D-MODULES ON INFINITE DIMENSIONAL VARIETIES

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Contents

1. Introduction

1.1. The goal of this foundational note is to develop the D-module formalism on indschemes of ind-infinite type.

1.2. The basic feature that we struggle against is that there are two types of infinite dimensionality at play: pro-infinite dimensionality and ind-infinite dimensionality. That is, we could have an infinite dimensional variety S that is the union $S = \cup_i S_i = \mathrm{colim}_i S_i$ of finite dimensional varieties, or T that is the projective limit $T = \lim_j T_j$ of finite dimensional varieties, e.g., a scheme of infinite type.

Any reasonable theory of D-modules will produce produce some kinds of de Rham homology and cohomology groups. We postulate as a basic principle that these groups should take values in discrete vector spaces, that is, we wish to avoid projective limits.

Then, in the ind-infinite dimensional case, the natural theory is the *homology* of S:

$$H_*(S) := \underset{i}{\mathrm{colim}}\, H_*(S_i)$$

while in the pro-infinite dimensional case, the natural theory is the *cohomology* of T:

$$H^*(T) := \underset{j}{\mathrm{colim}}\, H^*(T_j).$$

For indschemes that are infinite dimensional in both the ind and the pro directions, one requires a *semi-infinite* homology theory that is homology in the ind direction and cohomology in the pro direction.

Remark 1.2.1. Of course, such a theory requires some extra choices, as is immediately seen by considering the finite dimensional case. For example, for a smooth variety, we have a choice of normalization for the cohomological shifts.

1.3. Theories of semi-infinite homology have appeared in many places in the literature. We do not pretend to survey the literature on the subject here, but note that in the case of the loop group, it is well-known that semi-infinite cohomology, in the sense above, may be defined using the semi-infinite cohomology of Lie algebras.

We provide such a theory in large generality below. In fact, we develop two theories $D^!$ and D^* of derived categories of D-modules on indschemes of ind-infinite type. The theory $D^!$ is contravariant, and therefore carries a natural dualizing complex, and the theory D^* is covariant, and therefore is the place where cohomology is defined.

For *placid* indschemes (a technical condition defined below), the two categories are identified after a choice of *dimension theory*, and therefore allows us to define the *renormalized* or *semi-infinite* cohomology of the scheme. The extra choice of dimension theory here precisely reflects the numerical choice of cohomological shifts discussed above.

Remark 1.3.1*.* The main difference between our approach and other approaches taken in the literature is that we work systematically with derived categories of D-modules, rather than simply working with homology or with abelian categories. This is facilitated by our use of higher category theory, i.e., with the use of homotopy limits and colimits of DG categories.

1.4. **Overview.** In §2, we give very general definitions of our categories $D^!$ and D^* of D-modules: we define these categories for arbitrary *prestacks*. There's not much one can say in this level of generality, but it is convenient to work in this setting in order to note how formal the definitions are.

We note for reference below that $D^!$ is contravariant, while D^* is covariant. Moreover, for a prestack S, the DG category $D^!(S)$ admits a tensor product $\overset{!}{\otimes}$ and acts on $D^*(S)$ in a canonical way satisfying a version of the projection formula.

1.5. In §3, we develop the theory in the setting of (quasi-compact, quasi-separated) schemes.

The key technique here is to use Noetherian approximation, as developed in [Gro] and [TT]. Note that this idea is already essentially present in [KV]; the authors of *loc. cit.* credit it to Drinfeld.

1.6. In §4 we will introduce the notion of *placidity*. One can understand this condition as saying that the singularities of a scheme are of finite type in a precise sense.

The key point of placid schemes is that they admit a "renormalized dualizing complex" that lies in $D^*(S)$. This is notable because, as we recall, D^* is covariant: its natural functoriality (with respect to infinite type morphisms) is through pushforwards. Moreover, the functor of action on the renormalized dualizing complex gives an equivalence $D^!(S) \simeq D^*(S)$. In particular, one obtains a covariant structure on $D^!$ and a contravariant structure on D^* is the placid setting.

1.7. In §5, we discuss the holonomic theory for schemes of infinite type. There is nothing terribly unsurprising here.

Remark 1.7.1*.* This material could just as well be given for ℓ-adic sheaves.

1.8. In §6, we move to the setting of indschemes.

The key part is again a theory of placid indschemes with properties similar to the setting of placid schemes described above. It is here that dimension theories enter the story.

1.9. There are many pushforward and pullback functors constructed in the text below. The bulk of this text is really devoted to checking when certain functors are defined, when they coincide, when they are adjoint, when they satisfy base-change, etc.

Such a state of affairs can only lend itself to confusion for the reader, so to conclude this overview, we include a table describing which functors are defined when and what their basic properties are. The reader should refer to the body of this text for more information; this table is simply meant to be available for convenient reference.

We let $f : S \to T$ be a morphism of indschemes in this table.

Hypotheses	Functor	Adjunctions	Base-change properties	References
None	$f^! : D^!(T) \to D^!(S)$	See $f_{*,!-dR}$, $f_{*,ren}$	See $f_{*,!-dR}$, $f_{*,ren}$	§3.6, §**??**
None	$f_{*,dR} : D^*(S) \to D^*(T)$	See $f^{¡}$, $f^{*,ren}$	See $f^{¡}$, $f^{*,ren}$	§3.18, §**??**
f ind-finitely presented	$f_{*,!-dR} : D^!(S) \to D^!(T)$	Left adjoint to $f^!$ for f ind-proper; right adjoint to $f^!$ up to cohomological shift for f smooth	Always satisfies base-change with upper-! functors	§3.9, §6.3
f ind-finitely presented	$f^{¡} : D^*(T) \to D^*(S)$	Right adjoint to $f_{*,dR}$ for f proper; left adjoint to $f_{*,dR}$ up to cohomological shift for f smooth	Always satisfies base-change with lower-* functors	§3.22, §6.3
S and T placid and equipped with dimension theories	$f_{*,ren} : D^!(S) \to D^!(T)$	Right adjoint to $f^!$ for f placid and with the dimension theory of S induced from that of T by Construction 6.12.6	For f placid, satisfies base-change against the upper-! functors of finitely presented morphisms	§6.16, Prop. 6.18.1
S and T placid and equipped with dimension theories	$f^{!,ren} : D^*(T) \to D^*(S)$	Left adjoint to $f_{*,dR}$ for f placid and with the dimension theory of S induced from that of T by Construction 6.12.6	For f finitely presented, satisfies base-change against the lower-* functors of placid morphisms	§6.16, Prop. 6.18.1

TABLE 1. D-module functors in infinite type

1.10. **Mea culpa.** This theory is inadequate in that it completely ignores that most characteristic feature of the theory of D-modules: the forgetful functor to quasi-coherent sheaves. The reason is that one needs a theory of ind-coherent sheaves (c.f. [GR]) for (ind)schemes of (ind-)infinite type.

The methods used below are apparently inadequate for this purpose. The problem is that base-change between upper-! and lower-$*$ functors does not hold for Cartesian squares in the category of classical schemes as it does for D-modules: rather, one needs the square to be Cartesian in the category of derived schemes, and this immediate appearance of non-eventually coconnective derived schemes is in tension with our appeals to Noetherian approximation. However, at least in the placid case, the situation is okay: c.f. to [Gai3].

1.11. **Categorical conventions.** Our basic methodology in treating the above problems is the use of modern derived techniques. However, we (mostly) do not appeal to derived algebraic geometry.

We spell this out further in what follows.

1.12. We appeal frequently to higher category theory, and it is convenient to use higher category theory as the basic building block of our terminology. Therefore, by *category* we mean $(\infty, 1)$-category, and similarly by *groupoid* we mean ∞-groupoid. We let Cat and Gpd denote the corresponding categories. When we mean to specify that we are working with the classical notion of category, we use the term $(1, 1)$-category instead.

1.13. We work always over a field k of characteristic 0.

We let DGCat denote the category of DG categories (alias: stable categories enriched over k-modules). Let $\mathsf{Vect} = \mathsf{Vect}_k$ denote the DG category of k-modules.

Let DGCat_{cont} denote the category of cocomplete[1] DG categories and continuous functors: i.e., DG categories admitting all colimits, and functors commuting with all colimits. We freely use the linear algebra of such categories (tensor products, duality and all that) from [Gai1].

1.14. Let AffSch denote the $(1, 1)$-category of (classical, i.e., non-derived) affine schemes. Let PreStk denote the category of (classical) prestacks: by definition, this is the category of functors $\mathsf{AffSch}^{op} \to \mathsf{Gpd}$. Recall that e.g. the categories of schemes and indschemes are by definition full subcategories of PreStk.

1.15. For S a scheme of finite type, we let $D(S)$ denote the DG category of D-modules on S. We refer to [GR], where this construction is given in detail in a format convenient for our purposes.

1.16. **Acknowledgements.** This material has been strongly influenced by [BD] §7, [Dri] and [KV]. We also thank Dennis Gaitsgory for many helpful discussions about this material; in particular, the idea of systematically distinguishing between $D^!$ and D^*, our very starting point, is due to him. Finally, we thank Dario Beraldo for helpful conversations on this material.

2. D-modules on prestacks

2.1. In this section, we define $D^!$ and D^* for general prestacks.

There is not much to say in this level of generality: we work in this generality because the definitions are most natural like this, and simply to point out that it can be done. The later sections of this text are then dedicated to studying the special cases of schemes and indschemes.

2.2. Let $\mathsf{AffSch}^{f.t.} \subseteq \mathsf{AffSch}$ denote the subcategory of finite type affine schemes.

Note that $\mathsf{AffSch} \simeq \mathsf{Pro}(\mathsf{AffSch}^{f.t.})$: this is essentially the statement that a classical commutative algebra is the union of its finite type subalgebras.

2.3. **Definition of $D^!$.** We define:

$$D^! : \mathsf{AffSch}^{op} \to \mathsf{DGCat}_{cont}$$

as the left Kan extension of the functor $D : \mathsf{AffSch}^{f.t.,op} \to \mathsf{DGCat}_{cont}$ attaching to a finite type affine scheme S its category of D-modules and attaches to a morphism $f : S \to T$ the corresponding upper-! functor.

We extend this definition to:

[1] We freely ignore cardinality issues in what follows, but here cocomplete should be taken to mean presentable; we recall that the difference is a set-theoretic condition that is always satisfied in the examples used below.

$$D^! : \mathsf{PreStk}^{op} := \mathsf{Hom}(\mathsf{AffSch}, \mathsf{Gpd}) \to \mathsf{DGCat}_{cont}$$

by right Kan extension.

Remark 2.3.1. Here is what the above definition says more concretely:

- For a classical commutative algebra A, write $A = \cup_i A_i$ with A_i finite type. Note that for $A_i \subseteq A_j$, we have a map $\mathrm{Spec}(A_j) \to \mathrm{Spec}(A_i)$, and therefore a !-pullback functor for D-modules. Then $D^!(\mathrm{Spec}(A))$ is computed as:

$$D^!(\mathrm{Spec}(A)) := \mathrm{colim}\, D(\mathrm{Spec}(A_i)) \in \mathsf{DGCat}_{cont}.$$

- For a nice enough stack $\mathcal{Y}$, choose a hypercovering by affine schemes $Y_n = \mathrm{Spec}(A_n)$, so $\mathcal{Y} = |Y_\bullet| = \mathrm{colim}_{[n]\in \mathbf{\Delta}^{op}} Y_n$.
  Then $D^!(\mathcal{Y}) = \lim_{[n]\in\mathbf{\Delta}} D^!(Y_n)$.
- More generally, for any prestack $\mathcal{Y}$, we can formally write $\mathcal{Y} = \mathrm{colim}_{i\in\mathcal{I}} Y_i$ for some diagram $i \mapsto Y_i \in \mathsf{AffSch}$. We then have:

$$D^!(\mathcal{Y}) = \lim_{i\in\mathcal{I}^{op}} D^!(Y_i).$$

Less precisely, one should think that a $D^!$-module $\mathcal{F}$ on $\mathcal{Y}$ is defined by its compatible system of restrictions to affine schemes Y mapping to $\mathcal{Y}$, and a "typical" $D^!$-module on such a Y is pulled back along some $Y \to Z$ with $Z \in \mathsf{AffSch}^{f.t.}$.

Notation 2.3.2. For $f : \mathcal{Y} \to \mathcal{Z}$ a morphism of prestacks, we have a tautological map:

$$D^!(\mathcal{Z}) \to D^!(\mathcal{Y}) \in \mathsf{DGCat}_{cont}$$

which we denote by $f^!$. Note that there is no risk for confusion in this notation, since if f is a map between prestacks locally of finite type, this functor corresponds with the usual functor $f^!$.

2.4. For any prestack $\mathcal{Y}$, note that $D^!(\mathcal{Y})$ is a symmetric monoidal category. Indeed, for any prestacks $\mathcal{Y}$ and $\mathcal{Z}$, there is a tautological functor:

$$D^!(\mathcal{Y}) \otimes D^!(\mathcal{Z}) \xrightarrow{-\boxtimes-} D^!(\mathcal{Y} \times \mathcal{Z})$$

which is an equivalence if $\mathcal{Y}, \mathcal{Z} \in \mathsf{AffSch}$ (indeed: this claim immediately to the case $\mathcal{Y}, \mathcal{Z} \in \mathsf{AffSch}^{f.t.}$).

Then pairing $\mathcal{F} \overset{!}{\otimes} \mathcal{G} := \Delta^!(\mathcal{F} \boxtimes \mathcal{G})$ defines the desired symmetric monoidal structure, where $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is the diagonal map. We remark that $\overset{!}{\otimes}$ commutes with colimits in each variable.

2.5. **Definition of D^*.** We now define the second category of D-modules on a prestack. The definition is formally dual to the definition of $D^!$.

Namely, we define:

$$D^* : \mathsf{AffSch} \to \mathsf{DGCat}_{cont}$$

as the right Kan extension of the functor $D : \mathsf{AffSch}^{f.t.} \to \mathsf{DGCat}_{cont}$ attaching to a finite type affine scheme S its category of D-modules and attaches to a morphism $f : S \to T$ the corresponding lower-$*$ functor.

We extend this definition to:

$$D^* : \mathsf{PreStk} := \mathsf{Hom}(\mathsf{AffSch}, \mathsf{Gpd}) \to \mathsf{DGCat}_{cont}$$

by left Kan extension.

Remark 2.5.1. Again, here is the concrete interpretation of this definition:

- For a classical commutative algebra $A = \cup_i A_i$, we have:

$$D^*(\mathrm{Spec}(A)) := \lim D(\mathrm{Spec}(A_i)) \in \mathsf{DGCat}_{cont}$$

  where the structure functors are lower-$*$ functors.
- For any prestack $\mathcal{Y}$, we can write $\mathcal{Y} = \mathrm{colim}_{i \in \mathcal{I}} Y_i$ for some diagram $i \mapsto Y_i \in \mathsf{AffSch}$. We then have:

$$D^*(\mathcal{Y}) = \mathrm{colim}\, D^*(Y_i).$$

Less precisely, one should think that a typical D^*-module on $\mathcal{Y}$ is pushed forward along some map $Y \to \mathcal{Y}$ with $Y \in \mathsf{AffSch}$, and a D^*-module on such a Y is defined by the knowledge of its compatible system of push forwards along maps $Y \to Z$ with $Z \in \mathsf{AffSch}^{f.t.}$.

2.6. **Locally finite type case.** When $\mathcal{Y}$ is a prestack locally of finite type, we have canonical identifications $D^!(\mathcal{Y}) = D(\mathcal{Y})$, where we recall that $D(\mathcal{Y})$ is defined in [GR]. For $\mathcal{Y}$ a *scheme* of finite type, it is easy to identify this category with $D^*(\mathcal{Y})$ as well (by a descent argument).

2.7. **The projection formula.** Next, we discuss the relationship between $D^!$ and D^*.

By the projection formula in the finite type setting, $D^!(\mathcal{Y})$ acts on $D^*(\mathcal{Y})$ for any prestack $\mathcal{Y}$. More precisely, $D^*(\mathcal{Y})$ is a module for $D^!(\mathcal{Y})$ in DGCat_{cont}. We discuss this a bit heuristically here, and give a more precise construction in §2.8.

Indeed, in the case where $\mathcal{Y} = S \in \mathsf{AffSch}$, this action is characterized by the formula:

$$f_{*,dR}(f^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}) = \mathcal{F} \overset{!}{\otimes} f_{*,dR}(\mathcal{G})$$

for $f : S \to T$ with $T \in \mathsf{AffSch}^{f.t.}$, $\mathcal{F} \in D^!(T) = D(T)$, and $\mathcal{G} \in D^*(S)$. In the case of general $\mathcal{Y}$, this action is characterized by the formula:

$$\mathcal{F} \overset{!}{\otimes} g_{*,dR}(\mathcal{G}) = g_{*,dR}(g^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{G})$$

for $g : S \to \mathcal{Y}$ with $S \in \mathsf{AffSch}$, $\mathcal{F} \in D^!(\mathcal{Y})$, and $\mathcal{G} \in D^*(S)$.

Moreover, for $f : \mathcal{Y} \to \mathcal{Z}$ any morphism, $f_{*,dR} : D^*(\mathcal{Y}) \to D^*(\mathcal{Z})$ is (canonically) a morphism of $D^!(\mathcal{Y})$-module categories. We remark that this structure encodes the projection formula.

2.8. Here is a more precise construction of the above structures.

Let $\mathcal{C}$ (temporarily) denote the category whose objects are pairs $\mathcal{A} \in \mathsf{ComAlg}(\mathsf{DGCat}_{cont})$ and $\mathcal{M}$ a module for $\mathcal{A}$ in DGCat_{cont}, and where morphisms $(\mathcal{A}, \mathcal{M}) \to (\mathcal{B}, \mathcal{N})$ are pairs of a symmetric monoidal and continuous functor $\mathcal{A} \to \mathcal{B}$ plus $\mathcal{N} \to \mathcal{M}$ a continuous morphism of $\mathcal{A}$-module categories (where the $\mathcal{A}$-module category structure on $\mathcal{N}$ is induced by $\mathcal{A} \to \mathcal{B}$).

Now observe that there is a functor $D : \mathsf{Sch}^{f.t.,op} \to \mathcal{C}$ sending S to $(D(S), D(S))$ equipped with upper-! functoriality in the first variable and lower-* functoriality in the second variable. Indeed, this follows from the formalism of correspondences from [GR].

