THE ARINKIN-GAITSGORY TEMPEREDNESS CONJECTURE

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Abstract. Arinkin and Gaitsgory defined a category of *tempered* D-modules on Bun_G that is conjecturally equivalent to the category of quasi-coherent (not ind-coherent!) sheaves on $\mathrm{LocSys}_{\check{G}}$. However, their definition depends on the auxiliary data of a point of the curve; they conjectured that their definition is independent of this choice. Beraldo has outlined a proof of this conjecture that depends on some technology that is not currently available. Here we provide a short, unconditional proof of the Arinkin-Gaitsgory conjecture.

Contents

1. Introduction

1.1. Statement of the main theorem.

1.1.1. Let X be a geometrically connected, smooth, and projective curve over a field k of characteristic 0. Let G be a split reductive group over k. Let Bun_G denote the moduli space of G-bundles on X, and let $D(\mathrm{Bun}_G)$ denote the DG category of D-modules on Bun_G.

Let $\check{G}$ denote the Langlands dual group to G, and let $\mathrm{LocSys}_{\check{G}}$ denote the moduli space of $\check{G}$-bundles on X with connection.

1.1.2. Let us begin by recalling some context from geometric Langlands.

Recall the geometric Langlands conjecture:

$$D(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \tag{1.1.1}$$

which was given in this form by [AG], following Beilinson-Drinfeld.

The right hand side has a subcategory $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$, and the left hand side should have a parallel such subcategory. Following [AG], we refer to this putative subcategory of $D(\mathrm{Bun}_G)$ as the subcategory of *tempered* D-modules on Bun_G.

There are various (not obviously equivalent) proposals for the tempered subcategory. One was given in [AG] §12, using derived geometric Satake. It is dependent on a choice of point $x \in X(k)$; we denote the resulting subcategory as $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$. As in [AG], a geometric Langlands equivalence (1.1.1) that is equivalent with derived Satake at x will necessarily match $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ with $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$.

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1.1.3. We can now state our main theorem.

Theorem 1.1.3.1. *The subcategory* $D(\mathrm{Bun}_G)^{x-\mathrm{temp}} \subseteq D(\mathrm{Bun}_G)$ *is independent of the choice of point* x.

This result was proposed in [AG] Conjecture 12.7.5.

1.2. Relation to work of Beraldo.

1.2.1. A strategy of proof for Theorem 1.1.3.1 was outlined by Dario Beraldo already in 2015, yielding deeper results. We describe the ingredients for his approach below.

1.2.2. Roughly speaking, Beraldo's approach proceeds as follows.

Beraldo has explained that a Ran space (or *factorizable*) version of derived Satake would provide additional symmetries of $D(\mathrm{Bun}_G)$, refining Gaitsgory's spectral action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$. Specifically, in [Ber4], has constructed a monoidal category $\mathbb{H}(\mathrm{LocSys}_{\check{G}})$ receiving a monoidal functor from $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$, and has conjectured that the action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ extends to $\mathbb{H}(\mathrm{LocSys}_{\check{G}})$. He has observed that such an extension would yield Theorem 1.1.3.1, and that such an extension should follow from factorizable derived Satake (see [Ber1] §1.4.2 for related discussion, and [Ber2] for a precise assertion in the Betti setting).

1.2.3. Unfortunately, the factorizable derived Satake theorem has been slow to appear. It was claimed more than a decade ago by Gaitsgory-Lurie, and again more recently by Justin Campbell and the second author, where it is currently work in progress. In particular, at the time we are writing this, a definition of the spectral side has not yet appeared publicly in written form. So the full derivation of the action of Beraldo's $\mathbb{H}$ has remained somewhat heuristic.

1.2.4. Our purpose here is to provide a simple, unconditional proof of Theorem 1.1.3.1, sidestepping Beraldo's category $\mathbb{H}$ and factorizable Satake.

In particular, our argument does not resolve Beraldo's deep conjecture regarding the action of $\mathbb{H}$ on $D(\mathrm{Bun}_G)$. This remains an open problem, for which Beraldo's suggestion of using factorizable Satake (once available) continues to appear to be the most plausible strategy. Our work also does not settle other[1] applications of Beraldo's conjecture.

