CONJECTURES OF ARTHUR AND RAMANUJAN FOR UNRAMIFIED AUTOMORPHIC FORMS: ANNOUNCEMENT

SAM RASKIN

Abstract. This text is a contribution to the 2026 ICM Proceedings. We announce forthcoming work with Gaitsgory and Lafforgue using recent progress on geometric Langlands to understand the spectral theory of unramified cusp forms in positive characteristic, building on recent progress in geometric Langlands. Our main result is the Arthur-Ramanujan conjecture on the magnitudes of Hecke eigenvalues. We also connect Arthur's multiplicity formula to geometric Langlands.

This text is a summary of forthcoming joint work with Dennis Gaitsgory and Vincent Lafforgue.

1. Introduction

In the early days of the Langlands program, Langlands [Lan] showed that the (generalized) Ramanujan conjecture – a problem purely on the spectral theory of automorphic forms – followed from his general principle of functoriality. Ever since, the Ramanujan conjecture has been a benchmark of progress in the subject, serving to measure how advances in the Langlands program have improved our understanding of automorphic forms.

Beyond GL_n, the ultimate form of the Ramanujan conjecture is suggested in Arthur's influential work [Art].

This note is an announcement of forthcoming work on the Arthur-Ramanujan conjecture for unramified automorphic forms over function fields. Here we build on previous work [GR3] on automorphic forms over function fields, which itself grew from recent works [GR1, ABCCFGLRR, CCFGLRR, ABCFGLRR, GR2] establishing the geometric Langlands conjecture in characteristic 0 and works [AGKRRV1, AGKRRV2, AGKRRV3, GR3] developing a geometric Langlands program in characteristic $p > 0$ with rich connections to the arithmetic Langlands program.

A major part of the work under discussion is the development of a *categorified*[1] version of Arthur's multiplicity formula, albeit in the ℓ-adic context. This analysis makes the previous works [Zhu, AGKRRV1] on the spectral decomposition of the space of unramified automorphic forms more explicit. This material also clarifies the relationship between the geometric Arthur parameters of [AG] and the actual Arthur parameters of [Art].

We highlight one other perspective on this work. Recent years have seen geometric Langlands influencing arithmetic Langlands, cf. [Laf2, Zhu, FS, AGKRRV1], with the style of question and conjecture in the former subject finding precise counterparts in arithmetic. In particular, the new perspectives offer an alternative to Langlands functoriality, allowing one to work with a single reductive group at a time. The present note (and the work it discusses) can also be considered as

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[1]Though we are only replacing numbers with vector spaces.

a test of this newer perspective, evaluating its applicability to classical problems in the spectral theory of automorphic forms.

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1.1. **Background on automorphic forms.**

1.1.1. In what follows, we fix $X_\circ/\mathbf{F}_q$ a geometrically connected smooth projective curve over a finite field. We let $k = \overline{\mathbf{F}}_q$ and write X for the base-change of $X_\circ$ to k.

We let G be a split reductive[2] group and let $\check{G}$ denote its Langlands dual group as considered over $\mathbf{C}$. We let B denote a Borel of G with radical N and Cartan $T = B/N$. We let Z_G (resp. $Z_{\check{G}}$) denote the center of G (resp. $\check{G}$). For a disconnected group H, we let H° denote its neutral connected component.

We let Λ (resp. $\check{\Lambda}$) denote the coweight (resp. weight) lattice of G. We let $\rho \in \frac{1}{2}\Lambda$ denote the half sum of positive coroots.

Throughout, we assume that the (mild) hypotheses of [AGKRRV1, §14.4.1, D.1.1] on the characteristic of our ground field are satisfied.

1.1.2. *G-bundles.* With apologies, we discuss some technical information about twists. The reader might skip this material at first pass.

The simple case is when there is a T-bundle on $X_\circ$, which we abusively denote as $\rho(\Omega)$, with the property that the induced line bundle $\check{\alpha}_i(\rho(\Omega)) \simeq \Omega_{X_\circ}$ for any simple root $\check{\alpha}_i : T \to \mathbf{G}_m$. For example, if $X_\circ$ admits a square root of its canonical bundle (as is always possible over k), we may take $\rho(\Omega) := 2\rho(\Omega_{X_\circ}^{\frac{1}{2}})$. Alternatively, if the center of G is connected, we may take $\rho(\Omega) := \tilde{\rho}(\Omega_X^1)$ where $\tilde{\rho}$ is a sum of fundamental coweights.

In the above setup, we write Bun_G for the moduli stack of G-bundles on X, which is naturally defined over $\mathbf{F}_q$. We abuse notation slightly in writing $\mathrm{Bun}_G(\mathbf{F}_q)$ for the groupoid of G-bundles on $X_\circ$. The above assumptions ensure that $\rho(\Omega_X) \in \mathrm{Bun}_T(\overline{\mathbf{F}}_q)$ is naturally defined over $\mathbf{F}_q$.

In general, the map $2\rho : \mathbf{G}_m \to T$ induces a map $2\rho|_{\mu_2} : \mu_2 \to Z_G$. We let

$$\sqrt{\Omega} \in \mathrm{Bun}_{\mathbf{B}\mu_2} := \mathcal{M}aps(X, \mathbf{B}^2\mu_2)$$

[2]Our convention in this note is that *reductive* groups are by assumption connected, while we use the term *linearly reductive* group to refer to possibly disconnected reductive groups.

as the image of $\Omega^1_X \in \mathrm{Bun}_{\mathbf{G}_m}$ under the boundary map

$$\mathrm{Bun}_{\mathbf{G}_m} \to \mathrm{Bun}_{\mathbf{B}\mu_2}$$

corresponding to the extension $\mu_2 \to \mathbf{G}_m \xrightarrow{t \mapsto t^2} \mathbf{G}_m$.

We then let Bun_G denote the fiber product:

$$\begin{tikzcd} \mathrm{Bun}_G \arrow[rr] \arrow[d] & & \mathrm{Bun}_{G^{\mathrm{ad}}} \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "\sqrt{\Omega}"] & \mathrm{Bun}_{\mathbf{B}\mu_2} \arrow[r, "2\rho|_{\mu_2}"] & \mathrm{Bun}_{\mathbf{B}Z_G} \end{tikzcd}$$

Expressed in this way, Bun_G is naturally defined over $\mathbf{F}_q$, and we understand $\mathrm{Bun}_G(\mathbf{F}_q)$ accordingly. Working over k, the map $\sqrt{\Omega}$ is non-canonically isomorphic to the trivial point of $\mathrm{Bun}_{\mathbf{B}\mu_2}$, so the only possible difference is with $\mathbf{F}_q$-rational structures.

We understand e.g. $\mathrm{Bun}_B(\mathbf{F}_q)$ and $\mathrm{Bun}_T(\mathbf{F}_q)$ similarly, observing that Z_G lies in the center of both and allows us to perform the same constructions. Note that $\mathrm{Bun}_T(\mathbf{F}_q)$ by design has a canonical point $\rho(\Omega)$.

1.1.3. *Automorphic forms.* Our object of study is $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$, the space of functions $f : \mathrm{Bun}_G(\mathbf{F}_q) \to \mathbf{C}$ with finite support.

In the case when G has infinite center, it is convenient to fix a character

$$\chi : \mathrm{Bun}_{Z^\circ}(\mathbf{F}_q) \to \mathbf{C}^\times$$

for Z° the identity component of the center $Z = Z_G$ of G, which induces a homomorphism $\mathcal{A}ut^{\mathrm{unr}}_{Z^\circ,c} \to \mathbf{C}$ from the group algebra of $\mathrm{Bun}_{Z^\circ}(\mathbf{F}_q)$. In this case, we let $\mathcal{A}ut^{\mathrm{unr}}_{G,c,\chi}$ denote

$$\mathcal{A}ut^{\mathrm{unr}}_{G,c} \underset{\mathcal{A}ut^{\mathrm{unr}}_{Z^\circ,c}}{\otimes} \mathbf{C},$$

the space of unramified automorphic forms with nebentypus χ.

1.1.4. Recall that connected components of Bun_{Z° are indexed by Λ_{Z°, the coweight lattice of Z°. Moreover, each component contains an $\mathbf{F}_q$-rational point, while the $\mathbf{F}_q$-points of the identity component are a finite abelian group.

Therefore, the homomorphism

$$\mathrm{Bun}_{Z^\circ}(\mathbf{F}_q) \xrightarrow{\chi} \mathbf{C}^\times \xrightarrow{|\cdot|} \mathbf{R}^{>0}$$

factors uniquely through a map

$$\mathrm{Bun}_{Z^\circ}(\mathbf{F}_q) \twoheadrightarrow \Lambda_{Z^\circ} \dashrightarrow \mathbf{R}^{>0}.$$

Let G^{ab} be the abelianization of G. Because $\Lambda_{Z^\circ} \to \Lambda_{G^{\mathrm{ab}}}$ is an isogeny and $\mathbf{R}^{>0}$ is uniquely divisible, the above is equivalent to a homomorphism $\mu_{\chi,0} : \Lambda_{G^{\mathrm{ab}}} \to \mathbf{R}^{>0}$; we let μ_χ denote the resulting homomorphism $\Lambda_G \to \Lambda_{G^{\mathrm{ab}}} \to \mathbf{R}^{>0}$.

We can encode $\mu_{\chi,0}$ as an element of

$$\check{\Lambda}_{G^{\mathrm{ab}}} \otimes \mathbf{R}^{>0} \subseteq \check{\Lambda}_{G^{\mathrm{ab}}} \otimes \mathbf{C}^\times = Z^\circ_{\check{G}}(\mathbf{C})$$

i.e., as a positive element of the torus $Z^\circ_{\check{G}}$. We denote the resulting element by z_χ.

Example 1.1.1. χ is a finite order character if and only if $z_\chi = 1$.

1.1.5. Recall that there is a subspace $\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp}} \subseteq \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ of *cuspidal* automorphic forms. This space consists of those automorphic forms whose constant terms vanish for each parabolic $P \subsetneq G$.

We define $\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$ as

$$\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp}} \underset{\mathcal{A}ut^{\mathrm{unr}}_{Z^\circ,c}}{\otimes} \mathbf{C}.$$

It is well-known that $\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$ is finite-dimensional.