Then we obtain a $\mathsf{AffSch}^{op} \to \mathcal{C}$ sending S to the pair $(D^!(S), D^*(S))$ as the left Kan extension of the [GR] functor. Indeed, one computes filtered colimits in $\mathcal{C}$ as a colimit in the first variable and a limit in the second variable.

This treats the projection formula in the case of schemes, and the same method works for general prestacks.

3. D-modules on schemes

3.1. In this section, we treat $D^!$ and D^* in the special case of (quasi-compact quasi-separated) schemes.

The main idea is to use Noetherian approximation (c.f. §3.2) to reinterpret $D^!$ and D^* on such schemes. This will give us a handle on (possibly non-affine) morphisms of finite presentation, which allows us to increase the functoriality of this functors.

3.2. **Noetherian approximation.** We begin with a brief review of the theory of Noetherian approximation (alias: Noetherian descent). This theory is due to [Gro] and [TT].

3.3. Let S be a quasi-compact quasi-separated base scheme. Let $\mathsf{Sch}_{/S}^{f.p.}$ denote the category of schemes finitely presentated (in particular: quasi-separated) over S. If S is Noetherian we will also use the notation $\mathsf{Sch}_{/S}^{f.t.}$ because in this case finite type is equivalent to finite presentation.

We say an S-scheme T is *almost affine* if for every $S' \to S$ of finite presentation every map $T \to S'$ factors as $T \to T' \to S'$ where $T \to T'$ is affine and $T' \to S'$ is finitely presented. Let $\mathsf{Sch}_{/S}^{\mathrm{al.aff}}$ denote the category of almost affine S-schemes.

Let $\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}_{/S}^{f.p.})$ denote the full subcategory of $\mathsf{Pro}(\mathsf{Sch}_{/S}^{f.p.})$ consisting of objects T that arise as filtered limits $T = \lim T_i$ of finitely presented S-schemes under affine structural morphisms $T_j \to T_i$. We recall that projective limits of such systems exist and that if each T_i is affine over S then T is as well. Clearly such limits commute with base-change.

3.4. The main result of Noetherian approximation is the following, due to [Gro] §8 and [TT] Appendix C.

Theorem 3.4.1. *(1) The right Kan extension:*

$$\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}_{/S}^{f.p.}) \to \mathsf{Sch}_{/S}$$

of the embedding $\mathsf{Sch}_{/S}^{f.p.} \hookrightarrow \mathsf{Sch}_{/S}$ is defined and is fully-faithful. This right Kan extension maps into $\mathsf{Sch}_{/S}^{al.aff}$. If S is Noetherian and affine, then the essential image of this functor is all schemes over S that are quasi-compact and quasi-separated (in particular, quasi-compact quasi-separated k-schemes are almost affine).

(2) Suppose $T = \lim T_i$ is a filtered limit with each T_i finitely presented over S and $T_j \to T_i$ affine. Then if T' is a finitely presented T-scheme there exists an index i and a T_i-scheme T_i' of finite presentation such that $T' = T_i' \times_{T_i} T$ (as a T-scheme).

Moreover, if the map $T' \to T$ has any (finite) subset of the properties of being (e.g.) smooth, flat, proper, or surjective, then $T_i' \to T_i$ may be taken to have the same properties.

(3) Suppose $T = \lim_{i \in \mathcal{I}^{op}} T_i$ as in (2). Then if $T \to S$ is an affine morphism, then there exists $i_0 \in \mathcal{I}$ such that for every $i \in \mathcal{I}_{i_0/}$, $T_i \to S$ is affine.

(4) Suppose that $T = \lim T_i$ as in (2) and $U \subseteq T$ is a quasi-compact open subscheme. Then for some index $i \in \mathcal{I}$ and open $U_i \subseteq T_i$ we have $U = U_i \times_{T_i} T$ (as T-schemes).

Remark 3.4.2. We note that (3) appears in [TT] as Proposition C.6, where it is stated only in the case that S is affine. However, this immediately generalizes, since S is assumed quasi-compact and therefore admits a finite cover by affines.

3.5. We will also use the following technical result.

Proposition 3.5.1. *Suppose that $T = \lim_{i\in\mathcal{I}} T_i$ is a filtered limit of schemes under affine structure maps. Let $\alpha_i : T \to T_i$ denote the structure maps. Then passing to cotangent complexes, the canonical map:*

$$\operatorname*{colim}_{i\in\mathcal{I}} \alpha_i^*(\Omega^1_{T_i}) \to \Omega^1_T \in \mathsf{QCoh}(T)^{\leqslant 0}$$

is an equivalence. (Here e.g. Ω^1_T denotes the whole cotangent complex).

Proof. Let DGSch denote the category of DG schemes. Note that filtered limits of derived schemes under affine structural maps exists as well, and satisfy the same properties as in the non-derived case: namely, if $T = \lim T_i$ in DGSch is a filtered limit under affine structural maps of affine S-schemes, then T is affine over S as well. In particular, we deduce that $\mathsf{Sch} \subseteq \mathsf{DGSch}$ is closed under such limits.

Now the result follows immediately from the description of the cotangent complex in terms of square-zero extensions in derived algebraic geometry.

□

3.6. **$D^!$-modules.** Let Sch_{qcqs} denote the category of quasi-compact quasi-separated k- schemes. Let $S \in \mathsf{Sch}_{qcqs}$ be fixed.

By Theorem 3.4.1, we can write $S = \lim S_i$ with $S_i \in \mathsf{Sch}^{f.t.}$ and all structure morphisms affine.

Proposition 3.6.1. *The canonical morphism:*

$$\operatorname*{colim}_i D(S_i) \to D^!(S) \in \mathsf{DGCat}_{cont} \tag{3.6.1}$$

is an equivalence.

Remark 3.6.2. This claim is immediate from Theorem 3.4.1 (3) if S is affine.

The proof is deferred to §3.13. In the meantime, we will give some preliminary constructions on the left hand side of (3.6.1).

3.7. First, we claim that the left hand side of (3.6.1) is independent of the choice of way of writing $S = \lim S_i$ with the above properties.

By Theorem 3.4.1, Sch_{qcqs} is a full subcategory of $\mathsf{Pro}(\mathsf{Sch}^{f.t.})$. We define the functor $\widetilde{D}^! : \mathsf{Sch}^{op}_{qcqs} \to \mathsf{DGCat}_{cont}$ as the left Kan extension of the functor $D : \mathsf{Sch}^{f.t.,op} \to \mathsf{DGCat}_{cont}$.

Remark 3.7.1. Suppose that $\mathcal{C}^0$ is an (essentially small) category and $\mathcal{C} \subseteq \mathsf{Ind}(\mathcal{C}^0)$ is a full subcategory containing $\mathcal{C}^0$. Suppose that we are given $F : \mathcal{C} \to \mathcal{D}$ a functor that is the left Kan extension of its restriction to $\mathcal{C}^0$. Then for any filtered colimit $X = \operatorname{colim}_i X_i \in \mathcal{C}$ in $\mathsf{Ind}(\mathcal{C}^0)$, we have $F(X) = \operatorname{colim} F(X_i)$. Indeed, by definition:

$$F(X) = \operatorname*{colim}_{X' \to X, X' \in \mathcal{C}^0} F(X').$$

But this also computes the left Kan extension from $\mathcal{C}^0$ to $\mathsf{Ind}(\mathcal{C}^0)$. Therefore, this claim reduces to the case $\mathcal{C} = \mathsf{Ind}(\mathcal{C}^0)$, where it is well-known.

Applying this remark in our setting, we see that $\widetilde{D}^!(S)$ computes the left hand side of (3.6.1). Therefore, we need to show that the map:

$$\widetilde{D}^!(S) \to D^!(S)$$

is an equivalence.

Notation 3.7.2. For $f : S \to T \in \mathsf{Sch}_{qcqs}$, we let $f^! : \widetilde{D}^!(T) \to \widetilde{D}^!(S)$ denote the induced functor: this is hardly an abuse, since we will eventually be identifying this functor with $f^! : D^!(T) \to D^!(S)$.

3.8. **Correspondences.** Next, we extend the functoriality of $\widetilde{D}^!$.

Let $\mathsf{Sch}^{f.t.}_{corr}$ be the (1,1)-category of finite type schemes under correspondences. By [GR], we have the functor $D : \mathsf{Sch}^{f.t.}_{corr} \to \mathsf{DGCat}_{cont}$ that attaches to a finite type scheme T its category $D(T)$ of D-modules and to a correspondence $T \xleftarrow{\alpha} H \xrightarrow{\beta} S$ (i.e., a map $T \to S$ in Sch_{corr}) attaches the functor $\beta_{*,dR}\alpha^!$.

Let $\mathsf{Sch}_{qcqs,corr;all,f.p.}$ denote the category of quasi-compact quasi-separated schemes under correspondences of the form:

$$\begin{tikzcd} & H \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ T & & S \end{tikzcd}$$

where $H \in \mathsf{Sch}_{qcqs}$, β is finitely presented and α is arbitrary. Note that $\mathsf{Sch}_{qcqs,corr;all,f.p}$ contains $\mathsf{Sch}^{f.p.}_{corr}$ as a full subcategory. It also contains Sch^{op}_{qcqs} as a non-full subcategory where morphisms are correspondences where the right arrow is an isomorphism.

We define the functor:

$$\widetilde{D}^{!,enh} : \mathsf{Sch}_{qcqs,corr;all,f.p.} \to \mathsf{DGCat}_{cont}$$

by left Kan extension from $\mathsf{Sch}^{f.t.}_{corr}$.

Proposition 3.8.1. *The restriction of $\widetilde{D}^{!,enh}$ to Sch^{op}_{qcqs} canonically identifies with the functor $\widetilde{D}^! : \mathsf{Sch}^{op}_{qcqs} \to \mathsf{DGCat}_{cont}$.*

The proof will be given in §3.11.

3.9. We assume Proposition 3.8.1 until §3.10 so that we can discuss its consequences.

For $f : T \to S$ a map of quasi-compact quasi-separated schemes, the induced functor $\widetilde{D}^{!,enh}(S) = \widetilde{D}^!(S) \to \widetilde{D}^{!,enh}(T) = \widetilde{D}^!(T)$ coincides with $f^!$. If f is finitely presented we will denote the corresponding functor $\widetilde{D}^!(T) \to \widetilde{D}^!(S)$ by $f_{*,!-dR}$ (to avoid confusion with the functor $f_{*,dR} : D^*(T) \to D^*(S)$ defined in §3.18 below). We refer to the functor $f_{*,!-dR}$ as the "!-dR $*$-pushforward functor."

Note that the formalism of correspondences implies that we have base-change between $*$-pushforward and !-pullback for Cartesian squares.

Remark 3.9.1. Suppose that $f : T \to S$ is finitely presented. One can compute the functor $f_{*,!-dR}$ "algorithmically" as follows. Let f be obtained by base-change from $f' : T' \to S'$ a map of schemes of finite type via a map $S \to S'$. Write $S = \lim S_i$ where structure maps are affine and each S_i is a finite type S'-scheme. Then $T = \lim T_i$ for $T_i := S_i \times_{S'} T'$. Let $\alpha_i : S \to S_i$, $\beta_i : T \to T_i$ and $f_i : T_i \to S_i$ be the tautological maps.

Then for $\mathcal{F} \in D(T_i)$ we have $f_{*,!-dR}(\beta_i^!(\mathcal{F})) = \alpha_i^! f_{i,*,dR}(\mathcal{F})$, which completely determines the functor $f_{*,!-dR}$.

One readily deduces the following result from [GR].

Proposition 3.9.2. *If $f : S \to T$ is a proper (in particular, finitely presented) morphism of quasi-compact quasi-separated schemes, then $f^!$ is canonically the right adjoint to $f_{*,!-dR}$. This identification is compatible with the correspondence structure: e.g., given a Cartesian diagram:*

$$\begin{tikzcd} S' \arrow[r, "\psi"] \arrow[d, "f'"] & S \arrow[d, "f"] \\ T' \arrow[r, "\varphi"] & T \end{tikzcd}$$

with f proper, the identification:

$$f_{*,dR}\varphi^! \xrightarrow{\simeq} \psi^! f'_{*,!-dR}$$

arising from the correspondence formalism is given by the adjunction morphism.

Similarly, we have the following.

Proposition 3.9.3. *If $f : S \to T$ is a smooth finitely presented map of quasi-compact quasi-separated schemes, then $f^![-2 \cdot d_{S/T}]$ is left adjoint to $f_{*,!-dR}$. Here $d_{S/T}$ is the rank of $\Omega^1_{S/T}$ regarded as a locally constant function on S.*

Remark 3.9.4. By a *locally constant function* $T \to \mathbb{Z}$ on a scheme T, we mean a morphism of $T \to \mathbb{Z}$ with $\mathbb{Z}$ considered as the indscheme $\coprod_{n\in\mathbb{Z}} \operatorname{Spec}(k)$.

If T is quasi-compact quasi-separated and therefore a pro-finite type scheme $T = \lim T_i$ (under affine structure maps), then, by Noetherian approximation, any locally constant function on T arises by pullback from one on some T_i. In other words, if we define $\pi_0(T)$ as the profinite set $\lim_i \pi_0(T_i)$, then locally constant functions on T are equivalent to continuous functions on $\pi_0(T)$.

Remark 3.9.5. Recall that there is an automatic projection formula given the correspondence framework. Indeed, for $f : S \to T$ a finitely presented map of quasi-compact quasi-separated schemes, $\mathcal{F} \in \widetilde{D}^!(T)$ and $\mathcal{G} \in \widetilde{D}^!(S)$, we have a canonical isomorphisms:

$$f_{*,!-dR}\big(f^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{G}\big) \simeq \mathcal{F} \overset{!}{\otimes} f_{*,!-dR}(\mathcal{G})$$

coming base-change for $\mathcal{F} \boxtimes \mathcal{G} \in \widetilde{D}^!(T \times S)$ and the Cartesian diagram:

$$\begin{tikzcd} S \arrow[r, "\Gamma_f"] \arrow[d, "f"] & T \times S \arrow[d, "\mathrm{id}_T \times f"] \\ T \arrow[r, "\Delta_T"] & T \times T \end{tikzcd}$$

where Γ_f is the graph of f and Δ_T is the diagonal.

By the finite type case, these isomorphisms are given by the adjunctions of Proposition 3.9.2 and 3.9.3 when f is proper or smooth.

3.10. In the proof of Proposition 3.8.1 we will need the following technical result.

Let T be a quasi-compact quasi-separated scheme. Consider the category $\mathcal{C}_T$ of correspondences:

$$\mathcal{C}_T := \{\, S \xleftarrow{\alpha} H \xrightarrow{\beta} T \ \mid \beta \text{ finitely presented, } S \in \mathsf{Sch}^{f.t.} \text{ and } H \in \mathsf{Sch}_{qcqs}\}.$$

Here, as usual, compositions are given by fiber products.

Note that $\mathcal{C}_T$ contains as a non-full subcategory $\mathsf{Sch}^{f.t.,op}_{T/}$ of maps $\Upsilon : T \to S$ with $S \in \mathsf{Sch}^{f.t.}$, where given such a map we attach the correspondence $S \xleftarrow{\Upsilon} T \xrightarrow{\mathrm{id}_T} T$.

Lemma 3.10.1. *The embedding* $\mathsf{Sch}^{f.t.}_{T/} \to \mathcal{C}_T$ *is cofinal.*

Proof. Fix a correspondence $(S \xleftarrow{\alpha} H \xrightarrow{\beta} T) \in \mathcal{C}_T$. Translating Lurie's ∞-categorical Quillen Theorem A to this setting, we need to show the contractibility of the category $\mathcal{C}$ of commutative diagrams:

$$\begin{tikzcd} H \arrow[r, "\delta"] \arrow[d, "\beta"] & H' \arrow[d, "\beta'"] \arrow[dr, "\epsilon"] & \\ T \arrow[r, "\Upsilon"] & T' & S \end{tikzcd}$$

such that the square on the left is Cartesian, $H', T' \in \mathsf{Sch}^{f.t.}$ and $\epsilon \circ \delta = \alpha$. Here a morphism from one such diagram (denoted with subscripts "1") to another such diagram (denoted with subscripts "2") is given by maps $f : T'_1 \to T'_2$ and $g : H'_1 \to H'_2$ such that the following diagram commutes and all squares are Cartesian:

$$\begin{tikzcd} H \arrow[r, "\delta_1"'] \arrow[rr, bend left, "\delta_2"] \arrow[d, "\beta"] & H'_1 \arrow[r, "g"'] \arrow[d, "\beta'_1"] & H'_2 \arrow[d, "\beta'_2"] \arrow[dr, "\epsilon_2"] & \\ T \arrow[r, "\Upsilon_1"] \arrow[rr, bend right, "\Upsilon_2"'] & T'_1 \arrow[r, "f"] & T'_2 & S. \end{tikzcd}$$

First, we observe that the category $\mathcal{C}$ is non-empty. Indeed, because β is finitely presented we can find $T \to T' \in \mathsf{Sch}^{f.t.}$ and $\beta' : H' \to T'$ so that H is obtained from H' by base-change. Noting that H can be written as a limit under affine transition maps of H' obtained in this way and S is finite type, we see that $H \to S$ must factor though some H' obtained in this way.

To see that $\mathcal{C}$ is contractible, note that $\mathcal{C}$ admits non-empty finite limits (because Sch admits finite limits) and therefore $\mathcal{C}^{op}$ is filtered.

□

3.11. We now prove Proposition 3.8.1.

Proof of Proposition 3.8.1. We have an obvious natural transformation $\widetilde{D}^! \to \widetilde{D}^{!,enh}|_{\mathsf{Sch}^{op}_{qcqs}}$. It suffices to see that this natural transformation is an equivalence when evaluated on any fixed $T \in \mathsf{Sch}_{qcqs}$.

With the notation of §3.10, $\widetilde{D}^{!,enh}$ is by definition the colimit over $(S \xleftarrow{\alpha} H \xrightarrow{\beta} T) \in \mathcal{C}_T$ of the category $D(S)$. By Lemma 3.10.1, this coincides with the colimit over diagrams where β is an isomorphism, as desired.

□

Remark 3.11.1. Neither Lemma 3.10.1 nor Proposition 3.8.1 is particular to schemes, but rather a general interaction between pro-objects in a category with finite limits and correspondences.

3.12. **Descent.** Next, we discuss descent for $\widetilde{D}^!$.