1.3. Outline of the argument.

1.3.1. The main ideas of our argument proceed as following.

1.3.2. For our point x, let $\mathcal{H}_x^{sph}$ denote the associated (derived) spherical Hecke category. There is a certain object $\mathfrak{AT}_x \in \mathcal{H}_x^{sph}$, which we call the *anti-tempered unit* following [Ber5].

By definition, $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ is the kernel of the corresponding Hecke functor:

$$\mathfrak{AT}_x \star - : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G).$$

1.3.3. The point x can be varied in the above description.

Specifically, there is a functor:

$$\mathfrak{AT}_X : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$$

whose fiber at x is the original functor $\mathfrak{AT}_x$, and similarly for any other point.

[1] See e.g. [Ber3] for discussion of how an action of $\mathbb{H}$ in the setting of [AGK$^+$1] (and particularly [AGK$^+$2]) would yield (arithmetic) Arthur parameters for unramified automorphic representations.

1.3.4. Roughly speaking, our idea is that (in a suitable sense) the functor $D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$ yields objects that are locally constant along X, so the kernels of $\mathfrak{AT}_x$ and $\mathfrak{AT}_X$ coincide.

This is easier to explain in a slightly different context – that of sheaves with nilpotent singular support of [AGK$^+$1]. With notation as in *loc. cit.*, the corresponding Hecke functor:

$$\mathfrak{AT}_X : \mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathsf{Shv}(\mathrm{Bun}_G \times X)$$

maps into $\mathsf{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathsf{qLisse}(X)$ by universality of the anti-tempered unit and the Nadler-Yun theorem [AGK$^+$1] Theorems 10.2.8 and 10.5.2 (which are following [NY]). If e.g. we worked with complex curves, this would mean that the functors $\mathfrak{AT}_x$ and $\mathfrak{AT}_y$ are the same up to choosing a path between x and y, and the Tannakian formalism applies in general.[2]

In the D-module setting, we use Gaitsgory's spectral action from [Gai2] to essentially reduce to considering Hecke eigensheaves, and then proceed from there. The reduction is in a similar spirit to [AGK$^+$1] §14.3-4.

Remark 1.3.4.1. With that said, this note is logically independent of [AGK$^+$1]. Indeed, all of the ingredients in our argument were already available when Arinkin-Gaitsgory formulated their conjecture.

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2. Preliminary material

Below, we collect some notation and basic constructions.

We assume the reader is generally familiar with commonly used tools in de Rham geometric Langlands, referring to [Gai3] for an introduction to these ideas.

In what follows, X is a geometrically connected, smooth, projective curve over k. For $x \in X(k)$, we let $i_x : \mathrm{Spec}(k) \to X$ denote the corresponding embedding. We let $\mathrm{Ran} = \mathrm{Ran}_X$ denote the Ran space of X.

2.1. **Hecke functors.** We recall some preliminary constructions with Hecke functors parametrized by points of X.

Below, we work over powers of the curve and Ran space. For our point $x \in X(k)$, we let $\mathfrak{L}_x^+ G$ (resp. $\mathfrak{L}G$, resp. $\mathrm{Gr}_{G,x}$) denote the arc group (resp. loop group, resp. affine Grassmannian) based at this point. For a finite set I, let $\mathfrak{L}_{X^I}^+ G$ (resp. $\mathfrak{L}_{X^I} G$, resp. Gr_{G,X^I}) denote the standard corresponding space over X^I.

2.1.1. For a finite set I, let $\mathcal{H}_{X^I}^{sph} := D(\mathrm{Gr}_{G,X^I})^{\mathfrak{L}_{X^I}^+ G}$. Similarly, we let $\mathcal{H}_{\mathrm{Ran}}^{sph}$ denote the Ran space version of the spherical Hecke category, and $\mathcal{H}_x^{sph}$ for the spherical category at a point x.

We recall that $\mathcal{H}_{\mathrm{Ran}}^{sph}$ is a monoidal DG category acting canonically on $D(\mathrm{Bun}_G)$. We denote the product on $\mathcal{H}_{\mathrm{Ran}}^{sph}$ and its action on $D(\mathrm{Bun}_G)$ by $- \star -$.

