is a canonical map comparing these two functors. One checks that it's an equivalence by first noting that both sides commute with colimits —for (9.10.1) this follows from Lemma 9.9.2, and for the functor of §8.24, this is Proposition 8.24.1 —thereby reducing to checking this on the single object $\mathsf{Rep}(\Gamma)$, where it follows from Lemma 9.8.1.

9.11. **The formal completion.** Let $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^{\wedge}_{\mathrm{LocSys}_\Gamma(\mathcal{D}_x)}$ denote the formal completion of $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ along $\mathrm{LocSys}_\Gamma(\mathcal{D}_x)$. I.e., this is the prestack with S-points being the groupoid of maps $S \to \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ with a lift to $\mathrm{LocSys}_\Gamma(\mathcal{D}_x)$ on S^{red}.

9.12. First, note that because $\mathrm{LocSys}_\Gamma(\mathcal{D}_x) \to \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ is ind-locally closed, we have:

$$\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^{\wedge}_{\mathrm{LocSys}_\Gamma(\mathcal{D}_x)} \underset{\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^{\wedge}_{\mathrm{LocSys}_\Gamma(\mathcal{D}_x)} = \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^{\wedge}_{\mathrm{LocSys}_\Gamma(\mathcal{D}_x)}.$$

In particular, since pushforward for sheaves of categories satisfies arbitrary base-change, we have a fully-faithful embedding:

$$\mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^{\wedge}_{\mathrm{LocSys}_\Gamma(\mathcal{D}_x)}} \subseteq \mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)}$$

defined by pushforward.

9.13.   We can now formulate the main result of this section.

Theorem 9.13.1. *The composite functor:*

$$\Phi : \mathsf{ShvCat}_{/\operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^\wedge_{\operatorname{LocSys}_\Gamma(\mathcal{D}_x)}} \to \mathsf{ShvCat}_{/\operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)} \to \mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}})$$

is fully-faithful.

The remainder of this section is devoted to the proof of this result. We will proceed as follows:

First, in §9.14, we give a more explicit description of $\operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^\wedge_{\operatorname{LocSys}_\Gamma(\mathcal{D}_x)}$ in terms of usual finite type geometry, which allows us to apply known results from [Gai6] to it. Then in §9.16-9.19 we give a dévissage argument to reduce the computation to a more manageable one. Finally, in §9.20, we provide the main non-formal computation, which is an application of Theorem 8.13.1.

9.14. **Log connections.** First, we give a more explicit description $\operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^\wedge_{\operatorname{LocSys}_\Gamma(\mathcal{D}_x)}$.

Let t be a coordinate at x. We have a map:

$$\operatorname{Lie}(\Gamma)/G \to \operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$$

$$\xi \mapsto d + \xi \cdot \frac{dt}{t} \tag{9.14.1}$$

which one can easily see does not depend on the choice of coordinate t.

Let $\operatorname{Lie}(\Gamma)^\wedge_0$ denote the formal completion of $\operatorname{Lie}(\Gamma)$ at 0.

Lemma 9.14.1. *The induced map:*

$$\operatorname{Lie}(\Gamma)^\wedge_0/\Gamma \to \operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^\wedge_{\operatorname{LocSys}_\Gamma(\mathcal{D}_x)}$$

is an isomorphism.

Proof. A cotangent complex computation shows that $\operatorname{Lie}(\Gamma)^\wedge_0/\Gamma \to \operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ is formally étale. It follows that $\operatorname{Lie}(\Gamma)^\wedge_0/\Gamma \to \operatorname{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)^\wedge_{\operatorname{LocSys}_\Gamma(\mathcal{D}_x)}$ is formally étale as well, so that to check that it is an isomorphism, it suffices to do it at the reduced level, where it is clear.

$\square$

9.15.   In what follows, we will use §9.14 and 1-affineness of $\operatorname{Lie}(\Gamma)^\wedge_0/\Gamma$ (c.f. [Gai6]) to identify Φ with the corresponding functor:

$$\mathsf{QCoh}(\operatorname{Lie}(\Gamma)^\wedge_0/\Gamma)\text{--}\mathsf{mod} \to \mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}}).$$

9.16. **Reductions.** First, we claim that Φ commutes with limits and colimits.

For limits, the result follows from the observation that pullback of sheaves of categories commutes with limits (and certainly pushforward does).

However, for colimits this claim is less obvious. If we knew that the formal completion of the space $\mathcal{Z}_I$ from §9.6 along $\operatorname{LocSys}_\Gamma(\mathcal{D})_{X^I_{dR} \times \{x\}}$ were 1-affine, we would be okay. However, without this, it is not a priori clear that the pushforward of sheaves of categories in the definition of Φ commutes with colimits.

Rather than proving this 1-affineness, we directly show the following, which will be sufficient for our purposes:

Lemma 9.16.1. *The functor Φ commutes with colimits, and is a functor of categories tensored over DGCat_{cont}.*

Proof. By proper descent for IndCoh and formal smoothness of $\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma$, we have:

$$\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma) \simeq \mathsf{IndCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathsf{IndCoh}(\underbrace{\mathbb{B}\Gamma \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \cdots \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \mathbb{B}\Gamma}_{n \text{ times}}) \quad (9.16.1)$$

Moreover, note that each of the functors in this limit is $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$-linear and admits a $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$-linear left adjoint.

Therefore, for every $\mathcal{C} \in \mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{-}\mathsf{mod}$, we have:

$$\mathcal{C} \simeq \lim_{[n]\in\mathbf{\Delta}} \mathcal{C} \underset{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{-}\mathsf{mod}}{\otimes} \mathsf{IndCoh}(\underbrace{\mathbb{B}\Gamma \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \cdots \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \mathbb{B}\Gamma}_{n \text{ times}})$$

Since Φ preserves limits, we get a similar expression for $\Phi(\mathcal{C})$. Moreover, since each structure map in resulting limit still admits a left adjoint, we obtain the expression:

$$\Phi(\mathcal{C}) \simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{op}} \Phi\big(\mathcal{C} \underset{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{-}\mathsf{mod}}{\otimes} \mathsf{IndCoh}(\underbrace{\mathbb{B}\Gamma \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \cdots \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \mathbb{B}\Gamma}_{n \text{ times}}).$$

Now observe that each term in this colimit is obtained by pushforward from $\mathbb{B}\Gamma \to \mathrm{Lie}(\Gamma)_0^\wedge/\Gamma$. Therefore, each term:

$$\mathcal{C} \underset{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{-}\mathsf{mod}}{\otimes} \mathsf{IndCoh}(\underbrace{\mathbb{B}\Gamma \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \cdots \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \mathbb{B}\Gamma}_{n \text{ times}})$$

commutes with colimits in $\mathcal{C}$ (since the functor (9.10.1) commutes with colimits: recall that the point here was the 1-affineness of $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I}$).

These observations give the claim.

$\square$

9.17.   In what follows, $\underline{\mathsf{Hom}}$ will indicate the corresponding DGCat_{cont}-valued Hom functors.

We need to show that for $\mathcal{C}, \mathcal{D} \in \mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{-}\mathsf{mod}$, the map:

$$\underline{\mathsf{Hom}}_{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{-}\mathsf{mod}}(\mathcal{C}, \mathcal{D}) \to \underline{\mathsf{Hom}}_{\mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}})}(\Phi(\mathcal{C}), \Phi(\mathcal{D}))$$

is an equivalence.

In §9.18, we reduce to checking this in the case when $\mathcal{C} = \mathsf{Rep}(\Gamma) = \mathsf{QCoh}(\mathbb{B}\Gamma)$.[40] In §9.19, we reduce to checking this in the case when $\mathcal{D}$ is pushed forward from $\mathbb{B}\Gamma$. Finally, we check the result in this particular case in §9.20.

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[40]Of course, the relevant module structure here is defined by pullback along $\mathbb{B}\Gamma = \{0\}/\Gamma \hookrightarrow \mathrm{Lie}(\Gamma)_0^\wedge/\Gamma$.

9.18. Reduction to the case $\mathcal{C} = \mathsf{Rep}(\Gamma)$. First, observe that $\mathsf{Rep}(\Gamma)$ is dualizable as an object of $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$–$\mathsf{mod}$. Indeed, this is a formal consequence of the 1-affineness of $\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma$ and of the morphism $\mathbb{B}\Gamma \to \mathrm{Lie}(\Gamma)_0^\wedge/\Gamma$.

Moreover, by descent for IndCoh (c.f. (9.16.1)), $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$ can be written as a colimit in $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$–$\mathsf{mod}$ of terms of the form:

$$\mathsf{Rep}(\Gamma) \underset{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)}{\otimes} \mathcal{C}_0, \quad \mathcal{C}_0 \in \mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{–}\mathsf{mod}.$$

From these two observations, it follows formally that objects of the form $\mathcal{C} = \mathsf{Rep}(\Gamma) \otimes \mathcal{C}_0$ for $\mathcal{C}_0 \in \mathsf{DGCat}_{cont}$ generate $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$–$\mathsf{mod}$ under colimits.

Moreover, we can then rewrite:

$$\underline{\mathsf{Hom}}_{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{–}\mathsf{mod}}(\mathsf{Rep}(\Gamma) \otimes \mathcal{C}_0, \mathcal{D}) = \underline{\mathsf{Hom}}_{\mathsf{DGCat}_{cont}}(\mathcal{C}_0, \underline{\mathsf{Hom}}_{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{–}\mathsf{mod}}(\mathsf{Rep}(\Gamma), \mathcal{D})).$$

Since Φ is DGCat_{cont}-linear, and since we can perform the same manipulation in $\mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}})$, we reduce to the case $\mathcal{C}_0 = \mathsf{Vect}$ as desired.

9.19. Reduction to the case that $\mathcal{D}$ is monodromy-free. Next, we reduce to the case that $\mathcal{D}$ is obtained by pushforward from $\mathbb{B}\Gamma$. Note that the pushforward (i.e., restriction) functor:

$$\mathsf{Rep}(\Gamma)\text{–}\mathsf{mod} \to \mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{–}\mathsf{mod}$$

generates the target under limits. Indeed, applying proper descent for IndCoh again, we can express $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$ as a limit of terms in the image of this functor, and moreover, formation of this limit commutes with tensoring over $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$ since all the structure functors admit continuous $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$-linear left adjoints.

The argument now follows, since we have already observed that Φ commutes with arbitrary limits.

9.20. Computation via factorization modules. We now explicitly perform the computation in the above case, i.e., we wish to show that:

$$\underline{\mathsf{Hom}}_{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{–}\mathsf{mod}}(\mathsf{Rep}(\Gamma), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathsf{Hom}}_{\mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}})}(\mathsf{Rep}(\Gamma), \Phi(\mathcal{D})) \qquad (9.20.1)$$

where $\mathcal{D}$ is a $\mathsf{Rep}(\Gamma)$-module category (being considered as a $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$-module category by restriction).

9.21. The left hand side. First, we see from self-duality of $\mathsf{Rep}(\Gamma)$ (as a $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$-module category) that:

$$\underline{\mathsf{Hom}}_{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)\text{–}\mathsf{mod}}(\mathsf{Rep}(\Gamma), \mathcal{D}) \simeq \mathsf{Rep}(\Gamma) \underset{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)}{\otimes} \mathcal{D} =$$

$$\mathsf{Rep}(\Gamma) \underset{\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)}{\otimes} \mathsf{Rep}(\Gamma) \underset{\mathsf{Rep}(\Gamma)}{\otimes} \mathcal{D} = \mathsf{QCoh}(\mathbb{B}\Gamma \underset{\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma}{\times} \mathbb{B}\Gamma) \underset{\mathsf{Rep}(\Gamma)}{\otimes} \mathcal{D} = \qquad (9.21.1)$$

$$\pi_{2,*}(\mathcal{O}_{\mathbb{B}\Gamma \underset{\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma}{\times} \mathbb{B}\Gamma})\text{–}\mathsf{mod}(\mathcal{D}).$$

Here at the end we are considering $\pi_{2,*}(\mathcal{O}_{\mathbb{B}\Gamma \underset{\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma}{\times} \mathbb{B}\Gamma})$ as a commutative algebra in $\mathsf{Rep}(\Gamma)$, and taking modules in $\mathcal{D}$ with respect to its $\mathsf{Rep}(\Gamma)$-module structure.

9.22. **The right hand side.** Next, we compute $\underline{\mathsf{Hom}}_{\mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}})}(\Phi(\mathcal{C}), \Phi(\mathcal{D}))$.

We proceed in two steps: first, we express this category as factorization modules for a factorization algebra, and then we compute the relevant category of factorization modules.

The first step is carried out in §9.23-9.24, and the second step is carried out in the remainder of this section.

9.23.   Let $* \in I$ be a pointed finite set, and let $\overset{o}{I}$ denote the complement of $*$ in I. Let:

$$\Phi(\mathcal{D})_{X_{dR}^I, x}, \mathsf{Rep}(\Gamma)_{X_{dR}^I, x} \in D(X^{\overset{o}{I}} \times \{x\})$$

denote the relevant objects corresponding to the factorization module category structures.

A morphism of factorization module $\mathsf{Rep}(\Gamma) \to \Phi(\mathcal{D})$ is equivalent to the data of functors:

$$F_{I,x} : \mathsf{Rep}(\Gamma)_{X_{dR}^I, x} \to \Phi(\mathcal{D})_{X_{dR}^I, x} \in D(X^{\overset{o}{I}} \times \{x\})$$

compatible with factorization module structures.