For a map $f : S \to T$ of schemes and $[n] \in \mathbf{\Delta}$ let $\mathrm{Cech}^n(S/T)$ be defined as:

$$\mathrm{Cech}^n(S/T) := \underbrace{S \underset{T}{\times} \ldots \underset{T}{\times} S}_{n \text{ times}}.$$

Of course, $[n] \mapsto \mathrm{Cech}^n(S/T)$ forms a simplicial scheme in the usual way.

We use the terminology of Voevodsky's h-topology, developed in the infinite type setting in [Ryd]. We simply recall that h-coverings are finitely presented[2] and include both the classes of fppf coverings and proper[3] coverings.

Proposition 3.12.1. *Let $f : S \to T$ be an h-covering of quasi-compact quasi-separated schemes. Then the canonical functor (induced by pullback):*

$$\widetilde{D}^!(T) \to \lim_{[n]\in\mathbf{\Delta}} \widetilde{D}^!(\mathrm{Cech}^n(S/T)) \tag{3.12.1}$$

is an equivalence.

Recall from [Ryd] Theorem 8.4 that the h-topology of Sch_{qcqs} is generated by finitely presented Zariski coverings[4] and proper coverings. Therefore, it suffices to verify Lemmas 3.12.2 and 3.12.3 below.

Lemma 3.12.2. *$\widetilde{D}^!$ satisfies proper descent, i.e., for every $f : T \to S$ a proper (in particular, finitely presented) surjective morphism of quasi-compact quasi-separated schemes the morphism* (3.12.1) *is an equivalence.*

Proof. We can find $f' : S' \to T'$ a proper covering between schemes of finite type and $T \to T'$ so that f is obtained by base-change. Let $T = \lim T_i$ where each T_i is a T'-scheme of finite type and structure maps are affine. Let $S_i := T_i \times_{T'} S'$.

We now decompose the map (3.12.1) as:

$$\widetilde{D}^!(T) = \operatorname*{colim}_{i\in\mathcal{I}} D(T_i) \xrightarrow{\simeq} \operatorname*{colim}_{i\in\mathcal{I}} \lim_{[n]\in\mathbf{\Delta}} D(\mathrm{Cech}^n(S_i/T_i)) \to$$
$$\lim_{[n]\in\mathbf{\Delta}} \operatorname*{colim}_{i\in\mathcal{I}} D(\mathrm{Cech}^n(S_i/T_i)) = \lim_{[n]\in\mathbf{\Delta}} D(\mathrm{Cech}^n(S/T)).$$

Here the isomorphism is by h-descent in the finite type setting.

Therefore, it suffices to see that the map:

$$\operatorname*{colim}_{i\in\mathcal{I}} \lim_{[n]\in\mathbf{\Delta}} D(\mathrm{Cech}^n(S_i/T_i)) \to \lim_{[n]\in\mathbf{\Delta}} \operatorname*{colim}_{i\in\mathcal{I}} D(\mathrm{Cech}^n(S_i/T_i))$$

is an isomorphism. It suffices to verify the Beck-Chevalley conditions in this case (c.f. [Lur] Proposition 6.2.3.19). For each $i \in \mathcal{I}$ and each map $[n] \to [m]$ in $\mathcal{I}$, the functor:

$$D(\mathrm{Cech}^m(S_i/T_i)) \to D(\mathrm{Cech}^n(S_j/T_j))$$

[2] More honestly: it seems there is a bit of disagreement in the literature whether h-coverings are required to be finitely presented or merely finite type. We are using the convention that they are finitely presented.

[3] We include "finitely presented" in the definition of proper.

[4] We explicitly note that these are necessarily finitely presented because we work only with quasi-compact quasi-separated schemes. That is, any open embedding of quasi-compact quasi-separated schemes is necessarily of finite presentation: the only condition to check is that it is a quasi-compact morphism, and any morphism of quasi-compact schemes is itself quasi-compact.

admits a left adjoint given by the !-dR $*$-pushforward as in Proposition 3.9.2. By base change between upper-! and !-dR $*$-pushfoward (Proposition 3.8.1), the Beck-Chevalley conditions are satisfied, since for every $j \to i$ in $\mathcal{I}$ and $[n] \to [m]$ in $\mathbf{\Delta}$ the diagram:

$$\begin{tikzcd} \mathrm{Cech}^m(S_i/T_i) \arrow[r] \arrow[d] & \mathrm{Cech}^n(S_j/T_j) \arrow[d] \\ \mathrm{Cech}^n(S_i/T_i) \arrow[r] & \mathrm{Cech}^m(S_j/T_j) \end{tikzcd}$$

is Cartesian.

□

Lemma 3.12.3. $\widetilde{D}^! : \mathsf{Sch}^{op}_{qcqs} \to \mathsf{DGCat}_{cont}$ *satisfies quasi-compact Zariski descent.*

Proof. It suffices to show for every S a quasi-compact quasi-separated scheme and $S = U \cup V$ a Zariski open covering of S by quasi-compact open subschemes that the canonical map:

$$\widetilde{D}^!(S) \to \widetilde{D}^!(U) \underset{\widetilde{D}^!(U\cap V)}{\times} \widetilde{D}^!(V)$$

is an equivalence.

Let $j_U : U \to S$, $j_V : V \to S$ and $j_{U\cap V} : U \cap V \to S$ denote the corresponding (finitely presented) open embeddings.

Define a functor:

$$\widetilde{D}^!(U) \underset{\widetilde{D}^!(U\cap V)}{\times} \widetilde{D}^!(V) \to \widetilde{D}^!(S)$$
$$\Big(\mathcal{F}_U \in \widetilde{D}^!(U), \mathcal{F}_V \in \widetilde{D}^!(V), \mathcal{F}_U|_{U\cap V} \simeq \mathcal{F}_V|_{U\cap V} =: \mathcal{F}_{U\cap V} \in \widetilde{D}^!(U \cap V)\Big) \mapsto$$
$$\mathrm{Ker}\Big(j_{U,*,!-dR}(\mathcal{F}_U) \oplus j_{V,*,!-dR}(\mathcal{F}_V) \to j_{U\cap V,*,!-dR}(\mathcal{F}_{U\cap V})\Big).$$

We claim that this functor is inverse to the above.

Note that e.g. $j_{U,*,!-dR} : \widetilde{D}^!(U) \to \widetilde{D}^!(S)$ is fully-faithful. Indeed, by Proposition 3.9.3 we have an adjunction between $j_U^!$ and $j_{U,*,!-dR}$. The counit:

$$j_U^! j_{U,*,!-dR} \to \mathrm{id}_{\widetilde{D}^!(U)}$$

is an equivalence by Remark 3.9.1 and the corresponding statement in the finite type setting. This shows that the composition:

$$\widetilde{D}^!(U) \underset{\widetilde{D}^!(U\cap V)}{\times} \widetilde{D}^!(V) \to \widetilde{D}^!(S) \to \widetilde{D}^!(U) \underset{\widetilde{D}^!(U\cap V)}{\times} \widetilde{D}^!(V)$$

is the identity.

For the other direction, note that we have a canonical map:

$$\mathrm{id}_{\widetilde{D}^!(S)} \to \mathrm{Ker}\left(j_{U,*,!-dR}\, j_U^! \oplus j_{V,*,!-dR}\, j_V^! \to j_{U\cap V,*,!-dR}\, j_{U\cap V}^!\right)$$

and it suffices to see that this map is an equivalence. But this again follows by reduction to the finite presentation case via Remark 3.9.1.

□

3.13. By Lemma 3.12.3, to prove Proposition 3.6.1, it suffices to observe the following result.

Lemma 3.13.1. *$D^!$ satisfies descent with respect to quasi-compact Zariski coverings.*

Proof. As in §3.8, for $f : \mathcal{Y} \to \mathcal{Z}$ an *affine* (in particular, schematic) and finitely presented morphism of prestacks, there is a functor $f_{*,!-dR} : D^!(\mathcal{Y}) \to D^!(\mathcal{Z})$ characterized by the fact that it satisfies base-change with the upper-! functors, and coincides with the usual pushforward functor in the finite type setting. Moreover, if f is an open embedding, then we have an adjunction $(f^!, f_{*,!-dR})$, and $f_{*,!-dR}$ is fully-faithful.

We prove the following two results by induction.

- (A_n): For $S \in \mathsf{AffSch}$ and $j : U \hookrightarrow S$ a quasi-compact open subscheme admitting a cover by n affine schemes, the restriction functor $j^! : D^!(S) \to D^!(U)$ admits a fully-faithful right adjoint $j_{*,!-dR}$. Formation of $j_{*,!-dR}$ commutes with base-change with respect to maps $T \to S \in \mathsf{AffSch}$.
- (B_n): For $S \in \mathsf{AffSch}$ and a cover $S = U \cap V$ by quasi-compact Zariski open subschemes with V affine and U admitting an open cover by n affine schemes, the functor:

$$D^!(S) \to D^!(U) \underset{D^!(U \cap V)}{\times} D^!(V)$$

  is an equivalence.

We have already observed that A_1 is true. Moreover, the statement A_n implies B_n by the same argument as in Lemma 3.12.3. Therefore, we should show that A_n and B_n imply A_{n+1}.

Chose a cover $U_1 \cup U_2$ with U_2 affine and U_1 admitting a covering by n affine schemes. By B_n, we have:

$$D^!(U) \simeq D^!(U_1) \underset{D^!(U_1 \cap U_2)}{\times} D^!(U_2).$$

Noting that $U_1 \cap U_2 \hookrightarrow U_1$ is an affine morphism and therefore this intersection admits a cover by n affines, we can construct this right adjoint as:

$$D^!(U_1) \underset{D^!(U_1 \cap U_2)}{\times} D^!(U_2) \to D^!(S)$$
$$\Big(\mathcal{F}_1 \in D^!(U_1), \mathcal{F}_2 \in D^!(U_2), \mathcal{F}_1|_{U_1 \cap U_2} \simeq \mathcal{F}_2|_{U_1 \cap U_2} =: \mathcal{F}_{12} \in D^!(U_1 \cap U_2)\Big) \mapsto$$
$$\operatorname{Ker}\Big(j_{1,*,!-dR}(\mathcal{F}_1) \oplus j_{2,*,!-dR}(\mathcal{F}_2) \to j_{12,*,!-dR}(\mathcal{F}_{12})\Big)$$

where j_1, j_2 and j_{12} respectively denote the embeddings of U_1, U_2 and U_{12} into S.

This completes the proof of our inductive statements.

By base-change, we obtain that for any quasi-compactopen embedding $j : \mathcal{U} \hookrightarrow \mathcal{Y}$, $j^! : D^!(\mathcal{Y}) \to D^!(\mathcal{U})$ admits a fully-faithful right adjoint $j_{*,!-dR}$ characterized by its compatibility with base-change, and then the proof of Lemma 3.12.3 gives the desired descent claim. □

3.14. Having proved Proposition 3.6.1, we no longer distinguish between $\widetilde{D}^!$ and $D^!$, denoting both by $D^!$.

3.15. Here is a useful corollary of Proposition 3.6.1.

Corollary 3.15.1. *For $S, T \in \mathsf{Sch}_{qcqs}$, the functor:*

$$D^!(S) \otimes D^!(T) \to D^!(S \times T)$$

is an equivalence.

Proof. This is immediate from the $\widetilde{D}^!$ perspective and the corresponding result for finite type schemes.

□

3.16. **Correspondences for prestacks.** We say that a morphism $f : \mathcal{Y} \to \mathcal{Z}$ is finitely presented if it is schematic, and if for any $S \to \mathcal{Z}$ with $S \in \mathsf{AffSch}$ the map $\mathcal{Y} \times_{\mathcal{Z}} S \to S$ is finitely presented.

As another corollary of Proposition 3.6.1, we obtain that there is a functor $f_{*,!-dR}$ compatible with base-change for any finitely presented morphism $f : \mathcal{Y} \to \mathcal{Z}$ of prestacks.

Remark 3.16.1. We emphasize that by *schematic*, we mean schematic in the sense of classical (i.e., non-derived) algebraic geometry, which is a more forgiving notion than that of derived algebraic geometry. This is relevant, say, for considering the embedding of 0 inside of the indscheme associated with an infinite-dimensional k-vector space, which is a classically schematic embedding but not a DG schematic embedding.

3.17. **Equivariant setting.** Suppose that S is a quasi-compact quasi-separated base scheme and $\mathcal{G} \to S$ is a quasi-separated quasi-compact group scheme over S.

Suppose that P is a quasi-compact quasi-separated S-scheme with an action of $\mathcal{G}$. In this case, the semisimplicial bar complex:

$$\ldots \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathcal{G} \underset{S}{\times} \mathcal{G} \underset{S}{\times} P \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathcal{G} \underset{S}{\times} P \rightrightarrows P \tag{3.17.1}$$

induces the diagram:

$$D^!(P) \rightrightarrows D^!(\mathcal{G} \underset{S}{\times} P) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} D^!(\mathcal{G} \underset{S}{\times} \mathcal{G} \underset{S}{\times} P) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \ldots.$$

and we define the *$\mathcal{G}$-equivariant derived category* $D^!(P)^{\mathcal{G}}$ of P to be the limit of this diagram.

Example 3.17.1. Suppose that $\mathcal{G}$ is constant, i.e., $\mathcal{G} = S \times \mathcal{G}_0$ for some quasi-compact quasi- separated group scheme $\mathcal{G}_0$ over $\mathrm{Spec}(k)$. Then, by Corollary 3.15.1, $D^!(\mathcal{G}_0)$ obtains a coalgebra structure in DGCat_{cont} in the usual way (e.g. the comultiplication is !-pullback along the multiplication for $\mathcal{G}_0$). As such, $D^!(\mathcal{G}_0)$ coacts on $D^!(P)$ and $D^!(P)^{\mathcal{G}}$ is the usual (strongly) $\mathcal{G}_0$-equivariant category, i.e., the limit of the diagram:

$$D^!(P) \rightrightarrows D^!(\mathcal{G}_0) \otimes D^!(P) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} D^!(\mathcal{G}_0) \otimes D^!(\mathcal{G}_0) \otimes D^!(P) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \ldots.$$

Let $\mathcal{P}_{\mathcal{G}} \to S$ be a $\mathcal{G}$-torsor, i.e., $\mathcal{G}$ acts on $\mathcal{P}_{\mathcal{G}}$ and after an appropriate fppf base-change $S' \to S$ we have a $\mathcal{G}$-equivariant identification:

$$\mathcal{P}_{\mathcal{G}} \underset{S}{\times} S' = \mathcal{G} \underset{S}{\times} S'.$$

We obtain a canonical functor:

$$\varphi : D^!(S) \to D^!(\mathcal{P}_{\mathcal{G}})^{\mathcal{G}}.$$

Proposition 3.17.2. *In the above setting the functor φ is an equivalence.*

Proof. By fppf descent (Proposition 3.12.1), we reduce to the case there $\mathcal{P}_{\mathcal{G}}$ is a trivial $\mathcal{G}$-bundle over T, i.e., $\mathcal{P}_{\mathcal{G}} = \mathcal{G} \times_S T$. Then the bar complex extends to a split simplicial object in the usual way from which we deduce the result.

□

Remark 3.17.3. If $\mathcal{P}_{\mathcal{G}} \to S$ is a $\mathcal{G}$-torsor, we will sometimes summarize the situation in writing $S = \mathcal{P}_{\mathcal{G}}/\mathcal{G}$.

3.18. **D^*-modules.** Next, we discuss the $*$-theory of D-modules.

We have the following analogue of Proposition 3.6.1.

Proposition 3.18.1. *For $S \in \mathsf{Sch}_{qcqs}$ written as $S = \lim_i S_i$ with $S_i \in \mathsf{Sch}^{f.t.}$ and all structure morphisms affine, the functor:*

$$D^*(S) \to \lim_{S \to S_i} D^*(S_i) \in \mathsf{DGCat}_{cont}$$

is an equivalence.

Proof. The proof is similar to the proof of Proposition 3.6.1:

As before, we denote the right hand side of the above by $\widetilde{D}^*(S)$ and note that it is independent of the choice of presentation $S = \lim S_i$.

The statement again reduces to an appropriate descent type statement. Namely, for every $S \in \mathsf{Sch}_{qcqs}$ with cover $S = U \cup V$ for U, V quasi-compact open subschemes, we claim that the functors:

$$D^*(U) \underset{D^*(U \cap V)}{\oplus} D^*(V) \to D^*(S)$$

$$\widetilde{D}^*(U) \underset{\widetilde{D}^*(U \cap V)}{\oplus} \widetilde{D}^*(V) \to \widetilde{D}^*(S)$$

are equivalences.

The key point again is that e.g. the pushforward functor $j_{*,dR} : D^*(U) \to D^*(S)$ (resp. $\widetilde{D}^*(U) \to \widetilde{D}^*(S)$) is fully-faithful and admits a right adjoint compatible with base-change. This is an easy reduction to the finite type case for $\widetilde{D}^*$, and is proved for D^* by the same method as how the analogous statement was proved for $D^!$ (c.f. the proof of Proposition 3.13.1).

□

3.19. Recall that for S a finite type scheme the category $D(S)$ is self-dual under Verdier duality and for a map $f : T \to S$ between finite type schemes the functor dual to $f^!$ is $f_{*,dR}$. Therefore, for S a quasi-compact quasi-separated scheme we obtain the following from [Gai1].

Proposition 3.19.1. *If $D^!(S)$ is a dualizable category, then its dual is canonically identified with $D^*(S)$.*

Note that in this case this is an identification of $(D^!(S), \overset{!}{\otimes})$-module categories. Moreover, the functor dual to $f^!$ continues to be $f_{*,dR}$.

3.20. **Constant sheaf.** For T quasi-compact quasi-separated, there is a canonical *constant sheaf* $k_T \in D^*(T)$ constructed as follows.

For any $S \in \mathsf{Sch}^{f.t.}$ and $\alpha : T \to S$, we define an object "$\alpha_{*,dR}(k_T)$" $\in D(S) = D^*(S)$ by the formula:

$$\text{“}\alpha_{*,dR}(k_T)\text{”} := \underset{\substack{T \xrightarrow{\beta} T' \xrightarrow{\Upsilon} S \\ T' \in \mathsf{Sch}^{f.t.}, \Upsilon \circ \beta = \alpha}}{\operatorname{colim}} \Upsilon_{*,dR}(k_{T'}).$$

For any triangle:

$$\begin{tikzcd} T \arrow[r, "\alpha'"] \arrow[d, "\alpha"'] & S' \\ S \arrow[ur, "f"'] & \end{tikzcd}$$

with S and $S' \in \mathsf{Sch}^{f.t.}$, we have a canonical isomorphism:

$$\text{“}\alpha'_{*,dR}(k_T)\text{”} \xrightarrow{\simeq} f_{*,dR}(\text{“}\alpha_{*,dR}(k_T)\text{”})$$

and therefore we obtain the object $k_T \in D^*(T)$ (with each $\alpha_{*,dR}(k_T) = \text{“}\alpha_{*,dR}(k_T)\text{”}$) as desired.