Remark 1.1.2. We remind that for function fields, for any of the other usual definitions of automorphic form (e.g., L^2, etc.), cuspidal automorphic forms are always compactly (i.e., finitely) supported mod center. Therefore, the above definition coincides with any other definition of unramified cuspidal automorphic form with fixed central character.

1.1.6. Recall that for each irreducible representation $V = V^\lambda$ of $\check{G}$ and each closed point $x \in X_\circ$, there is a corresponding Hecke operator

$$T_{V,x} : \mathcal{A}ut^{\mathrm{unr}}_{G,c} \to \mathcal{A}ut^{\mathrm{unr}}_{G,c}.$$

More generally, by the Satake isomorphism, there is an action of the complexified representation ring

$$K_0(\mathsf{Rep}(\check{G}))_{\mathbf{C}} \simeq \Gamma(\check{G}//\check{G}, \mathcal{O})$$

on $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$.

Example 1.1.3. If $G = \check{G} = GL_n$ and $V = \Lambda^i V_{\mathrm{std}}$ is the ith exterior power of the standard representation, then

$$(T_{V,x} f)(\mathcal{E}) = q_x^{-\frac{i(n-i)}{2}} \sum_{\substack{\mathcal{E} \subseteq \mathcal{E}' \subseteq \mathcal{E}(x) \\ \dim \mathcal{E}'/\mathcal{E} = i}} f(\mathcal{E}')$$

for $f \in \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ and q_x the order of the residue field at x. Informally, Hecke operators are like Laplacians, weighted sums of neighboring values, but the notion of *neighbor* depends on x and λ.

Remark 1.1.4. When we speak of the *spectral theory* of automorphic forms, we roughly mean that we wish to understand (in whatever sense) the eigenvalues of Hecke operators acting on the space of automorphic forms, as well as the multiplicities of the eigenspaces.

These Hecke operators descend to $\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$. We say $f \in \mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$ is a *Hecke eigenfunction* if it is an eigenvector for all operators $T_{V,x}$ (in particular, $f \neq 0$).

For f a Hecke eigenfunction, the Satake isomorphism determines for each fixed x a point $\gamma_{x,f} \in \check{G}//\check{G}(\mathbf{C})$, the *Hecke eigenvalue* of f at x.

1.2. **The Ramanujan conjecture.**

1.2.1. Our first main result recovers a form of Arthur's correction to the Ramanujan conjecture. This involves the so-called *Arthur* SL_2, which requires some additional notation.

Let $e \in \check{\mathfrak{g}}$ be a nilpotent element and let $i_e : SL_2 \to \check{G}$ be its Jacobson-Morozov SL_2.

We let $H_e \subseteq \check{G}$ denote the centralizer of i_e, which we remind is a linearly reductive group. We let $\lambda_e : \mathbf{G}_m \subseteq SL_2 \to \check{G}$ denote the underlying character. We let $\check{M}_e$ be the centralizer of λ_e.

For any representation V of $\check{G}$, we let $V = \oplus_{n \in \mathbf{Z}} V_n$ be the grading of V induced by λ_e.

1.2.2. We now have:

Theorem A. *Let $f \in \mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$ be a cuspidal Hecke eigenform. Then there is a nilpotent element $e \in \check{\mathfrak{g}}$, unique up to conjugacy, with the following property:*

(AR_{e,z_χ}) *For any closed point $x \in X_\circ$ with residue field of order q_x, the Hecke eigenvalue $\gamma_{x,f}$ (cf. §1.1.6) admits a lift $\tilde{\gamma}_{x,f}$ to $\lambda_e(\sqrt{q}) \cdot H_e(\mathbf{C}) \subseteq \check{M}_e(\mathbf{C})$ so that for every irreducible highest weight representation V^λ of $\check{G}$, any eigenvalue η of $\tilde{\gamma}_{x,f}$ acting on V^λ_n (notation as in §1.2.1) satisfies*

$$|\eta| = q_x^{\frac{n}{2}} \cdot \lambda(z_\chi)^{\deg(x)}.$$

Remark 1.2.1. By tensoring with characters of $\mathrm{Bun}_{G^{\mathrm{ab}}}(\mathbf{F}_q)$, we can reduce to the case where χ is a finite order character. Then $z_\chi = 1$ and the above formula simplifies.

Remark 1.2.2. When $e = 0$, we refer to the condition AR_{e,z_χ} as the *naive Ramanujan* condition.

Remark 1.2.3. In the case $G = GL_n$, the above (and more) was proved in [Laf1], and moreover it was shown that the naive Ramanujan condition always holds. It is known e.g. from [HPS] that for general reductive groups, one sees cusp forms with non-zero e.

Remark 1.2.4. Ultimately, our argument reduces to Deligne's theory of weights [Del] and L. Lafforgue's theorem [Laf1] that irreducible local systems have geometric origin up to Tate twist, both of which are closely tied to the Ramanujan conjecture for GL_n.

Remark 1.2.5. For a fixed point $x \in X_\circ$, the question of uniqueness of e is a standard[3] argument using Jacobson-Morozov; in particular, global uniqueness follows. So we focus on existence in what follows.

Remark 1.2.6. For other recent progress on the Ramanujan conjecture beyond GL_n in the setting of function fields, see [ST, CH].

Remark 1.2.7. The approach described in this note uses an enormous amount of input from geometric Langlands in the form of [GR3], which deduces a weak version of the geometric Langlands conjecture in characteristic p from its D-module counterpart. It is interesting to consider whether a softer approach might be viable. I understand that Dario Beraldo has been contemplating such approaches, trying to use the geometric methods considered in [BLR] to understand Arthur parameters and the so-called *genuine* D-module structure on the Drinfeld sheaf arising via derived Satake.

1.3. **Generic automorphic forms.** Recall that for GL_n, cusp forms admit Whittaker models. The non-existence of Whittaker models for general G is expected to be related to the failure of naive Ramanujan for general G. Theorem B below confirms this expectation in our setting.

1.3.1. We let Bun_N^Ω denote the fiber product $\mathrm{Bun}_B \times_{\mathrm{Bun}_T} \{\rho(\Omega)\}$ and take $\mathrm{Bun}_N^\Omega(\mathbf{F}_q)$ accordingly, per the conventions of §1.1.2 (and indeed, the main purpose of those twists was to allow us to form $\mathrm{Bun}_N^\Omega(\mathbf{F}_q)$). There is a canonical Whittaker map

$$\psi : \mathrm{Bun}_N^\Omega \to \mathbf{A}^1$$

defined over $\mathbf{F}_q$, cf. [GR1] §1.3.

[3]Here is the outline: by SL_2-representation theory, the $\check{M}_e$-orbit through e is always dense in $\check{\mathfrak{g}}_2$, the degree 2 space for the grading from an SL_2. It follows that the conjugacy class of $\check{\lambda}_e$ determines that of e. Then it is clear that $\check{\lambda}_e$ is determined uniquely by (AR_{e,z_χ}).

Fixing a primitive pth root of unity $\zeta_p \in \mathbf{C}^\times$, we obtain a *Whittaker functional*

$$\mathrm{c}_\psi : \mathcal{A}ut^{\mathrm{unr}}_{G,c} \to \mathbf{C}$$
$$f \mapsto \sum_{x \in \mathrm{Bun}_N^\Omega(\mathbf{F}_q)} f(\mathfrak{p}_N(x)) \zeta_p^{\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p} \psi(x)}$$

where $\mathfrak{p}_N : \mathrm{Bun}_N^\Omega \to \mathrm{Bun}_G$ is the natural map and the sum is understood as weighted by automorphisms of points as usual, i.e., we integrate against the standard *groupoid counting* measure.

1.3.2. We now have:

Theorem B. *In the setting of Theorem A, let* $(\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi})_{\gamma_f} \subseteq \mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$ *be the subspace of Hecke eigenfunctions with the same Hecke eigenvalues as* f *(plus* 0*). Then* $e = 0$ *in Theorem A if and only if the map* $\mathrm{c}_\psi : (\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi})_{\gamma_f} \to \mathbf{C}$ *is non-zero, i.e., if and only if there is a Hecke eigenfunction with non-zero Whittaker coefficient.*

2. Between $\overline{\mathbf{Q}}_\ell$ and $\mathbf{C}$

The Langlands program seeks to parametrize automorphic representations for a given global field in terms of representations of the speculative *Langlands group*. However, for function fields, we can replace the Langlands group with the *Weil group* at the expense of seeing *continuous* ℓ*-adic representations*, which are best adapted to $\overline{\mathbf{Q}}_\ell$-valued automorphic forms. Here we adjust accordingly.

With Langlands parameters at hand, the motto of the section is that Arthur-Ramanujan is immediate from irreducibility of the Arthur parameter and Deligne and L. Lafforgue's results on weights. In other words, in this section, we translate Arthur-Ramanujan from the Archimedean world to a fine question about Langlands parameters (which have ℓ-adic nature).

2.1. Notation.

2.1.1. Let $\mathbf{e} := \overline{\mathbf{Q}}_\ell$. It is convenient to fix an isomorphism $\iota : \mathbf{e} \simeq \mathbf{C}$. But the reader can imagine that we are redefining $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$ as $\mathbf{e}$-valued functions and so on, replacing $\mathbf{C}$ with $\mathbf{e}$ in any definitions from §1 that did not explicitly mention the Archimedean norm.

Note that we have a preferred choice of $\sqrt{q} \in \mathbf{e}^\times$ via ι. The reader might imagine we first choose a square root of q in $\mathbf{e}$ and then chose ι in such a way that $\iota(\sqrt{q}) > 0$.

2.1.2. We recall now that [Laf2, Xue] defined the action of the (commutative) *excursion algebra* on $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$, extending the action of the Hecke algebra. Here it is important, at least with current tools, to work over $\mathbf{e}$ rather than $\mathbf{C}$.

We can refine our considerations of systems of Hecke eigenvalues from before by considering *excursion eigenfunctions* in place of Hecke eigenfunctions.