By self-duality of $\mathsf{Rep}(\Gamma)_{X_{dR}^I, x}$, this is the same as the specification of objects:

$$\mathcal{K}_{I,x} \in \mathsf{Rep}(\Gamma)_{X_{dR}^I, x} \underset{D(X^I \times \{x\})}{\otimes} \Phi(\mathcal{D})_{X_{dR}^I, x}.$$

satisfying some compatibilities.

9.24.   What do the compatibilities of the functors $F_{I,x}$ translate to in terms of the kernels $\mathcal{K}_{I,x}$?

A typical compatibility for the functors is that each time we write $I = I_1 \coprod I_2$ with $* \in I_2$, the induced functor:

$$[\mathsf{Rep}(\Gamma)_{X_{dR}^{I_1}} \otimes \mathsf{Rep}(\Gamma)_{X_{dR}^{I_2}, x}] \underset{D(X^I \times \{x\})}{\otimes} D([X^{I_1} \times (X^{\overset{o}{I_2}} \times \{x\})]_{disj}) \simeq$$

$$\mathsf{Rep}(\Gamma)_{X_{dR}^I, x} \underset{D(X^I \times \{x\})}{\otimes} D([X^{I_1} \times (X^{\overset{o}{I_2}} \times \{x\})]_{disj}) \to$$

$$\mathcal{D}_{X_{dR}^I, x} \underset{D(X^I \times \{x\})}{\otimes} D([X^{I_1} \times (X^{\overset{o}{I_2}} \times \{x\})]_{disj}) \simeq$$

$$[\mathsf{Rep}(\Gamma)_{X_{dR}^{I_1}} \otimes \mathcal{D}_{X_{dR}^{I_2}, x}] \underset{D(X^I \times \{x\})}{\otimes} D([X^{I_1} \times (X^{\overset{o}{I_2}} \times \{x\})]_{disj})$$

be the restriction of the functor:

$$\mathrm{id}_{\mathsf{Rep}(\Gamma)_{X_{dR}^{I_1}}} \otimes F_{I_2, x}.$$

In terms of kernels, this says $\mathcal{K}_{I,x}$ restricted to this disjoint locus is isomorphic to the restriction of $\Delta_*(\mathcal{O}_\Gamma)_{X_{dR}^{I_1}} \boxtimes \mathcal{K}_{I_2, x}$. Here $\Delta_*(\mathcal{O}_\Gamma) \in \mathsf{Rep}(\Gamma \times \Gamma)$ is the regular representation considered as a commutative algebra among Γ-bimodules, and:

$$\Delta_*(\mathcal{O}_\Gamma)_{X_{dR}^{I_1}} \in \mathsf{Rep}(\Gamma)_{X_{dR}^{I_1}} \underset{D(X^{I_1})}{\otimes} \mathsf{Rep}(\Gamma)_{X_{dR}^{I_1}} = \mathsf{Rep}(\Gamma \times \Gamma)_{X_{dR}^{I_1}}$$

is the corresponding part of the associated factorization algebra.

Extending this reasoning further, we find that a morphism of factorization module categories as above is exactly the same data as a factorization module in $\mathsf{Rep}(\Gamma) \otimes \mathcal{D}$ for the commutative factorization algebra defined by the regular representation $\Delta_*(\mathcal{O}_\Gamma)$. Here we are considering $\mathsf{Rep}(\Gamma) \otimes \mathcal{D}$ as a $\mathsf{Rep}(\Gamma) \otimes \mathsf{Rep}(\Gamma)$-module category in the obvious way.

9.25. Computation of the local conformal blocks. From Theorem 8.25.1, we obtain that:

$$\underline{\mathsf{Hom}}_{\mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\Gamma)^{\mathrm{fact}})}(\mathsf{Rep}(\Gamma), \Phi(\mathcal{D}))$$

is modules over the local conformal blocks $H_\nabla(\overset{o}{\mathcal{D}}_x, \Delta_*(\mathcal{O}_\Gamma)^{\mathrm{fact}})$, where our ambient factorization category is $\mathsf{Rep}(\Gamma \times \Gamma)^{\mathrm{fact}}$.

We will now compute these conformal blocks more explicitly.

Construction 9.25.1. Define a prestack $\mathcal{Y} \to X$ as parametrizing $y \in X$, a pair σ_1, σ_2 of Γ-local systems on $\mathcal{D}_y$, and an isomorphism $\sigma_1 \simeq \sigma_2$ on $\mathcal{D}_y \backslash \{x\}$: the S-families definition is given as above using gauge forms.

Note that $\mathcal{Y}$ naturally descends to $\mathcal{Y}_{X_{dR}} \to X_{dR}$ and has a natural map $\mathcal{Y}_{X_{dR}} \to \mathbb{B}\Gamma \times \mathbb{B}\Gamma \times X_{dR}$ encoding the pair of local systems. Moreover, this latter map is affine.

It follows that $\mathcal{Y}$ defines a modification of $\Delta_*(\mathcal{O}_\Gamma)$ as a D-algebra in $\mathsf{Rep}(\Gamma) \otimes \mathsf{Rep}(\Gamma) \otimes D(X)$, and the definition of $\mathcal{Y}$ makes it clear that this is the universal modification.

9.26. Therefore, the local conformal blocks of $\Delta_*(\mathcal{O}_\Gamma)$ at x is computed as the fiber of $\mathcal{Y}$.

This fiber parametrizes a pair of Γ-local systems on $\mathcal{D}_x$ identified on $\overset{o}{\mathcal{D}}_x$. By formal étaleness of the map (9.14.1), we have:

$$\mathbb{B}\Gamma \underset{\mathrm{Lie}(\Gamma)/\Gamma}{\times} \mathbb{B}\Gamma \xrightarrow{\simeq} \mathrm{LocSys}_\Gamma(\mathcal{D}_x) \underset{\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_\Gamma(\mathcal{D}_x)$$

9.27. Combining the above, we obtain that the left and right hand sides of (9.20.1) are isomorphic; tracing the constructions, it is easy to see that such an isomorphism is effected by the map in (9.20.1).

10. Fusion and nearby cycles

10.1. Suppose that $\mathcal{C}$ is a $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma)$-module category. We have seen that we can recover this structure using the induced chiral $\mathsf{Rep}(\Gamma)$-module structure on $\mathcal{C}$. However, the procedure is somewhat inexplicit: we merely showed that the functor is fully-faithful.

Arkhipov-Bezrukavnikov's interpretation of Gaitsgory's central sheaves [Gai1] suggests a more explicit construction, and below we will compare the two, as will be essential for the comparison with their functor.

More precisely, note that the factorization functor $\mathsf{Whit}^{sph,\mathrm{fact}} \to \mathsf{Whit}_{acc}^{\frac{\infty}{2},\mathrm{fact}}$ and the Casselman-Shalika equivalence $\mathsf{Whit}^{sph,\mathrm{fact}} \simeq \mathsf{Rep}(\check{G})^{\mathrm{fact}}$ makes $\mathsf{Whit}_{x,acc}^{\frac{\infty}{2}}$ into a chiral $\mathsf{Rep}(\check{G})^{\mathrm{fact}}$-module category (supported at x). We show in Proposition 10.2.1 that it lies in the full subcategory:

$$\mathsf{QCoh}(\check{\mathfrak{g}}_0^\wedge/\check{G})-\mathsf{mod} \subseteq \mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\check{G})^{\mathrm{fact}}).$$

Then from §10.3 to the end of the section, we compare this structure to a similar structure produced in [AB].

10.2. **The accessible semi-infinite Whittaker category and fusion.** First, we verify the following compatibility.

Proposition 10.2.1. $\mathsf{Whit}^{\frac{\infty}{2}}_{x,acc}$ *lies in the full subcategory:*

$$\mathsf{QCoh}(\check{\mathfrak{g}}_0^\wedge/\check{G})\text{--}\mathsf{mod} \subseteq \mathsf{FactModCat}_{un,x}(\mathsf{Rep}(\check{G})^{\mathrm{fact}}).$$

Here $\mathsf{Whit}^{\frac{\infty}{2}}_{x,acc}$ *is considered as a chiral module category (supported at x) first over* $\mathsf{Whit}^{\frac{\infty}{2},\mathrm{fact}}_{acc}$, *then over* $\mathsf{Whit}^{sph,\mathrm{fact}} \simeq \mathsf{Rep}(\check{G})^{\mathrm{fact}}$ *by restriction.*

Proof. We have an adjunction $\mathsf{Whit}^{sph}_x \rightleftarrows \mathsf{Whit}^{\frac{\infty}{2}}_{x,acc}$ with the right adjoint being conservative by definition. Therefore, the right adjoint is monadic.

Moreover, this adjunction is compatible with chiral $\mathsf{Rep}(\check{G})$-module structures (since the adjunction works factorizably). Since $\mathsf{Whit}^{sph}_x \simeq \mathsf{Rep}(\check{G})$ obviously lies in our full subcategory, this implies the claim. Indeed, the induced monad on Whit^{sph}_x is a morphism in $\mathsf{QCoh}(\check{\mathfrak{g}}_0^\wedge/\check{G})\text{--}\mathsf{mod}$, so modules over it is a $\mathsf{QCoh}(\check{\mathfrak{g}}_0^\wedge/\check{G})$-module category. $\qquad\square$

10.3. **Comparison with Arkhipov-Bezrukavnikov.** Here is the main idea of the [AB] construction.

For each representation $V \in \mathsf{Rep}(\Gamma)$, we obtain a vector bundle $\mathcal{E}_V$ on $\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma$ by pullback from Γ. Moreover, this vector bundle carries a canonical endomorphism s_V: indeed, this is true already on $\mathrm{Lie}(\Gamma)/\Gamma$ by the usual Tannakian formalism.

Remark 10.3.1. Note that s_V is *locally nilpotent* in the sense that if we invert it (i.e., form $\mathrm{colim}_{\mathbb{Z}^{\geqslant 0}} \mathcal{E}_V \xrightarrow{s_V} \mathcal{E}_V \xrightarrow{s_V} \ldots$), we get the zero object. This follows from the fact that its restriction to the reduced locus of $\mathrm{Lie}(\Gamma)_0^\wedge/\Gamma$ is canonically zero.

Now for $\mathcal{C}$ as above, $\mathcal{E}_V$ defines an endofunctor of $\mathcal{C}$, and this endofunctor carries a canonical locally nilpotent self-natural transformation induced by s_V. The Arkhipov-Bezrukavnikov idea is that this same natural transformation arises via nearby cycles (equipped with its monodromy endomorphism), using the $\mathsf{Rep}(\Gamma)$-chiral module structure on $\mathcal{C}$.

In §10.4-10.7, we will give a comparison of this type.

10.4. **Nearby cycles construction.** We first recall the construction of nearby cycles, following [Bei].

Let $X = \mathbb{A}^1$, $x = 0 \in X$ and $U = \mathbb{A}^1\backslash x$ with embedding $j : U \hookrightarrow X$. Let $\mathcal{J}_n \in D(U)^\heartsuit$ be the Jordan block of length n.

Suppose that $\widetilde{\mathcal{C}}$ is a $D(X)$-module category, with $\overset{!}{\otimes}$ denoting the action functor. Let $\widetilde{\mathcal{C}}_U$ denote the subcategory of objects $*$-extended from U, and let $\mathcal{C}$ denote the subcategory of objects supported at $0 \in \mathbb{A}^1$.

Suppose $\mathcal{F} \in \widetilde{\mathcal{C}}_U$, and that the left adjoint $i^{*,dR}$ to the embedding $\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ is defined on $j_{*,dR}(\mathcal{F})$. Note that the same then holds for $j_{*,dR}(\mathcal{J}_n \overset{!}{\otimes} \mathcal{F})$ for any n. Following [Bei], we then define the *(unipotent) nearby cycles* of $\mathcal{F}$ as:

$$\Psi^{un}(\mathcal{F}) := \underset{n}{\mathrm{colim}}\, i^{*,dR} j_{*,dR}(\mathcal{J}_n \overset{!}{\otimes} \mathcal{F}) \in \mathcal{C}.$$

Note that $\Psi^{un}(\mathcal{F})$ carries a canonical endomorphism $N = N_\mathcal{F}$ coming from the monodromy of $\mathcal{J}_n$, which is "locally nilpotent" (i.e., $\Psi^{un}(\mathcal{F})[N^{-1}] = 0$).

10.5.   Suppose that $\mathcal{C}$ is a $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)^\wedge_0/\Gamma)$-module category, and let $\widetilde{\mathcal{C}} \in D(X)$–$\mathsf{mod}$ denote the category encoding the "over $X \times x$" part of the chiral module structure. So $\widetilde{\mathcal{C}}_U = \mathsf{Rep}(\Gamma) \otimes \mathcal{C}$, and $\widetilde{\mathcal{C}}_x = \mathcal{C}$.

Proposition 10.5.1. *Suppose $\mathcal{C}$ is compactly generated.*[41]
    Then for any $\mathcal{F} \in \mathcal{C}$ and $V \in \mathsf{Rep}(\Gamma)$, $\Psi^{un}(V \boxtimes \mathcal{F}) \in \mathcal{C}$ is defined, and canonically isomorphic to $\mathcal{E}_V \otimes \mathcal{F}$. The monodromy action on $\Psi^{un}(V \boxtimes \mathcal{F})$ corresponds to the endomorphism s_V from above.

    The proof of this result is postponed to §10.7.