Letting $p_T : T \to \operatorname{Spec}(k)$ denote the structure map, the continuous functor $p_T^{*,dR} : \mathsf{Vect} \to D^*(T)$ sending k to k_T is readily seen to be the left adjoint to $p_{T,*,dR}$.

3.21. **Correspondences.** Next, we extend the functoriality of D^* as in §3.8.

Let $\mathsf{Sch}_{qcqs,corr;f.p.,all}$ denote the category of quasi-compact quasi-separated schemes under correspondences of the form:

$$\begin{tikzcd} & H \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ T & & S \end{tikzcd}$$

where $H \in \mathsf{Sch}_{qcqs}$, α is finitely presented and β is arbitrary. Note that $\mathsf{Sch}_{qcqs,corr;f.p.,all}$ contains $\mathsf{Sch}^{f.t.}_{corr}$ as a full subcategory. It also contains Sch_{qcqs} as a non-full subcategory where morphisms are correspondences where the left arrow is an isomorphism.

We define the functor:

$$D^{*,enh} : \mathsf{Sch}_{qcqs,corr;f.p.,all} \to \mathsf{DGCat}_{cont}$$

by right Kan extension from $\mathsf{Sch}^{f.t.}_{corr}$.

Like Proposition 3.8.1, the following is immediate from Lemma 3.10.1.

Proposition 3.21.1. *The restriction of $D^{*,enh}$ to Sch_{qcqs} canonically identifies with the functor $D^*|_{\mathsf{Sch}_{qcqs}} : \mathsf{Sch}_{qcqs} \to \mathsf{DGCat}_{cont}$.*

3.22. For $f : T \to S$ a map of quasi-compact quasi-separated schemes, the induced functor:

$$D^{*,enh}(T) = D^*(T) \to D^{*,enh}(S) = D^*(S)$$

coincides with $f_{*,dR}$. If f is finitely presented we will denote the corresponding functor $D^*(S) \to D^*(T)$ by $f^{¡}$ to avoid confusion with the functor $f^! : D^!(T) \to D^!(S)$. Note that the formalism of correspondences implies that we have base-change between $*$-pushforward and ¡-pullback for Cartesian squares.

Remark 3.22.1. Suppose that $f : T \to S$ is finitely presented. One can compute the functor $f^{¡}$ as follows. In the notation of Remark 3.9.1, for $\mathcal{F} \in D(S)$ we have $\alpha_{i,*,dR} f^{¡}(\mathcal{F}) = f_i^! \beta_{i,*,dR}(\mathcal{F})$ by base-change, computing $f^{¡}(\mathcal{F})$ in $D(T) = \lim D(T_i)$ as promised.

One deduces from Remark 3.22.1 the following result.

Proposition 3.22.2. *If $f : S \to T$ is a finitely presented proper morphism of quasi-compact quasi-separated schemes, then $f^{!}$ is canonically the right adjoint to $f_{*,dR}$.*

Similarly, we have:

Proposition 3.22.3. *If $f : S \to T$ is a finitely presented smooth map of quasi-compact quasi-separated schemes, then $f^{!}[-2d_{S/T}]$ is left adjoint to $f_{*,dR}$, with $d_{S/T}$ as in Proposition 3.9.3.*

3.23. **Descent.** Next, we discuss descent for D^*.

Proposition 3.23.1. *For $f : S \to T$ an h-covering of quasi-compact quasi-separated schemes the functor:*

$$D^*(T) \to \lim_{n \in \mathbf{\Delta}} D^*(\mathrm{Cech}^n(S/T))$$

induced by the functors $f_n^{!}$ with $f_n : \mathrm{Cech}^n(S/T) \to T$ the canonical map is an equivalence.

Proof. Because f is finite presentation we can apply Noetherian approximation to find $f' : S' \to T'$ an h-covering between schemes of finite type and $T \to T'$ so that f is obtained by base-change. Let $T = \lim T_i$ where each T_i is a T'-scheme of finite type (and structure maps are affine) and let $S_i := T_i \times_{T'} S'$.

Then each $S_i \to T_i$ is an h-covering between finite type schemes. Note that $\mathrm{Cech}^n(S/T) = \lim \mathrm{Cech}^n(S_i/T_i)$.

Now we have:

$$D^*(T) = \lim_{i \in \mathcal{I}^{op}} D(T_i) \xrightarrow{\simeq} \lim_{i \in \mathcal{I}^{op}} \lim_{[n] \in \mathbf{\Delta}} D(\mathrm{Cech}^n(S_i/T_i)) = \lim_{[n] \in \mathbf{\Delta}} \lim_{i \in \mathcal{I}^{op}} D(\mathrm{Cech}^n(S_i/T_i)) = \lim_{[n] \in \mathbf{\Delta}} D^*(\mathrm{Cech}^n(S/T)).$$

Here the indicated isomorphism is by usual h-descent for finite type schemes and Proposition 3.21.1. $\square$

Variant 3.23.2. One can similarly show that the functor:

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}} D^*(\mathrm{Cech}^n(S/T)) \to D^*(T)$$

defined by de Rham pushforwards is an equivalence for $S \to T$ an h-covering. Indeed: it is easy to verify for Zariski coverings (the argument is basically the same as for Lemma 3.12.3), and for proper coverings, it follows automatically from Proposition 3.23.1.

This is the statement that should properly be thought of as dual to Proposition 3.12.1.

3.24. **Equivariant setting.** Suppose that we are in the setting of §3.17, i.e., $\mathcal{G}$ is a group scheme over S that acts on an S-scheme P.

In this case, (3.17.1) defines the *coequivariant derived category*:

$$D^*(P)_{\mathcal{G}} := \operatorname{colim}\Big(\ldots \substack{\textstyle\rightarrow\\[-0.5ex]\textstyle\rightarrow\\[-0.5ex]\textstyle\rightarrow\\[-0.5ex]\textstyle\rightarrow} D^*(\mathcal{G} \underset{S}{\times} \mathcal{G} \underset{S}{\times} P) \rightrightarrows\!\!\!\rightarrow D^*(\mathcal{G} \underset{S}{\times} P) \rightrightarrows D^*(P) \Big) \tag{3.24.1}$$

with the colimit computed in DGCat_{cont}.

The analogue of Proposition 3.17.2 holds in this setting: if $P \to S$ is an $\mathcal{G}$-torsor, we obtain a functor:

$$D^*(P)_{\mathcal{G}} \to D^*(S)$$

that is an equivalence by essentially the same argument as in *loc. cit*, but using Variant 3.23.2 of Proposition 3.23.1.

4. Placidity

4.1. In this section, we introduce the notion of *placidity* and discuss its consequences.

Recall from §1.6 that placidity is a technical condition on the singularities of a scheme S allowing us to identify $D^!(S)$ with $D^*(S)$.

For a morphism $f : S \to T$ of placid schemes, we let $f_{*,ren} : D^!(S) \to D^!(T)$ and $f^{!,ren} : D^!(T) \to D^!(S)$ denote the corresponding functors, obtained through the above identification.

In general, these renormalized functors are very badly behaved, e.g., the pairs $(f^!, f_{*,ren})$ and $(f^{!,ren}, f_{*,dR})$ do not satisfy base-change.

In §4.10, we introduce a notion of placid morphism, which is something like a pro-smooth morphism. Proposition 4.11.1 (generalized to the indschematic setting by Proposition 6.18.1) says that for placid morphism, $f^!$ is left adjoint to $f_{*,ren}$, and similarly, $f^{!,ren}$ is left adjoint $f_{*,dR}$. Here the dimension shifts implicit in the infinite dimensional setting work out to eliminate the usual cohomological shifts needed to make such statements in the finite dimensional setting.

Moreover, Proposition 4.11.1 implies that there are good base-change properties for placid morphisms.

4.2. **Definition of placidity.** We now give the definition of placidity.

Definition 4.2.1. For $T \in \mathsf{Sch}$ we say an expression $T = \lim_{i \in \mathcal{I}^{op}} T_i$ is a *placid presentation of* T if:

1. The indexing category $\mathcal{I}$ is filtered.
2. Each T_i is finite type over k.
3. For every $i \to j$ in $\mathcal{I}$ the corresponding map $T_j \to T_i$ is an affine smooth covering.

We say that $T \in \mathsf{Sch}$ is *placid* if it admits a placid presentation.

Example 4.2.2. As is well known from the theory of group schemes, any affine group scheme is placid (we need the characteristic zero assumption on k here).

Example 4.2.3. Suppose that S is a finite type scheme and $\mathcal{G} \to S$ is a projective limit under smooth surjective affine maps of smooth S-group schemes. Suppose that $\mathcal{P}_{\mathcal{G}} \to S$ is a $\mathcal{G}$-torsor in the sense of §3.17. Then $\mathcal{P}_{\mathcal{G}}$ is placid.

Example 4.2.4. For a Cartesian square:

$$\begin{tikzcd} S_2 \arrow[r] \arrow[d] & T_2 \arrow[d] \\ S_1 \arrow[r] & T_1 \end{tikzcd}$$

with T_1 finite type, S_1 and T_2 placid, the scheme S_2 is necessarily placid.

Indeed, for $S_1 = \lim_{i \in \mathcal{I}^{op}} S_{1,i}$ and $T_2 = \lim_{j \in \mathcal{J}^{op}} T_{2,j}$ placid presentations by T_1-schemes, we have:

$$S_2 = \lim_{(i,j) \in \mathcal{I}^{op} \times \mathcal{J}^{op}} S_{1,i} \underset{T_1}{\times} T_{2,j}.$$

Obviously all structure maps are smooth affine covers, so this is a placid presentation of S_2.

Remark 4.2.5. By Noetherian descent, if S is placid and $T \to S$ is finite presentation, then T is placid as well. Moreover, there always exist placid presentations $S = \lim_{i \in \mathcal{I}^{op}} S_i$, $T = \lim_{i \in \mathcal{I}^{op}} T_i$ and compatible morphisms $T_i \to S_i$ inducing $T \to S$, and such that, for every $i \to j \in \mathcal{I}$, the diagram:

$$\begin{tikzcd} T_j \arrow[r] \arrow[d] & S_j \arrow[d] \\ T_i \arrow[r] & S_i \end{tikzcd}$$

is Cartesian.

Remark 4.2.6. By [Gro] Corollary 8.3.7, given a placid presentation $T = \lim_i T_i$, each structure morphism $T \to T_i$ is surjective on schematic points.

Remark 4.2.7. A placid scheme is automatically quasi-compact and quasi-separated.

Remark 4.2.8. As explained in [Dri], many natural schemes are Nisnevich (in particular: étale) locally placid, but not themselves placid. Much of the material that follows generalizes without difficulty to the case of Nisnevich locally placid schemes. However, we do not pursue this level of generality here because for our later applications, only placid schemes (and indschemes) occur.

4.3. If T is a placid scheme with placid presentation $T = \lim_{i \in \mathcal{I}^{op}} T_i$ then we have:

$$D^*(T) = \operatorname*{colim}_{i \in \mathcal{I}} D(T_i) \tag{4.3.1}$$

where the structure functors are the $*$-pullback functors (defined because the maps $T_j \to T_i$ are smooth). For $i \in \mathcal{I}^{op}$ and $f_i : T \to T_i$ the corresponding structure map, we let $f_i^{*,dR}$ denote the functor $D^*(T_i) \to D^*(T)$ left adjoint to $f_{i,*,dR}$.

In particular, we see that $D^*(T)$ is compactly generated and therefore canonically dual to $D^!(T)$, which is also compactly generated. (Note that in the $D^!$-case, compact objects are !-pullbacks of compact objects from finite type schemes, where for D^* they are $*$-pullbacks).

Similarly, we obtain:

$$D^!(T) = \lim_{i \in \mathcal{I}^{op}} D(T_i) \tag{4.3.2}$$

where the structure functors are the right adjoints to the $f_i^!$ functors, i.e., shifted de Rham cohomology functors (again, these are adjoint by smoothness).

Remark 4.3.1. It follows from the identification of D^* as a colimit that for placid $T = \lim_{i \in \mathcal{I}^{op}} T_i$ as above and $\mathcal{F} \in D^*(T)$, the canonical map:

$$\operatorname*{colim}_{i \in \mathcal{I}} f_i^{*,dR} f_{i,*,dR}(\mathcal{F}) \to \mathcal{F} \tag{4.3.3}$$

is an equivalence.

4.4. Let T be a quasi-compact quasi-separated scheme.

Let $\mathsf{Pres}(T)$ denote the 1-category whose objects are placid presentations $(\mathcal{I}, \{T_i\}_{i \in \mathcal{I}})$ of T and where morphisms $(\mathcal{I}, \{T_i^1\}_{i \in \mathcal{I}}) \to (\mathcal{J}, \{T_j^2\}_{j \in \mathcal{J}})$ are given by a datum:

$$F : \mathcal{I} \to \mathcal{J} \text{ and } \{f_i : T_i^1 \to T_{F(i)}^2\}_{i \in \mathcal{I}} \text{ compatible morphisms of schemes under } T.$$

One easily shows that $\mathsf{Pres}(T)$ is filtered.

4.5. Fix two placid presentations $(\mathcal{I}, \{T_i^1\}_{i\in\mathcal{I}})$ and $(\mathcal{J}, \{T_j^2\}_{j\in\mathcal{J}})$ of a scheme T. We will make use of the following observation.

Lemma 4.5.1. *For every $j \in \mathcal{J}$ and every factorization $T \to T_i^1 \to T_j^2$ for $i \in \mathcal{I}$, the morphism $T_i^1 \to T_j^2$ is smooth.*

Proof. Suppose x is a geometric point of T. For each $i' \in \mathcal{I}$, let $x_{i'}$ denote the corresponding geometric point of $T_{i'}^1$.

Applying Proposition 3.5.1, we obtain:

$$\operatorname{Coker}\left(x_i^*(\Omega^1_{T_i^1/T_j^2}) \to x^*(\Omega^1_{T/T_j^2})\right) = \operatorname*{colim}_{i'\in\mathcal{I}_{i/}} \operatorname{Coker}\left(x_i^*(\Omega^1_{T_i^1/T_j^2}) \to x_{i'}^*(\Omega^1_{T_{i'}^1/T_j^2})\right) = \operatorname*{colim}_{i'\in\mathcal{I}_{i/}} x_{i'}^*(\Omega^1_{T_{i'}^1/T_i^1}).$$

Because the structure maps $T_j \to T_i$ are smooth the right hand side is a filtered limit of vector spaces concentrated in degree 0 and therefore is concentrated in degree 0 as well.

On cohomology we obtain a long exact sequence with segments:

$$\ldots \to H^{i-1}\Big(\operatorname*{colim}_{i'\in\mathcal{I}_{i/}} x_{i'}^*(\Omega^1_{T_{i'}^1/T_i^1})\Big) \to H^i\Big(x_i^*(\Omega^1_{T_i^1/T_j^2})\Big) \to H^i\Big(x^*(\Omega^1_{T/T_j^2})\Big) \to \ldots.$$

The left term is zero for $i \neq 1$ and the right term is zero for $i \neq 0$. But $x_i^*(\Omega^1_{T_i^1/T_j^2})$ is tautologically concentrated in degrees $\leqslant 0$, so it is concentrated in degree 0 as desired. $\square$

4.6. **Dimensions.** We digress briefly to fix some terminology regarding dimensions.

Let T be a finite type scheme. We define the dimension function $\dim_T : T \to \mathbb{Z}^{\geqslant 0}$ to be the locally constant function that on a connected component is constant with value the Krull dimension of that connected component (i.e., the maximal dimension of an irreducible component of this connected component).

For $f : T \to S$ a map between finite type schemes, we let $\dim_{T/S} : T \to \mathbb{Z}$ be the locally constant function $\dim_T - f^*(\dim_S)$.

Example 4.6.1. If $f : T \to S$ is a smooth dominant morphism, then $\dim_{T/S}$ is the rank of the vector bundle $\Omega^1_{T/S}$.

Therefore, for a Cartesian diagram of finite type schemes:

$$\begin{tikzcd} T' \arrow[r, "\psi"] \arrow[d, "g"] & T \arrow[d, "f"] \\ S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

with φ and ψ both dominant smooth morphisms, $\dim_{T'/S'} = \psi^*(\dim_{T/S})$. In particular, this identity holds whenever φ is a smooth covering map.

Counterexample 4.6.2. We need not have $\dim_{T/S} = d_{S/T} := \operatorname{rank}(\Omega^1_{T/S})$ if $f : T \to S$ is smooth but not dominant.

For example, let $S = \mathbb{A}^2 \coprod_0 \mathbb{A}^1$ be a line and a plane glued along a point, and let $T = \mathbb{G}_m \times \mathbb{A}^1$ mapping to S via the composition:

$$\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \hookrightarrow \mathbb{A}^2 \coprod_0 \mathbb{A}^1.$$

Then $d_{S/T}$ the constant function 1, while $\dim_{S/T}$ is the constant function $\dim_T - \dim_S = 2-2 = 0$.

Remark 4.6.3. By Remark 4.2.5, we see from Example 4.6.1 that $\dim_{T/S}$ can be defined as a locally constant function $T \to \mathbb{Z}$ for any finitely presented morphism $T \to S$ of placid schemes by Noetherian descent.

Given a pair of finitely presented morphisms $T \xrightarrow{f} S \to V$ of placid schemes, this construction satisfies the basic compatibility:

$$\dim_{T/V} = \dim_{T/S} - f^*(\dim_{S/V}). \tag{4.6.1}$$

4.7. **Renormalized dualizing sheaf.** Suppose that T is placid scheme. We will now define the *renormalized dualizing sheaf* $\omega_T^{ren} \in D^*(T)$.

Fix a placid presentation $T = \lim_{i\in \mathcal{I}^{op}} T_i$ of T. Because each structure map $\varphi_{ij} : T_j \to T_i$ is a smooth covering, we have canonical identifications:

$$\varphi_{ij}^{*,dR}(\omega_{T_i}[-2\cdot \dim_{T_i}]) = \omega_{T_j}[-2\cdot(\dim_{T_j})].$$

Therefore we have a uniquely defined sheaf ω_T^{ren} characterized by the fact that it is the $*$-pullback of $\omega_{T_i}[-2\cdot\dim_{T_i}]$ from any T_i to T.