For any such excursion eigenfunction f, the main theorem of [Laf2] associates a continuous semi-simple[4] representation

$$\sigma = \sigma_f : \mathcal{W}_X \to \check{G}(\mathbf{e})$$

[4]This means that for every parabolic $\check{P}$ of $\check{G}$ so that σ factors through $\check{P}(\mathbf{e})$, up to conjugation, σ factors through $\check{M}(\mathbf{e})$ for $\check{M} \subseteq \check{P}$ the Levi subgroup.

where $1 \to \pi_1^{\text{ét}}(X) \to \mathcal{W}_X \to \mathbf{Z} \to 0$ is the *Weil group* of X, i.e., $\mathcal{W}_X = \pi_1^{\text{ét}}(X_\circ) \times_{\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{F}_q))} \mathbf{Z}$ where $1 \in \mathbf{Z}$ is the Frobenius automorphism of $\overline{\mathbf{F}}_q/\mathbf{F}_q$; as usual, the Weil group is topologized as an extension of the discrete group $\mathbf{Z}$ by the profinite geometric fundamental group.

We refer to σ_f as the *Langlands parameter* of f.

2.2. **A refinement.**

2.2.1. We prove the following result.

Theorem C. *For any $f \in \mathcal{A}ut^{\text{unr}}_{G,\text{cusp},\chi}$ an excursion eigenfunction, there is a homomorphism $i_e : SL_2 \to \check{G}$ with centralizer $H_e \subseteq \check{G}$ and a continuous homomorphism*

$$\tilde{\sigma} = \tilde{\sigma}_f : \mathcal{W}_X \to H_e(\mathbf{e})$$

such that:

(1) The representation

$$\tilde{\sigma}^{\text{Arth}} : \mathcal{W}_X \times SL_2(\mathbf{e}) \to \check{G}(\mathbf{e})$$
$$(g_1, g_2) \mapsto \tilde{\sigma}(g_1) \cdot i_e(g_2) = i_e(g_2) \cdot \tilde{\sigma}(g_1)$$

is irreducible, i.e., it does not factor through $\check{P}(\mathbf{e})$ for any proper parabolic $\check{P} \subsetneq \check{G}$.

(2) The Langlands parameter σ_f of f is recovered as the composition

$$\mathcal{W}_X \xrightarrow{g \mapsto (g, \mathrm{diag}(|g|^{\frac{1}{2}}, |g|^{-\frac{1}{2}}))} \mathcal{W}_X \times SL_2(\mathbf{e}) \xrightarrow{\tilde{\sigma}^{\text{Arth}}} \check{G}(\mathbf{e}) \tag{2.2.1}$$

where $|\cdot|^{\frac{1}{2}} : \mathcal{W}_X \to \mathbf{e}^\times$ is the composition $\mathcal{W}_X \to \mathbf{Z} \xrightarrow{n \mapsto (\sqrt{q})^n} \mathbf{e}^\times$.

Remark 2.2.1. Of course, $\tilde{\sigma}^{\text{Arth}}$ is the *Arthur parameter* of f (by definition, at least from our perspective).

2.3. **Theorem C implies Theorem A.** By [AGKRRV1] §25.4.7-8, irreducibility of $\tilde{\sigma}^{\text{Arth}} = \tilde{\sigma}_f^{\text{Arth}}$ means that it must be *ι-pure of weight w* for some $w \in \Lambda_{Z^\circ_{\check{G}}} \otimes \mathbf{R}$ – here we heavily rely on [Laf1]. Concretely, being ι-pure of weight w means that for every highest weight representation V^λ of $\check{G}$, the induced action of $\mathcal{W}_X$ (being a subgroup of $\mathcal{W}_X \times SL_2$ in a trivial way) on V^λ is ι-pure of weight $(w, \lambda) \in \mathbf{R}$ (pairing the real-valued coweight w of $\check{G}$ with the integral weight λ of $\check{G}$). We remark that there is no SL_2-factor in [AGKRRV1], but this does not meaningfully change the argument.

Let $\mathcal{W}_X \times SL_2(\mathbf{e})$ act on such V^λ. Let $V^\lambda = \oplus_n V^\lambda_n$ be decomposition into weight spaces with respect to the $\mathbf{G}_m$ in SL_2.

By the above purity and (2.2.1), for a closed point $x \in X_\circ$, the Frobenius element $\mathrm{Fr}_x \in \mathcal{W}_X$ acts via σ on V^λ in a way preserving V^λ_n, and with all eigenvalues $\omega \in \mathbf{e} \overset{\iota}{\simeq} \mathbf{C}$ for this action satisfying:

$$|\omega| = q_x^{(w,\lambda)+\frac{n}{2}}.$$

The Hecke compatibility for Lafforgue's decomposition states that the image of $\tilde{\gamma}_{x,f} := \sigma(\mathrm{Fr}_x) \in \check{M}_e(\mathbf{e}) \subseteq \check{G}(\mathbf{e})$ in $\check{G}//\check{G}(\mathbf{e})$ is $\gamma_{x,f}$.

Easy analysis with the centers implies that under the map $\mathbf{R} \xrightarrow{r \mapsto q^r} \mathbf{R}^{>0}$, $w \in \Lambda_{Z^\circ_{\check{G}}} \otimes \mathbf{R}$ maps to $z_\chi \in \Lambda_{Z^\circ_{\check{G}}} \otimes \mathbf{R}^{>0} = \check{\Lambda}_{G^{\text{ab}}} \otimes \mathbf{R}^{>0}$, so $\lambda(z_\chi) = q^{(w,\lambda)}$; Theorem A obviously follows.

2.4. **Application to L-functions.** We briefly note that the above ideas can trivially be applied to constrain the poles of L-functions for G in terms of the group theory of $\check{G}$. We spell out the details here, for which there are no surprises. (This material might be considered in reference to the Sato-Tate conjecture and related ideas.)

Namely, let f be a cuspidal eigenfunction for the excursion operators, and assume for simplicity that $z_\chi = 1$ (i.e., the nebentypus of f has finite order). For each irreducible representation V^λ, let $L(t, f, \lambda)$ be the corresponding L-function, where we use the standard substitution $t = q^{-s}$ for function fields.

Theorem D. *There is a linearly reductive subgroup $\Gamma_f \subseteq H_e \subseteq \check{G}$ that* governs the zeros and poles of the L-functions $L(t, f, -)$ *in the following sense:*

Any zero or pole $t = \omega$ of $L(t, f, \lambda)$ occurs only when $|\omega| = q^{-\frac{n}{2}}$ for some $n \in \mathbf{Z}$.

Moreover, for fixed n, the corresponding number of poles minus the number of zeros, both counted with multiplicities, is exactly:

$$\delta_{n,\lambda} - 2\delta_{n-1,\lambda} + \delta_{n-2,\lambda} - (2g-2)\dim(V_{n-1})$$

for g the genus of X and $\delta_{n,\lambda} := \dim((V_n^\lambda)^{\Gamma_f})$ the dimension of the space of Γ_f-invariants in V_n^λ. In particular, the number of poles is $\leq \delta_{n,\lambda} + \delta_{n-2,\lambda}$. (We remind that z_χ was assumed equal to 1.)

Moreover, the normalizer of Γ_f in H_e is not contained in any proper Levi subgroup of $\check{G}$.

Proof. Take Γ_f to be the geometric monodromy group of $\tilde{\sigma}_f$. From §2.3, the argument is standard, using [Del] and Grothendieck-Lefschetz.

□

3. The Arthur filtration and fake discrete series

To proceed, we need to better understand the *algebraic spectral theory* of $\mathbf{e}$-valued automorphic forms.

Theorem C asks us to define an Arthur parameter $\tilde{\sigma}_f^{\mathrm{Arth}}$ for cuspidal f. In particular, we should assign a dual nilpotent orbit e to f. In this section, we will define such a nilpotent orbit.

This construction is obtained from a more refined construction that we call the *Arthur filtration* on the space of automorphic forms, which arises naturally via geometric Langlands. We introduce an algebraic counterpart to the discrete series and present conjectures connecting it to the analytic theory.

We warn that at this point in the paper, we need more input from the geometric Langlands literature, making the discussion less self-contained.

3.1. **Review of [AGKRRV].**

3.1.1. Let $\mathsf{Shv}(\mathrm{Bun}_G)$ be the category of $\mathbf{e}$-sheaves on Bun_G. Recall its subcategory $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ from [AGKRRV1], which we sometimes abbreviate as $\mathsf{Shv}_{\mathcal{N}ilp}$. We remind that it is compactly generated and has perverse compact generators by [GR3] Theorem 1.1.7. Moreover, $\mathsf{Shv}_{\mathcal{N}ilp}$ is closed under the Hecke action. Finally, pushforward along the geometric Frobenius defines an endofunctor $\Phi_{\mathrm{Bun}_G} : \mathsf{Shv}(\mathrm{Bun}_G) \to \mathsf{Shv}(\mathrm{Bun}_G)$ preserving $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$.

Theorem 3.1.1 ([AGKRRV3]). *The categorical trace of $\Phi|_{\mathsf{Shv}_{\mathcal{N}ilp}}$ is concentrated in cohomological degree 0 and identifies with $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$ compatibly with Deligne-Grothendieck's sheaves-to-functions construction.*

Here we remind that for $\mathcal{C}$ a dualizable (e.g., compactly generated) DG category with an endofunctor T, there is a natural complex $\mathrm{tr}_{\mathcal{C}}(T)$ in Vect, the DG category of (complexes of) vector spaces. We refer to [BZN, GKRV] for a detailed introduction to the formalism of categorical traces.

3.1.2. Recall from [AGKRRV1, AGKRRV3] that the Hecke action on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ can naturally be paired with Theorem 3.1.1 to recover V. Lafforgue's excursion operators. (We are terse here because we will recall finer statements in §4.)

3.2. **Construction of the Arthur filtration.** Next, using derived Satake, [AG] §12.8.3 defines a subcategory $\mathsf{Shv}(\mathrm{Bun}_G)_I \subseteq \mathsf{Shv}(\mathrm{Bun}_G)$ for every closed $I \subseteq \check{\mathcal{N}}/\check{G}$ ($\check{\mathcal{N}} \subseteq \check{\mathfrak{g}}$ is the nilpotent cone). We let $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)_I = \mathsf{Shv}_{\mathcal{N}ilp} \cap \mathsf{Shv}(\mathrm{Bun}_G)_I$ be the resulting subcategory.

Remark 3.2.1. After the initial work of Arinkin-Gaitsgory, this categorical filtration is considered many places in the geometric Langlands literature. See [BLR, Lys, FR2] for some recent examples.