[2] In particular, this sketch provides a genuine argument in the $\mathsf{Shv}_{\mathrm{Nilp}}$ setting, whether constructible (as in [AGK$^+$1]) or not (as in [BZN], [NY]); the Betti case may also be deduced directly from Beraldo's ideas via [Ber2]. It should also be possible to adapt [Ber2] to the constructible [AGK$^+$1] setting, but this has not yet been done as far as we know.

2.1.2. Let $\mathcal{F} \in \mathcal{H}^{sph}_{X^I}$ be given.

On the one hand, $\mathcal{F}$ defines an object of $\mathcal{H}^{sph}_{\mathrm{Ran}}$, so a Hecke functor $\mathcal{F} \star - : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$. There is also a closely related functor:

$$\mathrm{Hecke}_{\mathcal{F}} : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X^I)$$

constructed as follows. We have a standard Hecke action functor:

$$\mathcal{H}^{sph}_{X^I} \otimes D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G).$$

Considering the left hand side as a $(D(X^I), \overset{!}{\otimes})$-module (via the action on the first functor), this action lifts uniquely:

$$\begin{array}{ccc} \mathcal{H}^{sph}_{X^I} \otimes D(\mathrm{Bun}_G) & \dashrightarrow & D(\mathrm{Bun}_G) \otimes D(X^I) \simeq D(\mathrm{Bun}_G \times X^I) \\ & \searrow & \downarrow {\scriptstyle \mathrm{id} \otimes C^{\bullet}_{dR}(X^I, -)} \\ & & D(\mathrm{Bun}_G). \end{array}$$

of $D(X^I)$-module cateories. Finally, inserting $\mathcal{F}$ on the first tensor factor (in the dotted arrow above) gives the desired functor $\mathrm{Hecke}_{\mathcal{F}}$.

We explicitly note that composing $\mathrm{Hecke}_{\mathcal{F}}$ with de Rham cohomology along X^I gives $\mathcal{F} \star -$.

2.1.3. We remind the category $\mathsf{Rep}(\check{G})_{X^I}$ from [Ras] §6, and the construction of the *naive Satake functor*:

$$\mathcal{S}_{X^I} : \mathsf{Rep}(\check{G})_{X^I} \to \mathcal{H}^{sph}_{X^I}.$$

Similarly, we let:

$$\mathcal{S}_{\mathrm{Ran}} : \mathsf{Rep}(\check{G})_{\mathrm{Ran}} \to \mathcal{H}^{sph}_{\mathrm{Ran}}$$

denote the Ran space version, constructed out of the above functors.

2.1.4. We will need the following technical notion in what follows.

Definition 2.1.4.1. The subcategory $\mathcal{H}^{sph,aULA}_{X^I} \subseteq \mathcal{H}^{sph}_{X^I}$ of *almost ULA* objects the full (non-cocomplete) subcategory generated under finite colimits and direct summands by applying $\mathcal{S}_{X^I}$ to objects of $\mathsf{Rep}(\check{G})_{X^I}$ ULA over X^I. The subcategory $\mathcal{H}^{sph,qULA}_{X^I} \subseteq \mathcal{H}^{sph}_{X^I}$ of *quasi-ULA* objects is the full subcategory generated under filtered colimits by almost ULA objects.

Remark 2.1.4.2. We refer to [Ras] Appendix A and §6 for a convenient discussion of ULA objects in this setting.

Remark 2.1.4.3. Recall that e.g., the skyscraper sheaf $\delta_1 \in \mathcal{H}^{sph}_x$ at the origin $1 \in \mathrm{Gr}_{G,x}$ is not compact; rather, it is *almost compact* in the technical sense. For similar reasons, the standard spherical sheaves over X^I are not literally ULA over X^I; we use the term *almost ULA* in parallel with *almost compact*.

2.2. **Intermediate results.**

We now formulate two intermediate results, from which we easily deduce Theorem 1.1.3.1.

2.2.1. *Local constancy.* Let $\mathcal{F} \in \mathcal{H}_X^{sph}$ be given. For $x \in X(k)$, let $\mathcal{F}_x \in \mathcal{H}_x^{sph}$ denote the !-fiber of $\mathcal{F}$ at x.

We let:

$$\mathrm{Hecke}_{\mathcal{F}} : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$$

denote the following functor.