10.6. **Review of** [FG1]**.** We will need to use some results from [FG1]. These results will essentially be about relating the Kashiwara-Malgrange V-filtration to nearby cycles, so we will need to work in a classical setting for which such ideas make sense (though it would be interesting to find a more derived version of what follows).
    Let $\mathfrak{z} \in D(X)$ with $\mathfrak{z}[-1] \in D(X)^\heartsuit$ be a classical commutative D_X-algebra (considered as a right D-module) that is flat over X. Let $\widetilde{\mathfrak{z}}$ be another such classical D_X-algebra equipped with a D_X-algebra map $\widetilde{\mathfrak{z}} \hookrightarrow \mathfrak{z}$ that is an isomorphism when restricted to U.
    We will refer to $\mathfrak{z}$-modules with compatible (right) D_X-module structures as $\mathfrak{z}[D_X]$-modules.
    Suppose that $\mathcal{V} \in D(X)$ is a $\mathfrak{z}[D_X]$-module that is finite rank projective as a mere $\mathfrak{z}$-module (so $\mathcal{V}$ is in cohomological degree -1, like $\mathfrak{z}$ itself).
    We have the following result, which is a generalization of results from [FG1] (see below).

Proposition 10.6.1. *There is a unique finite rank projective $\widetilde{\mathfrak{z}}$-module $\widetilde{\mathcal{V}}$ with an isomorphism $\widetilde{\mathcal{V}}|_U \simeq \mathcal{V}|_U$ of $\widetilde{\mathfrak{z}}|_U = \mathfrak{z}|_U$-modules such that the induced connection on $\widetilde{\mathcal{V}}|_U$ has a pole of order 1 (i.e., for t a local coordinate at $x \in X$, the operator $t\partial_t$ preserves $\widetilde{\mathcal{V}} \subseteq j_*j^*(\widetilde{\mathcal{V}})$).*
    Moreover, there is a canonical isomorphism between $i^{*,dR}j_{*,dR}(\mathcal{V}) \in \widetilde{\mathfrak{z}}_x$–$\mathsf{mod}$ and the two step complex:

$$\widetilde{\mathcal{V}}_x \xrightarrow{t\partial_t} \widetilde{\mathcal{V}}_x$$

(concentrated in cohomological degrees -1 and -2, since $\mathcal{V}$ is in degree -1). Here $i : x \hookrightarrow X$ and $\widetilde{\mathfrak{z}}_x := i^!_x\widetilde{\mathfrak{z}}$. Moreover, $i^{*,dR}$ is the partially defined left adjoint to the functor $\widetilde{\mathfrak{z}}_x$–$\mathsf{mod} \hookrightarrow \widetilde{\mathfrak{z}}[D_X]$–$\mathsf{mod}$ (so satisfies a different universal property from the usual $*$-pullback of D-modules).

Remark 10.6.2. The existence statement is [FG1] Lemma 1.4. The calculation of $i^{*,dR}j_{*,dR}(\mathcal{V})$ on -1st cohomology is [FG1] Proposition 3.4, but we will be redone in what follows.

Proof. We use t as a coordinate at $x \in X$.
    Let $N \in \mathfrak{z}$–mod_x be given. The main observation is that we can compute the complex of maps $j_*(\mathcal{V}) \to i_{*,dR}(N) = N((t))dt/N[[t]]dt$ of $\mathfrak{z}$-modules (*not* $\mathfrak{z}[D_X]$-modules) explicitly as follows:
    Let $\mathfrak{z}_x = i^!\mathfrak{z}$, which is a commutative algebra (in cohomological degree 0) with $\mathfrak{z}$–$\mathsf{mod}_x = \mathfrak{z}_x$–$\mathsf{mod}$. Let $\mathfrak{z}^\ell := \mathfrak{z} \otimes_{\mathcal{O}_X} \omega_X^{-1} \in \mathsf{ComAlg}(\mathsf{QCoh}(X)^\heartsuit)$ be the left D-module underlying $\mathfrak{z}$. Let $\mathfrak{z}^\ell_{K_x}$ (resp. $\mathfrak{z}^\ell_{O_x}$) be the sections of $\mathfrak{z}^\ell$ over $\overset{o}{\mathcal{D}}_x$ (resp. $\mathcal{D}_x$), i.e., $\mathfrak{z}^\ell \otimes_{\mathcal{O}_X} k((t))$ (resp. $\mathfrak{z}^\ell \otimes_{\mathcal{O}_X} k[[t]]$).
    Note that there is a canonical map $\mathfrak{z}^\ell_{O_x} \to \mathfrak{z}_x[[t]]$ coming from the D_X-algebra structure: indeed, the target is the t-adic completion of the source. Inverting t, we obtain a map $\mathfrak{z}^\ell_{K_x} \to \mathfrak{z}_x((t))$, which we observe lifts to $\widetilde{\mathfrak{z}}_x((t))$. In particular, $\mathfrak{z}^\ell_{K_x}$ acts on $N((t))$.
    We then claim that the above complex of morphisms identifies canonically with:

$$\mathcal{V}_{K_x}^{\ell,\vee} \underset{\mathfrak{z}_{K_x}^{\ell}}{\otimes} N((t))$$

where the superscript ℓ means we tensor with ω_X^{-1}, and the subscript K_x indicates that we tensor over $\mathfrak{z}^{\ell}$ with $\mathfrak{z}_{K_x}^{\ell}$. To see the displayed equation, write $j_*(\mathcal{V}) = \mathrm{colim}_n \widetilde{\mathcal{V}}(n \cdot x)$ to see that it identifies with:

$$\lim_n \widetilde{\mathcal{V}}^{\ell,\vee}(-n \cdot x) \underset{\widetilde{\mathfrak{z}}^{\ell}}{\otimes} N((t))/N[[t]] = \lim_n \widetilde{\mathcal{V}}^{\ell,\vee} \underset{\widetilde{\mathfrak{z}}^{\ell}}{\otimes} N((t))/t^n N[[t]] = \widetilde{\mathcal{V}}^{\ell,\vee} \underset{\widetilde{\mathfrak{z}}^{\ell}}{\otimes} \lim_n N((t))/t^n N[[t]] =$$

$$\widetilde{\mathcal{V}}^{\ell,\vee} \underset{\widetilde{\mathfrak{z}}^{\ell}}{\otimes} N((t)) = \widetilde{\mathcal{V}}^{\ell,\vee} \underset{\widetilde{\mathfrak{z}}^{\ell}}{\otimes} \mathfrak{z}_{K_x}^{\ell} \underset{\mathfrak{z}_{K_x}^{\ell}}{\otimes} N((t)) = \mathcal{V}_{K_x}^{\ell,\vee} \underset{\mathfrak{z}_{K_x}^{\ell}}{\otimes} N((t))$$

as desired (where the second equality follows from dualizability of $\widetilde{\mathcal{V}}^{\ell}$ over $\widetilde{\mathfrak{z}}^{\ell}$).

Then using the de Rham resolution:

$$j_{*,dR}(\mathcal{V}) \underset{\mathcal{O}_X}{\otimes} T_X \underset{\mathcal{O}_X}{\otimes} D_X \to j_{*,dR}(\mathcal{V}) \underset{\mathcal{O}_X}{\otimes} D_X$$

of $j_{*,dR}(\mathcal{V})$, we find that $\underline{\mathrm{Hom}}_{\mathfrak{z}[D_X]-\mathsf{mod}}(j_{*,dR}(\mathcal{V}), i_{*,dR}(N))$ is computed by a two-step complex:

$$\mathcal{V}_{K_x}^{\ell,\vee} \underset{\mathfrak{z}_{K_x}^{\ell}}{\otimes} N((t)) \to \mathcal{V}_{K_x}^{\ell,\vee} \underset{\mathfrak{z}_{K_x}^{\ell}}{\otimes} N((t))dt$$

where the differential is induced by the connection on $j_*(\mathcal{V}^{\ell,\vee})$.

Using the logarithmic property of the connection on $\widetilde{\mathcal{V}}$, we can filter this complex by:

$$t^i \widetilde{\mathcal{V}}^{\ell,\vee} \underset{\mathfrak{z}_{O_x}^{\ell}}{\otimes} N[[t]] \to t^{i-1} \widetilde{\mathcal{V}}^{\ell,\vee} \underset{\mathfrak{z}_{O_x}^{\ell}}{\otimes} N[[t]]dt.$$

If $\Gamma \in \mathrm{End}(\widetilde{\mathcal{V}}_x^{\ell,\vee})$ is the residue of the connection (which we recall is given by the action of $t\partial_t$ on this fiber, and is nilpotent), then on the ith graded piece of the above map we have $(\Gamma + i \cdot \mathrm{id}_{\widetilde{\mathcal{V}}^{\ell,\vee}}) \otimes \mathrm{id}_{N[[t]]}$, which is invertible for $i \neq 0$. Therefore, the above complex is canonically quasi-isomorphic to:

$$\mathcal{V}_x^{\ell,\vee} \underset{\mathfrak{z}_x}{\otimes} N = (\widetilde{\mathcal{V}}^{\ell,\vee} \underset{\mathfrak{z}_{O_x}^{\ell}}{\otimes} N[[t]])/t \to (t^{-1} \widetilde{\mathcal{V}}^{\ell,\vee} \underset{\mathfrak{z}_{O_x}^{\ell}}{\otimes} N[[t]]dt)/\widetilde{\mathcal{V}}^{\ell,\vee} \underset{\mathfrak{z}_{O_x}^{\ell}}{\otimes} N[[t]]dt = \widetilde{\mathcal{V}}_x^{\ell,\vee} \underset{\mathfrak{z}_x}{\otimes} N$$

where the differential is $\Gamma \otimes \mathrm{id}_N$.

Since this last complex is also the complex of $\mathfrak{z}_x$-module maps from $\widetilde{\mathcal{V}}_x \xrightarrow{t\partial_t} \widetilde{\mathcal{V}}_x$ to N, we obtain the desired claim.

$\square$

We obtain the following standard consequence.

Corollary 10.6.3. *In the above setting, we have* $\Psi^{un}(j_{*,dR}(\mathcal{V})) = \widetilde{\mathcal{V}}_x$, *with the monodromy operator* N *given by the residue of connection.*

10.7. We now prove Proposition 10.5.1.

Proof of Proposition 10.5.1.

Step 1. First, consider the case where $\mathcal{C} = \mathsf{QCoh}(\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma)$ and $\mathcal{F} = \omega_{\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma}$ the dualizing sheaf. (Note that $\mathcal{F}$ is in the heart of the canonical t-structure.) We should show that $\Psi^{un}(V \boxtimes \mathcal{F})$ is the fiber of $\mathcal{E}_V$ at x.

Indeed, for V finite-dimensional, this follows after base-change to the space of gauge forms from Corollary 10.6.3, and by equivariance, we obtain the desired isomorphism after descending to the space of local systems as well. For V arbitrary, this follows from the commutation of Ψ^{un} with colimits.

Step 2. Next, let $\mathcal{C} = \mathsf{QCoh}(\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma) \otimes \mathcal{D}$ for some $\mathcal{D} \in \mathsf{DGCat}_{cont}$ compactly generated, and $\mathcal{F} = \omega_{\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma} \boxtimes \mathcal{G}$ for some $\mathcal{G} \in \mathcal{D}$.

This case reduces to the case where $\mathcal{G}$ is compact, since nearby cycles tautologically commutes with colimits. Then, since:

$$- \boxtimes \mathcal{G} : \mathsf{QCoh}(\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma) \to \mathsf{QCoh}(\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma) \otimes \mathcal{D}$$

is a left adjoint, the corresponding functor of $D(X)$-module categories commutes[42] with nearby cycles.

Step 3. Finally, let $\mathcal{C}$ be arbitrary (and compactly generated), and let $\mathcal{F} \in \mathcal{C}$ be given.

We have the action functor $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma) \otimes \mathcal{C} \to \mathcal{C}$, which admits $\mathsf{QCoh}(\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma)$-linear right adjoint. Therefore, the induced functor of $D(X)$-module categories commutes with nearby cycles.

Applying the previous step to $\omega_{\mathrm{Lie}(\Gamma)^{\wedge}_0/\Gamma} \boxtimes \mathcal{F}$, we now obtain the claim.

$\square$

10.8. **Compatibility with Arkhipov-Bezrukavnikov.** We obtain following result.

Theorem 10.8.1. *The action of $\mathsf{QCoh}(\check{\mathfrak{g}}^{\wedge}_0/\check{G})$ on $\mathsf{Whit}^{\frac{\infty}{2}}_{x,acc}$ induced by Arkhipov-Bezrukavnikov theory coincides with that arising via fusion, i.e., via Proposition 10.2.1.*

In order to do this, we need to recall the Arkhipov-Bezrukavnikov construction of the $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$-action. This is done in §10.9-10.14

10.9. In what follows, we let $\mathsf{VB}(\mathcal{Y})$ denote the 1-category of finite rank vector bundles on a stack $\mathcal{Y}$.

10.10. Recall that for every representation $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ finite-dimensional (i.e., $V \in \mathsf{VB}(\mathbb{B}\check{G})$), the corresponding object $\mathcal{E}_V$ of $\mathsf{VB}(\check{\mathfrak{g}}/\check{G})$ carries a canonical endomorphism as an object of this category. Roughly speaking, at a point $\xi \in \check{\mathfrak{g}}$, this endomorphism is given by ξ.

More precisely, we use the following construction. It suffices to construction a $\check{G}$-equivariant map of $\mathrm{Sym}(\check{\mathfrak{g}}^{\vee})$-modules:

$$V \otimes \mathrm{Sym}(\check{\mathfrak{g}}^{\vee}) \to V \otimes \mathrm{Sym}(\check{\mathfrak{g}}^{\vee}),$$

or equivalently, a $\check{G}$-equivariant map:

$$V \to V \otimes \mathrm{Sym}(\check{\mathfrak{g}}^{\vee}).$$

---

[42]The only non-trivial part in the commutation with nearby cycles the commutation with the partially-defined left adjoin $i^{*,dR}$.