A priori the definitions depend on a choice of point $x \in X$, but using [FR1], one sees that at least for $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ the resulting subcategories do not depend on this point; in particular, each $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)_I$ is preserved under Frobenius.

By design, for $I_1 \subseteq I_2$, $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)_{I_1} \subseteq \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)_{I_2}$ and the functor preserves compact objects.

When $I = \overline{\mathcal{O}}_e$ is the closure of a nilpotent orbit through e, we use the notation $\leq e$ in place of I. Similarly, when $I = \partial\mathcal{O}_e = \overline{\mathcal{O}}_e \setminus \mathcal{O}_e$ is the boundary to the nilpotent orbit, we use the notation $< e$ in place of I.

3.2.1. We define $\mathrm{fil}_I\, \mathcal{A}ut^{\mathrm{unr}}_{G,c} \in \mathsf{Vect}$ as the trace of Frobenius acting on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)_I$, and let $\mathrm{fil}_{\leq e}\, \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ be understood accordingly.

By definition, $\mathrm{fil}_I\, \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ is a *complex* of vector spaces. We let $\mathrm{gr}_e\, \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ be the (homotopy) cokernel (i.e., cone) of the morphism $\mathrm{fil}_{<e}\, \mathcal{A}ut^{\mathrm{unr}}_{G,c} \to \mathrm{fil}_{\leq e}\, \mathcal{A}ut^{\mathrm{unr}}_{G,c}$. The following technical theorem plays a key role. We discuss its proof later in this text.

Theorem E. *For every e, $\mathrm{gr}_e\, \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ is concentrated in cohomological degree 0.*

In other words, a priori we have a filtration on $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$ indexed by nilpotent orbits *in the derived category*, but the theorem says this is actually a filtration in the most traditional sense.

Remark 3.2.2. A standard analogy says that the homological subtleties for coherent sheaves considered in [AG] are analogous to functional analytic subtleties (e.g., non-temperedness) considered classically; the above construction is certainly in this spirit.

Example 3.2.3. For G simply-connected (to simplify slightly), one can use [Fær] and the Atiyah-Bott formula [GL] to see that for the principal nilpotent orbit e^{princ}, $\mathrm{gr}_{e^{\mathrm{princ}}}\, \mathcal{A}ut^{\mathrm{unr}}_{G,c} = \mathcal{A}ut^{\mathrm{unr}}_{G,c}/\,\mathrm{fil}_{<e^{\mathrm{princ}}}\, \mathcal{A}ut^{\mathrm{unr}}_{G,c} \simeq \mathbf{e}$ via the map $(f \in \mathcal{A}ut^{\mathrm{unr}}_{G,c}) \mapsto \sum_{x \in \mathrm{Bun}_G(\mathbf{F}_q)} f(x)$ where the (finite, by design) sum is groupoid-weighted as in §1.3.1.

For $G = SL_2$, this completely characterizes the Arthur filtration, and this calculation directly verifies Theorem E in this case.

3.2.2. *Properties of the filtration.* By design, the categorical filtration on $\mathsf{Shv}_{\mathcal{N}ilp}$ is preserved under Hecke functors. Therefore, by §3.1.2, the Arthur filtration is stable under the action of the excursion algebra, in particular by Hecke operators.

Specializing further, the Arthur filtration is preserved under the action of $\mathrm{Bun}_{Z^\circ}(\mathbf{F}_q)$, and a simple variant on Theorem E says that the filtration is with respect to the convolution action of $\mathcal{A}ut^{\mathrm{unr}}_{Z^\circ,c}$. Therefore, we obtain similar filtrations on each space $\mathcal{A}ut^{\mathrm{unr}}_{G,c,\chi}$.

3.3. Fake discrete series.

3.3.1. *Motivation.* Suppose G is semi-simple for convenience. Then $\mathrm{Bun}_G(\mathbf{F}_q)$ has finite volume with respect to the Tamagawa measure, so the constant function $1_{\mathrm{Bun}_G(\mathbf{F}_q)}$ gives a canonical 1-dimensional summand of $L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C})$. More generally, residues of Eisenstein series provide non-cuspidal discrete series representations appearing in L^2.

Arthur's philosophy [Art] says that for multiplicity formulae, it is necessary to consider all discrete series representations, not merely the cuspidal ones. We will eventually see this play out in the ℓ-adic setting. Motivated by Example 3.2.3, we now propose a fix to $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$ which contains the constant function, is adapted to ℓ-adic coefficients, and which we call *fake discrete series.*

3.3.2. Recall from [Laf2, Prop. 8.23] that $\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp}}$ is the *Hecke-finite* submodule of $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$, i.e., the submodule of $f \in \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ whose span under all Hecke operators is finite-dimensional; the same applies for $\mathcal{A}ut^{\mathrm{unr}}_{G,c,\chi}$. (One can use excursion operators as well as Hecke operators for this purpose.)

The Arthur filtration is stable under Hecke operators, so $\mathrm{gr}_\bullet \mathcal{A}ut^{\mathrm{unr}}_{G,c,\chi} = \oplus_e \mathrm{gr}_e \mathcal{A}ut^{\mathrm{unr}}_{G,c,\chi}$ is acted on by the Hecke algebra.

Definition 3.3.1. The *fake discrete series (resp. of type e)* $\mathcal{A}ut^{\mathrm{unr}}_{G,\text{fake-disc},\chi}$ (resp. $\mathcal{A}ut^{\mathrm{unr}}_{G,\text{fake-disc},\chi,e}$) is the Hecke-finite submodule of $\mathrm{gr}_\bullet \mathcal{A}ut^{\mathrm{unr}}_{G,c,\chi}$ (resp. $\mathrm{gr}_e \mathcal{A}ut^{\mathrm{unr}}_{G,c,\chi}$).

Clearly the Hecke (or excursion) eigenvalues occurring in $\mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$ also occur in $\mathcal{A}ut^{\mathrm{unr}}_{G,\text{fake-disc},\chi}$.

Theorem F. *$\mathcal{A}ut^{\mathrm{unr}}_{G,\text{fake-disc},\chi}$ is finite-dimensional.*

3.3.3. *Extension of Arthur-Ramanujan.* We now have the following result extending Theorem C:

Theorem G. *For any $f \in \mathcal{A}ut^{\mathrm{unr}}_{G,\text{fake-disc},\chi,e}$ an eigenfunction for the excursion algebra, there is a continuous representation $\tilde{\sigma} = \tilde{\sigma}_f : \mathcal{W}_X \to H_e(\mathbf{e})$ satisfying the conclusions of Theorem C (interpreting the excursion eigenvalue of f as its Langlands parameter, as* [Laf2] *does for cuspidal eigenfunctions).*

Corollary 3.3.2. *Theorems A and D hold for fake discrete series, not only for cusp forms.*

Corollary 3.3.3. *For $f \in \mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$, the nilpotent e from Theorem A is the unique nilpotent such that $f \in \mathrm{fil}_{\leq e} \mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$ and $f \notin \mathrm{fil}_{< e} \mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$.*

Example 3.3.4. For G simply-connected (to simplify) and $e = e^{\mathrm{princ}}$, Example 3.2.3 asserts that $\mathcal{A}ut_{G,e^{\mathrm{princ}}} = \mathcal{A}ut^{\mathrm{unr}}_{G,\text{fake-disc},e^{\mathrm{princ}}} \simeq \mathbf{e}$ (with generator corresponding to the constant function $1_{\mathrm{Bun}_G(\mathbf{F}_q)}$ in a suitable sense). In this case, $H_{e^{\mathrm{princ}}}$ is the trivial group and so the Arthur parameter is $\mathcal{W}_X \times SL_2(\mathbf{e}) \to SL_2(\mathbf{e}) \xrightarrow{i_{e^{\mathrm{princ}}}} \check{G}(\mathbf{e})$, as is standard.

Remark 3.3.5. Note that if we have an irreducible representation $\tilde{\sigma}$ as in Theorem C, then the center of H_e must be finite modulo the center of $\check{G}$; indeed, otherwise there is a proper Levi $\check{M}$ through which $SL_2 \times H_e \to \check{G}$ factors. In particular, the above result rules out many nilpotents from contributing to the (fake) discrete spectrum.

3.4. **Actual discrete series (conjectures).**

3.4.1. We formulate some conjectures relating $\mathcal{A}ut^{\mathrm{unr}}_{G,\text{fake-disc},\chi}$ to the *actual* discrete series representations. Given Theorem A and the spectral decomposition for Eisenstein series [KO1] (and its extension as announced in [KO2, Thm. 1]), one hopes that these conjectures are within reach.

3.4.2. We have:

Conjecture 3.4.1. *Let G be semi-simple.*[5]

1. *The Arthur filtration is defined over $\mathbf{Q}$. In other words, if $\mathcal{A}ut^{\mathrm{unr}}_{G,\mathbf{Q},c}$ denotes the space of $\mathbf{Q}$-valued functions and $\mathrm{fil}_{\leq e}\,\mathcal{A}ut^{\mathrm{unr}}_{G,\mathbf{Q},c} := \mathcal{A}ut^{\mathrm{unr}}_{G,\mathbf{Q},c} \cap \mathrm{fil}_{\leq e}\,\mathcal{A}ut^{\mathrm{unr}}_{G,c}$, then $\mathrm{fil}_{\leq e}\,\mathcal{A}ut^{\mathrm{unr}}_{G,c} = \mathbf{e} \otimes_{\mathbf{Q}} \mathrm{fil}_{\leq e}\,\mathcal{A}ut^{\mathrm{unr}}_{G,\mathbf{Q},c}$ (which of course implies the same on gr_e).*
2. *Given (1), we can define an Arthur filtration $\mathrm{fil}_{\leq e}\, L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C})$ as the closure of*
$$\mathbf{C} \otimes_{\mathbf{Q}} \mathrm{fil}_{\leq e}\,\mathcal{A}ut^{\mathrm{unr}}_{G,\mathbf{Q},c} \subseteq L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C}).$$
Because we are working with Hilbert spaces, this filtration necessarily splits into summands $L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C})_e$ so $\mathrm{fil}_{\leq e}\, L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C}) \simeq \oplus_{e' \leq e} L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C})_{e'}$.
Then we conjecture that the natural map
$$\mathbf{C} \underset{\mathbf{Q}}{\otimes} \mathrm{gr}_e\,\mathcal{A}ut^{\mathrm{unr}}_{G,\mathbf{Q},c} \to L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C})_e$$
gives an isomorphism on Hecke-finite submodules.