By construction, the composition:

$$D(\mathrm{Bun}_G) \xrightarrow{\mathrm{Hecke}_{\mathcal{F}}} D(\mathrm{Bun}_G \times X) \xrightarrow{(\mathrm{id} \times i_x)^!)} D(\mathrm{Bun}_G)$$

is the usual Hecke functor:

$$\mathcal{F}_x \star - : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$$

defined by $\mathcal{F}_x$.

2.2.2. With the above preliminary constructions out of the way, we can state:

Theorem 2.2.2.1. *Suppose $\mathcal{F} \in \mathcal{H}_X^{sph}$ is quasi-ULA. Then* $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}}) = \mathrm{Ker}(\mathcal{F}_x \star -)$.

This is the main technical result of the present paper; its proof is given in §3.

2.2.3. *Projectors.* We follow terminology from [Ber5].

Define the *tempered unit (at x)* $\mathbb{1}_x^\tau \in \mathcal{H}_x^{sph}$ as follows. We recall the *derived Satake theorem* of [BF], which asserts:

$$D(\mathrm{Gr}_{G,x})^{\mathfrak{L}_x^+ G} \simeq \mathsf{IndCoh}_{\mathrm{Nilp}}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \subseteq \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}).$$

There are adjoint functors:

$$\Xi : \mathsf{QCoh}(\mathbb{B}\check{G})^{\mathbb{S}^2}) \rightleftarrows \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) : \Psi.$$

Moreover, the unit object in $\mathcal{H}_x^{sph}$ corresponds to the trivial representation $\mathsf{triv} \in \mathsf{Rep}(\check{G})^\heartsuit = \mathsf{IndCoh}_{\mathrm{Nilp}}((\mathbb{B}\check{G})^{\mathbb{S}^2})^\heartsuit$. We then take $\mathbb{1}_x^\tau$ to correspond to $\Xi\Psi(\mathsf{triv})$.

2.2.4. By definition, there is a canonical map:

$$\mathbb{1}_x^\tau \to \delta_1 \in \mathcal{H}_x^{sph}.$$

We then define the *anti-tempered unit (at x)* as:

$$\mathfrak{AT}_x := \mathrm{Ker}(\mathbb{1}_x^\tau \to \delta_1).$$

By definition, an object $\mathcal{G} \in D(\mathrm{Bun}_G)$ lies in $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ if and only if $\mathfrak{AT}_x \star \mathcal{G} = 0$.

2.2.5. We now have the following basic observation.

Lemma 2.2.5.1. *There is a canonical object $\mathfrak{AT} \in \mathcal{H}_X^{sph,qULA}$ (not depending on the choice of point $x \in X(k)$) with !-fiber $\mathfrak{AT}_x \in \mathcal{H}_x^{sph}$ at x.*

Proof. This essentially follows from the universality of the construction of $\mathfrak{AT}_x$. We include more details below.

Let $\widehat{\mathcal{D}}$ be *some* formal disc. Let Aut denote the group indscheme of its automorphisms. Let $\mathrm{Aut}^\star \subseteq \mathrm{Aut}$ denote the group subscheme of automorphisms fixing the closed point of $\widehat{\mathcal{D}}$; we remind that $\mathrm{Aut}^\star \to \mathrm{Aut}$ is an isomorphism modulo nilpotent ideals. The group Aut acts strongly on $\mathcal{H}_x^{sph}$.

By a standard construction, any Aut-equivariant object $\mathcal{F}_0$ of $\mathcal{H}^{sph}$ (the spherical category corresponding to $\widehat{\mathcal{D}}$) gives rise to an object $\mathcal{F} \in \mathcal{H}_X^{sph}$. We claim any resulting such objects are quasi-ULA; indeed, $(\mathcal{H}^{sph})^{\mathrm{Aut}}$ is generated under colimits by objects in the heart of its t-structure, and the heart of its t-structure is exactly $\mathsf{Rep}(\check{G})^{\heartsuit}$, and these objects map to the standard (almost ULA) objects of $\mathcal{H}_X^{sph}$ (cf. [Gai1] Proposition 1).