We claim that ω_T^{ren} canonically does not depend on the choice of placid presentation. Indeed, this follows from Lemma 4.5.1 and by filteredness of $\mathsf{Pres}(T)$.

Example 4.7.1. Let T be finite type. Then $\omega_T^{ren} \in D^*(T) = D(T)$ identifies with $\omega_T[-2\cdot\dim_T]$.

Example 4.7.2. Suppose T admits a placid presentation $T = \lim T_i$ with each T_i smooth. Then $\omega_T^{ren} = k_T$.

4.8. Suppose that T is a placid scheme. We define the functor:

$$\eta_T : D^!(T) \to D^*(T)$$

by action on ω_T^{ren}.

Proposition 4.8.1. *The functor η_T is an equivalence.*

Proof. Choose $T = \lim_{i\in\mathcal{I}^{op}} T_i$ a placid presentation. We claim that the functor:

$$\eta_T : D^!(T) := \operatorname*{colim}_{i\in\mathcal{I}} D(T_i) \to D^*(T) \overset{(4.3.1)}{=} \operatorname*{colim}_{i\in\mathcal{I}} D(T_i).$$

is the colimit of the shifted identity functors $\mathrm{id}_{D(T_i)}[-2\cdot\dim_{T_i}]$. Indeed, the colimit of these functors is a morphism of $D^!(T)$-module categories and sends $\omega_T \in D^!(T)$ to $\omega_T^{ren} \in D^*(T)$.

Now the result obviously follows from this identification.

□

Example 4.8.2. If T is finite type then η_T is the composite equivalence $D^!(T) := D(T) =: D^*(T)$ shifted by $-2\dim_T$.

4.9. **Renormalized functors.** Let $f : T \to S$ a map of placid schemes.

We let $f_{*,ren} : D^!(T) \to D^!(S)$ denote the induced functor so that we have the commutative diagram:

$$\begin{tikzcd} D^!(T) \arrow[r, "f_{*,ren}"] \arrow[d, "\simeq"', "\eta_T"] & D^!(S) \arrow[d, "\simeq"', "\eta_S"] \\ D^*(T) \arrow[r, "f_{*,dR}"] & D^*(S). \end{tikzcd}$$

In the same way we obtain the functor $f^{!,ren} : D^*(S) \to D^*(T)$ fitting into a commutative diagram:

$$\begin{tikzcd} D^*(S) \arrow[r, "f^{!,ren}"] & D^*(T) \\ D^!(S) \arrow[r, "f^!"] \arrow[u, "\eta_S", "\simeq"'] & D^!(T) \arrow[u, "\eta_T", "\simeq"'] \end{tikzcd}$$

Note that we have a canonical isomorphism

$$f^{!,ren}(\omega_S^{ren}) = \omega_T^{ren} \tag{4.9.1}$$

because:

$$f^{!,ren}(\omega_S^{ren}) = f^!(\omega_S) \overset{!}{\otimes} \omega_T^{ren} = \omega_T \overset{!}{\otimes} \omega_T^{ren} = \omega_T^{ren}.$$

Example 4.9.1 (Renormalized functors in finite type: $D^!$)*.* Suppose $f : T \to S$ is a map between finite type schemes. We identify $D^!(S)$ and $D^!(T)$ with $D(S)$ and $D(T)$ in the canonical way.

Then the functor $f_{*,ren} : D(T) \to D(S)$ identifies with $f_{*,dR}[-2 \cdot \dim_{T/S}]$. In particular, if f is smooth and dominant, then $(f^!, f_{*,ren})$ form an adjoint pair of functors.

Note that in this setting the functor $f_{*,!-dR}$ coincides with the (non-renormalized) functor $f_{*,dR}$.

Warning 4.9.2. If $f : S \to T$ is a closed embedding of placid schemes, then $f_{*,ren}$ is not left adjoint to $f^!$ (c.f. Example 4.9.1). In fact, if f is a closed embedding of infinite codimension, then $f_{*,ren}$ does not preserve compact objects and therefore does not admit a continuous right adjoint at all. Moreover, the composition $f^! f_{*,ren}$ is typically zero in this case.

Warning 4.9.3. Given a Cartesian diagram:

$$\begin{tikzcd} T_1 \arrow[r, "\psi"] \arrow[d, "\phi"] & S_1 \arrow[d, "f"] \\ T_2 \arrow[r, "g"] & S_2 \end{tikzcd}$$

of finite type schemes, we find that:

$$f^! g_{*,ren} = f^! g_{*,dR}[-2 \cdot \dim_{T_2/S_2}] = \psi_{*,dR}\varphi^![-2 \cdot \dim_{T_2/S_2}]$$

while $\psi_{*,ren}\varphi^! = \psi_{*,dR}\varphi^![-2 \cdot \dim_{T_1/S_1}]$. Since dimensions do not always behave well under base-change, we see that base-change does not always hold between renormalized pushforward and upper-!.

Example 4.9.4 (Renormalized functors in finite type: D^*)*.* Suppose $f : T \to S$ is a map between finite type schemes. We identify $D^*(S)$ and $D^*(T)$ with $D(S)$ and $D(T)$ in the canonical way.

Then the functor $f^{!,ren} : D(S) \to D(T)$ identifies with $f^!(-)[-2\dim_{T/S}]$. Note that if f is smooth and dominant, then $f^{!,ren}$ identifies canonically with $f^{*,dR}$.

The functor $f^{¡}$ coincides with the (non-renormalized) functor $f^!$.

Remark 4.9.5*.* We emphasize explicitly that the "renormalization" here has nothing to do with the renormalized de Rham cohomology functor from [DG]. Rather, the terminology is taken from [Dri] §6.8.

4.10. **Placid morphisms.** We will now further analyze the renormalized functors under certain very favorable circumstances.

We say a morphism $f : S \to T$ of placid schemes is *placid* if, for any placid presentations $S = \lim_{i \in \mathcal{I}^{op}} S_i$, $T = \lim_{j \in \mathcal{J}^{op}} T_j$, for every $j \in \mathcal{J}$ there exists $i \in \mathcal{I}$ with the morphism $S \to T \to T_j$ factoring as $S \to S_i \to T_j$ and with $S_i \to T_j$ a smooth covering.

Obviously, if this holds for one pair of placid presentations then it holds for any.

Example 4.10.1*.* By Noetherian descent and Remark 4.2.6, a smooth morphism of finite presentation which is surjective on geometric points is placid.

Example 4.10.2*.* Suppose that $S = \lim_{i \in \mathcal{I}^{op}} S_i$ and $T = \lim_{i \in \mathcal{I}^{op}} T_i$ are placid presentations, and suppose that we are given compatible smooth coverings $f_i : S_i \to T_i$ inducing $f : S \to T$. Then f is a placid morphism. We emphasize that by *compatible* we mean that the relevant squares commute, not that they are Cartesian.

Remark 4.10.3*.* For categorical arguments, it is convenient to use the following formulation of this definition.

Let $\mathsf{Sch}^{f.t.}_{sm-cov}$ denote the category of finite type schemes where we only allow smooth coverings as morphisms. Let:

$$\mathsf{Pro}^{\text{aff}}(\mathsf{Sch}^{f.t.}_{sm-cov}) \subseteq \mathsf{Pro}(\mathsf{Sch}^{f.t.}_{sm-cov})$$

denote the full subcategory where we only allow objects obtained as projective limits under morphisms that are affine (in addition to being a priori smooth coverings).

Then the functor:

$$\mathsf{Pro}^{\text{aff}}(\mathsf{Sch}^{f.t.}_{sm-cov}) \to \mathsf{Pro}^{\text{aff}}(\mathsf{Sch}^{f.t.}) = \mathsf{Sch}_{qcqs}$$

is a (non-full) embedding of categories. Indeed, this is a general feature: (non-full) embeddings of $(1,1)$-categories induce embeddings on Ind or Pro categories, since filtered limits and colimits of injections in Set are still injections. Moreover, its essential image are placid schemes, and a morphism lies in this non-full subcategory if and only if it is placid.

Observe that $\mathsf{Pro}^{\text{aff}}(\mathsf{Sch}^{f.t.}_{sm-cov}) \to \mathsf{Sch}_{qcqs}$ commutes with filtered projective limits with affine structure maps, i.e., this functor is the right Kan extension of its restriction to $\mathsf{Sch}^{f.t.}_{sm-cov}$. Indeed, $\mathsf{Sch}_{qcqs} \subseteq \mathsf{Pro}(\mathsf{Sch}^{f.t.})$ commutes with such filtered projective limits, and $\mathsf{Pro}^{\text{aff}}(\mathsf{Sch}^{f.t.}_{sm-cov}) \subseteq \mathsf{Pro}(\mathsf{Sch}^{f.t.}_{sm-cov})$ does too. Moreover, $\mathsf{Pro}(\mathsf{Sch}^{f.t.}_{sm-cov}) \to \mathsf{Pro}(\mathsf{Sch}^{f.t.})$ tautologically commutes with filtered limits, proving the claim.

Warning 4.10.4*.* Against the usual conventions for terminology in algebraic geometry, placid morphisms are not intended as a relative form of placidity.

Indeed, we can only speak about placid morphisms between between schemes already known to be placid. Moreover, for a placid scheme S, the structure map $S \to \operatorname{Spec}(k)$ may not be placid.

The terminology is rather taken by analogy with the definition of placid schemes, as in Remark 4.10.3.

Counterexample 4.10.5. It may be tempting to think of placid morphisms as being analogous to being a smooth covering morphisms, since this condition is equivalent for finite type schemes. The following example is meant to show the geometric limitations of this line of thought.

Let $\mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^1 \times \mathbb{A}^{n-1}$ by:

$$\Big(\lambda, (x_1, \ldots, x_n)\Big) \mapsto \Big(\lambda, (x_1 - \lambda \cdot x_2, \ldots, x_{n-1} - \lambda \cdot x_n)\Big).$$

Each of these morphisms is a smooth covering. Moreover, these morphisms are compatible as n varies, and therefore induce a placid morphism (of infinite type):

$$\begin{gathered}\mathbb{A}^1 \times \mathbb{A}^\infty \to \mathbb{A}^1 \times \mathbb{A}^\infty \\ (\lambda, x_1, x_2, \ldots) \mapsto (\lambda, x_1 - \lambda \cdot x_2, x_2 - \lambda \cdot x_3, \ldots).\end{gathered}$$

where we use the notation $\mathbb{A}^\infty = \lim_n \mathbb{A}^n$, the limit taken under structure maps $\mathbb{A}^m \to \mathbb{A}^n$ $(m \geqslant n)$ of projection onto the first n-coordinates.

Then for $0 \neq \lambda \in k$, the fiber of this map at $(\lambda, 0, 0, \ldots, 0)$ is a copy of $\mathbb{A}^1$, realized as the loci of points:

$$(\lambda, x_1, \lambda^{-1} \cdot x_1, \lambda^{-2} \cdot x_1, \ldots)$$

with $x_1 \in \mathbb{A}^1$ arbitrary.

However, the fiber at $(0, 0, 0, \ldots)$ is just the point scheme $\mathrm{Spec}(k)$, realized as the locus $(0, 0, 0, \ldots)$.

Lemma 4.10.6. *Given a Cartesian diagram:*

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "f"] & T_1 \end{tikzcd}$$

of placid schemes with g finite presentation and f a placid morphism, the morphism φ is placid as well.

Proof. Let $S_1 = \lim_i S_{1,i}$ and $T_1 = \lim_j T_{1,j}$ be placid presentations. We take a compatible placid presentation $T_2 = \lim_j T_{2,j}$ as in Remark 4.2.5.

Note that:

$$S_2 = \lim_j \lim_i S_{1,i} \underset{T_{1,j}}{\times} T_{2,j}$$

where we really only take the limit under i such that the map $S_1 \to T_{1,j}$ factors (necessarily uniquely) through $S_{1,i}$.

For a pair of morphisms $(i_1 \to i_2)$ and $(j_1 \to j_2)$, we claim that the induced map:

$$S_{1,i_2} \underset{T_{1,j_2}}{\times} T_{2,j_2} \to S_{1,i_1} \underset{T_{1,j_1}}{\times} T_{2,j_1}$$

is an affine smooth covering. Indeed, we have $T_{2,j_2} = T_{1,j_2} \times_{T_{1,j_1}} T_{2,j_1}$ so that the left hand side of the above is $S_{1,i_2} \times_{T_{1,j_1}} T_{2,j_1}$. Because $S_{1,i_2} \to S_{1,i_1}$ is an affine smooth covering, we obtain the claim.

Therefore, the terms $S_{1,i} \times_{T_{1,j}} T_{2,j}$ define a placid presentation of S_2. But each map:

$$S_{1,i} \underset{T_{1,j}}{\times} T_{2,j} \to T_{2,j}$$

is a smooth covering because each $S_{1,i} \to T_{1,j}$ is assumed to be, completing the proof. $\square$

The following results from the argument above.

Corollary 4.10.7. *Suppose that we have a Cartesian square:*

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "f"] & T_1 \end{tikzcd}$$

of placid schemes with g finite presentation and f a placid morphism. Then $\dim_{S_2/S_1} = \varphi^*(\dim_{T_2/T_1})$.

Proof. Let $S_1 = \lim_i S_{1,i}$, $T_1 = \lim_j T_{1,j}$ and $T_2 = \lim_j T_{2,j}$ be as in the proof of Lemma 4.10.6. As in *loc. cit.*, we have a placid presentation of S_2 with terms:

$$S_{1,i} \underset{T_{1,j}}{\times} T_{2,j}.$$

Fixing and index j_0, as in *loc. cit.*, we have:

$$S_{1,i} \underset{T_{1,j}}{\times} T_{2,j} = S_{1,i} \underset{T_{1,j_0}}{\times} T_{2,j_0}.$$

for every morphism $j_0 \to j$. Therefore, the morphisms $S_{1,i} \underset{T_{1,j}}{\times} T_{2,j} \to S_{1,i}$ are obtained one from another by base-change, so that $\dim_{S_2/S_1}$ is defined as the pullback of the function:

$$\dim_{S_{1,i} \underset{T_{1,j}}{\times} T_{2,j}/S_{1,i}}$$

for any choice of indices. But because our maps are smooth coverings, this function is the pullback of $\dim_{T_{2,j}/T_{1,j}}$, giving the result. $\square$

4.11. For our purposes, the key feature of placid morphisms is given by the following proposition.

Proposition 4.11.1.
1. *For a placid morphism $f : S \to T$ of placid schemes, the left adjoint $f^{*,dR}$ to $f_{*,dR} : D^*(S) \to D^*(T)$ is defined.*
2. *For a placid morphism $f : S \to T$ of placid schemes, there is a canonical identification $f^{!,ren} \simeq f^{*,dR} : D^*(T) \to D^*(S)$.*

   More precisely, with Sch_{pl} denoting the category of placid schemes under placid morphisms, there is a canonical identification of functors:

$$(D^*, f^{*,dR}) \simeq (D^*, f^{!,ren}) : \mathsf{Sch}_{pl}^{op} \to \mathsf{DGCat}_{cont}$$

   inducing the identity over the maximal subgroupoid of Sch_{pl}^{op}.
3. *For a placid morphism $f : S \to T$ of placid schemes, the functor $f^! : D^!(T) \to D^!(S)$ admits a right adjoint, and this right adjoint is functorially identified with $f_{*,ren}$ in the sense above.*

(4) For a Cartesian square of placid schemes:

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "f"] & T_1 \end{tikzcd} \tag{4.11.1}$$

with f placid and g finitely presented, the canonical morphisms:

$$f^{!,ren}g_{*,dR} \to \psi_{*,dR}\varphi^{!,ren}$$
$$f^! g_{*,ren} \to \psi_{*,ren}\varphi^!$$

arising from the adjunctions above are equivalences.

We begin with the following general remarks.

Let $\mathsf{DGCat}_{cont}^{ladj}$ denote the category of cocomplete DG categories under k-linear functors that admit continuous right adjoints. Let $\mathsf{DGCat}_{cont}^{radj}$ denote the category of cocomplete DG categories under k-linear functors that admit left adjoints.

We have an obvious equivalence $\mathsf{DGCat}_{cont}^{ladj} \simeq \mathsf{DGCat}_{cont}^{radj,op}$ given by passing to the adjoint functor.

One easily verifies:

Lemma 4.11.2. *The category $\mathsf{DGCat}_{cont}^{ladj}$ admits colimits, and the functor $\mathsf{DGCat}_{cont}^{ladj} \to \mathsf{DGCat}_{cont}$ preserves these colimits. Similarly, $\mathsf{DGCat}_{cont}^{radj}$ admits limits, and the functor $\mathsf{DGCat}_{cont}^{radj} \to \mathsf{DGCat}_{cont}$ commutes with limits.*

Proof. The content is that given a diagram $i \mapsto \mathcal{C}_i$ of cocomplete DG categories under structure functors admitting continuous right adjoints, a functor $\mathcal{C} := \operatorname{colim}_i \mathcal{C}_i \to \mathcal{D}$ admits a continuous right adjoint if and only if each $\mathcal{C}_i \to \mathcal{D}$ does. But this is obvious, since $\mathcal{C}$ is then also the limit of the $\mathcal{C}_i$ under the right adjoint functors.

□

Proof of Proposition 4.11.1. Recall from Remark 4.10.3 that Sch_{pl} is the full subcategory:

$$\mathsf{Pro}^{\mathrm{aff}}(\mathsf{Sch}_{sm-cov}^{f.t.}) \subseteq \mathsf{Pro}(\mathsf{Sch}_{sm-cov}^{f.t.}).$$

Moreover, because $\mathsf{Sch}_{pl} \to \mathsf{Sch}_{qcqs}$ is the right Kan extension of its restriction to $\mathsf{Sch}_{sm-cov}^{f.t.}$, we see that $D^*|_{\mathsf{Sch}_{pl}}$ is the right Kan extension of $D^*|_{\mathsf{Sch}_{sm-cov}^{f.t.}} = D|_{\mathsf{Sch}_{sm-cov}^{f.t.}}$.

Moreover, note that $D^*|_{\mathsf{Sch}_{sm-cov}^{f.t.}}$ factors through $\mathsf{DGCat}_{cont}^{radj}$ by smoothness.