It would follow from this conjecture that there is canonical isomorphism of

$$\mathbf{C} \underset{\mathbf{Q}}{\otimes} \mathcal{A}ut^{\mathrm{unr}}_{G,\mathbf{Q},\text{fake-disc}} := (\mathrm{gr}_\bullet\,\mathcal{A}ut^{\mathrm{unr}}_{G,\mathbf{Q},c})^{\text{Hecke-finite}} \simeq L^2_{\mathrm{disc}}(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C})$$

between fake discrete series (at least, up to an isomorphism $\iota : \mathbf{e} \simeq \mathbf{C}$) and actual discrete series (which by definition is the maximal finite-dimensional subspace of L^2 closed under Hecke operators).

Remark 3.4.1. Let $L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C})_{\mathrm{Eis}}$ be the closed subspace topologically spanned by Eisenstein series from characters of $\Lambda = \mathrm{Bun}_T(\mathbf{F}_q)/\mathrm{Bun}^\circ_T(\mathbf{F}_q)$. In [KO1], it was shown that this space is a sum indexed by dual nilpotent orbits, and in [KO2], a similar result was announced for all of $L^2(\mathrm{Bun}_G(\mathbf{F}_q), \mathbf{C})$. We of course expect the two decompositions by dual nilpotent orbits to coincide.

Remark 3.4.2. By Theorem G, one can see Conjecture 3.4.1 (1) holds for the Arthur filtration on cuspidal functions. Hopefully the rest can be treated via induction by computing inner products of Eisenstein series as in Kazhdan-Okounkov.

4. Arithmetic local systems and Arthur's multiplicity formula

4.1. **Moduli spaces of local systems and automorphic forms.**

[5] One can generalize to arbitrary reductive G by taking $\Xi \subseteq \mathrm{Bun}_Z(\mathbf{F}_q)$ finite index and working with Ξ-invariant functions on $\mathrm{Bun}_G(\mathbf{F}_q)$, as in [Laf2], and we consider functions L^2 modulo the center. As noted already, up to tensoring by a character of $\mathrm{Bun}_{G^{\mathrm{ab}}}(\mathbf{F}_q)$, all automorphic forms with fixed nebentypus can be translated to have this Ξ-invariance property for some Ξ, so this is not a restriction in generality.

4.1.1. Let H be an affine algebraic group. In [AGKRRV1], we introduced two stacks of H-local systems on X, both defined over $\mathbf{e}$.

The first, $\mathrm{LS}_H^{\mathrm{restr}}$, has $\mathbf{e}$-points that are H-local systems on X, i.e., continuous representations $\pi_1^{\text{ét}}(X) \to H(\mathbf{e})$ up to conjugacy. It is a formal stack with many connected components – two local systems lie in the same connected component if and only if they have the same semi-simplification.

Pullback along the geometric Frobenius of X defines a Frobenius automorphism $\mathrm{Frob} = \mathrm{Frob}_{\mathrm{LS}_H^{\mathrm{restr}}}$ of $\mathrm{LS}_H^{\mathrm{restr}}$. Its (homotopy) fixed points are the stack $\mathrm{LS}_H^{\mathrm{arthm}}$, whose points are representations $\mathcal{W}_X \to H(\mathbf{e})$ up to conjugacy.

For an informal introduction to these ideas with more details, see [Ras] §2.

4.1.2. We have the following results.

Theorem 4.1.1 ([GR3])**.** *There is an open substack* $\mathrm{LS}_{\check{G}}^{\mathrm{restr},\prime} \subseteq \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ *that is a disjoint union of connected components with base-change* $\mathrm{LS}_{\check{G}}^{\mathrm{arthm},\prime} := \mathrm{LS}_{\check{G}}^{\mathrm{arthm}} \times_{\mathrm{LS}_{\check{G}}^{\mathrm{restr}}} \mathrm{LS}_{\check{G}}^{\mathrm{restr},\prime}$ *so that:*

- *There is a geometric Langlands equivalence*
$$\mathbf{L}_G : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr},\prime})$$
satisfying standard compatibilities: compatibility with Hecke functors, with Whittaker coefficients, and with geometric Eisenstein series (see [GR3] *for more details).*
  On the right hand side,
$$\mathcal{N}ilp^{\mathrm{spec}} \subseteq T^*[-1]\,\mathrm{LS}_{\check{G}}^{\mathrm{restr}} = \{\sigma \in \mathrm{LS}_{\check{G}}^{\mathrm{restr}}, \varphi \in \mathrm{C}_{\text{ét}}(X, \check{\mathfrak{g}}_\sigma)\}$$
is the spectral nilpotent cone (cf. [AG, §11]*) and* $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}$ *indicates* IndCoh *with singular support in* $\mathcal{N}ilp^{\mathrm{spec}}$*, as in* [AG]*.*
- *There is an arithmetic Langlands equivalence*
$$\mathcal{A}ut_{G,c}^{\mathrm{unr}} \simeq \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm},\prime}, \omega)$$
for ω *the dualizing sheaf.*

The first part of this result is proved in [GR3] by a reduction to characteristic 0 geometric Langlands [GR1, ABCCFGLRR, CCFGLRR, ABCFGLRR, GR2]. Of course, we expect
$$\mathrm{LS}_{\check{G}}^{\mathrm{restr},\prime} = \mathrm{LS}_{\check{G}}^{\mathrm{restr}} \tag{4.1.1}$$
(and this is so for GL_n), but at the moment, the above is all that we can prove.

As explained in [AGKRRV1, BLR], the second part of this theorem is a consequence of the first part using the results of [AGKRRV3].

4.2. **The spectral filtration.** Given $I \subseteq \check{\mathcal{N}}/\check{G}$ closed as before, we have a corresponding closed $\mathcal{N}ilp_I^{\mathrm{spec}} \subseteq \mathcal{N}ilp^{\mathrm{spec}}$.

Because the functor $\mathbf{L}_G$ in (4.1.1) is equivariant with respect to the derived Satake equivalence, we obtain

Corollary 4.2.1. *(1) For every nilpotent* e*,* $\mathbf{L}_G$ *identifies the categories*
$$\mathbf{L}_G : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)_I \simeq \mathsf{IndCoh}_{\mathcal{N}ilp_I^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr},\prime}).$$

(2) There are canonical equivalences

$$\mathrm{fil}_I\, \mathcal{A}ut^{\mathrm{unr}}_{G,c} \simeq \mathrm{tr}(\mathrm{Frob}^*, \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_I}(\mathrm{LS}^{\mathrm{restr},\prime}_{\check{G}}))$$

and

$$\mathrm{gr}_e\, \mathcal{A}ut^{\mathrm{unr}}_{G,c} \simeq \mathrm{tr}(\mathrm{Frob}^*, \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_e}(\mathrm{LS}^{\mathrm{restr},\prime}_{\check{G}}))$$

where

$$\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_e}(\mathrm{LS}^{\mathrm{restr},\prime}_{\check{G}}) := \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_{\leq e}}(\mathrm{LS}^{\mathrm{restr},\prime}_{\check{G}})/\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_{< e}}(\mathrm{LS}^{\mathrm{restr},\prime}_{\check{G}})$$

is the corresponding quotient category.

In sum, to understand the graded terms in the Arthur filtration, it is enough to understand the trace of Frobenius acting on the spectral categories $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_e}(\mathrm{LS}^{\mathrm{restr},\prime}_{\check{G}})$, which is a direct summand of the trace without the $\prime$.

Our goal in the remainder of this section is to explain an *explicit* calculation of these traces in the spirit of the Arthur multiplicity formula.

4.3. **Generic forms.**

4.3.1. Let us take a moment to explicitly describe $\mathrm{fil}_0\, \mathcal{A}ut^{\mathrm{unr}}_{G,c}$.

First, Corollary 4.2.1 gives an isomorphism

$$\mathrm{fil}_0\, \mathcal{A}ut^{\mathrm{unr}}_{G,c} \simeq \Gamma(\mathrm{LS}^{\mathrm{arthm},\prime}_{\check{G}}, \mathcal{O}). \tag{4.3.1}$$

As in [AGKRRV1, §24.2], one can take

$$\mathcal{E}xc := \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \mathcal{O})$$

as the definition of the excursion algebra.[6] From this point of view, its action on $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$ comes via traces from the Nadler-Yun action of $\mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ ([AGKRRV1, Main Thm. 14.3.2]).

Theorem H. *Let* Poinc $\in \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ *be obtained by pull-push using the diagram*

$$\mathbf{e}^\times \xleftarrow{\zeta_p^{\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(-)}} \mathbf{F}_q \xleftarrow{\psi} \mathrm{Bun}^{\Omega}_N(\mathbf{F}_q) \xrightarrow{\mathfrak{p}_N} \mathrm{Bun}_G(\mathbf{F}_q).$$

(We remark here that $\mathrm{Bun}^{\Omega}_N(\mathbf{F}_q)$ *is finite.)*

Then Poinc *lies in* $\mathrm{fil}_0\, \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ *and corresponds under* (4.3.1) *to* $1 \in \Gamma(\mathrm{LS}^{\mathrm{arthm},\prime}_{\check{G}}, \mathcal{O})$.

It follows that $\mathrm{fil}_0\, \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ is generated by Poinc using excursion operators. It follows directly that if $f \in \mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp},\chi}$ is an eigenfunction for the excursion algebra, then it lies in fil_0 if and only if $c_\psi(f) \neq 0$, which is a refinement of Theorem B.

Remark 4.3.1. Under the construction [GR1] of the geometric Langlands functor, the structure sheaf of $\mathrm{LS}_{\check{G}}$ corresponds to the vacuum Poincaré sheaf; a suitable variant of this is true in the restricted setting as well, cf. [GR3, §1.1.3]. Morally, the above result follows by taking trace of Frobenius on this assertion, but there are technical complications: "suitable variant" is doing a lot of work in the previous sentence. Still, we are able to prove Theorem H; details will appear in future work.