Next, we observe that we have a projection $\pi : \mathrm{Aut}^\star \to \mathbb{G}_m$ with pro-unipotent kernel. Moreover, every object of $\mathcal{H}^{sph}$ is automatically equivariant with respect to the kernel π; indeed, by pro-unipotence, this can be checked on generators, and then it follows in the previous paragraph. Moreover, this same logic shows every object is Aut-monodromic, or equivalently (after a choice of coordinate), $\mathbb{G}_m$-monodromic for $\mathbb{G}_m$ acting by loop rotation.

We now observe that $\mathcal{H}^{sph}$ carries a canonical endofunctor corresponding to $\Xi\Psi$ on the spectral side. One readily checks that $\Xi\Psi$ is (canonically) strongly Aut-equivariant by using [BF], noting that their form of derived Satake describes the loop equivariant category, so can be understood to be Aut-equivariant in a suitable sense by the above. This concludes the argument.

□

We now observe that Theorem 1.1.3.1 follows immediately from Lemma 2.2.5.1 and Theorem 2.2.2.1.

Remark 2.2.5.2. To avoid the subtleties involved in the above argument, one could also proceed as follows. First, by [Ber5] Theorem 1.4.8, $\mathrm{Ker}(\mathfrak{AT}_x \star -) = \mathrm{Ker}(\mathcal{WS}_0 \star -)$ for $\mathcal{WS}_0$ as in *loc. cit.*, i.e., one takes the unit spherical Whittaker sheaf in $\mathsf{Whit}_x^{sph} := D(\mathrm{Gr}_{G,x})^{\mathfrak{L}_x N,\psi}$ and $*$-averages it to $D(\mathrm{Gr}_G)^{\mathfrak{L}_x^+ G}$. This description of $\mathcal{WS}_0$ manifestly extends to define a quasi-ULA (even almost ULA) object $\mathcal{WS}_{0,X} \in \mathcal{H}_X^{sph}$, to which we could then apply Theorem 2.2.2.1.

2.3. **Gaitsgory's spectral action.** We now review the main results of [Gai2]; see also [Gai3] §4.3-4.5 and §11.1.

First, there is a canonical symmetric monoidal functor:

$$\mathrm{Loc} : \mathsf{Rep}(\check{G})_{\mathrm{Ran}} \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

from *loc. cit.* It admits a fully faithful continuous right adjoint (cf. *loc. cit.*); therefore, the restriction functor:

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})\text{–}\mathsf{mod} \to \mathsf{Rep}(\check{G})_{\mathrm{Ran}}\text{–}\mathsf{mod}$$

is fully faithful. (Here modules are taken in the symmetric monoidal category DGCat_{cont} of cocomplete DG categories).

On the other hand, there is an action of $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$ on $D(\mathrm{Bun}_G)$ that is constructed as:

$$\mathsf{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathcal{S}_{\mathrm{Ran}}} \mathcal{H}_{\mathrm{Ran}}^{sph} \curvearrowright D(\mathrm{Bun}_G).$$

Theorem 2.3.0.1 (Gaitsgory, [Gai2], [Gai3] Theorem 4.5.2)**.** *The above action of* $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$ *on* $D(\mathrm{Bun}_G)$ *factors through a (necessarily unique) action of* $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ *via the localization functors.*

Remark 2.3.0.2. Related results in other contexts have also recently been obtained: see [NY], [AGK$^+$1], [FS]. In these other contexts, the proofs are more conceptual.

We again use $- \star -$ to denote the action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ on $D(\mathrm{Bun}_G)$.

3. Proof of Theorem 2.2.2.1

As above, the proof of Theorem 1.1.3.1 reduces to Theorem 2.2.2.1. The purpose of this section is to prove the latter result.

3.1. **Setup.** It is clear that $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}}) \subseteq \mathrm{Ker}(\mathcal{F}_x \star -)$. So it remains to show the converse. We therefore fix $\mathcal{G} \in \mathrm{Ker}(\mathcal{F}_x \star -) \subseteq D(\mathrm{Bun}_G)$ and aim to show that $\mathcal{G} \in \mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})$.