This map is given by dualizing the action map $\check{\mathfrak{g}} \otimes V \to V$ and composing:

$$V \to V \otimes \check{\mathfrak{g}}^\vee \to V \otimes \mathrm{Sym}(\check{\mathfrak{g}}^\vee).$$

10.11.   We will appeal to the following version of a result of [AB].

Theorem 10.11.1 (Arkhipov-Bezrukavnikov)**.** *Suppose* $\mathcal{C} \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ *a monoidal DG category, and* $F : \mathsf{QCoh}(\check{\mathfrak{g}}/\check{G}) \to \mathcal{C}$ *is a monoidal functor. Suppose moreover that* $F|_{\mathsf{VB}(\check{\mathfrak{g}}/\check{G})}$ *factors through a monoidal 1-category*[43] *in* $\mathcal{C}$.

Then F *can be reconstructed from* $F|_{\mathsf{VB}(\check{\mathfrak{g}}/\check{G})}$, *i.e., the appropriate restriction functor*[44] *is fully-faithful.*

Moreover, $F_{\mathsf{VB}(\check{\mathfrak{g}}/\check{G})}$ *can be recovered from the composite monoidal functor:*

$$\mathsf{VB}(\mathbb{B}\check{G}) \to \mathsf{VB}(\check{\mathfrak{g}}/\check{G}) \xrightarrow{F_{\mathsf{VB}(\check{\mathfrak{g}}/\check{G})}} \mathcal{C}$$
$$V \mapsto F(V)$$

plus the data of the canonical endomorphisms of the objects $F(V)$ *(i.e., the endomorphisms induced by §10.10).*

Proof. This result is just a mild reinterpretation of what occurs in [AB]. This being the case, we prove the result by referring to the relevant parts of *loc. cit.*

We can recover F from $F|_{\mathsf{VB}(\check{\mathfrak{g}}/\check{G})}$ by [Lur] Theorem 1.3.3.8.

Let $\overset{o}{\mathcal{C}} \subseteq \mathcal{C}$ be the monoidal category generated by the essential image of $F|_{\mathsf{VB}(\check{\mathfrak{g}}/\check{G})}$. By assumption, this is a 1-category, and obviously it is an additive 1-category. Then apply [AB] Proposition 4 to obtain the second part of the theorem.

$\square$

10.12.   Now recall the construction of [AB]: they construct a monoidal functor $\mathsf{VB}(\check{\mathfrak{g}}/\check{G}) \to H_{\mathrm{aff}} := D(\mathrm{Fl}^{\mathrm{aff}}_{G,x})^I$ using Gaitsgory's nearby cycles construction [Bei] and [Gai2].

This functor maps into the heart of the (perverse) t-structure, so extends canonically to the bounded derived category $\mathsf{Perf}(\check{\mathfrak{g}}/\check{G})$ of the (exact) category $\mathsf{VB}(\check{\mathfrak{g}}/\check{G})$, and then by ind-extension we obtain a functor from $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$.

Then H_{aff} acts on $\mathsf{Whit}(\mathrm{Fl}^{\mathrm{aff}}_G) \simeq \mathsf{Whit}^{\frac{\infty}{2}}_x$ by convolution, so $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$ acts. Moreover, their work shows that $\mathsf{Whit}(\mathrm{Fl}^{\mathrm{aff}}_G)$ is isomorphic to $\mathsf{QCoh}(\check{\mathfrak{n}}/\check{B})$ as a $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$-module category. According to Lemma 11.6.1 below, the subcategory $\mathsf{Whit}^{\frac{\infty}{2}}_{acc,x} \subseteq \mathsf{Whit}^{\frac{\infty}{2}}_x$ corresponds to the subcategory $\mathsf{QCoh}(\check{\mathfrak{n}}^\wedge_0/\check{B})$ of quasi-coherent sheaves set-theoretically supported on $\mathbb{B}\check{B}$.

10.13.   We claim that the monoidal functor:[45]

$$\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G}) \to \mathrm{End}_{\mathsf{DGCat}_{cont}}(\mathsf{Whit}^{\frac{\infty}{2}}_{acc,x})$$

satisfies the hypothesis of Theorem 10.11.1.

Indeed, the right hand side can be identified with $\mathsf{QCoh}(\check{\mathfrak{n}}^\wedge_0/\check{B} \times \check{\mathfrak{n}}^\wedge_0/\check{B})$. We can identify this QCoh with IndCoh via tensoring with the dualizing sheaf by formal smoothness.

---

[43]In other words, there are no negative Exts between objects in this subcategory.

[44]From monoidal functors defined on $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$ satisfying our hypotheses to monoidal functors from $\mathsf{VB}(\check{\mathfrak{g}}/\check{G})$.

[45]The same works for $\mathsf{Whit}^{\frac{\infty}{2}}_x$, but is less relevant for us.

Then our monoidal functor is given by !-pullback to $\check{\mathfrak{n}}_0^{\wedge}/\check{B}$ and then applying IndCoh pushforward along the diagonal map. The restriction to $\mathsf{VB}(\check{\mathfrak{g}}/\check{G})$ then maps, up to shift, to the heart of the "inductive" t-structure from [GR1] §7, so maps through a 1-category.

10.14. **Proof of Theorem 10.8.1.** By the above, we can recover the action of $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$ on $\mathsf{Whit}_{acc,x}^{\frac{\infty}{2}}$ from action of the sheaves $\mathcal{E}_V$ ($V \in \mathsf{VB}(\mathbb{B}\check{G})$) plus the isomorphisms between the action of $\mathcal{E}_{\mathrm{triv}}$ and the identity functor (for triv the trivial representation), plus the isomorphisms between $\mathcal{E}_{V \otimes W}$ and the composition of $\mathcal{E}_V$ and $\mathcal{E}_W$.

Recall that the action of $\mathcal{E}_V$ in Arkhipov-Bezrukavnikov theory comes from a nearby cycles construction with the corresponding spherical sheaf on the affine Grassmannian, and the canonical endomorphism of this endofunctor corresponds to the monodromy of the nearby cycles. Since we are dealing with Whittaker categories, we can instead do fusion against the corresponding Whittaker sheaf on the affine Grassmannian. Indeed, because !-averaging from $G(O_x)$-equivariance to Whittaker equivariance is a left adjoint, it commutes with nearby cycles. Finally, since we are working over a single point, $D^!(\mathrm{Fl}_x^{\frac{\infty}{2}})$ is $G(K_x)$-equivariantly equivalent to $D(\mathrm{Fl}_{G,x}^{\mathrm{aff}})$. Now Proposition 10.5.1 gives the desired identification.

Matching the identifications between the action of $\mathcal{E}_{\mathrm{triv}}$ with the identity is easy: it is a simple ULAness argument that we omit.

Finally, matching up the isomorphisms between $\mathcal{E}_{V \otimes W}$ and $\mathcal{E}_V \otimes_{\mathcal{O}_{\check{\mathfrak{g}}/\check{G}}} \mathcal{E}_W$ follows directly using the 2-dimensional nearby cycles from [Gai2] §4. Indeed, these are a tool that make sense for a general $\mathsf{Rep}(\check{G})$-module category, providing a correspondence between the nearby cycles against $V \otimes W$ and the iterated nearby cycles for V and W. In the Arkhipov-Bezrukavnikov setup, these are tautologically used to produce the isomorphism we are after. It is a simple check that in the general setting of a D_X-algebra $\mathfrak{z}$ as above, the isomorphism matches up under Proposition 10.5.1, c.f. [FG1] Lemma 3.5.

11. PROOF OF THE MAIN THEOREM

11.1.   In this section, we draw together the work from the rest of this paper to prove Theorem 5.7.1, i.e., that the functor $i^{\frac{\infty}{2},!,enh}$ is an equivalence.

11.2. **Outline of this section. WRITE**

11.3.   We begin with the next result.

Lemma 11.3.1. *The functor* $i_{X^I}^{\frac{\infty}{2},!,enh} : \mathsf{Whit}_{acc,X^I}^{\frac{\infty}{2}} \to \Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,X^I}))$ *preserves ULA objects (c.f. Theorem 5.10.1).*

Proof. By the proof of Proposition 5.15.1, translates of the unit object of $\mathsf{Whit}_{acc,X^I}^{\frac{\infty}{2}}$ are ULA generators for the category. Noting that $i_{X^I}^{\frac{\infty}{2},!,enh}$ commutes with these translations (which are $D(X^I)$-linear automorphisms of the categories in question, so formally preserve ULAness), it suffices to see that the image of the unit is ULA.

Of course, this object is the vacuum representation $\Upsilon_{\check{\mathfrak{n}}}$ itself, i.e., the corresponding factorization module over X^I. ULAness of the vacuum in this case[46] is proved[47] in [Ras2] Proposition 7.13.1. $\qquad\square$

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[46]Note that this is not true for a general factorization algebra. We remind that the proof in this case relies on the especially simple nature of $\Upsilon_{\check{\mathfrak{n}}}$, and ultimately on good properties of the factorization category $I \mapsto \mathsf{Rep}(\check{B})_{X^I}$ established in [Ras2] §6.

[47]In a slightly different setting, but the same proof goes through.

11.4. **Reduction to a point.** We now show that the main theorem reduces to its version over the fiber of a single point $x \in X$. First, note that this case implies the theorem for $I = \{*\}$, i.e., over the first power of the curve. Indeed, this follows from étale local constancy of everything in this case.

Now by Lemma 11.3.1, the functor $i_{X^I}^{\frac{\infty}{2},!,enh}$ admits a $D(X^I)$-linear (in particular, continuous) right adjoint (c.f. [Ras2] Proposition B.7.1). We need to test whether the unit for this adjunction is an equivalence. In fact, one can test whether any natural transformation of $D(X^I)$-linear functors is an equivalence over any stratification, in particular the "partition" stratification.[48] Since $i_{X^I}^{\frac{\infty}{2},!,enh}$ factorizes, we are reduced to the $I = \{*\}$-case, as desired.

11.5. **Reduction to the case $Z(G)$ connected.** Recall that there exists a short exact sequence $1 \to G \to \widetilde{G} \to \mathcal{T} \to 1$ for $\mathcal{T}$ a torus and $\widetilde{G}$ reductive with connected center, and such that $\widetilde{G}$ is equipped with a Cartan and Borel $\widetilde{T} \subseteq \widetilde{B}^-$ with $\widetilde{B}^- \cap G = B^-$, $\widetilde{T} \cap G = T$.[49] Note that the embedding $B^- \hookrightarrow \widetilde{B}^-$ induces an isomorphism on unipotent radicals. Suppose the theorem is true for $\widetilde{G}$.

Then it is easy to see:

$$\mathsf{Whit}_{\widetilde{G},X^I}^{\frac{\infty}{2}} \simeq \mathsf{Whit}_{G,X^I}^{\frac{\infty}{2}} \underset{D(\mathrm{Gr}_{T,X^I})}{\otimes} D(\mathrm{Gr}_{\widetilde{T},X^I})$$

$$\Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{\widetilde{T},X^I})) \simeq \Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,X^I})) \underset{D(\mathrm{Gr}_{T,X^I})}{\otimes} D(\mathrm{Gr}_{\widetilde{T},X^I}).$$

In particular, $\mathsf{Whit}_{\widetilde{G},X^I}^{\frac{\infty}{2}}$ and $\Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,X^I}))$ embed as full subcategories of the corresponding categories for $\widetilde{G}$, giving the desired claim.

11.6. **Our subcategories match up under Arkhipov-Bezrukavnikov.** We now have the following basic result about Arkhipov-Bezrukavnikov theory.

Lemma 11.6.1. *Under the equivalence $\mathsf{Whit}_x^{\frac{\infty}{2}} \simeq \mathsf{Whit}'(D(\mathrm{Fl}_G^{\mathrm{aff}})) \simeq \mathsf{QCoh}(\check{\mathfrak{n}}/\check{B})$ of [AB], the subcategories $\mathsf{Whit}_{acc,x}^{\frac{\infty}{2}}$ and $\mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B})$ match up.*

Indeed, this follows immediately from Theorem 7.3.1 of [FG2].[50]

11.7. **The kernel.** We wish to reinterpret our comparison functor as being given by a kernel so that we can analyze this kernel using t-structures.

Since we are working with the formal completions, this requires a bit of generalities first.

Remark 11.7.1. We emphasize in what follows that we're working with derived schemes: this is especially relevant when looking at formulae involving fiber products.

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[48]For precision: we mean the stratification indexed by surjections $p : I \twoheadrightarrow J$, where the corresponding strata is the image of $[X^J]_{disj}$ under the diagonal map.

[49]E.g., if $G = SL_n$, we can take $\widetilde{G} = GL_n$ mapping to $\mathcal{T} = \mathbb{G}_m$ by the determinant. The general case is proved using an argument with root data: one finds that $\widetilde{G}$ actually has the form $G \times \mathcal{T}'/Z(G)$ for $\mathcal{T}'$ a torus with $Z(G)$ embedded into it.

[50]Note however that there is a potentially confusing typo in the beginning of §7.1 from *loc. cit.*: the right hand side of the second equation break should be about the base-change of $D(\mathrm{Fl}_G^{\mathrm{aff}})$ to $\mathbb{B}\check{B}$ along the embedding $\mathbb{B}\check{B} \hookrightarrow \check{\mathfrak{n}}/\check{B}$.

11.8.   Recall that a prestack $\mathcal{Y}$ is *quasi-perfect* if $\mathsf{QCoh}(\mathcal{Y})$ is compactly generated by perfect objects.[51] Here $\mathcal{F} \in \mathsf{QCoh}(\mathcal{Y})$ is *perfect* if its restriction to any affine scheme is.