[6] In any case, there are different possible meanings of *excursion algebra* in the literature. The important properties are that it should act on $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$ and should contain explicit elements acting by Lafforgue's excursion operators.

4.4. **Philosophy.** Here is the main motivation, where we explicitly grant ourselves the freedom of imprecision.

At first approximation, the Langlands program suggests that (suitable) automorphic forms are in bijection with semi-simple Langlands parameters $\mathcal{W}_X \to \check{G}(\mathbf{e})$ (up to conjugacy).

At second approximation, Arthur's conjectures say that automorphic forms are in bijection with semi-simple Arthur parameters $\mathcal{W}_X \times SL_2(\mathbf{e}) \to \check{G}(\mathbf{e})$.

At third approximation, Arthur's conjectures say that automorphic forms are in bijection with semi-simple Arthur parameters $\tilde{\sigma}^{\mathrm{Arth}} : \mathcal{W}_X \times SL_2(\mathbf{e}) \to \check{G}(\mathbf{e})$ up to a correction involving a sign character $\epsilon^{\mathrm{Arth}} : S_{\tilde{\sigma}} \to \{\pm 1\}$ for $S_{\tilde{\sigma}} \subseteq \check{G}$ the centralizer of $\tilde{\sigma}^{\mathrm{Arth}}$, see [Art, Eqn. (8.4)] for the definition. To motivate later constructions, we remark that one can check that ϵ^{Arth} is trivial if the homomorphism $i_e : SL_2 \to \check{G} \to \check{G}^{\mathrm{ad}}$ factors through PGL_2 (i.e., the nilpotent e is even).

Translated into the perspective of [Zhu, AGKRRV1], the first approximation would suggest $\mathcal{A}ut_{G,c}^{\mathrm{unr}} \simeq \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \mathcal{O})$.

The second approximation might *naively* translate to the assertion that $\mathcal{A}ut_{G,c}^{\mathrm{unr}}$ is in bijection with the moduli stack of $\check{G}$-valued representations of $\mathcal{W}_X \times SL_2$, which is easily seen (cf. [AGKRRV1, Prop. 3.5.4]) to be $\coprod_e \mathrm{LS}_{H_e}^{\mathrm{arthm}}$.

Less naively, the second approximation should be translated to the assertion that $\mathrm{gr}_e \mathcal{A}ut_{G,c}^{\mathrm{unr}} \simeq \Gamma(\mathrm{LS}_{H_e}^{\mathrm{arthm}}, \mathcal{O})$. This is equivariant for the action of the excursion algebra *when one regards* $\mathrm{LS}_{H_e}^{\mathrm{arthm}}$ *as mapping to* $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ *via* (2.2.1).

One ought to regard any confusion about the "naive second approximation" and the "actual second approximation" as resulting from an excess of familiarity with the L^2 situation, where any filtration canonically splits as a direct sum, cf. Conjecture 3.4.1.

Finally, the third approximation should be translated to the assertion that

$$\mathrm{gr}_e \mathcal{A}ut_{G,c}^{\mathrm{unr}} \simeq \Gamma(\mathrm{LS}_{H_e}^{\mathrm{arthm}}, \mathcal{L}_e) \tag{4.4.1}$$

where $\mathcal{L}_e$ is a line bundle whose square is $\mathcal{O}$ and whose restriction along $\mathbf{B}S_{\tilde{\sigma}} \to \mathrm{LS}_{H_e}^{\mathrm{arthm}}$ corresponds to the 1-dimensional representation defined by ϵ^{Arth}.

We develop this picture in the remainder of this section. The material that follows implementing this idea is the most technical part of this work. Some readers may prefer at this point to skip §4.5, skim or skip §4.6, and proceed to §5.

4.5. **Spectral Whittaker coefficients.** We now detail the construction of *spectral Whittaker functors*. The initial idea is due to V. Lafforgue, who considered the case of the principal nilpotent element (in connection to the constant sheaf on Bun_G). The generalization to even nilpotent elements is considered in [BZSV, §18.5]. *The material of this subsection is joint between myself, Ben-Zvi, and Venkatesh.*

4.5.1. *Notation concerning nilpotent elements.* Let $i_e : SL_2 \to \check{G}$ be a Jacobson-Morozov SL_2 associated to a nilpotent e. The adjoint representation $\check{\mathfrak{g}}$ inherits a grading $\check{\mathfrak{g}} = \oplus \check{\mathfrak{g}}_n$.

We let $V_e := \check{\mathfrak{g}}_1$ considered as a representation of H_e. Recall that V_e carries a canonical symplectic form:

$$v, w \in V_e \mapsto \check{\kappa}\big(\mathrm{Lie}(i_e)(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), [v,w]\big)$$

for $\check{\kappa}$ a (fixed once and for all) non-degenerate invariant form on $\check{\mathfrak{g}}$ and $\mathrm{Lie}(i_e) : \mathfrak{sl}_2 \to \check{\mathfrak{g}}$ the derivative of i_e; this form is clearly preserved by the action of H_e.

4.5.2. *Clifford gerbes.* Let Q be a perfect complex in degrees $[-1,1]$ with a non-degenerate quadratic form q, which we sometimes consider as a derived stack, and suppose the Euler characteristic of Q is even (for convenience), equivalently, $H^0(Q)$ is even-dimensional.

There is a corresponding Clifford algebra $\mathrm{Cl}(Q,q)$. One can consider the corresponding category of modules M over $\mathrm{Cl}(Q,q)$ for which $H^{-1}(Q)$ acts locally nilpotently on $H^*(M)$. As is standard, this is an invertible DG category; any choice of Lagrangian $L \to Q$ provides an equivalence between this category and Vect. Moreover, again by standard arguments, the square of this gerbe is canonically trivialized (parallel to the $\{\pm 1\}$ in Arthur's setting).

There is a variant: if we instead ask that $H^1(Q)$ acts locally nilpotently, a version of Kashiwara's lemma implies that we obtain a canonically equivalent category. Where relevant, we refer to this incarnation of the gerbe as the *right nilpotent* version, but by default assume the *left nilpotent* convention of the previous paragraph.

The above works as well in families.[7]

We sometimes refer to invertible DG categories (or families of such) as *gerbes*, maybe to the chagrin of some specialists.

4.5.3. For a local system $\sigma \in \mathrm{LS}^{\mathrm{restr}}_{H_e}$, there is a corresponding lisse sheaf $V_{e,\sigma}$ on X associated to the representation V_e. By design, $V_{e,\sigma}$ carries a symplectic form, so its shifted étale cohomology $C_{\text{ét}}(X, V_{e,\sigma})[1]$ (i.e., the *derived version of* $H^1_{\text{ét}}$) carries a non-degenerate symmetric bilinear form by Poincaré duality, i.e, it is naturally a quadratic space (or rather: complex).

The construction of §4.5.2 associates a gerbe $\mathcal{G}_{e,\sigma}$ to the above quadratic space. This construction works well in families, so we obtain a gerbe $\mathcal{G}_e$ over $\mathrm{LS}^{\mathrm{restr}}_{H_e}$.

We let $\mathsf{IndCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e})$ and $\mathsf{QCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e})$ denote the corresponding gerbe-twisted forms of IndCoh and QCoh. Informally, this means that we consider modules over the above sheaves of Clifford algebras rather than over just the structure sheaf.

4.5.4. The key construction is the existence of a canonical functor

$$\mathrm{coeff}^{\mathrm{spec}}_e : \mathsf{IndCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \to \mathsf{IndCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e})$$

called the *spectral Whittaker coefficient*. The construction works in any sheaf theory, e.g., de Rham or Betti, but restricted local systems are the context relevant for arithmetic purposes.

4.5.5. To outline the construction, we use some language of matrix factorizations below. Given a stack $f : \mathcal{Y} \to \mathbf{A}^1$, recall that $f^{-1}(0)$ is acted on by the DG group scheme $\Omega\mathbf{A}^1$, so $\mathsf{IndCoh}(\Omega\mathbf{A}^1)$ acts on $\mathsf{IndCoh}(f^{-1}(0))$. Recall [Pre] that the category of *matrix factorizations* $\mathrm{MF}(\mathcal{Y}, f)$ is the tensor product

$$\mathsf{IndCoh}(f^{-1}(0)) \underset{\mathsf{IndCoh}(\Omega\mathbf{A}^1)}{\otimes} \mathsf{IndCoh}(\Omega\mathbf{A}^1)/\mathsf{QCoh}(\Omega\mathbf{A}^1).$$

In particular, there is a tautological quotient functor

$$\mathsf{IndCoh}(f^{-1}(0)) \to \mathrm{MF}(\mathcal{Y}, 0). \tag{4.5.1}$$

When $\mathbf{G}_m$ acts on $\mathcal{Y}$ and f is $\mathbf{G}_m$-equivariant (with respect to some power of the scaling action on $\mathbf{A}^1$), we let $\mathrm{MF}^{\mathbf{G}_m}(\mathcal{Y}, f) := \mathrm{MF}(\mathcal{Y}, f)^{\mathbf{G}_m}$ be the $\mathbf{G}_m$-equivariant objects, i.e., the category of *graded* matrix factorizations.

[7] Not in families, one easily finds that the gerbe for Q is canonically identified with that for the non-derived quadratic space $H^0(Q)$.

Now $\mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes V_e}$ is the moduli of $\sigma \in \mathrm{LS}_{H_e}$ plus a section of $C_{\text{ét}}(V_{e,\sigma})[1]$. In particular, $\mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes V_e} \to \mathrm{LS}^{\mathrm{restr}}_{H_e}$ is a quadratic bundle with quadratic form $q_e : \mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes V_e} \to \mathbf{A}^1$.

Let $\check{G}_{\geq 1} \subseteq \check{G}$ be the unipotent subgroup with Lie algebra $\check{\mathfrak{g}}_{\geq 1} = \oplus_{n \geq 1} \check{\mathfrak{g}}_n$ (the grading determined by λ_e), and similarly for $\check{G}_{\geq 2}$. Let Heis_{V_e} be the pushout of $\check{G}_{\geq 1}$ along the Whittaker homomorphism $\check{G}_{\geq 2} \to \mathbf{A}^1$ obtained by exponentiating the map $\check{\mathfrak{g}}_{\geq 2} \to k$ of $\check{\kappa}$-pairing with $\mathrm{Lie}(i_e)(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$. Observe that this Heisenberg extension is encoded by a homomorphism $\psi : V_e \to \mathbf{BG}_a$, which is H_e-equivariant with respect to the trivial action on the target, so gives $H_e \ltimes V_e \to \mathbf{BG}_a$.