As in Example 4.9.4, the corresponding functor:

$$D|_{\mathsf{Sch}_{sm-cov}^{f.t.}} \to \mathsf{DGCat}_{cont}^{radj} \simeq \mathsf{DGCat}_{cont}^{ladj,op}$$

identifies with $(D, f^{!,ren})|_{\mathsf{Sch}_{sm-cov}^{f.t.}}$, i.e., the functor attaching to a scheme of finite type its category of D-modules, and to a smooth surjective morphism of schemes the corresponding renormalized pullback functor.[5]

By Lemma 4.11.2, the right Kan extension of this functor also factors through $\mathsf{DGCat}_{cont}^{radj}$, proving (1). Moreover, it follows that the corresponding functor to $\mathsf{Sch}_{pl}^{op} \to \mathsf{DGCat}^{ladj}$ encoding the left adjoints is the left Kan extension of $(D, f^{!,ren})|_{\mathsf{Sch}_{sm-cov}^{f.t.,op}}$.

We have an equivalence:

[5]This identification is treated formally in the homotopical setting in [GR].

$$(D, f^{!,ren})|_{\mathsf{Sch}^{f.t.,op}} \simeq (D, f^!)|_{\mathsf{Sch}^{f.t.,op}}$$

computed termwise on a finite type scheme S as η_S^{-1}. Moreover, $(D^!, f^!)$ is the left Kan extension of the left hand side.

For a placid scheme S with placid presentation $S = \lim S_i$, we have:

$$\eta_S = \operatorname*{colim}_i \eta_{S_i} : D^!(S) = \operatorname*{colim}_i D^!(S_i) \to \operatorname*{colim}_i D^*(S_i) = D(S)$$

the colimit on the right taken under renormalized pullback functors (equivalently: $*$-dR pullback). Indeed, this was already observed in the proof of Proposition 4.8.1.

Therefore, we see that $(D^!, f^{!,ren})$ is the left Kan extension of $(D, f^{!,ren})|_{\mathsf{Sch}^{f.t.}_{sm-cov}}$, as desired. This completes the proof of (2).

Note that (3) is a formal consequence of (2). Therefore, it remains to show (4).

Suppose we are given a Cartesian square (4.11.1). It obviously suffices to show either of the base-change morphisms is an equivalence, so we treat the map $f^{!,ren} g_{*,dR} \to \psi_{*,dR}\varphi^{!,ren}$.

First, suppose that T_1 and T_2 are finite type.

We take a placid presentation $S_1 = \lim_i S_{1,i}$. We can assume each $S_{1,i}$ is a T_1-scheme by Noetherian approximation.

Because $S_1 \to T_1$ is placid, each $S_{1,i} \to T_1$ is a smooth covering. Define $S_{2,i} = S_{1,i} \times_{T_1} T_2$.

We use the notation:

$$\begin{tikzcd} S_2 \arrow[r, "\beta_i"] \arrow[d, "\psi"] & S_{2,i} \arrow[r, "\varphi_i"] \arrow[d, "\psi_i"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "\alpha_i"] & S_{1,i} \arrow[r, "f_i"] & T_1. \end{tikzcd} \tag{4.11.2}$$

We now have:

$$f_{*,dR} f^{!,ren} g_{*,dR} = \operatorname*{colim}_i f_{i,*,dR} f_i^{!,ren} g_{*,dR} = \operatorname*{colim}_i f_{i,*,dR}\psi_{i,*,dR}\varphi_i^{!,ren} =$$
$$\operatorname*{colim}_i g_{*,dR}\varphi_{i,*,dR}\varphi_i^{!,ren} = g_{*,dR}\varphi_{*,dR}\varphi^{!,ren} = f_{*,dR}\psi_{*,dR}\varphi^{!,ren}$$

Here the first and fourth equalities follows from filteredness of our index category and the adjunctions. The base-change in our second equality follows from the usual smooth base-change theorem in the finite type setting.

Applying the above argument to the left square of (4.11.2) and applying (finite dimensional) smooth base-change to the right square, we see that the map:

$$\alpha_{i,*,dR} f^{!,ren} g_{*,dR} \to \alpha_{i,*,dR}\psi_{*,dR}\varphi^{!,ren}$$

is always an equivalence. But this suffices to see our base-change by definition of D^*.

We now treat the case of general g of finite presentation. Suppose that we have a diagram:

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[r, "\theta"] \arrow[d, "g"] & T_2' \arrow[d, "g'"] \\ S_1 \arrow[r, "f"] & T_1 \arrow[r, "\varepsilon"] & T_1' \end{tikzcd} \tag{4.11.3}$$

with both squares Cartesian, the schemes T_i' of finite type, and the maps θ and ε placid.

Then we have base-change maps:

$$f^{!,ren}\varepsilon^{!,ren}g'_{*,dR} \to f^{!,ren}g_{*,dR}\theta^{!,ren} \to \psi_{*,dR}\varphi^{!,ren}\theta^{!,ren}.$$

By our earlier analysis, the first map is an equivalence by considering the right square of (4.11.3), and the composite map is also an equivalence by considering the outer square of (4.11.3). Therefore, we see that the map:

$$f^{!,ren}g_{*,dR}\theta^{!,ren} \to f^{!,ren}g_{*,ren}\theta^{!,ren}$$

is an equivalence. Varying T'_1 over some placid presentation of T_1, the corresponding functors $\theta^{!,ren}$ generate $D^*(T_2)$, so this suffices.

□

4.12. As a consequence of Proposition 4.11.1, we show that some features from Examples 4.9.1 and 4.9.4 survive to greater generality.

Proposition 4.12.1. *For $f : T \to S$ a finitely presented morphism of placid schemes, we have canonical identifications:*

$$f^{¡}[-2 \cdot \dim_{T/S}] = f^{!,ren} : D^*(S) \to D^*(T)$$
$$f_{*,!-dR}[-2 \cdot \dim_{T/S}] = f_{*,ren} : D^!(T) \to D^!(S).$$

where $\dim_{T/S}$ *is defined as in §4.6.*

Proof. Let $S = \lim S_i$ be a placid presentation, and by Remark 4.2.5, we may assume we have a placid presentation $T = \lim T_i$ so that we have maps $f_i : T_i \to S_i$ with each $i \to j$ inducing a Cartesian diagram, and with f obtained by base-change from each of the f_i. Note that $\dim_{T/S}$ is then obtained by pullback from each $\dim_{T_i/S_i}$.

We use the notation:

$$\begin{array}{ccc} T & \xrightarrow{\psi_i} & T_i \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f_i} \\ S & \xrightarrow{\varphi_i} & S_i. \end{array}$$

For the first part, note that by (4.3.3) and Example 4.9.4, we have:

$$f^{¡} = \operatorname*{colim}_i \psi_i^{*,dR}\psi_{i,*,dR}f^{¡} = \operatorname*{colim}_i \psi_i^{*,dR}f_i^{¡}\varphi_{i,*,dR} = \operatorname*{colim}_i \psi_i^{*,dR}f_i^{!,ren}\varphi_{i,*,dR}[2 \cdot \dim_{T/S}].$$

By Proposition 4.11.1, $\psi_i^{*,dR} = \psi_i^{!,ren}$. Therefore, we compute the above as:

$$\operatorname*{colim}_i \psi_i^{!,ren}f_i^{!,ren}\varphi_{i,*,dR}[2 \cdot \dim_{T/S}] = \operatorname*{colim}_i f^{!,ren}\varphi_i^{!,ren}\varphi_{i,*,dR}[2 \cdot \dim_{T/S}] = f^{!,ren}[2 \cdot \dim_{T/S}]$$

by again applying (4.3.3) and the identification $\varphi_i^{!,ren} = \varphi_i^{*,dR}$.

For the second part, note that we have functorial base change isomorphisms:

$$\varphi_i^! f_{i,*,ren} \simeq f_{*,ren}\psi_i^!$$

by Proposition 4.11.1. By Example 4.9.1, $f_{i,*,!-dR}[-2 \cdot \dim_{T/S}] = f_{i,*,ren}$. Moreover, these cohomological shifts are compatible with varying i, so we obtain the result by definition of $f_{*,!-dR}$.

□

Corollary 4.12.2. *Suppose we are given a Cartesian square:*

$$\begin{tikzcd} S_2 \arrow[r, "\varphi"] \arrow[d, "\psi"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "f"] & T_1 \end{tikzcd}$$

with S_1 and T_2 placid schemes, f and g placid morphisms, and T_1 finite type. Then the canonical morphisms:

$$f^{!,ren} g_{*,dR} \to \psi_{*,dR} \varphi^{!,ren}$$
$$f^! g_{*,ren} \to \psi_{*,ren} \varphi^!$$

are equivalences.

Proof. Note that we have already seen in Example 4.2.4 that S_2 is actually a placid scheme.

It tautologically suffices to prove that the first base-change morphism is an equivalence.

We form the diagram:

$$\begin{tikzcd} S_2 \arrow[r, "i"] \arrow[d, "\psi"] & S_1 \times T_2 \arrow[r, "p_2"] \arrow[d, "\mathrm{id}_{S_1} \times g"] & T_2 \arrow[d, "g"] \\ S_1 \arrow[r, "\Delta_f"] & S_1 \times T_1 \arrow[r, "p_2"] & T_1. \end{tikzcd}$$

Here Δ_f is the graph of f. Note that each of these squares is Cartesian. In particular, i is a finitely presented morphism. We are reduced to proving the base-change result for each of these squares separately.

For the right square, the result is essentially obvious: it follows from the compatibility of push-forward with products of schemes.

For the left square, note that the base-change result holds with the upper-¡ functor in place of the renormalized upper-! functor by the correspondence formalism. Therefore, the result follows from Proposition 4.12.1.

□

5. Holonomic D-modules

5.1. In this section, we discuss the holonomic theory in infinite type.

Remark 5.1.1. Since many "standard" D-modules in infinite type are not compact (e.g., the delta D-module concentrated at a point in an infinite type placid scheme), it is convenient to break conventions with the usual D-module theory and allow some non-compact D-modules to be counted as holonomic.

5.2. **Holonomic D-modules.** Let S be a scheme of finite type. Let $D_{coh,hol}(S)$ denote the full subcategory of $D_{coh}(S)$ (the compact objects in $D(S)$) composed of those coherent complexes with holonomic cohomologies, defined in the usual way. Let $D_{hol}(S) \subseteq D(S)$ denote the full subcategory:

$$D_{hol}(S) := \mathsf{Ind}(D_{coh,hol}(S)) \subseteq D(S).$$

We refer to objects of $D_{hol}(S)$ simply as holonomic objects.[6]

For $f : S \to T$ a map of finite type schemes, the usual theory of D-modules implies that the functors $f_{*,dR}$ and $f^!$ preserve the subcategories of holonomic objects.

For S a quasi-compact quasi-separated scheme, we obtain the categories:

$$D^!_{hol}(S) \text{ and } D^*_{hol}(S)$$

defined by a Kan extension, as in the case of $D^!$ and D^*. We have obvious functors $D^!_{hol}(S) \to D^!(S)$ and $D^*_{hol}(S) \to D^*(S)$, the latter being fully-faithful. For S placid, we can express $D^*_{hol}(S)$ as a limit as for $D^*(S)$, and therefore we see that $D^*_{hol}(S) \to D^*(S)$ is fully-faithful in this case as well. We refer to subobjects of $D^*(S)$ lying in $D^*_{hol}(S)$ as holonomic objects, and similarly for $D^!$ when S is placid.

We have upper-! and lower-* functors for $D^!_{hol}(S)$ and $D^*_{hol}(S)$ respectively, compatible with the forgetful functors.

Proposition 5.2.1. *For $f : S \to T$ a morphism of quasi-compact quasi-separated schemes, the morphism $f_{*,dR} : D^*_{hol}(S) \to D^*_{hol}(T)$ admits a left adjoint $f^{*,dR}$.*

If T is placid and f is finitely presented, then the morphism $f^! : D^!_{hol}(T) \to D^!_{hol}(S)$ admits a left adjoint $f_!$.

Moreover, in each of the above settings, these left adjoints are well-behaved with respect to maps to non-holonomic objects as well, i.e., the partially-defined left adjoints to $f_{*,dR} : D^*(S) \to D^*(T)$ and $f^! : D^!(T) \to D^!(S)$ are defined on holonomic objects, and these left adjoints preserve the holonomic subcategories (and therefore are computed by the above functors). Of course, we are assuming f finitely presented and T placid when discussing $f_!$.

We will prove this in §5.4 below.

5.3. We digress to prove the following lemma.

Lemma 5.3.1. *Let $\mathcal{I}$ be an indexing category with $\mathcal{I}^{op}$ filtered. Let $(i \mapsto \mathcal{C}_i)$ and $(i \mapsto \mathcal{D}_i)$ are two $\mathcal{I}$-shaped diagrams of cocomplete categories under continuous functors, with structure functors:*

$$\psi_\alpha : \mathcal{C}_i \to \mathcal{C}_j \qquad \psi_i : \mathcal{C} := \lim_{j \in \mathcal{I}} \mathcal{C}_j \to \mathcal{C}_i$$
$$\varphi_\alpha : \mathcal{D}_i \to \mathcal{D}_j \qquad \varphi_i : \mathcal{D} := \lim_{j \in \mathcal{I}} \mathcal{D}_j \to \mathcal{D}_i$$

for $\alpha : i \to j$ in $\mathcal{I}$ and for $i \in \mathcal{I}$.

Suppose $G_i : \mathcal{C}_i \to \mathcal{D}_i$ are compatible functors with induced functor:

$$G : \mathcal{C} \to \mathcal{D}.$$

If each functor G_i admits a left adjoint F_i, then G admits a left adjoint $F : \mathcal{D} \to \mathcal{C}$ such that, for every $j \in \mathcal{I}$, we have:

$$\psi_j F = \operatorname*{colim}_{(\alpha : i \to j) \in (\mathcal{I}_{/j})^{op}} \psi_\alpha F_i \varphi_i.$$

[6]We note that, of course, this condition completely ruins all the nice finiteness conditions that "usual" (coherent) holonomic complexes satisfy, e.g., finiteness of de Rham cohomology. This loss is obvious necessary for the infinite dimensional setting.

Proof. For $j \in \mathcal{I}$ fixed, note that for any diagram:

$$i' \xrightarrow{\beta} i \xrightarrow{\alpha} j$$

we have the natural map:

$$\varphi_\beta \to \varphi_\beta G_{i'} F_{i'} \to G_i \psi_\beta F_{i'}.$$

By adjunction, this gives rise to a map:

$$F_i \varphi_\beta \to \psi_\beta F_{i'}.$$

Composing on the left with ψ_α and on the right with $\varphi_{i'}$, we obtain the map:

$$\psi_\alpha F_i \varphi_\beta \varphi_{i'} = \psi_\alpha F_i \varphi_i \to \psi_{\alpha \circ \beta} F_{i'} \varphi_{i'} = \psi_\alpha \psi_\beta F_{i'} \varphi_{i'}.$$

Expressing this in the obvious homotopy-compatible way, we obtain a diagram of functors:

$$(\alpha : i \to j) \in (\mathcal{I}_{/j})^{op} \mapsto \psi_\alpha F_i \varphi_i.$$

Define the functor:

$$\text{“}\psi_j F\text{”} := \operatorname*{colim}_{(\alpha:i \to j) \in (\mathcal{I}_{/j})^{op}} \psi_\alpha F_i \varphi_i.$$

As j varies, we see by filteredness that these functors are homotopy compatible, and therefore we obtain a functor $F : \mathcal{D} \to \mathcal{C}$ with the property that we have functorial identifications:

$$\psi_j F = \text{“}\psi_j F\text{”}$$

with “$\psi_j F$” as above.

For every $j \in \mathcal{I}$, we have the map:

$$\psi_j FG = \operatorname*{colim}_{(\alpha:i \to j) \in (\mathcal{I}_{/j})^{op}} \psi_\alpha F_i \varphi_i G = \psi_\alpha F_i G_i \psi_i \to \psi_\alpha \psi_i = \psi_j.$$

As $j \in \mathcal{I}$ varies, these maps are homotopy compatible and therefore we obtain the counit map:

$$FG \to \mathrm{id}_{\mathcal{C}}.$$

Similarly, for every $j \in \mathcal{I}$, we have the map:

$$\varphi_j = \operatorname*{colim}_{(\alpha:i \to j) \in (\mathcal{I}_{/j})^{op}} \varphi_j = \operatorname*{colim}_{(\alpha:i \to j) \in (\mathcal{I}_{/j})^{op}} \varphi_\alpha \varphi_i \to \operatorname*{colim}_{(\alpha:i \to j) \in (\mathcal{I}_{/j})^{op}} \varphi_\alpha G_i F_i \varphi_i =$$
$$\operatorname*{colim}_{(\alpha:i \to j) \in (\mathcal{I}_{/j})^{op}} G_j \psi_\alpha F_i \varphi_i = G_j \psi_j F = \varphi_j GF.$$

As j varies, these maps are homotopy compatible and therefore give the unit map:

$$\mathrm{id}_{\mathcal{D}} \to GF.$$

One readily checks that the unit and counit maps constructed above actually define an adjunction. $\square$

5.4. We now prove the proposition above.

Proof of Proposition 5.2.1. For any map $f : S \to T$, it is easy to see that we can arrange to have $S = \lim_{i \in \mathcal{I}^{op}} S_i$, $T = \lim_{i \in \mathcal{I}^{op}}$ filtered systems of finite type schemes under affine maps and with compatible maps $f_i : S_i \to T_i$ inducing f in the limit (note that we do not assume any diagrams are Cartesian). Therefore, the existence of the left adjoint $f^{*,dR}$ follows immediately from Lemma 5.3.1.

Let us see that these objects map in the obvious way to non-holonomic objects. For $\alpha : i \to j$, let $\varphi_i : S \to S_i$, $\varphi_\alpha : S_j \to S_i$, $\psi_i : T \to T_i$, $\psi_\alpha : T_j \to T_i$ denote the structure maps.