In this case, $\mathsf{QCoh}(\mathcal{Y})$ has an obvious canonical self-duality (as an object of DGCat_{cont}).

For $f : \mathcal{Y} \to \mathcal{Z}$ a morphism of quasi-perfect prestacks, we let $f^{ren}_* : \mathsf{QCoh}(\mathcal{Y}) \to \mathsf{QCoh}(\mathcal{Z})$ denote the functor dual to the pullback.

Example 11.8.1. Quasi-compact quasi-separated schemes are quasi-perfect. The functor f^{ren}_* is the usual pushforward functor in this case.

Example 11.8.2. The following results are shown in [GR1] §7. Suppose $\mathcal{Y} = S^{\wedge}_T$ is the formal completion of a quasi-compact quasi-separated scheme S along a finitely presented closed embedding $T \hookrightarrow S$. Then $S^{\wedge}_T$ is quasi-perfect.

Moreover, given a commutative diagram:

$$
\begin{array}{ccc}
T_1 & \longrightarrow & S_1 \\
\downarrow & & \downarrow{\scriptstyle g} \\
T_2 & \longrightarrow & S_2
\end{array}
$$

with induced morphism $f : S_{1,T_1^{\wedge}} \to S_{2,T_2^{\wedge}}$, the functor f^{ren}_* is computed via the commutative diagram:

$$
\begin{array}{ccc}
\mathsf{QCoh}(S_{1,T_1^{\wedge}}) & \lhook\joinrel\longrightarrow & \mathsf{QCoh}(S_1) \\
\downarrow{\scriptstyle f^{ren}_*} & & \downarrow{\scriptstyle g_*} \\
\mathsf{QCoh}(S_{2,T_2^{\wedge}}) & \lhook\joinrel\longrightarrow & \mathsf{QCoh}(S_2).
\end{array}
$$

The horizontal arrows are the canonical embeddings, realizing $\mathsf{QCoh}(S_{i,T_i^{\wedge}})$ as the subcategory of $\mathsf{QCoh}(S_1)$ of objects set-theoretically supported on T_1.

Moreover, using the results of *loc. cit.*, one finds in this case that f^{ren}_* restricted to compact objects coincides with f_*, the right adjoint to f^*: since f^{ren}_* is continuous, this characterizes it uniquely.

11.9.   We have a functor:

$$F : \mathsf{QCoh}(\check{\mathfrak{n}}^{\wedge}_0/\check{B}) \to \mathsf{QCoh}(\check{\mathfrak{n}}^{\wedge}_0/\check{B})$$

defined by:

$$\mathsf{QCoh}(\check{\mathfrak{n}}^{\wedge}_0/\check{B}) \overset{[\mathrm{AB}]}{\simeq} \mathsf{Whit}^{\frac{\infty}{2}}_{acc,x} \xrightarrow{i_x^{\frac{\infty}{2},!,enh}} \mathsf{QCoh}(\check{\mathfrak{n}}^{\wedge}_0/\check{B}) \subseteq \Upsilon_{\check{\mathfrak{n}}}\text{--}\mathsf{mod}^{\mathrm{fact}}_{un}(D(\mathrm{Gr}_{T,x}))$$

that we are trying to show is an equivalence.

11.10.   Observe that F is a morphism of $\mathsf{QCoh}(\check{\mathfrak{g}}^{\wedge}_0/\check{G})$-module categories. Indeed, this follows from the fact that $i_x^{\frac{\infty}{2},!,enh}$ is a morphism of factorization $\mathsf{Rep}(\check{G})^{\mathrm{fact}}$-module categories by Theorem 4.15.1, and the comparison of Theorem 10.8.1.

Note also that $\mathsf{QCoh}(\check{\mathfrak{n}}^{\wedge}_0/\check{B})$ is self-dual *as a* $\mathsf{QCoh}(\check{\mathfrak{g}}^{\wedge}_0/\check{G})$-*module category*: this follows from the corresponding facts about[52] $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$-module categories, which in turn follows from rigidity of $\mathsf{QCoh}(\check{\mathfrak{g}}/\check{G})$.

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[51]There is no requirement that every perfect object be compact, however.

[52]Note the absense of the formal completion.

We deduce that there exists a *kernel*:[53]

$$\mathcal{K} \in \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B}) \underset{\mathsf{QCoh}(\check{\mathfrak{g}}_0^\wedge/\check{G})}{\otimes} \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B}) \xrightarrow{\simeq} \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B})$$

such that:

$$F(-) = p_{2,*}^{ren}(\mathcal{K} \otimes p_1^*(-)).$$

We will analyze this kernel in the remainder of this section, in order to deduce that F is an equivalence.

Remark 11.10.1. In what follows, it is helpful to remember that we are trying to show that $\mathcal{K} \simeq \Delta_*^{ren}(\mathcal{O}_{\check{\mathfrak{n}}_0^\wedge/\check{B}})$ for Δ the (relative) diagonal map.

11.11.   Note that $\mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B})$ carries a canonical t-structure, called the *inductive* t-structure in [GR1] §7. It is readily characterized by compatibility with filtered colimits and the fact that the pushforward from the reduced locus (which is $\check{B}\backslash\check{G}/\check{B}$ here) is t-exact.

We have the following key result.

Proposition 11.11.1. $\mathcal{K}$ *lies in the heart of the (inductive) t-structure on* $\mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B})$.

We will prove this after some generalities in §11.15.

11.12. **Group actions and t-structures.** Let $\mathcal{C}$ be a cocomplete DG category equipped with a t-structure compatible with filtered colimits.

Let Γ be an affine algebraic group. Note that $\mathsf{QCoh}(\Gamma)$ is a monoidal category under convolution. By duality, it is a coalgebra object in DGCat_{cont} as well. Modules for the monoidal structure are the same as comodules for the coalgebra structure. We say that Γ *acts weakly* on $\mathcal{C}$ to mean $\mathcal{C}$ is equipped with a co/module structure.

A weak Γ-action on $\mathcal{C}$ is *compatible with this t-structure* if the coaction functor:

$$\mathcal{C} \to \mathcal{C} \otimes \mathsf{QCoh}(\Gamma)$$

is t-exact. Here $\mathcal{C} \otimes \mathsf{QCoh}(\Gamma) = \mathcal{O}_\Gamma\mathsf{-mod}(\mathcal{C})$ is equipped with the obvious t-structure: connective objects are generated under colimits by objects induced from $\mathcal{C}^{\leq 0}$, i.e., by objects of the form $\mathcal{F}\boxtimes\mathcal{O}_\Gamma$ for $\mathcal{F} \in \mathcal{C}^{\leq 0}$.

Lemma 11.12.1. *In the above setting,* $\mathcal{C}^{\Gamma,w} := \mathsf{Hom}_{\mathsf{QCoh}(\Gamma)\mathsf{-mod}}(\mathsf{Vect}, \mathcal{C})$ *admits a unique t-structure such that* $\mathsf{Oblv} : \mathcal{C}^{\Gamma,w} \to \mathcal{C}$ *is t-exact.*

Proof. Equipping $\mathcal{C} \otimes \mathsf{QCoh}(\Gamma)^{\otimes n}$ with a t-structure as above, one finds that each functor in the semi-cosimplicial diagram defining $\mathcal{C}^{\Gamma,w}$ is t-exact. This immediately gives the result.

$\square$

Lemma 11.12.2. *A weak Γ-action on $\mathcal{C}$ is compatible with the t-structure if and only if* act $: \mathcal{C} \otimes \mathsf{QCoh}(\Gamma) \to \mathcal{C}$ *is t-exact.*

Proof. Recall that one can compute the functor $\mathrm{act}(- \boxtimes \mathcal{O}_\Gamma) : \mathcal{C} \to \mathcal{C}$ as the composition:

$$\mathcal{C} \xrightarrow{\mathrm{coact}} \mathcal{C} \otimes \mathsf{QCoh}(\Gamma) \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes \Gamma(\Gamma,-)} \mathcal{C} \otimes \mathsf{Vect} = \mathcal{C}$$

---

[53]The isomorphism here follows from 1-affineness: see [Gai6].

(this is a standard argument with the Beck-Chevalley formalism).

Suppose act is t-exact. Then we obtain that the composition above is t-exact. But the latter functor is conservative and t-exact, so we obtain that coact must be t-exact.

Conversely, suppose that coact is t-exact. Then the composition above is the composition of two t-exact functors, so is t-exact. By definition of the t-structure on $\mathcal{C} \otimes \mathsf{QCoh}(\Gamma)$, this implies that act is right t-exact. But act is obviously left t-exact (being right adjoint to a t-exact functor), so is t-exact. □

11.13. Suppose that $\check{G}$ acts weakly on $\mathcal{C}$ and that is equipped with a t-structure compatible with the $\check{G}$-action. By Lemma 11.12.2, the induced $\check{B}$-action is also compatible with t-structures.

Some notation: let $\mathrm{Av}_*^{\check{B} \to \check{G}, w}$ denote the right adjoint to the forgetful functor $\mathcal{C}^{\check{G}, w} \to \mathcal{C}^{\check{B}, w}$. Also, recall that $\mathsf{Rep}(\check{B})$ acts on $\mathcal{C}^{\check{B}, w}$; we denote the action by $*$. For $\check{\lambda}$ a coweight, we let $\ell^{\check{\lambda}}$ denote the corresponding representation of $\check{T}$ (or $\check{B}$).

Proposition 11.13.1. *If* $\mathcal{F} \in \mathcal{C}^{\check{B}, w}$ *satisfies* $\mathrm{Av}_*^{\check{B} \to \check{G}, w}(\ell^{\check{\lambda}} * \mathcal{F}) \in \mathcal{C}^{\check{G}, w, \heartsuit}$ *for all* $\check{\lambda} \in \check{\Lambda}$ *sufficiently negative, then* $\mathcal{F} \in \mathcal{C}^{\heartsuit}$.

Proof. For $\check{\lambda}$ a weight, let $\mathcal{L}^{\check{\lambda}}$ denote the corresponding line bundle on $\check{G}/\check{B}$.

Let D be an ample divisor on $\check{G}/\check{B}$. Note that $\mathcal{O}(D) = \mathcal{L}^{-\check{\lambda}}$ for some regular dominant $\check{\lambda}$.[54] We can assume that $\check{\lambda}$ is sufficiently large in the sense of our hypothesis.

Let $j : U \hookrightarrow \check{G}/\check{B}$ denote the affine open subvariety $U := (\check{G}/\check{B}) \setminus D$ of $\check{G}/\check{B}$. By affineness and Lemma 11.12.2, the functor $\mathcal{C}^{B, w} \to \mathcal{C}$ given by convolution with $j_*(\mathcal{O}_U)$ is conservative and t-exact.[55]

We have embeddings $\mathcal{O}(D) \hookrightarrow \mathcal{O}(2D) \hookrightarrow \ldots \hookrightarrow \mathcal{O}(nD) \hookrightarrow \ldots$ in $\mathsf{QCoh}(\check{G}/\check{B})$ and the colimit is $j_*(\mathcal{O}_U)$. Now observe that convolution with $\mathcal{O}(nD)$ computes the functor $\mathrm{Av}_*^{\check{B} \to \check{G}, w}(\ell^{-n\check{\lambda}} * -)$. Therefore, our hypothesis on $\mathcal{F}$ implies that convolution with $j_*(\mathcal{O}_U)$ sends $\mathcal{F}$ to an object in degree 0, which implies the claim on $\mathcal{F}$ by conservativeness and exactness of $\mathcal{F}$. □

11.14. A normalization. Before proceeding, we need the following normalization.

Below, we will consider $\mathsf{Whit}_x^{\frac{\infty}{2}}$ as a $D(\mathrm{Gr}_{T,x}) = \mathsf{Rep}(\check{T})$-module category not under usual translations, but with cohomological shifts built in: that is, $\delta_{\check{\lambda}(t)}$ acts by translating by $\check{\lambda}(t)$ on $\mathrm{Fl}_x^{\frac{\infty}{2}}$ and applying a cohomological shift $[(2\rho, \check{\lambda})]$. Note that this makes sense factorizably as well: c.f. §2.26.

A further normalization, for clarity: when we speak of translation by $\check{\lambda}(t)$, this sends the identity in $\mathrm{Fl}_x^{\frac{\infty}{2}}$ to $\check{\lambda}(t)$. We mention this, since we will think of $\mathrm{Fl}_x^{\frac{\infty}{2}}$ as $G(K_x)/N(K_x)T(O_x)$, i.e., we quotient on the right, and then it might be regarded as more natural to take the inverted action. But to be clear, we are not doing this, but are using commutativity of T to turn this right action into a left one.

Because of the cohomological shifts incorporated above, note that $i_{X^I}^{\frac{\infty}{2}, !, enh}$ is a morphism of $D(\mathrm{Gr}_{T,X^I})$-module categories (recall that there were such cohomological shifts in the definition of $i_{X^I}^{\frac{\infty}{2}, !, enh}$ in the first place). Moreover, this Gr_T-linearity is compatible with factorization.

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[54]Of course, this is the usual sign issue in Borel-Weil-Bott. Recall the basic reason we need this sign: the canonical line bundle $\Omega_{\check{G}/\check{B}}^{top}$ is anti-ample, and comes from the representation $\Lambda^{top}(\check{\mathfrak{g}}/\check{\mathfrak{b}})^\vee$, which has weight $2\check{\rho}$.

[55]For example: we can take our divisor D as the sum of the codimension 1 N^--Schubert varieties. Then U is the big cell N^- and then convolution with $j_*(\mathcal{O}_U)$ computes $\mathrm{Av}_*^{N^-, w}$.