An exercise in Heisenberg groups shows that q_e coincides with the composition:

$$\mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes V_e} \to \mathrm{LS}_{\mathbf{BG}_a} \to H^2_{\text{ét}}(X) \simeq \mathbf{A}^1. \tag{4.5.2}$$

This expression induces a null-homotopy of the composition:

$$\mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes \check{G}_{\geq 1}} \to \mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes V_e} \xrightarrow{q_e} \mathbf{A}^1.$$

Next, observe that $\mathbf{G}_m$ acts canonically on $\mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes \check{G}_{\geq 1}}$ – its double cover $(\mathbf{G}_m)_{SL_2}$ acts on $\check{G}_{\geq 1}$ encoding the grading on $\check{\mathfrak{g}}_{\geq 1}$, but the $\mu_2 \subseteq (\mathbf{G}_m)_{SL_2}$ maps to H_e so acts trivially on the above stack, so the action descends to $\mathbf{G}_m = (\mathbf{G}_m)_{PGL_2} = (\mathbf{G}_m)_{SL_2}/\mu_2$. The projection to $\mathrm{LS}_{H_e \ltimes V_e}$ is $\mathbf{G}_m$-equivariant for the obvious $\mathbf{G}_m$-action on a (quadratic) bundle.

By a variant of [Pre, Thm. 9.3.4], $\mathrm{MF}^{\mathbf{G}_m}(\mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes V_e}, q_e)$ is the category of $\mathbf{Z}/2$-graded modules over the Clifford (super)algebra, so has a functor (forgetting the $\mathbf{Z}/2$-grading) to $\mathsf{IndCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e})$. Then the spectral Whittaker functor is formed using upper-! and lower-* along the correspondence

$$\mathrm{LS}^{\mathrm{restr}}_{\check{G}} \longleftarrow \mathrm{LS}^{\mathrm{restr}}_{H_e \ltimes \check{G}_{\geq 1}}/\mathbf{G}_m \longrightarrow q_e^{-1}(0)/\mathbf{G}_m,$$

the projection (4.5.1), and this forgetful functor.

4.6. **Properties of spectral Whittaker coefficients.**

4.6.1. We have the following structural properties:

Theorem I. *(1) The functor* $\mathrm{coeff}^{\mathrm{spec}}_e$ *maps* $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_{\leq e}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ *to* $\mathsf{QCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e}) \subseteq \mathsf{IndCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e})$.

(2) $\mathrm{coeff}^{\mathrm{spec}}_e$ *vanishes on* $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_{<e}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

(3) The restricted functor $\mathrm{coeff}^{\mathrm{spec}}_e|_{\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_{\leq e}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}$ *preserves compact objects.*

4.6.2. It follows that we obtain a functor

$$\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_e}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) = \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_{\leq e}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})/\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_{<e}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \to \mathsf{QCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e}).$$

We denote the resulting functor by $\overline{\mathrm{coeff}}^{\mathrm{spec}}_e$. By Theorem I, $\mathrm{coeff}^{\mathrm{spec}}_e$ preserves compact objects, so induces a morphism on traces.

Theorem J. *The induced map*

$$\mathrm{tr}(\overline{\mathrm{coeff}}^{\mathrm{spec}}_e) : \mathrm{tr}(\mathrm{Frob}, \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_e}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})) \to \mathrm{tr}(\mathrm{Frob}, \mathsf{QCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e}))$$

is an isomorphism.

In the above theorem, the left hand side is naturally acted on by $\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \mathcal{O})$ and the right hand side is acted on by $\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{H_e}, \mathcal{O})$. The isomorphism is equivariant for the action of $\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \mathcal{O})$ using the map $\mathrm{LS}^{\mathrm{arthm}}_{H_e} \to \mathrm{LS}^{\mathrm{arthm}}_{\check{G}}$ given by (2.2.1). The translation using q occurs ultimately because of the distinguished role of $H^2_{\text{ét}}(X) \simeq \mathbf{e}(-1)$ (e.g., the bilinear form on $H^1_{\text{ét}}(V_{e,\sigma})$ was valued in $H^2_{\text{ét}}(X) \simeq \mathbf{e}(-1)$, see also (4.5.2)).

4.6.3. *Application to traces.* If there were no gerbe $\mathcal{G}_e$, we would calculate the trace of Frobenius on $\mathsf{QCoh}_{\mathcal{G}_e}(\mathrm{LS}^{\mathrm{restr}}_{H_e})$ as functions on $\mathrm{LS}^{\mathrm{arthm}}_{H_e}$.

Because of the gerbe, the trace is given as global sections of a line bundle $\mathcal{L}_e$ on $\mathrm{LS}^{\mathrm{arthm}}_{H_e}$ instead.

This line bundle arises as follows. Given a point of $\mathrm{LS}^{\mathrm{arthm}}_{H_e}$, i.e., a point $\sigma \in \mathrm{LS}^{\mathrm{restr}}_{H_e}$ with an isomorphism $\alpha : \Phi^*_X(\sigma) \simeq \sigma$, we obtain an isomorphism

$$\mathcal{G}_{e,\sigma} \overset{\alpha}{\simeq} \mathcal{G}_{e,\Phi^*_X(\sigma)} \simeq \mathcal{G}_{e,\sigma}$$

using the evident Frobenius equivariance of $\mathcal{G}_e$. This is an isomorphism of μ_2-gerbes, so is given by a μ_2-torsor, i.e., a line bundle $\mathcal{L}_e$ whose tensor square is trivialized. See §4.6.5 for a more explicit description of $\mathcal{L}_e$.

4.6.4. *The main point.* Unconditionally, we deduce from the discussion of this section that

$$\mathrm{gr}_e\, \mathcal{A}ut^{\mathrm{unr}}_{G,c} \simeq \Gamma(\mathrm{LS}^{\mathrm{arthm},\prime}_{H_e}, \mathcal{L}_e). \tag{4.6.1}$$

Here $\mathrm{LS}^{\mathrm{arthm},\prime}_{H_e}$ are those local systems whose image in $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}$ lie in $\mathrm{LS}^{\mathrm{restr},\prime}_{\check{G}}$.

4.6.5. Explicitly, for $\sigma \in \mathrm{LS}^{\mathrm{arthm}}_{H_e}(\mathbf{e})$ with automorphism group S_σ, one can restrict $\mathcal{L}_e$ along $\mathbf{B}S_\sigma \to \mathrm{LS}^{\mathrm{arthm}}_{H_e}$ to obtain a character $\epsilon : S_\sigma \to \mu_2 = \{\pm 1\}$. This character can be explicitly described as follows.

The map ϵ can be explicitly described as follows. One forms $\mathcal{H} := H^1_{\text{ét}}(X, V_{e,\sigma})$, which is a quadratic space. There is an action of S_σ on $\mathcal{H}$ preserving the quadratic form. Moreover, there is a Frobenius automorphism, which lies in the general orthogonal group – it preserves the quadratic form up to a factor of q.

Therefore, we have a map $S_\sigma \times \mathbf{Z} \to GO(\mathcal{H})$, GO being the generalized orthogonal group. Pullback of $GSpin(\mathcal{H})$ defines a central extension $1 \to \mu_2 \to E \to S_\sigma \times \mathbf{Z} \to 1$. The commutator for this central extension gives a canonical map $S_\sigma \to \mu_2$, which is easily seen to be this character ϵ (essentially because the Clifford gerbe construction defines the spin group).

The following is not obvious from the definitions, but results by elementary manipulations.

Lemma K. *For semi-simple σ, the character $\epsilon : S_\sigma \to \mu_2$ coincides with Arthur's character ϵ^{Arth} referenced in §4.4.*

4.6.6. *Method of proof of Theorem J.* This discussion is both cursory and technical, but it feels appropriate to give the flavor of the proof of this key result.

The most important point is the construction of a *filtration*[8] on $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}_e}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ whose associated graded is $\mathsf{QCoh}^{\Rightarrow}(\mathcal{N}ilp^{\mathrm{spec},\sim}_e)$, where the notation needs some explaining.

Below, for $n \geq 0$, we let $\check{\mathfrak{g}}_{\geq n} = \oplus_{m \geq n} \check{\mathfrak{g}}_m$ denote the sum of the corresponding λ_e-weight spaces and let $\check{G}_{\geq n} \subseteq \check{G}$ denote the corresponding connected subgroup.

We notice that for $C_e \subseteq \check{G}$ the centralizer of e (not $i_e!$), $\mathcal{N}ilp^{\mathrm{spec}}_e = \mathrm{LS}^{\mathrm{restr}}_{C_e}$ parametrizes symmetric monoidal, right t-exact functors $\mathsf{Rep}(C_e) = \mathsf{QCoh}(\mathbf{B}C_e) \to \mathsf{qLisse}(X)$, cf. [AGKRRV1]. We have $\mathbf{B}C_e = \mathring{\check{\mathfrak{g}}}_{\geq 2}/\check{G}_{\geq 0}$ where $\mathring{\check{\mathfrak{g}}}_{\geq 2} := \mathrm{Ad}_{\check{G}_{\geq 0}}(e) \subseteq \check{\mathfrak{g}}_{\geq 2}$ is the open orbit. We can form $\mathbf{B}C^{\sim}_e$, the formal completion of $\mathring{\check{\mathfrak{g}}}_{\geq 2}/\check{G}_{\geq 0}$ in $\check{\mathfrak{g}}_{\geq 1}/\check{G}_{\geq 0}$, and then define $\mathcal{N}ilp^{\mathrm{spec},\sim}_e$ accordingly (with $\mathbf{B}C_e$ replaced by $\mathbf{B}C^{\sim}_e$ in the above).

[8]This story has a *-1-shifted microlocal* character, though it is all done quite explicitly.

Finally, observe that $\mathbf{G}_m$ acts via λ_e on $\mathbf{B}C_e^{\sim}$, hence on $\mathcal{N}ilp_e^{\mathrm{spec},\sim}$, so we can form *sheared* quasi-coherent sheaves $\mathsf{QCoh}^{\Rightarrow}$, cf. [AG, §A.2] or [BZSV, §6].