Note that e.g. $D^*_{hol}(T) \to D^*(T)$ is continuous. Therefore, for $\mathcal{F} \in D^*_{hol}(T)$ and $\mathcal{G} \in D^*(S)$, we have:

$$\mathrm{Hom}_{D^*(T)}(f^{*,dR}(\mathcal{F}), \mathcal{G}) = \lim_i \lim_{\alpha: i \to j} \mathrm{Hom}_{D(T_i)}(\psi_{\alpha,*,dR} f_j^{*,dR} \varphi_{j,*,dR}(\mathcal{F}), \psi_{i,*,dR}(\mathcal{G})) =$$
$$\lim_i \mathrm{Hom}_{D(T_i)}(f_i^{*,dR} \varphi_{i,*,dR}(\mathcal{F}), \psi_{i,*,dR}(\mathcal{G})) = \mathrm{Hom}_{D(T_i)}(\varphi_{i,*,dR}(\mathcal{F}), f_{i,*,dR}\psi_{i,*,dR}(\mathcal{G})) =$$
$$\mathrm{Hom}_{D(T_i)}(\varphi_{i,*,dR}(\mathcal{F}), \varphi_{i,*,dR} f_{*,dR}(\mathcal{G})) = \mathrm{Hom}_{D^*(T)}(\mathcal{F}, f_{*,dR}(\mathcal{G}))$$

For f finite presentation, we can take placid presentations $S = \lim S_i$ and $T = \lim T_i$ as in Remark 4.2.5: by base-change, the upper-! functors are compatible with the shifted lower-$*$ functors expressing D^* as a limit (using placidity), so Lemma 5.3.1 again applies. The same argument as above treats maps to non-holonomic objects.

□

5.5. We also have the following observation.

Proposition 5.5.1. *If S is placid, then η_S identifies the subcategories $D^!_{hol}(S)$ and $D^*_{hol}(S)$.*

Proof. Suppose $\mathcal{F} \in D^!(S)$. We will show that $\mathcal{F} \in D^!_{hol}(S)$ if and only if $\eta_S(\mathcal{F}) \in D^*_{hol}(\mathcal{F})$.

Let $S = \lim_i S_i$ be a placid presentation of S and let $\alpha_i : S \to S_i$ denote the structure maps.

By definition, $\eta_S(\mathcal{F})$ is in $D^*_{hol}(\mathcal{F})$ if and only if $\alpha_{i,*,ren}(\mathcal{F}) \in D_{hol}(S_i)$ for every i. By (4.3.3) and Proposition 4.11.1, we have:

$$\mathcal{F} = \operatorname*{colim}_i \alpha_i^! \alpha_{i,*,ren}(\mathcal{F})$$

giving the result.

To see that for $\mathcal{F} = D^!_{hol}(S)$ we have $\alpha_{i,*,ren}(\mathcal{F}) \in D_{hol}(S_i)$, note that $D^!_{hol}(S)$ is tautologically generated under colimits by objects $\alpha_j^!(\mathcal{F}_j)$, for $\mathcal{F}_j \in D_{hol}(S_j)$. By filteredness of our indexing category, we can compute $\alpha_{i,*,ren}\alpha_j^!(\mathcal{F}_j)$ as a colimit of objects obtained by pushing and pulling along correspondences $S_i \leftarrow S_k \to S_j$ (coming from correspondences $i \to k \leftarrow j$ in the indexing category).

□

Corollary 5.5.2. *For $f : S \to T$ a morphism of placid schemes, the functors $f_{*,ren}$ and $f^{!,ren}$ preserve holonomic objects in $D^!$ and D^* respectively.*

6. D-modules on indschemes

6.1. In this section, we generalize the earlier material to treat the theory of D-modules on indschemes, and especially on placid indschemes.

The principal new feature is that for treating renormalized functors, we need a choice of *dimension theory*, which was only implicit in the discussion in the schemes case.

6.2. **Indschemes.** We say that $T \in \mathsf{PreStk}$ is a (classical) indscheme if $T = \operatorname{colim}_{i \in \mathcal{I}} T_i$ in PreStk where $\mathcal{I}$ is filtered, $T_i \in \mathsf{Sch}_{qcqs} \subseteq \mathsf{PreStk}$ and each structure map $T_i \to T_j$ is a closed embedding (recall that in this case $T \in \mathsf{Stk} \subseteq \mathsf{PreStk}$).

6.3. **Correspondences.** We say a morphism $f : T \to S$ of indschemes is *finitely presented* if f is schematic and its base-change by any scheme is a finitely presented morphism.

Exactly parallel to Propositions 3.8.1 and 3.21.1 one shows that $D^!$ and D^* upgrade (via Kan extensions) to functors $D^{!,enh}$ and $D^{*,enh}$ from the categories of indschemes under correspondences where the "right" (resp. "left") map is finitely presented.

For $f : S \to T$ finitely presented we have the corresponding functors $f_{*,!-dR} : D^!(S) \to D^!(T)$ and $f^{\mathsf{i}} : D^*(T) \to D^*(S)$. If f is additionally assumed proper or smooth, we again have the usual adjunctions.

6.4. **Reasonable indschemes.** The following definition is taken from [BD] §7.

Definition 6.4.1. A subscheme $S \subseteq T$ is a *reasonable subscheme* of T if S is a quasi-compact quasi-separated closed subscheme such that, for every closed subscheme S' of T containing S, the closed embedding $S \hookrightarrow S'$ is finitely presented.

T is a *reasonable* indscheme if T is the colimit of its reasonable subschemes.

Example 6.4.2. Every quasi-compact quasi-separated scheme is reasonable when regarded as an indscheme.

Example 6.4.3. Every indscheme of ind-finite type is reasonable.

Example 6.4.4. For an ind-pro finite set T, considered as an indscheme in the obvious way, a subset $S \subseteq T$ is reasonable if and only if it is compact and open in the ind-pro topology.

Terminology 6.4.5. Because of Example 6.4.4, we sometimes refer to reasonable subschemes as "compact open" subschemes. We especially use this terminology in the group setting, where we speak of compact open subgroups, meaning group subschemes that are reasonable as subschemes.

Lemma 6.4.6. *Suppose T is a reasonable indscheme and $f : S \to T$ a finitely presented morphism of indschemes. Then S is a reasonable indscheme, and for every reasonable subscheme $T' \subseteq T$, $f^{-1}(T') \subseteq S$ is a reasonable subscheme.*

Proof. Fix a reasonable subscheme $T' \subseteq T$. It suffices to show that $f^{-1}(T') \subseteq S$ is a reasonable subscheme.

First, suppose that $T' \subseteq T'' \subseteq T$ is a reasonable subscheme of T. We will show that $f^{-1}(T') \hookrightarrow f^{-1}(T'')$ is a finitely presented closed embedding.

Note that $f^{-1}(T') \to T'$ is finitely presented because f is, and similarly for T''. Moreover, $f^{-1}(T') \to T''$ is finitely presented, since it factors as $f^{-1}(T') \to T' \to T''$ with the latter morphism being finitely presented because T' is reasonable.

Therefore, since $f^{-1}(T') \to f^{-1}(T'')$ sits in the diagram:

$$f^{-1}(T') \to f^{-1}(T'') \to T''$$

with the composite morphism and the right morphism finitely presented, the morphism $f^{-1}(T') \to f^{-1}(T'')$ is finitely presented as well (the relevant "two out of three" principle appears in [Gro] Proposition 1.6.2).

To see that this suffices: suppose that $f^{-1}(T') \subseteq S' \subseteq T$ is closed subscheme. We can take T'' as above we $S' \to T$ factoring through T''. Therefore, we have:

$$f^{-1}(T') \subseteq S' \subseteq f^{-1}(T'').$$

That $f^{-1}(T') \to f^{-1}(T'')$ is finite presentation means that the ideal sheaf of $f^{-1}(T')$ is finitely generated over the structure sheaf of $f^{-1}(T'')$. Therefore, we see that it is finitely generated over the structure sheaf of S' as well, so that our closed embedding $f^{-1}(T') \subseteq S'$ is itself finitely presented.

∎

6.5. The key feature of reasonable indschemes is the following. Suppose $T = \mathrm{colim}_{i\in\mathcal{I}} T_i$ as in the definition.

Then every $\alpha : T_i \to T_j$ is a finitely presented closed embedding and therefore $\alpha^! : D^!(T_j) \to D^!(T_i)$ admits the left adjoint $\alpha_{*,!-dR}$ and $\alpha_{*,dR} : D^*(T_i) \to D^*(T_j)$ admits the right adjoint $\alpha^{¡}$.

Therefore, we have:

$$\begin{aligned} D^!(T) &= \operatorname*{colim}_{i\in\mathcal{I}} D^!(T_i) \\ D^*(T) &= \lim_{i\in\mathcal{I}^{op}} D^*(T_i) \end{aligned} \tag{6.5.1}$$

where on the left we use functors $\alpha_{*,!-dR}$ and on the right we use functors $\alpha^{¡}$.

We deduce that for T and S reasonable indschemes we have canonical equivalences:

$$D^!(T \times S) = D^!(T) \otimes D^!(S). \tag{6.5.2}$$

6.6. **Descent.** We say a morphism $f : T \to S$ of indschemes is an *h-covering* if its base-change by any affine scheme is an h-covering.

Proposition 6.6.1. *Let $f : S \to T$ be an h-covering of indschemes. Then the canonical functor:*

$$D^!(T) \to \lim_{[n]\in\mathbf{\Delta}} D^!(\mathrm{Cech}^n(S/T))$$

given by $!$-pullback is an equivalence.

Proof. This is obvious from Proposition 3.12.1: it just amounts to commuting limits with limits. More generally, it holds for any h-covering (in the above sense) of prestacks.

∎

Similarly, we have the following result under more restrictive hypotheses.

Proposition 6.6.2. *Let $f : S \to T$ be an h-covering of reasonable indschemes. Then the canonical functor:*

$$D^*(T) \to \lim_{[n]\in\mathbf{\Delta}} D^*(\mathrm{Cech}^n(S/T))$$

given by ¡-pullback is an equivalence.

Proof. As above, this follows from Proposition 3.23.1 by commuting limits with limits, using the presentation (6.5.1) of D^*.

∎

6.7. **Equivariant setting.** We now render the material of §3.17 and §3.24 to the indscheme setting.

Suppose that S is an indscheme and $\mathcal{G} \to S$ is a group indscheme over S.

Suppose P is an indscheme with a morphism $P \to S$ and an action of $\mathcal{G}$. We define the equivariant derived category $D^!(P)^{\mathcal{G}}$ as the limit of the diagram formed using (3.17.1):

$$D^!(P)^{\mathcal{G}} := \lim \left(D^!(P) \rightrightarrows D^!(\mathcal{G} \underset{S}{\times} P) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} D^!(\mathcal{G} \underset{S}{\times} \mathcal{G} \underset{S}{\times} P) \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \ldots \right)$$

Similarly, we define the coequivariant derived category by (3.24.1).

Now suppose that $\mathcal{P}_{\mathcal{G}} \to S$ is an indscheme with a $\mathcal{G}$-action as above and that $\mathcal{P}_{\mathcal{G}}$ is a $\mathcal{G}$-torsor in the sense that, for every closed subscheme S' of S, the fiber product $\mathcal{P}_{\mathcal{G}} \times_S S'$ is a $\mathcal{G} \times_S S'$-torsor in the sense of §3.17: after an fppf base-change in S', $\mathcal{P}_{\mathcal{G}} \times_S S' \to S'$ is $\mathcal{G}$-equivariantly isomorphic to $\mathcal{G}$.

Proposition 6.7.1. *The pullback functor:*

$$D^!(S) \to D^!(\mathcal{P}_{\mathcal{G}})^{\mathcal{G}}$$

is an equivalence.

The pushforward functor:

$$D^*(\mathcal{P}_{\mathcal{G}})_{\mathcal{G}} \to D^*(S)$$

is an equivalence if S is reasonable, and $\mathcal{G}$ is a union $\mathcal{G} = \cup \mathcal{G}_i$ where the $\mathcal{G}_i$ are closed group indschemes in $\mathcal{G}$ with the property that $\mathcal{G}_i \times_S S' \to \mathcal{G} \times_S S'$ is a reasonable subscheme for every reasonable subscheme $S' \subseteq S$.

Proof. For the first functor, we commute limits with limits to dévissage to the case where S is a quasi-compact quasi-separated scheme. Then the result follows as in Proposition 3.17.2: by Proposition 6.6.1 we reduce to the case of a trivial $\mathcal{G}$-bundle where it follows by using split simplicial objects.

The second functor is analyzed similarly: commuting colimits with colimits, we reduce to the case where S is a quasi-compact quasi-separated scheme.

Note that $\mathcal{P}_{\mathcal{G}}$ must be induced as a torsor from some $\mathcal{G}_i$-torsor for some i_0. Therefore, $\mathcal{P}_{\mathcal{G}}$ is reasonable: it is a union of the induced $\mathcal{G}_i$-torsors for $i \to i_0$, and these are obviously reasonable subschemes. Therefore, we can apply Proposition 6.6.2 to again reduce to the case of a trivial torsor. □

Remark 6.7.2. When our indschemes are reasonable, Example 3.17.1 translates verbatim to the present setting by using (6.5.2).

Remark 6.7.3. We will sometimes use the notational convention of Remark 3.17.3 in the above setting as well.

6.8. **Placidity.** We now give an indscheme analogue of the notion of placidity.

Definition 6.8.1. We say that $T \in \mathsf{IndSch}$ is a *placid indscheme* if T is reasonable and every reasonable subscheme of T is placid.

Remark 6.8.2. By Remark 4.2.5, we see that T is placid if and only if we can write $T = \operatorname{colim}_{i \in \mathcal{I}} T_i$ as in the definition of indscheme so that each T_i is placid and a reasonable subscheme of T.

Remark 6.8.3. By (6.5.1) and §4.3, for T placid the categories $D^!(T)$ and $D^*(T)$ are compactly generated and canonically dual.

The following is the indscheme analogue of Example 4.2.3.

Example 6.8.4. Suppose that S is a placid indscheme and $\mathcal{G} \to S$ is a group indscheme over S. Suppose moreover that for every closed subscheme S' of S the fiber product $\mathcal{G} \times_S S' \to S'$ is a group scheme that can be written as a projective limit under smooth maps of group schemes $\mathcal{G}_i$ smooth and affine over S'. Then $\mathcal{G}$ is a placid indscheme.

More generally, if $\mathcal{P}_{\mathcal{G}} \to S$ is a $\mathcal{G}$-torsor over S in the sense of §6.7 then $\mathcal{P}_{\mathcal{G}}$ is a placid indscheme. Indeed, we reduce to showing that if S as above is actually a placid scheme, then $\mathcal{P}_{\mathcal{G}} \to S$ is a placid morphism. But $\mathcal{P}_{\mathcal{G}}$ is the projective limit of the induced $\mathcal{G}_i$-torsors, giving the result.

6.9. **Fiber products.** We digress somewhat to give the following technical result, which we will need in [Ras].

Proposition 6.9.1. *Let $S_1 \to S_2$ and $T \to S_2$ be morphisms of indschemes.*

(1) If S_1 and S_2 are finite type schemes, then the canonical morphisms:

$$D^!(T) \underset{D(S_2)}{\otimes} D(S_1) \to D^!(T \underset{S_2}{\times} S_1)$$
$$D^*(T) \underset{D(S_2)}{\otimes} D(S_1) \to D^*(T \underset{S_2}{\times} S_1)$$

of ! and ¡-pullback respectively are equivalences.

(2) If S_1 is a placid indscheme and S_2 is a finite type scheme and T is an arbitrary indscheme, then:

$$D^!(T) \underset{D(S_2)}{\otimes} D^!(S_1) \to D^!(T \underset{S_2}{\times} S_1)$$

is an equivalence.

We will deduce Proposition 6.9.1 from the following two lemmas from the finite dimensional setting.

Lemma 6.9.2. *Let $S_1 \to S_2$ and $T \to S_2$ morphisms of finite type schemes, the canonical morphism:*

$$D(T) \underset{D(S_2)}{\otimes} D(S_1) \to D(T \underset{S_2}{\times} S_1)$$

is an equivalence.

This result is well-known, and follows easily e.g. from the 1-affineness of the prestacks S_{dR} for S finite type: see [Gai2] for the terminology and for this result.

Lemma 6.9.3. *For $f : S \to T$ a morphism of finite type schemes, $D(S)$ is dualizable as a $D(T)$-module category.*

Proof. We will show that $D(S)$ is self-dual as a $D(T)$-module category.

Let Δ_f denote the diagonal embedding $S \to S \times_T S$.

We have the evaluation:

$$D(S) \underset{D(T)}{\otimes} D(S) \simeq D(S \underset{T}{\times} S) \xrightarrow{\Delta_f^!} D(S) \xrightarrow{f_{*,dR}} D(T)$$

and coevaluation:

$$D(T) \xrightarrow{f^!} D(S) \xrightarrow{\Delta_{f,*,dR}} D(S \underset{T}{\times} S) \simeq D(S) \underset{D(T)}{\otimes} D(S).$$

One readily checks by base-change that these define a duality datum as required.

□

Proof of Proposition 6.9.1. For (1): the category $D(S_1)$ is dualizable as a $D(S_2)$-module category. Therefore, tensoring over $D(S_2)$ with $D(S_1)$ commutes with limits of categories. Applying the definition of $D^!$, the result then immediately follows from the finite type case.

Similarly, to prove (2) it suffices to show that $D^!(S_1)$ is dualizable as a $D(S_2)$-module category. Using the methods of [Gai1], this follows from the finite type case combined with (6.5.1).

□

6.10. **Dimension theories.** Let T be a placid indscheme. We use the notation of §4.6 here.

Definition 6.10.1. A *dimension theory* $\tau = \tau^T$ on T is a rule that assigns to every reasonable subscheme S of T a locally constant function:

$$\tau_S : S \to \mathbb{Z}$$

such that for any pair of reasonable subschemes $S' \subseteq S \subseteq T$ we have:

$$\tau_{S'} = \tau_S|_{S'} + \dim_{S'/S}. \tag{6.10.1}$$

Example 6.10.2. By Remark 4.6.3, every placid scheme T carries a canonical dimension theory normalized by the condition that $\dim_T$ be identically zero.

Example 6.10.3. Let T be an indscheme of ind-finite type. Then a reasonable subscheme of T is just a closed finite type subscheme S, and the rule $\tau_S := \dim_S$ is a dimension theory on T.

Remark 6.10.4. If $T = \cup_i S_i$ is written as a union of reasonable subschemes, it suffices to define the τ_{S_i} satisfying the compatibility (6.10.1). Indeed, this again follows from Remark 4.6.3.

Example 6.10.5. By Remark 6.10.4, the product $T_1 \times T_2$ of indschemes T_i equipped with dimension theories τ^{T_i} inherits a canonical dimension theory $\tau^{T_1 \times T_2}$ such that, for every pair $S_i \subseteq T_i$, $i = 1, 2$ of reasonable subschemes, we have:

$$\tau^{T_1 \times T_2}_{S_1 \times S_2} = p_1^*(\tau^{T_1}_{S_1}) + p_2^*(\tau^{T_2}_{S_2})$$

with p_i^* denoting the restriction of a function along the projection.