11.15. **Proof of Proposition 11.11.1.** We now show that the kernel lies in the heart of the t-structure.

Proof of Proposition 11.11.1. To see that $\mathcal{K}$ lies in the heart of the t-structure, it suffices to see that its restriction $\widetilde{\mathcal{K}}$ to:

$$\mathbb{B}\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B}$$

does.

We will see this by applying Proposition 11.13.1 with $\mathcal{C} = \mathsf{QCoh}(\mathrm{Spec}(k) \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B})$. By *loc. cit.*, it suffices to show that for every dominant coweight $\check{\lambda}$, the pushforward:

$$q_*(p_1^*(\ell^{-\check{\lambda}}) \otimes \widetilde{\mathcal{K}}) \in \mathsf{QCoh}(\mathbb{B}\check{G} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B}) \tag{11.15.1}$$

lies in the heart of the t-structure. Here p_1 is the projection to $\mathbb{B}\check{B}$, and q is the projection:

$$q : \mathbb{B}\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B} \to \mathbb{B}\check{G} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B}.$$

The object (11.15.1) can be considered as the kernel of a functor:

$$\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B})$$

of $\mathsf{QCoh}(\check{\mathfrak{g}}_0^\wedge/\check{G})$-module categories. Its relation to F is described as follows.

First, let i denote the map $\mathbb{B}\check{B} \to \check{\mathfrak{n}}_0^\wedge/\check{B}$. Then the functor defined by (11.15.1) is given by applying the functor $i_*q^* : \mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B})$, tensoring with $\pi^*(\ell^{-\check{\lambda}})$, and then applying the functor F, where π is the projection $\check{\mathfrak{n}}_0^\wedge/\check{B} \to \mathbb{B}\check{T}$.

By assumption, F intertwines the operation of tensoring with $\pi^*(\ell^{-\check{\lambda}})$.

Moreover, we claim that the composition $\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B}) \xrightarrow{F} \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B})$ is computed as the tautological functor $\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B})$, defined by the kernel $\varphi_*(\mathcal{O}_{\mathbb{B}\check{B}}) \in \mathsf{QCoh}(\mathbb{B}\check{G} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B})$ where $\varphi : \mathbb{B}\check{B} \to \mathbb{B}\check{G} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B}$ is the obvious map. Indeed, since F is a morphism of categories over $\check{\mathfrak{g}}_0^\wedge/\check{G}$, it is a functor of categories over $\mathbb{B}\check{G}$, so we see we only need to compute the image of the trivial representation under F, and this is the structure sheaf of $\mathbb{B}\check{B} \subseteq \check{\mathfrak{n}}_0^\wedge/\check{B}$ by construction.

We deduce that $q_*(p_1^*(\ell^{-\check{\lambda}}) \otimes \widetilde{\mathcal{K}})$ is isomorphic to the kernel $\varphi_*(\ell^{-\check{\lambda}})$. We claim that this object lies in the heart of the t-structure for $\check{\lambda}$ dominant. Indeed, φ is given by the composition:

$$\mathbb{B}\check{B} \to \mathbb{B}\check{G} \to \mathbb{B}\check{G} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B}$$

and the latter map is a closed embedding, so the claim follows from Borel-Weil-Bott.

$\square$

11.16. By Proposition 11.11.1, we have:

$$\mathcal{K} \in \mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B})^\heartsuit.$$

Note that we can replace our stack by its underlying classical substack here, since this operation preserves the heart of the t-structure on QCoh.

We now aim to show that $\mathcal{K}$ is *scheme*-theoretically supported on the diagonally embedded copy of $\check{\mathfrak{n}}_0^\wedge/\check{B}$. Note that this only makes sense in the abelian category, not in the derived category.

Note that we have a fiber square of *classical* stacks:

$$
\begin{array}{ccc}
\check{\mathfrak{n}}_0^\wedge & \xrightarrow{\;\Delta\;} & (\check{\mathfrak{n}}_0^\wedge/\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B})^{cl} \\[2mm]
\downarrow & & \downarrow \\[2mm]
\mathbb{B}\check{B} & \xrightarrow{\;\Delta\;} & \mathbb{B}\check{B} \underset{\mathbb{B}\check{G}}{\times} \mathbb{B}\check{B} = \check{B}\backslash\check{G}/\check{B}
\end{array}
$$

(i.e., this is not a fiber square of derived stacks).

11.17. In some more generality that Y is a classical Artin stack equipped with a representable projection $p : Y \to \check{B}\backslash\check{G}/\check{B}$, and $\mathcal{F} \in \mathsf{QCoh}(Y)^\heartsuit$ is given. We wish to give a criterion to check that $\mathcal{F}$ is scheme-theoretically supported on $p^{-1}(\mathbb{B}\check{B})^{cl}$.[56]

For $\check{\lambda}$ a dominant coweight, let $\mathcal{V}^{\check{\lambda}}$ denote the vector bundle on Y induced from the highest weight representation $V^{\check{\lambda}}$ from the structure map:

$$Y \to \check{B}\backslash\check{G}/\check{B} \to \mathbb{B}\check{G}.$$

Similarly, for $\check{\lambda}$ a coweight, let $\mathcal{L}_1^{\check{\lambda}}$ and $\mathcal{L}_2^{\check{\lambda}}$ denote the line bundles induced from the two projections:

$$Y \to \check{B}\backslash\check{G}/\check{B} \to \mathbb{B}\check{B}.$$

We have canonical surjective maps:

$$\kappa_i^{\check{\lambda}} : \mathcal{V}^{\check{\lambda}} \to \mathcal{L}_i^{w_0(\check{\lambda})}, \quad i = 1, 2.$$

First, we claim that $p^{-1}(\mathbb{B}\check{B})^{cl}$ is exactly the (classically defined) locus where these two quotients of $\mathcal{V}^{\check{\lambda}}$ coincide.[57] Indeed, this follows from the Plücker relations for the flag variety.

11.18. We want a version of the above that takes a quasi-coherent sheaf $\mathcal{F}$ into account.

We claim the following:

Proposition 11.18.1. *A quasi-coherent sheaf $\mathcal{F} \in \mathsf{QCoh}(Y)^\heartsuit$ is scheme-theoretically supported on $p^{-1}(\mathbb{B}\check{B})^{cl}$ if (and only if) the quotients:*

$$\mathcal{V}^{\check{\lambda}} \underset{\mathcal{O}_Y}{\otimes} \mathcal{F} \to \mathcal{L}_i^{w_0(\check{\lambda})} \underset{\mathcal{O}_Y}{\otimes} \mathcal{F}, \quad i = 1, 2$$

coincide (i.e., are isomorphic as quotients).

This follows immediately from the next result.

---

[56]The notation cl here is just used to emphasize that we are working with classical stacks, though it's hard to make a mistake in this setting since we're only using the abelian category of quasi-coherent sheaves.

[57]This should be understood in the usual sense from Grothendieck's theory of the quot scheme.

Lemma 11.18.2. *Suppose Y is a classical stack, $\mathcal{V}$ is a finite rank vector bundle on Y, and $\kappa_i : \mathcal{V} \to \mathcal{L}_i$ are quotients. Let $Z \subseteq Y$ be the classical substack where these quotients agree.*

Then $\mathcal{F} \in \mathsf{QCoh}(Y)^{\heartsuit}$ is scheme-theoretically supported on Z if (and only if) the quotients:

$$\mathcal{V} \underset{\mathcal{O}_Y}{\otimes} \mathcal{F} \to \mathcal{L}_i \underset{\mathcal{O}_Y}{\otimes} \mathcal{F}, \quad i = 1, 2$$

coincide.

Proof. The claim is smooth (even flat) local on Y, so we can assume Y is an affine scheme $Y = \mathrm{Spec}(A)$ and that all vector bundles in sight are trivialized.

Let n be the rank of $\mathcal{V}$, so our quotients are defined by row vectors $\begin{pmatrix} f_1 & \dots & f_n \end{pmatrix}$ and $\begin{pmatrix} g_1 & \dots & g_n \end{pmatrix}$, where the condition that these be *quotients* means that each of the two maps $Y \to \mathbb{A}^n$ should factor through $\mathbb{A}^n \backslash 0$. Further Zariski localizing (using the standard affine covering of $\mathbb{P}^{n-1}$), we may WLOG assume that f_1 is invertible, and then without changing the isomorphism class of our quotient, we may assume that $f_1 = 1$.

Note that the ideal defining Z is generated by the elements $g_1 f_i - g_i$, $i = 2, \dots, n$.

Now the two maps:

$$\mathcal{F}^{\oplus n} \to \mathcal{F}$$

are also defined using the row vectors above. If they coincide as quotients, there is an isomorphism $\alpha : \mathcal{F} \xrightarrow{\simeq} \mathcal{F}$ such that the diagram commutes, i.e., we should have:

$$\alpha\left(\sum_{i=1}^n f_i s_i\right) = \sum_{i=1}^n g_i s_i$$

for all $(s_i)_{i=1}^n \in \mathcal{F}^{\oplus n}$.

For $s \in \mathcal{F}$ fixed, letting $s_1 = s$ and $s_j = 0$ for $j \neq 1$, we see that $\alpha(s) = g_1 s$. In turn taking $s_i = s$ for i fixed and all $s_j = 0$ for $j \neq s$, we obtain $\alpha(f_i s) = g_i s$, but we have just seen that $\alpha(f_i s) = g_1 f_i s$, so $(g_1 f_i - g_i)s = 0$, which is what was to be shown.

(For the converse, which we will not need, note that g_1 is invertible on Z, since by assumption, the g_i generate the unit ideal of A, and each $g_i|_Z$ is divisible by $g_1|_Z$. Therefore, taking α as above to be defined as multiplication by g_1 really is an isomorphism if $\mathcal{F}$ is scheme-theoretically supported on Z.)

$\square$

11.19.   To specialize to our setting, take $Y = (\check{\mathfrak{n}}/\check{B} \times_{\check{\mathfrak{g}}/\check{G}} \check{\mathfrak{n}}/\check{B})^{cl}$ mapping to $\check{B}\backslash\check{G}/\check{B}$ in the obvious way. Our $\mathcal{K}$ can be considered as an object of $\mathsf{QCoh}(Y)^{\heartsuit}$ *set*-theoretically supported on $\check{B}\backslash\check{G}/\check{B} \subseteq Y$.

We want to apply Proposition 11.18.1 in this setting to see that $\mathcal{K}$ is *scheme*-theoretically supported on the inverse image of $\mathbb{B}\check{B}$ (which is the diagonally embedded copy of $\check{\mathfrak{n}}/\check{B}$).

Translating from kernels to functors, we should verify the following condition:

Let $\mathcal{V}^{\check{\lambda}}$ and $\mathcal{L}^{w_0(\check{\lambda})}$ denote the vector bundles on $\check{\mathfrak{n}}_0^{\wedge}/\check{B}$ obtained from the projections to $\mathbb{B}\check{G}$ and $\mathbb{B}\check{T}$ (so we are changing the notation slightly from §11.18, since $\check{\mathfrak{n}}_0^{\wedge}/\check{B}$ is not our Y). We have a canonical map $\mathcal{V}^{\check{\lambda}} \to \mathcal{L}^{w_0(\check{\lambda})}$.

We have an isomorphism:

$$F(\mathcal{V}^{\check{\lambda}} \otimes -) \xrightarrow{\simeq} \mathcal{V}^{\check{\lambda}} \otimes F(-)$$

since F is a functor of categories over $\check{\mathfrak{g}}_0^{\wedge}/\check{G}$. Then to verify the condition of Proposition 11.18.1, we should produce a commutative diagram:

$$F(\mathcal{V}^{\check{\lambda}} \otimes -) \xrightarrow{\;\simeq\;} \mathcal{V}^{\check{\lambda}} \otimes F(-) \qquad (11.19.1)$$

$$F(\mathcal{L}^{w_0(\check{\lambda})} \otimes -) \xdashrightarrow{\;\simeq\;} \mathcal{L}^{w_0(\check{\lambda})} \otimes F(-).$$

This should be understood as a diagram of $\mathsf{QCoh}(\check{\mathfrak{g}}_0^{\wedge}/\check{G})$-linear functors.

In fact, we already have an isomorphism in the bottom row from the $\mathsf{Rep}(\check{T})$-linearity of F.[58] All that is left is to verify that this diagram commutes.

We will verify this property in what follows.

11.20. Geometric input. We first make some observations about the $\mathsf{Whit}^{\frac{\infty}{2}}$ and the functor $i^{\frac{\infty}{2},!,enh}$, which will provide the substance for the verification above.

11.21. Let $\check{\lambda}$ be a dominant coweight.

Let $\mathcal{W}_x^{\check{\lambda}} \in \mathsf{Whit}_x^{\frac{\infty}{2}}$ be obtained from $V^{\check{\lambda}}$ via the composition:

$$\mathsf{Rep}(\check{G}) \simeq \mathsf{Whit}_x^{sph} \to \mathsf{Whit}_x^{int} \to \mathsf{Whit}_x^{\frac{\infty}{2}}$$

(i.e., we take the corresponding spherical Whittaker sheaf and create an object of $\mathsf{Whit}_x^{\frac{\infty}{2}}$ from it in the only way we know how).