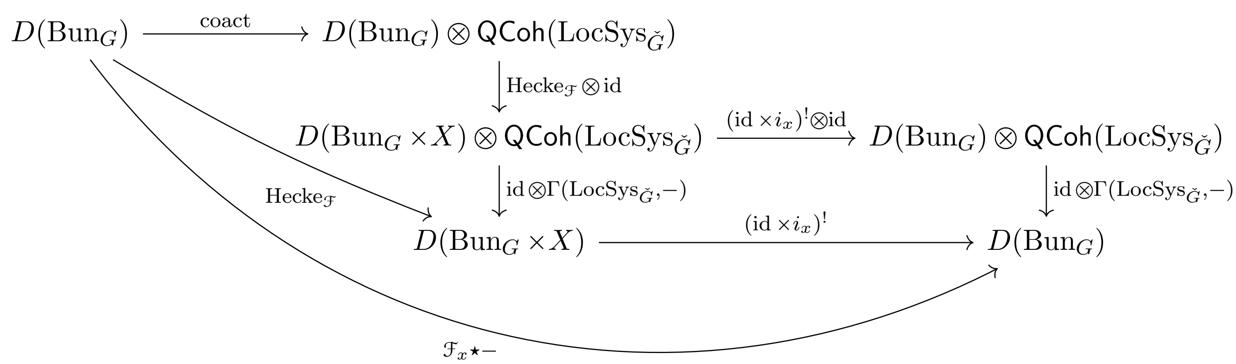
3.1.1. We have an action functor:

$$\mathrm{act} : \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G).$$

As the first factor is canonically self-dual, we obtain a functor:

$$\mathrm{coact} : D(\mathrm{Bun}_G) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes D(\mathrm{Bun}_G).$$

3.1.2. We now form the following commutative diagram, whose analysis is central to the argument.



We consider $\mathcal{G}$ as an object of the top left term. By assumption, it is mapped to 0 in the bottom right term. Our goal is to show that it maps to zero in the bottom left term. We will do so by showing the following:

- (§3.2) $\mathcal{G}$ maps to zero in the top term of the rightmost column of the diagram, i.e.:
$$((\mathrm{id} \times i_x^!) \otimes \mathrm{id})(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\,\mathrm{coact}(\mathcal{G}) = 0. \tag{3.1.1}$$
- (§3.3) $\mathcal{G}$ maps to zero in the middle term of the second column of the diagram, i.e.:
$$(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\,\mathrm{coact}(\mathcal{G}) = 0. \tag{3.1.2}$$

Clearly the latter claim suffices.

3.2. **Step 1.** We begin by establishing (3.1.1).

3.2.1. *Reduction.* We have the following standard observation.

Lemma 3.2.1.1. *Suppose that $\mathcal{Y}$ is a QCA algebraic stack in the sense of* [DG2] *and suppose that $\mathcal{C}$ is a DG category. Then an object:*

$$\mathcal{F} \in \mathcal{C} \otimes \mathsf{QCoh}(\mathcal{Y})$$

is zero if and only if for every $\mathcal{E} \in \mathsf{QCoh}(\mathcal{Y})$, we have:

$$\big(\mathrm{id} \otimes \Gamma(\mathcal{Y}, -)\big)(\mathcal{F} \otimes \mathcal{E}) = 0 \in \mathcal{C}.$$

Here we consider $\mathcal{C} \otimes \mathsf{QCoh}(\mathcal{Y})$ as a module category for $\mathsf{QCoh}(\mathcal{Y})$ in the evident way, writing the action the right.

Proof. More generally, for a dualizable DG category $\mathcal{D}$, and object:

$$\mathcal{F} \in \mathcal{C} \otimes \mathcal{D}$$

is zero if and only if $(\mathrm{id} \otimes \lambda)(\mathcal{F}) = 0 \in \mathcal{C}$ for every $\lambda \in \mathcal{D}^\vee$, as a functor $\mathsf{Vect} \xrightarrow{\mathcal{F}} \mathcal{C} \otimes \mathcal{D}$ is equivalent by duality to a functor $\mathcal{D}^\vee \to \mathcal{C}$. Now the claim follows from the existence of perfect self-duality for QCA stacks, cf. [DG2] §4.3.7.