Then consider $\mathsf{QCoh}_{\mathcal{G}_e}(\mathrm{LS}_{H_e}^{\mathrm{restr}})$ as filtered by considering it as *right nilpotent* modules over the Clifford algebra, cf. §4.5.2. The associated graded category is a sheared version of $\mathsf{QCoh}(\mathrm{LS}_{H_e}^{\mathrm{restr},\sim})$, where $\mathrm{LS}_{H_e}^{\mathrm{restr},\sim}$ is defined to parametrize maps $\mathsf{QCoh}(V_e^{\wedge}/H_e) \to \mathsf{qLisse}(X)$, i.e., it is the moduli of an H_e-local system and a section of $V_{e,\sigma}$ in the formal neighborhood of 0.

There are evident maps

$$V_e^{\wedge}/H_e \xrightarrow{\xi \mapsto \xi + e} (\check{\mathfrak{g}}_{\geq 1})^{\wedge}_{\mathring{\check{\mathfrak{g}}}_{\geq 2}}/\check{G}_{\geq 0} = \mathbf{B}C_e^{\sim}, \quad \mathrm{LS}_{H_e}^{\mathrm{restr},\sim} \to \mathcal{N}ilp_e^{\mathrm{spec},\sim}$$

that latter of which induces a pullback functor

$$\mathsf{QCoh}(\mathcal{N}ilp_e^{\mathrm{spec},\sim}) \to \mathsf{QCoh}(\mathrm{LS}_{H_e}^{\mathrm{restr},\sim}). \tag{4.6.2}$$

We prove that the spectral Whittaker coefficient is filtered for the filtrations considered above and on associated graded is a sheared version of (4.6.2). In other words, the spectral Whittaker coefficient is a quantum analogue of the simple, geometric functor (4.6.2) (where we can ignore $\sim$ if we restrict to even nilpotents).

We also show that (4.6.2) induces an isomorphism on traces of Frobenius. As a first approximation to this, note that the $\mathbf{G}_m$ action on $\mathcal{N}ilp_e^{\mathrm{spec}}$ is contracting with fixed points $\mathrm{LS}_{H_e}^{\mathrm{restr}}$. (The actual argument is in the spirit of Theorem L.)

Functoriality of traces then implies that both sides of Theorem J have filtrations, and that the map is an isomorphism on associated graded. We prove that although the filtrations are unbounded from below, they are bounded in each cohomological degree, so are complete; therefore, the gr calculation suffices.

5. More on arithmetic local systems

We now study the algebraic geometry of the moduli spaces of arithmetic local systems. We then indicate the proofs of Theorem G and Theorem E.

5.1. Vanishing results.

5.1.1. We have the following result, which can be thought of as treating the case of *even* nilpotent elements.

Theorem L. *For any linearly reductive group H, the derived global sections $\Gamma(\mathrm{LS}_H^{\mathrm{arthm}}, \mathcal{O})$ is concentrated in cohomological degree 0, i.e., $H^i(\mathrm{LS}_H^{\mathrm{arthm}}, \mathcal{O}) = 0$ for $i \neq 0$. Moreover, $H^0(\mathrm{LS}_H^{\mathrm{arthm}}, \mathcal{O})$ is reduced.*

The argument is similar to the proof of [AGKRRV1, Prop. 25.1.7]. Briefly, the idea is that $\mathrm{LS}_H^{\mathrm{arthm}}$ is smooth in a neighborhood of any *ι-pure* local system, so this locus is understandable. We reduce to this case using [Laf1], the weight filtration, and simple GIT.

5.1.2. *Generalization.* In the above setup, assume now that H is equipped with a symplectic representation V. As in §4, there is a line bundle $\mathcal{L}_V$ on $\mathrm{LS}_H^{\mathrm{arthm}}$ whose tensor square is $\mathcal{O}$.

Theorem M. *(1) $\mathrm{LS}_H^{\mathrm{arthm}}$ is the disjoint union of two open substacks $\mathrm{LS}_{H,V+}^{\mathrm{arthm}} \coprod \mathrm{LS}_{H,V-}^{\mathrm{arthm}}$ where a point σ lies in $\mathrm{LS}_{H,V+}^{\mathrm{arthm}}$ (resp. $\mathrm{LS}_{H,V-}^{\mathrm{arthm}}$) if and only if its semi-simplification does, which occurs if and only the resulting character $\epsilon_V : S_\sigma := \mathrm{Aut}(\sigma) \to \mu_2$ corresponding to $\mathcal{L}_V|_{\mathbf{B}S_\sigma}$ (cf. §4.6.5) is trivial (resp. non-trivial).*

(2) $\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{H,V-}, \mathcal{L}_V) = 0$.

(3) $\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{H,V+}, \mathcal{L}_V)$ *is concentrated in cohomological degree* 0 *and is a rank 1 projective module over* $\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{H,V+}, \mathcal{O})$ *(which is concentrated in cohomological degree* 0 *by Theorem L).*

The general methodology is the same as the proof of Theorem L, though finer analysis is needed.

5.1.3. *First applications.* First, we note that Theorem L and Theorem M combined with (4.6.1) clearly imply Theorem E.

Second, we observe that Theorem M combined with Lemma K, Conjecture 3.4.1, and the expectation (4.1.1) imply Arthur's multiplicity formula for unramified automorphic forms. In this sense, we see that geometric Langlands in characteristic p implies an ℓ-adic version of Arthur's multiplicity formula, and plausibly could be applied to the original formulation in the near future.

5.2. **Proof of Theorem G.**

5.2.1. First, we have the elementary lemma:

Lemma 5.2.1. *Let* A *be a finitely generated, reduced commutative algebra over the field* $\mathbf{e}$*. Let* P *be a projective* A*-module. Suppose there is an element* $v \in P$ *such that* $A \cdot v$ *is 1-dimensional over* $\mathbf{e}$*.*

Then there is a decomposition $A = \mathbf{e} \times \overline{A}$ *of commutative algebras such that the action of* A *on* v *factors through the first factor.*

5.2.2. To simplify the discussion, we assume G is a semi-simple group, so we can ignore the minor games with the nebentypus.

Now suppose $f \in \mathcal{A}ut^{\mathrm{unr}}_{G,\text{fake-disc},e}$ is an eigenfunction for the excursion algebra $\mathcal{E}xc_{\check{G}} := \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \mathcal{O})$.

By (4.6.1), the action of $\mathcal{E}xc_{\check{G}}$ on $\mathrm{gr}_e \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ extends to an action of $\mathcal{E}xc_{H_e} := \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{H_e}, \mathcal{O})$.

Using Vinberg's results [Vin], one can show that the map $\mathcal{E}xc_{\check{G}} \to \mathcal{E}xc_{H_e}$ is finite. Therefore, $\mathcal{E}xc_{H_e} \cdot f$ is finite-dimensional over $\mathbf{e}$; we can further diagonalize this action to reduce to the case where f is an eigenfunction for $\mathcal{E}xc_{H_e}$.

Using GIT, we can identify the points of $\mathrm{Spec}(\mathcal{E}xc_{H_e})$ with the space of *semi-simple* continuous representations $\mathcal{W}_X \to H_e(\mathbf{e})$ up to conjugacy. In particular, we obtain the semi-simple representation $\tilde{\sigma}_f : \mathcal{W}_X \to H_e(\mathbf{e})$, and our task is to show that it is irreducible.

By Theorem M, $\mathrm{gr}_e \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ is a projective module over $\mathcal{E}xc_{H_e}$ (it locally has ranks 0 and 1). By Theorem L, $\mathcal{E}xc_{H_e}$ is reduced.

Therefore, we can apply Lemma 5.2.1 to see that the homomorphism $\mathcal{E}xc_{H_e} \to \mathbf{e}$ defined by f corresponds to a (reduced) isolated point in $\mathrm{Spec}(\mathcal{E}xc_{H_e})$. (As an aside, Theorem F follows at this point.)

Now suppose $\tilde{\sigma}_f$ were not irreducible. The motto is that in this case we can *vary the Frobenius* of $\tilde{\sigma}_f$ to contradict the isolatedness above.

More precisely, there is a non-trivial homomorphism $\lambda : \mathbf{G}_m \to S_{\tilde{\sigma}_f} \subseteq H_e$, where $S_{\tilde{\sigma}_f}$ is the centralizer of $\tilde{\sigma}_f$.

For $t \in \mathbf{G}_m$, define ${}^t\tilde{\sigma}_f : \mathcal{W}_X \to H_e(\mathbf{e})$ by sending $(g \in \mathcal{W}_X) \mapsto \lambda(t^{\log|g|}) \cdot \tilde{\sigma}_f(g)$; here $\log|-|$ is just notation for the projection $\mathcal{W}_X \to \mathbf{Z}$. This defines a map $\mathbf{G}_m \to \mathrm{LS}^{\mathrm{arthm}}_{H_e} \to \mathrm{Spec}(\mathcal{E}xc_{H_e})$

sending $1 \in \mathbf{G}_m$ to $\tilde{\sigma}_f$. We claim the resulting composition is non-constant, contradicting the fact that $\tilde{\sigma}_f$ was an isolated point.

Indeed, this follows from the elementary lemma:

Lemma 5.2.2. *Suppose $\lambda : \mathbf{G}_m \to GL_n$ is a non-constant homomorphism. Suppose $T \in GL_n$ is a matrix commuting with $\lambda(t)$ for every t. Then for some $i \geq 0$, the polynomial function $t \mapsto \operatorname{tr} \Lambda^i(\lambda(t) \cdot T)$ is non-constant.*

Remark 5.2.3. There is a subtlety in this logic worth highlighting. Suppose we are in the simplest case where $e = 0$ and we wish to prove that a generic automorphic form satisfies naive Ramanujan. Then the above argument crucially uses the fact that $\mathrm{gr}_0 \mathcal{A}ut^{\mathrm{unr}}_{G,c} \to \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ is *injective*, i.e., essentially the full force of Theorem E. Therefore, even in this case, our argument needs our study of gr_e for non-zero nilpotents e.

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Yale University, Department of Mathematics, 219 Prospect St, New Haven, CT 06511

Email address: `sam.raskin@yale.edu`