Let $\mathrm{unit}^{w_0(\check{\lambda})}_{\mathsf{Whit}_x^{\frac{\infty}{2}}}$ denote the $w_0(\check{\lambda})$-translate of the unit object, where we are using the convention of §11.14 here.[59] We will construct a canonical map:

$$\varepsilon_x^{\check{\lambda}} : \mathcal{W}_x^{\check{\lambda}} \to \mathrm{unit}^{w_0(\check{\lambda})}_{\mathsf{Whit}_x^{\frac{\infty}{2}}} \in \mathsf{Whit}_{acc,x}^{\frac{\infty}{2}}. \qquad (11.21.1)$$

Note that the left hand side is obtained from the IC sheaf on $G(O)\backslash\overline{G(O_x)\check{\lambda}(t)G(O_x)}$ by projecting from $B(O_x)$-invariants $= B(O_x)$-coinvariants to $N(K_x)T(O_x)$-coinvariants (with respect to the right action) and then !-averaging from $G(O)$ to Whittaker (with respect to the left action).

The right hand side admits a similar interpretation, but we should start with the $*$-extension of $\omega[(-2\rho, \check{\lambda})]$ on $G(O)\backslash\overline{G(O_x)w_0(\check{\lambda})(t)B(O_x)}$ instead.

Now recall that:

$$G(O)\backslash\overline{G(O_x)w_0(\check{\lambda})(t)B(O_x)} \hookrightarrow G(O)\backslash\overline{G(O_x)\check{\lambda}(t)G(O_x)}$$

is an open embedding.[60]

Therefore, we have a map from our IC sheaf to our $*$-extension of ω, but we should cohomologically shift ω up by the dimension of each of these orbits, which is exactly $(2\rho, \check{\lambda})$. This gives a map in the category of left $G(O_x)$-equivariant and right $B(O_x)$-equivariant D-modules, and then we obtain the desired map in $\mathsf{Whit}_x^{\frac{\infty}{2}}$ by functoriality.

---

[58]More precisely, $\mathcal{K}$ descends to $\check{\mathfrak{n}}_0^{\wedge}/\check{B} \underset{\check{\mathfrak{g}}_0^{\wedge}/\check{G}\times\mathbb{B}\check{T}}{\times} \check{\mathfrak{n}}_0^{\wedge}/\check{B}$ for simple reasons.

[59]Since is easy to forget who this object is: it is the $*$-extension of the canonical Whittaker D-module on $N^-(K_x)G(O_x)w_0(\check{\lambda})(t)N(K_x) = N^-(K_x)G(O_x)w_0(\check{\lambda})(t)N(K_x)T(O_x) \subseteq G(K_x)$, with the cohomological shift $[(2\rho, w_0(\check{\lambda}))] = [(-2\rho, \check{\lambda})]$.

[60]If the reader thinks it should be $\check{\lambda}(t)$ instead of $w_0(\check{\lambda})(t)$, this is because we are used considering orbits on $G(K_x)/G(O_x)$, but we are switching left and right actions here and the inversion means *anti*dominant Iwahori orbits are the ones open in the corresponding $G(O_x)$-orbits (and these Iwahori orbits equal the corresponding $B(O_x)$-orbits).

11.22. Now observe that we can compute $i_x^{\frac{\infty}{2},!,enh}$ applied to both sides of (11.21.1): the left hand side goes to $\mathrm{Chev}_{\check{\mathfrak{n}},x}^{\mathrm{spec}}(V^{\check{\lambda}})$ by Theorem 4.15.1, and the right hand side goes to the $w_0(\check{\lambda})$-translate of the vacuum representation for $\Upsilon_{\check{\mathfrak{n}}}$ at x, i.e., to the factorization $\Upsilon_{\check{\mathfrak{n}}}$-module obtained from $\ell^{w_0(\check{\lambda})} \in \mathsf{Rep}(\check{B})$ via chiral induction from the corresponding Lie-$*$ module for $\check{\mathfrak{n}}_X$.

More descriptively: the left hand side is the chiral module version of the homological Chevalley complex for $\check{\mathfrak{n}}$ with coefficients in $V^{\check{\lambda}}$, and the right hand side is the same but for $\ell^{w_0(\check{\lambda})}$.

Our main geometric input is the fact that the map obtained from our geometric procedure coincides with the canonical map seen in the spectral picture, i.e., the map induced by the map $V^{\check{\lambda}} \to \ell^{w_0(\check{\lambda})} \in \mathsf{Rep}(\check{B})$. This follows from the proof of Theorem 4.15.1: see [Ras2] §7.25.

11.23. Finally, we note that the above all makes sense if we work over the curve X, instead of just at the single point x. We will need the above in this form actually. We use subscripts X instead of x to indicate that this is the case.

11.24. **Verification of the claim from §11.19.** We now verify that the diagram (11.19.1) commutes forgetting about the fact that these are $\mathsf{QCoh}(\check{\mathfrak{g}}_0^{\wedge}/\check{G})$-linear functors. That is, we instead show that the diagram commutes as a diagram of mere functors. Then we will explain why this was enough in §11.25.

We will do this by identifying each of the vertical arrows in this diagram with some other morphism of functors.

Namely, suppose we use [AB] to identify $\mathsf{Whit}_{acc,x}^{\frac{\infty}{2}}$ with $\mathsf{QCoh}(\check{\mathfrak{n}}_0^{\wedge}/\check{B})$, and we identify $\mathsf{QCoh}(\check{\mathfrak{n}}_0^{\wedge}/\check{B})$ with a full subcategory of $\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,x}))$. Then every vertex in the diagram (11.19.1) can be recovered from the corresponding functor:

$$\mathsf{Whit}_{acc,x}^{\frac{\infty}{2}} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}_{un}^{\mathrm{fact}}(D(\mathrm{Gr}_{T,x})).$$

We claim that the vertical legs of the diagram can each be identified with $i_x^{\frac{\infty}{2},!,enh}$ composed with the endofunctor of $\mathsf{Whit}_x^{\frac{\infty}{2}}$ given by taking nearby cycles against the moving points version of (11.21.1), i.e., the maps:

$$\varepsilon_{X\backslash x}^{\check{\lambda}} : \mathcal{W}_{X\backslash x}^{\check{\lambda}} \to \mathrm{unit}_{\mathsf{Whit}_{X\backslash x}^{\frac{\infty}{2}}}^{w_0(\check{\lambda})} \in \mathsf{Whit}_{acc,X\backslash x}^{\frac{\infty}{2}}.$$

For the left vertical arrow of (11.19.1), this is an extended chase through the constructions: up to identifying our semi-infinite categories with categories of Iwahori nature and easy checks that various functors commute with nearby cycles[61], this is the Arkhipov-Bezrukavnikov construction of the action using Gaitsgory's central sheaves [Gai1].[62]

For the right vertical arrow, this follows from Proposition 10.5.1, using the geometric input from §11.20-11.23 and the fact that semi-infinite restriction $i^{\frac{\infty}{2},!,enh}$ commutes with nearby cycles (which follows from Lemma 11.3.1).

---

[61]The basic (easy) fact for such verification is that any $D(X)$-linear functor $F : \mathcal{C} \to \mathcal{D} \in D(X)\text{–}\mathsf{mod}$ that admits a $D(X)$-linear right adjoint commutes with nearby cycles. In particular, this is the case if $\mathcal{C}$ is ULA and F preserves ULA objects.

[62]We apologize for not providing more details here. There are many small details to fill in, and we expect that the reader comfortable with the construction of [AB] and the categorical machinery used in this paper will not have difficulty with this, while the reader lacking either of those backgrounds will not benefit from their being included.

11.25. Finally, we explain why we did not need to worry about the $\mathsf{QCoh}(\check{\mathfrak{g}}_0^\wedge/\check{G})$-linearity in verifying the commutativity of the diagram (11.19.1).

Namely, we can regard this as a diagram of kernels, i.e., objects of $\mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B})^\heartsuit$.

Since the diagonal morphism:

$$\check{\mathfrak{n}}_0^\wedge/\check{B} \underset{\check{\mathfrak{g}}_0^\wedge/\check{G}}{\times} \check{\mathfrak{n}}_0^\wedge/\check{B} \to \check{\mathfrak{n}}_0^\wedge/\check{B} \times \check{\mathfrak{n}}_0^\wedge/\check{B}$$

is affine[63], the corresponding pushforward functor is injective on Homs in the abelian category, so we can verify the commutativity of the diagram downstairs instead.

11.26. **Completion of the proof.** We have now obtained that $\mathcal{K}$ is set-theoretically supported on $\check{\mathfrak{n}}_0^\wedge/\check{B}$, meaning that we can regard it as having been pushed forward from there. So we can change notation and consider $\mathcal{K}$ as an object of $\mathsf{QCoh}(\check{\mathfrak{n}}_0^\wedge/\check{B})$. Then the relation to F becomes that F is given by tensoring with $\mathcal{K}$.

Recall that F takes $i_*(\mathcal{O}_{\mathbb{B}\check{B}})$ to itself, where i is the map $\mathbb{B}\check{B} \to \check{\mathfrak{n}}_0^\wedge/\check{B}$. Therefore, we obtain:

$$i_*(\mathcal{O}_{\mathbb{B}\check{B}}) = F(i_*(\mathcal{O}_{\mathbb{B}\check{B}})) = \mathcal{K} \otimes i_*(\mathcal{O}_{\mathbb{B}\check{B}}) = i_* i^*(\mathcal{K}).$$

We claim that this forces $\mathcal{K}$ to be a line bundle, and thereby obtaining that F is an equivalence. It suffices to verify this after base-changing to $\check{\mathfrak{n}}_0^\wedge$. Therefore, it follows from the next lemma.

Lemma 11.26.1. *Suppose $\mathcal{K} \in \mathsf{QCoh}(\mathbb{A}_0^{n,\wedge})$ with $i^*(\mathcal{K}) \simeq k$, where $i : \mathrm{Spec}(k) \hookrightarrow \mathbb{A}_0^{n,\wedge}$ is the inclusion. Then $\mathcal{K} \simeq \mathcal{O}_{\mathbb{A}_0^{n,\wedge}}$.*

Proof. First, we claim that $\Gamma(\mathbb{A}_0^{n,\wedge}, \mathcal{K}) \in \mathsf{Vect}^\heartsuit$.

Indeed, let i_m denote the embedding $X_m := \mathrm{Spec}(k[x_1, \dots, x_n]/(x_1, \dots, x_n)^m) \hookrightarrow \mathbb{A}_0^{n,\wedge}$. Then $\mathcal{K} = \lim_m i_{m,*} i_m^*(\mathcal{K})$, so:

$$\Gamma(\mathbb{A}_0^{n,\wedge}, \mathcal{K}) = \lim_m \Gamma(X_m, i_m^*(\mathcal{K})).$$

Using iterated square-zero extensions (and the hypothesis that $i^*(\mathcal{K}) \in \mathsf{Vect}^\heartsuit$), we find that every term of this limit is in cohomological degree zero and all structure maps are surjective, giving the claim.

Now recall from [GR1] §7 that this means $\mathcal{K}$ lies in the heart of a t-structure, and this heart is the abelian category of derived I-adically complete $k[x_1, \dots, x_n]$ modules, where $I = (x_1, \dots, x_n)$. In particular, $\mathcal{O}_{\mathbb{A}_0^{n,\wedge}}$ also lies in the heart of this t-structure and is projective there, and $\mathcal{K} \to i_* i^*(\mathcal{K})$ is an epimorphism in this abelian category.

Therefore, the map $\mathcal{O}_{\mathbb{A}_0^{n,\wedge}} \to k \simeq i_* i^*(\mathcal{K})$ lifts to a map to $\mathcal{K}$, which is an isomorphism since it is such after applying i^*.

$\square$

Appendix A. Proof of Lemmas 9.8.1 and 9.9.2

A.1. The purpose of this section is to prove two technical results on regular local systems from §9. In each, we essentially want to describe the geometry of the (factorizable versions) of the spaces of local systems in terms of the categories $\mathsf{Rep}(\Gamma)_{X_{dR}^I}$ studied in [Ras2] §6.

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[63]Being obtained by base-change from the diagonal morphism for $\check{\mathfrak{g}}_0^\wedge/\check{G}$.

A.2. First, recall the setup of Lemma 9.8.1: we want to show that $I \mapsto \mathsf{QCoh}(\mathrm{LocSys}_{\Gamma}(\mathcal{D})_{X_{dR}^I})$ is the factorization category associated with the symmetric monoidal DG category $\mathsf{Rep}(\Gamma) \in \mathsf{DGCat}_{cont}$.

  We will first prove a version of Lemma 9.8.1 with affine schemes (or more generally, possibly non-connective commutative algebras) playing the role of $\mathbb{B}\Gamma$, and will deduce the lemma from here and from structural results on $\mathsf{Rep}(\Gamma)_{X^I}$ given in [Ras2].

A.3. Let $A \in \mathsf{Vect}$ be a commutative algebra.

  Recall from [Ras2] §6.12 (c.f. also [Ras1] §7) that there is a commutative factorization algebra:

$$I \mapsto A_{X_{dR}^I} \in \mathsf{ComAlg}(D(X^I))$$

associated to A, such that $A_X = A \otimes \omega_X$.

  Here each $A_{X_{dR}^I}$ is a commutative algebra for the $\overset{!}{\otimes}$ tensor product on $D(X^I)$. Therefore, we can take the associated category of modules:

$$A_{X_{dR}^I}\text{--}\mathsf{mod}(D(X^I)) \in D(X^I)\text{--}\mathsf{mod}.$$

The assignment $I \mapsto A_{X_{dR}^I}\text{--}\mathsf{mod}(D(X^I))$ defines a factorization category.