Remark 6.10.6. Dimension theories are étale local.

Remark 6.10.7. For T a group indscheme, the choice of dimension theory may be seen as analogous to the choice of a Haar measure in the p-adic setting.

Remark 6.10.8. See [Dri] for relevant material on dimension theories. In particular, questions of existence (and non-existence) are treated in some detail.

6.11. We now give something of a classification of the set of dimension theories.

Definition 6.11.1. A *locally constant function* $T \to \mathbb{Z}$ on an indscheme T is a morphism of indschemes $T \to \mathbb{Z} = \coprod_{n \in \mathbb{Z}} \mathrm{Spec}(k)$.

Remark 6.11.2. For $T = \mathrm{colim}\, T_i$, a locally constant function on T is equivalent to a compatible system of locally constant functions on the T_i. As in Remark 3.9.4, we can make sense of $\pi_0(T)$ as an ind-profinite set, and a locally constant function on T is equivalent to a continuous function $\pi_0(T) \to \mathbb{Z}$, with π_0 equipped with its natural topology as an ind-profinite set.

Clearly locally constant functions form an abelian group under addition. Moreover, they obviously act on the set of dimension theories on T: given $d : T \to \mathbb{Z}$ and τ a dimension theory on T, we obtain a new dimension theory $d + \tau$ with $(d+\tau)_S = d|_S + \tau_S$ for every reasonable subscheme S of T.

Proposition 6.11.3. *Suppose that S is a placid indscheme that admits a dimension theory. Then the set of dimension theories for S is a torsor for the set of locally constant functions $S \to \mathbb{Z}$, i.e., the above action of locally constant functions on dimension theories is a simply transitive action.*

Proof. The difference between two dimension theories obviously defines a locally constant function on S.

□

6.12. The following construction of dimension theories is useful in many situations.

Definition 6.12.1. A morphism $f : T \to S$ of placid indschemes is *healthy* if there exists a reasonable subscheme $S' \subseteq S$ such that:

(1) The inverse image of any closed subscheme $S' \subseteq S'' \subseteq S$ is a reasonable subscheme of T.
(2) For every closed subscheme $S' \subseteq S'' \subseteq S$, we have:

$$\dim_{T'/T''} = f'^{,*}(\dim_{S'/S''})$$

with $f' : T' \to S'$ the fiber product of f along S' and T'' the fiber along S''.

We say a subscheme $S' \subseteq S$ is f-healthy if it is reasonable and satisfies the above conditions (so f is healthy if and only if there exists an f-healthy subscheme of S).

Example 6.12.2. Every morphism $f : T \to S$ of placid schemes is healthy: S itself is f-healthy.

Counterexample 6.12.3. For $n \geqslant 0$, let S_n be the union of a line, a plane, up to an affine n-space all glued together along 0. Let $S = \mathrm{colim}\, S_n$. Let T_n be the union of n (ordered) lines glued along 0, mapping to S_n by embedding the rth irreducible component into $\mathbb{A}^r$ as a line into a vector space. Let $T = \mathrm{colim}_n\, T_n$. Then the resulting map $T \to S$ is not healthy.

Example 6.12.4. In §6.17, we will give a definition of placid morphism of placid indschemes such that every placid morphism is healthy.

Remark 6.12.5. Any reasonable subscheme containing an f-healthy subscheme is itself f-healthy.
In particular, we see that given two choices S'_1, S'_2 of f-healthy subschemes of S, there is always a third S'_3 containing both.

Our key use of this definition is the following construction.

Construction 6.12.6. For $f : T \to S$ a healthy morphism of placid indschemes, any dimension theory τ^S on S induces a unique dimension theory τ^T on T such that for any f-healthy reasonable subscheme $S' \subseteq S$, we have $\tau^T_{T'} = f^{',*}(\tau^S_{S'})$ for $f' : T' \to S'$ the base-change of f along $S' \hookrightarrow S$.

Indeed, that this construction can be performed follows immediately from Remarks 6.10.4 and 6.12.5.

Remark 6.12.7. Healthy morphisms are obviously preserved under compositions, and Construction 6.12.6 is obviously compatible with compositions.

6.13. As §6.12 generalizes Example 6.10.2, we now generalize Example 6.10.3.

We say a morphism $f : T \to S$ of reasonable indschemes is *ind-finitely presented* if $T = \operatorname{colim} T_i$ with each $T_i \to T$ a reasonable subscheme such that $T_i \to S$ factors through a reasonable subscheme S_i of S with $T_i \to S_i$ finite presentation.

We claim under this hypothesis that T inherits a canonical dimension theory τ^T from a dimension theory τ^S of S.

Indeed, for $T' \subseteq T$ a reasonable subscheme, the morphism $T' \to S$ factors through some reasonable subscheme $S' \subseteq S$, and $f' : T' \to S'$ is finite presentation by assumption. We take:

$$\tau^T_{T'} := \dim_{T'/S'} + f^{',*}(\tau^S_{S'}).$$

To simultaneously show that τ^T is well-defined and actually defines a dimension theory, take $T' \xhookrightarrow{i_1} T'' \subseteq T$ reasonable subschemes mapping via f' and f'' to reasonable subschemes $S' \xhookrightarrow{i_2} S'' \subseteq S$ respectively, and compute:

$$\begin{aligned}\tau^T_{T'} - i_1^*(\tau^T_{T''}) &:= \dim_{T'/S'} - i_1^*(\dim_{T''/S''}) + f^{',*}(\tau^S_{S'}) - f^{',*} i_2^*(\tau^S_{S''}) = \\ &= -f^{',*}(\dim_{S'/S''}) + \dim_{T'/T''} + f^{',*}(\dim_{S'/S''}) = \dim_{T'/T''}\end{aligned}$$

as desired, where we have used the expansions:

$$\begin{aligned}\dim_{T'/S'} &= \dim_{T'/S''} - f^{',*}(\dim_{S'/S''}) \\ i_1^*(\dim_{T''/S''}) &= \dim_{T'/S''} - \dim_{T'/T''}\end{aligned}$$

of (4.6.1).

Example 6.13.1. If T is a reasonable subscheme of a placid indscheme S, then the embedding $T \hookrightarrow S$ satisfies the hypotheses of this section. If τ^S is a dimension theory on S, the induced dimension theory τ^T on T constructed above is the "obvious" one, which to a reasonable subscheme $T' \subseteq T$ assigns the function $\tau^T_{T'} := \tau^S_{T'}$.

Warning 6.13.2. If $f : T \to S$ is a finitely presented morphism of placid schemes, the pullback constructed above of the dimension theory τ^S given in Example 6.10.2 is *not* (generally) the dimension theory on T constructed in Example 6.10.2: they differ by $\dim_{T/S}$.

6.14. **Renormalization.** Let T be a placid indscheme and let τ be a dimension theory on T. We will define the "τ-renormalized dualizing sheaf" $\omega^\tau_T \in D^*(T)$ below.

Let $i : S \hookrightarrow T$ be a reasonable subscheme. We formally define:

$$\text{“}i^!(\omega^\tau_T)\text{”} := \omega^{ren}_S[2\tau_S] \in D^*(S).$$

Suppose that for S as above $\iota : S' \to S$ is a reasonable subscheme (equivalently: of S or of T, or equivalently ι is a finitely presented closed embedding). Then we have canonical isomorphisms:

$$\iota^{¡}(\text{“}i^{¡}(\omega_T^\tau)\text{”}) = \iota^{¡}(\omega_S^{ren})[2\tau_S] = \iota^{!,ren}(\omega_S^{ren})[2\cdot(\tau_S+\dim_{S'/S})] = (\omega_{S'}^{ren})[2\cdot(\tau_S+\dim_{S'/S})] =: \text{“}(i\circ\iota)^{¡}(\omega_T^\tau)\text{”}$$

where the second equality is Proposition 4.12.1 and the third equality is (4.9.1).

These identifications are readily made homotopy compatible and therefore define ω_T^τ in $D^*(T)$ so that $\iota^{¡}(\omega_T^\tau) = \text{“}\iota^{¡}(\omega_T^\tau)\text{”}$ for all $\iota : S \hookrightarrow T$ as above.

6.15. Let T and τ be as in §6.14.

Let $\eta_T^\tau : D^!(T) \to D^*(T)$ denote the functor of action on ω_T^τ. We immediately deduce from Proposition 4.8.1 that η_T^τ is an equivalence.

6.16. Let $f : T \to S$ a morphism of placid indschemes equipped with dimension theories τ^T and τ^S.

Then as in §4.9 we obtain functors $f_{*,\tau} : D^!(T) \to D^!(S)$ and $f^{!,\tau} : D^!(S) \to D^!(T)$ so that we have the commuting diagram:

$$\begin{tikzcd}
D^!(T) \arrow[r, "f_{*,\tau}"] \arrow[d, "\eta_T^{\tau^T}", "\simeq"'] & D^!(S) \arrow[d, "\eta_S^{\tau^S}", "\simeq"'] \\
D^*(T) \arrow[r, "f_{*,dR}"] & D^*(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
D^*(S) \arrow[r, "f^{!,\tau}"] \arrow[d, "\eta_S^{\tau^S}", "\simeq"'] & D^*(T) \arrow[d, "\eta_T^{\tau^T}", "\simeq"'] \\
D^!(S) \arrow[r, "f^!"] & D^!(T).
\end{tikzcd}$$

Example 6.16.1. If $f : T \to S$ is a map of placid schemes, each equipped with their canonical dimension theories (see Example 6.10.2), then the functors constructed above are the renormalized functors of §4.9.

Notation 6.16.2. In light of Example 6.16.1, when the relative dimension theory τ is implicit we denote the functors $f_{\tau,ren}$ and $f^{!,ren}$ above simply by $f_{*,ren}$ and $f^{!,ren}$.

Fixing a map $f : T \to S$ of placid indschemes, we obtain a pullback map for locally constant functions and therefore an induced diagonal action of locally constant functions on S on the set of pairs (τ^T, τ^S) of dimension theories for T and S:

$$\Big(d : S \to \mathbb{Z}, (\tau^T, \tau^S)\Big) \mapsto (\tau^T + d\circ f, \tau^S + d).$$

Definition 6.16.3. A *relative dimension theory* for T and S is an equivalence class of pairs (τ^T, τ^S) of dimension theories for T and for S modulo the above action of locally constant functions on S.

Clearly the functors $f^{!,ren}$ and $f_{*,ren}$ only depend on the relative dimension theory defined by the pair (τ^T, τ^S).

Example 6.16.4. Let $f : T \to S$ be an ind-finitely presented morphism of placid indschemes with S equipped with dimension theory. By §6.13, we obtain a dimension theory on T and therefore a relative dimension theory for f.

As in Examples 4.9.1 and 4.9.4, the functors $f_{*,ren}$ and $f^{!,ren}$ canonically identify with $f_{*,!-dR}$ and $f^{¡}$ respectively.[7]

---

[7]Unlike Example 4.9.1, there are no cohomological shifts in this formula. There is no real discrepancy because of Warning 6.13.2.

6.17. Next, we extend the notion of placid morphism from §4.10 to the indscheme framework.

Definition 6.17.1. A morphism $f : T \to S$ of placid indschemes is *placid* if there exists a reasonable subscheme $S' \subseteq S$ such that:

(1) The inverse image of any closed subscheme $S' \subseteq S'' \subseteq S$ is a reasonable subscheme of T.
(2) For every closed subscheme $S' \subseteq S'' \subseteq S$, the morphism $T'' := S'' \times_S T \to S''$ is placid.

Remark 6.17.2. By Corollary 4.10.7, we immediately see that any placid morphism is healthy.

Example 6.17.3. If f is finitely presented, smooth and surjective on geometric points, then f is placid.

Example 6.17.4. Suppose that S is a placid indscheme and $\mathcal{G} \to S$ is a group indscheme satisfying the hypotheses of Example 6.8.4. Suppose $\mathcal{P}_{\mathcal{G}} \to S$ is a $\mathcal{G}$-torsor on S. Then $\mathcal{P}_{\mathcal{G}} \to S$ is placid. In particular, this morphism is healthy. Indeed, this follows by Example 6.8.4.

6.18. We have the following indschematic version of Proposition 4.11.1.

Proposition 6.18.1. *Let $f : T \to S$ be placid and suppose that S is equipped with a dimension theory. By Construction 6.12.6, this choice induces a dimension theory on T.*

(1) The functors:

$$f_{*,dR} : D^*(T) \to D^*(S)$$
$$f_{*,ren} : D^!(T) \to D^!(S)$$

admit left adjoints. Moreover, these left adjoints are canonically identified with $f^{!,ren}$ and $f^!$ respectively.

(2) Suppose that we are given a Cartesian diagram:

$$\begin{tikzcd} T' \arrow[r, "\varphi"] \arrow[d, "\psi"] & S' \arrow[d, "g"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

of placid indschemes with f placid and g finitely presented. Then φ is also placid, and the natural transformations:

$$f^{!,ren} g_{*,dR} \to \psi_{*,dR} \varphi^{!,ren}$$
$$f^! g_{*,ren} \to \psi_{*,ren} \varphi^!$$

are equivalences. Here we have equipped S' and T' with the dimension theories of §6.13 using the finitely presented maps g and ψ.

Proof. It suffices to prove each of these statements in the $D^!$-setting.

Then (1) then follows immediately Proposition 4.11.1 (say, by applying a simplified version of Lemma 5.3.1). So it remains to show (2).

Let S_0 be a reasonable subscheme of S satisfying the hypotheses of the definition of placid morphism for f. Then combining Lemmas 4.10.6 and 6.4.6., we find that its pullback to S' satisfies the same conditions for φ. In particular, we see that φ is placid.

We form the commutative cube:

$$\begin{tikzcd}
T_0' \arrow[rr, "\varphi_0"] \arrow[dr, "\iota'"] \arrow[dd, "\psi_0"] & & S_0' \arrow[dr, "i'"] \arrow[dd, "g_0"] & \\
& T' \arrow[rr] \arrow[dd] & & S' \arrow[dd] \\
T_0 \arrow[rr, "f_0"] \arrow[dr, "\iota"] & & S_0 \arrow[dr, "i"] & \\
& T \arrow[rr] & & S
\end{tikzcd}$$

where all faces are taken to be Cartesian squares. We equip these new schemes with the dimension theories obtained using Example 6.13.1.

Note that the dimension theories on the back square are not (necessarily) the canonical ones on placid schemes from Example 6.10.2.

Still, the relative dimension theories of T_0/S_0 and T_0'/S_0' are the same, so renormalized functors for these dimension theories coincide with those of §4.9.

Moreover, the dimension theories for S_0'/S_0 differs from the "canonical" one by $\dim_{S_0'/S_0}$, and similarly for T_0'/T_0. Note that this error term $\dim_{S_0'/S_0}$ pulls back to T_0' as $\dim_{T_0'/T_0}$ by Corollary 4.10.7.

We will use the notation e.g. $g_{0,*,ren}$ here for the renormalized functor corresponding to our given dimension theory, therefore differing by cohomological shifts from the so-named functor in §4.9.

In this notation, we see from the above discussion that we can apply Proposition 4.11.1 to deduce:

$$f_0^! g_{0,*,ren} \xrightarrow{\simeq} \psi_{0,*,ren}\varphi_0^!.$$

Because $D^!(S')$ is generated under colimits by D-modules of the form $i'_{*,!-dR}(\mathcal{F}) = i'_{*,ren}(\mathcal{F})$ as we increase S_0, it suffices to show that the natural transformation:

$$f^! g_{*,ren} i'_{*,ren} \to \psi_{*,ren}\varphi^! i'_{*,ren}$$

is an equivalence.

Similarly, since T is a union of the schemes T_0 as S_0 varies, it suffices to show that the natural transformation:

$$\iota^! f^! g_{*,ren} i'_{*,ren} \to \iota^! \psi_{*,ren}\varphi^! i'_{*,ren}$$

is an equivalence.

Now we compute:

$$\iota^! f^! g_{*,ren} i'_{*,ren} = f_0^! i^! i_{*,ren} g_{0,*,ren} = f_0^! g_{0,*,ren} \xrightarrow{\simeq} \psi_{0,*,ren}\varphi_0^! = \iota^! \iota_{*,ren}\psi_{0,*,ren}\varphi_0^! =$$
$$\iota^! \psi_{*,ren}\iota'_{*,ren}\varphi_0^! i'^{,!} i'_{*,ren} = \iota^! \psi_{*,ren}\iota'_{*,ren}\iota'^{,!}\varphi^! i'_{*,ren} = \iota^! \psi_{*,ren}\varphi^! i'_{*,ren}$$

as desired.

□

6.19. **Holonomic D-modules.** For T an indscheme, we define $D^!_{hol}(S)$ and $D^*_{hol}(S)$ by Kan extension, as in the definition of $D^!$ and D^*.

We have canonical forgetful functors:

$$D^!_{hol}(S) \to D^!(S) \text{ and } D^*_{hol}(S) \to D^*(S)$$

and compatible upper-! and lower-* functoriality, respectively. For S reasonable (resp. placid), $D^*_{hol}(S) \to D^*(S)$ (resp. $D^!_{hol}(S) \to D^!_{hol}(S)$) is fully-faithful.

Definition 6.19.1. A morphism $f : S \to T$ of reasonable indschemes is a *reasonable morphism* if there exists cofinal system $T = \cup T_i$ of reasonable subschemes such that $f^{-1}(T_i)$ is a reasonable subscheme in S (in particular: f is schematic).

Proposition 6.19.2. *If $f : S \to T$ is a reasonable morphism of reasonable indschemes, then the partially-defined left adjoint $f^{*,dR}$ to $f_{*,dR}$ is defined on holonomic objects in $D^*(T)$.*

Similarly, if f is a morphism of ind-finite presentation of placid indschemes, then the partially-defined left adjoint $f_!$ to $f^! : D^!(T) \to D^!(S)$ is defined on holonomic objects.

Proof. Follows from the combination of Proposition 5.2.1 and Lemma 5.3.1 by the same argument as in Proposition 5.2.1.

□

We have the following counterparts to Proposition 5.5.1 and its Corollary 5.5.2, proved by the same arguments.

Proposition 6.19.3. *For S a placid indscheme with a dimension theory τ, η^τ_S identifies $D^!_{hol}(S)$ with $D^*_{hol}(S)$.*

Corollary 6.19.4. *For S and T placid indschemes with a dimension theories and $f : S \to T$ a morphism, $f_{*,ren}$ and $f^{!,ren}$ preserve holonomic objects in $D^!$ and D^* respectively.*

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