A.4. On the other hand, $A\text{--}\mathsf{mod}$ is a rigid and symmetric monoidal DG category, and therefore we may form the factorization category:

$$I \mapsto A\text{--}\mathsf{mod}_{X_{dR}^I}.$$

By construction, we have a canonical functor:

$$A_{X_{dR}^I}\text{--}\mathsf{mod}(D(X^I)) \to A\text{--}\mathsf{mod}_{X_{dR}^I}. \tag{A.4.1}$$

Proposition A.4.1. *The functor* (A.4.1) *is an equivalence.*

Proof. First, note that $A_{X_{dR}^I}\text{--}\mathsf{mod}(D(X^I))$ is ULA as a category over X_{dR}^I, in the sense of [Ras2] Appendix B, and has ULA generator $A_{X_{dR}^I}$. Indeed, this follows since the (conservative) forgetful functor $A_{X_{dR}^I}\text{--}\mathsf{mod}(D(X^I)) \to D(X^I)$ admits a $D(X^I)$-linear right adjoint sending ω_{X^I} to $A_{X_{dR}^I}$.

  Moreover, by [Ras2] Lemma 6.16.2, the image of $A_{X_{dR}^I}\text{--}\mathsf{mod}(D(X^I))$ under (A.4.1) is ULA. Indeed, *loc. cit.* shows that for any rigid symmetric monoidal category $\mathcal{C}$, the unit object is ULA over X^I.

  Therefore, by [Ras2] Proposition B.8.1 and by factorization, to see that (A.4.1) is an equivalence, it suffices to check this for $I = \{*\}$, where the result is clear.

$\square$

A.5. We now deduce the appropriate version of this result for $\mathrm{LocSys}_{\Gamma}(\mathcal{D})$.

Proof of Lemma 9.8.1.

Step 1. First, we use the expression of $\mathrm{LocSys}_{\Gamma}(\mathcal{D})$ by gauge forms from §9.4. In the notation of *loc. cit.*, we obtain:

$$\mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I}) \xrightarrow{\simeq} \lim_{[n]\in\boldsymbol{\Delta}} \mathsf{QCoh}\big(\underbrace{G(O)_{X_{dR}^I} \underset{X_{dR}^I}{\times} \ldots \underset{X_{dR}^I}{\times} G(O)_{X_{dR}^I}}_{n \text{ times}} \underset{X_{dR}^I}{\times} \mathrm{Lie}(\Gamma)\otimes\Omega_X^1(O)_{X_{dR}^I}\big).$$

(A.5.1)

Step 2. Now observe that by étale descent for sheaves of categories, we may reduce to the case where $X = \mathbb{A}^1$. Let $x \in X(k)$ be the point 0.

Trivialize Ω_X^1 by a translation invariant 1-form, and similarly write $G(O)_{X_{dR}} = G(O_x) \times X_{dR}$ via translations. Wethen obtain that the action of $G(O)_{X_{dR}}$ on $\mathrm{Lie}(\Gamma)\otimes\Omega_X^1(O)_{X_{dR}}$ is obtain from the action of $G(O_x)$ on $\Omega_X^1(O_x)$ (i.e., 1-forms on the disc at x) by realizing these as constant D_X-schemes. We formally deduce the same over each X_{dR}^I.

Step 3. Applying (A.5.1) over a point, we obtain:

$$\mathsf{Rep}(\Gamma) = \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D}_x)) \simeq \lim_{[n]\in\boldsymbol{\Delta}} \mathsf{QCoh}\big(G(O_x)^n \times \mathrm{Lie}(\Gamma)\otimes\Omega_X^1(O_x)\big).$$

Since each structure functor is symmetric monoidal, we obtain:

$$\mathsf{Rep}(\Gamma)_{X_{dR}^I} \simeq \lim_{[n]\in\boldsymbol{\Delta}} \mathsf{QCoh}\big(G(O_x)^n \times \mathrm{Lie}(\Gamma)\otimes\Omega_X^1(O_x)\big)_{X_{dR}^I}.$$ (A.5.2)

Indeed, recall that for $\mathcal{C} \in \mathsf{ComAlg}(\mathsf{DGCat}_{cont})$, the associated category $\mathcal{C}_{X_{dR}^I}$ is a limit in DGCat_{cont} of terms of the form:

$$\mathcal{C}^{\otimes K}\otimes D(U)$$

where $U \subseteq X^I$ is a certin (variable) open subset, and K is a (variable) finite set. It follows that $\mathcal{C} \mapsto \mathcal{C}_{X_{dR}^I}$ commutes with sifted those limits in $\mathsf{ComAlg}(\mathsf{DGCat}_{cont})$ formed from diagrams of dualizable DG categoriesand with dualizable limit (to commute the appropriate limits with tensor products); this justifies (A.5.2), recalling that the simplex category is sifted.

By our earlier work on affine D_X-schemes and by Step 2, the right hand side of this equation identifies with the right hand side of (A.5.1), giving the result.

$\square$

A.6.   We now begin work on Lemma 9.9.2.

A.7.   As a first step, we reduce to showing only that $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}$ is 1-affine.

Indeed, for both results, it suffices to show that the morphism $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X_{dR}^I} \to X_{dR}^I$ is 1-affine (see [Ras1] §A.8 for the terminology here), since the morphism $X_{dR}^I \to \mathrm{Spec}(k)$ is 1-affine. This means that for every $S \to X_{dR}^I$ with S an affine scheme, we need to show that the corresponding fiber product is 1-affine.

Since every map $S \to X_{dR}^I$ lifts to X^I (by smoothness of X), it suffices to show that $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I} \to X^I$ is 1-affine. But this is equivalent to 1-affineness of the total space by [Ras1] Proposition A.11.6.

A.8.   Next, we observe the following consequence of Lemma 9.8.1.

Corollary A.8.1. *The prestack* $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}$ *is passable in the sense of* [Gai6] *§5, i.e.,* $\mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I})$ *is rigid monoidal, and the diagonal map for* $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}$ *is quasi-affine (in fact: affine in this case).*

Proof. The canonical map:

$$\mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I_{dR}} \underset{D(X^I)}{\otimes} \mathsf{QCoh}(X^I) \to \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I})$$

is an equivalence since X^I_{dR} and X^I are 1-affine.

The rigid monoidality now follows from the analysis of $\mathsf{Rep}(\Gamma)_{X^I_{dR}} \underset{D(X^I)}{\otimes} \mathsf{QCoh}(X^I)$ given in [Ras2]

Theorem 6.17.1. The affineness of the diagonal follows from the expression of $\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}$ as the quotient of of gauge forms by the gauge group.

$\square$

We now obtain the following result from [Gai6] Proposition 5.1.3:

Corollary A.8.2. *The functor:*

$$\Gamma : \mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}} \to \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I})\text{--mod}$$

is fully-faithful.

A.9.   Following [Gai6] §6, we will deduce 1-affineness from the following.

Proposition A.9.1. *The map:*

$$\underset{[n]\in\mathbf{\Delta}^{op}}{\mathrm{colim}} \mathsf{QCoh}\big(\underbrace{G(O)_{X^I} \underset{X^I}{\times} \ldots \underset{X^I}{\times} G(O)_{X^I}}_{n\ times} \underset{X^I}{\times} \mathrm{Lie}(\Gamma)\otimes\Omega^1_X(O)_{X^I}\big) \to \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}) \in \mathsf{DGCat}_{cont}$$

$$(\mathrm{A.9.1})$$

is an equivalence, where all the structure maps are pushforwards for quasi-coherent sheaves.

Proof that Proposition A.9.1 implies Lemma 9.9.2. We sketch the argument just to remind the reader how it goes, referring to [Gai6] Proposition 6.2.7 for details.

One has a functor:

$$\mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}} \to \mathsf{DGCat}_{cont}$$

given by forming an appropriate version of the colimit (A.9.1) with coefficients (e.g., the sheaf of categories $\mathsf{QCoh}_{/\mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}}}$ maps to (A.9.1)).

We claim that the composition:

$$\mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I})\text{--mod} \to \mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}} \to \mathsf{DGCat}_{cont}$$

is the forgetful functor, where the first functor is the left adjoint to global sections. Indeed, both functors are formed using colimits and tensor products, so it is easy to compute the composition as:

$$\big(\mathcal{C} \in \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I})\text{--mod}\big) \mapsto$$

$$\mathcal{C} \underset{\mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I})}{\otimes} \Big(\underset{[n]\in\mathbf{\Delta}^{op}}{\mathrm{colim}} \mathsf{QCoh}\big(\underbrace{G(O)_{X^I} \underset{X^I}{\times} \ldots \underset{X^I}{\times} G(O)_{X^I}}_{n\ times} \underset{X^I}{\times} \mathrm{Lie}(\Gamma) \otimes \Omega^1_X(O)_{X^I}\big)\Big) \overset{Prop.A.9.1}{\simeq} \mathcal{C}.$$

In particular, the left adjoint to $\Gamma : \mathsf{ShvCat}_{/\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I}} \to \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D})_{X^I})\text{--mod}$ is conservative, so we obtain the result by Corollary A.8.2.

$\square$

A.10.   It now remains to prove Proposition A.9.1.

A.11.   First, we prove a version of Proposition A.9.1 at a (closed) point $x \in X$.

Lemma A.11.1. *The map:*

$$\operatorname*{colim}_{[n]\in\mathbf{\Delta}^{op}} \mathsf{QCoh}\big(G(O_x)^n \times \mathrm{Lie}(\Gamma) \otimes \Omega^1_X(O_x)\big) \to \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D}_x)) \in \mathsf{DGCat}_{cont} \qquad \text{(A.11.1)}$$

is an equivalence.

Proof.

Step 1. First, we rewrite the left hand side of (A.11.1): since all of these schemes are affine, QCoh turns products into tensor products in DGCat_{cont}. Therefore, we can rewrite the left hand side of (A.11.1) as the *(weak) coinvariants of $\Gamma(O_x)$ acting on* $\mathsf{QCoh}(\mathrm{Lie}(\Gamma) \otimes \Omega^1_X(O_x))$, i.e., it is:

$$\mathsf{QCoh}(\mathrm{Lie}(\Gamma) \otimes \Omega^1_X(O_x)) \underset{\mathsf{QCoh}(\Gamma(O_x))}{\otimes} \mathsf{Vect} \in \mathsf{DGCat}_{cont}$$

where $\mathsf{QCoh}(\Gamma(O_x))$ is equipped with the convolution monoidal structure, and it acts on Vect through the action of $\Gamma(O_x)$ on $\mathrm{Spec}(k)$.

Step 2. Let $\mathcal{K}_1 \subseteq \Gamma(O_x)$ denote the first congruence subgroup, i.e., the kernel of the evaluation map to Γ.

We can rewrite the above (weak) coinvariants by first taking coinvariants with respect to $\mathcal{K}_1$ and then taking the coinvariants with respect to the residual (weak) action of Γ.

Step 3. Next, observe that the weak coinvariants of $\mathcal{K}_1$ acting on $\mathsf{QCoh}(\mathrm{Lie}(\Gamma) \otimes \Omega^1_X(O_x))$ is Vect equipped with the trivial weak Γ-action: indeed, this follows from the fact that Γ acts simply transitively on $\mathrm{Lie}(\Gamma) \otimes \Omega^1_X(O_x)$.

Step 4. Finally, it remains to see that the map:

$$\mathsf{Vect} \underset{\mathsf{QCoh}(\Gamma)}{\otimes} \mathsf{Vect} \to \mathsf{QCoh}(\mathrm{LocSys}_\Gamma(\mathcal{D}_x)) = \mathsf{Rep}(\Gamma)$$

is an equivalence. But this is exactly the main result of [Gai6] §7.

$\square$

A.12.   We now deduce Proposition A.9.1 (and with it, Lemma 9.9.2) from the above and from results of [Ras2].

Proof of Proposition A.9.1. First, we observe that the statement is étale local on X (e.g., using an easier version of Proposition 6.2.7 of [Gai6] to justify the descent), and therefore we can reduce to the case $X = \mathbb{A}^1$. Let $x \in X(k)$ again denote the point 0.

For every surjection $p : I \twoheadrightarrow J$, let $U(p) \subseteq X^I$ denote the corresponding open subscheme:

$$U(p) = \{x = (x_i) \in X^I \mid x_i \neq x_j \text{ if } p(i) \neq p(j)\}.$$

By [Ras2] Lemma 6.18.1, for $\mathcal{C} \in \mathsf{ComAlg}(\mathsf{DGCat}_{cont})$ compactly generated and rigid monoidal (in particular, canonically self-dual), we have an isomorphism:

$$\operatorname*{colim}_{I \overset{p}{\twoheadrightarrow} J \twoheadrightarrow K} \mathcal{C}^{\otimes K} \otimes \mathsf{QCoh}(U(p)) \xrightarrow{\simeq} \mathcal{C}_{X^I_{dR}}$$

where we recall the (irrelevant for our purposes) information that:

- The index category is the twisted arrow category of the partially-ordered set of partitions of I.
- This diagram is *dual* to the diagram defining $\mathcal{C}_{X^I}$, i.e., maps in the J-variable induced pushforward functors on the $\mathsf{QCoh}(U(p))$ factor, and maps in the K-variable induce functors *dual* to the tensor product functors $\mathcal{C}^{\otimes K} \to \mathcal{C}^{\otimes K'}$.

It follows that the assignment $\mathcal{C} \mapsto \mathcal{C}_{X^I} \in \mathsf{QCoh}(X^I)\text{–}\mathsf{mod}$ commutes with sifted colimits of diagrams where all terms are compactly generated rigid monoidal and the colimit is as well.

Since $\mathcal{C} = \mathsf{QCoh}(\Gamma(O_x))$, $\mathsf{QCoh}(\Omega^1_X(O_x))$ and $\mathsf{Rep}(\Gamma)$ are rigid monoidal, we obtain the claim from Lemma A.11.1.

$\square$

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