□

Therefore, it suffices to show that for any $\mathcal{E} \in \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$, we have:

$$(\mathrm{id} \otimes \Gamma(\mathrm{LocSys}_{\check{G}}, -))\Bigg(\Big(((\mathrm{id} \times i_x^!) \otimes \mathrm{id})(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id}) \,\mathrm{coact}(\mathcal{G})\Big) \otimes \mathcal{E}\Bigg) = 0. \tag{3.2.1}$$

3.2.2. We now manipulate the left hand side of (3.2.1).

We have:

$$(\mathrm{id} \otimes \Gamma(\mathrm{LocSys}_{\check{G}}, -))\Bigg(\Big(((\mathrm{id} \times i_x^!) \otimes \mathrm{id})(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id}) \,\mathrm{coact}(\mathcal{G})\Big) \otimes \mathcal{E}\Bigg) =$$

$$(\mathrm{id} \otimes \Gamma(\mathrm{LocSys}_{\check{G}}, -))((\mathrm{id} \times i_x^!) \otimes \mathrm{id})(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\Big(\mathrm{coact}(\mathcal{G}) \otimes \mathcal{E}\Big).$$

We now observe that coact is a morphism of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$-bimodules, considering $D(\mathrm{Bun}_G)$ as a bimodule via the spectral action and symmetric monoidality of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$. Therefore, we can rewrite the above as:

$$(\mathrm{id} \otimes \Gamma(\mathrm{LocSys}_{\check{G}}, -))((\mathrm{id} \times i_x^!) \otimes \mathrm{id})(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id}) \,\mathrm{coact}(\mathcal{E} \star \mathcal{G}).$$

By the big diagram of §3.1.2, this term coincides with:

$$\mathcal{F}_x \star (\mathcal{E} \star \mathcal{G}).$$

Therefore, it suffices to show that this term vanishes.

3.2.3. By the above, it remains to show that $\mathrm{Ker}(\mathcal{F}_x \star -) \subseteq D(\mathrm{Bun}_G)$ is a $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$-submodule category. Reformulating this using Theorem 2.3.0.1, it suffices to show that it is a $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$-submodule category. I.e., we wish to show that for any $\mathcal{V} \in \mathsf{Rep}(\check{G})_{\mathrm{Ran}}$, $\mathcal{F}_x \star \mathcal{V} \star \mathcal{G} = 0$.

As $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$ is generated as a monoidal category by its subcategory $\mathsf{Rep}(\check{G})_X$, we can assume $\mathcal{V}$ lies in this subcategory. By excision, we can treat separately the cases where $\mathcal{V}$ is $*$-extended from $\mathsf{Rep}(\check{G})_{X \setminus x}$ and $\mathsf{Rep}(\check{G})_x$. In the former case, it follows as $\mathcal{V}$ commutes with $\mathcal{F}_x$ (Hecke functors at different points obviously commute). In the latter case, it follows as $\mathcal{V}$ commutes with $\mathcal{F}_x$, e.g., by the existence of the *pointwise* symmetric monoidal structure on the derived Satake category established in [BF].

This concludes the proof of (3.1.1).

3.3. **Step 2.** We now prove (3.1.2). This requires some digressions.

3.3.1. *Lisse sheaves.* Suppose $\mathcal{Y}$ is an Artin stack.

We define $\mathsf{Lisse}_{\mathcal{Y}}(X) \subseteq \mathsf{QCoh}(\mathcal{Y}) \otimes D(X)$ to be the full DG subcategory generated by under colimits by (finite rank) vector bundles on $\mathcal{Y} \times X_{dR}$. We consider objects of $\mathsf{Lisse}_{\mathcal{Y}}(X)$ as *$\mathcal{Y}$-families of lisse D-modules on X.*

Let $x \in X(k)$. We abuse notation in letting $i_x^!$ denote the composition:

$$\mathsf{Lisse}_{\mathcal{Y}}(X) \hookrightarrow \mathsf{QCoh}(\mathcal{Y}) \otimes D(X) \xrightarrow{\mathrm{id} \otimes i_x^!} \mathsf{QCoh}(\mathcal{Y}) \otimes \mathsf{Vect} = \mathsf{QCoh}(\mathcal{Y}).$$

We will use the following result.

Proposition 3.3.1.1. *Suppose $\mathcal{Y}$ is locally almost of finite type and eventually coconnective. Then the functor $i_x^!$ is conservative.*

More generally, for any dualizable DG category $\mathcal{C}$, the functor:

$$\mathrm{id}_{\mathcal{C}} \otimes i_x^! : \mathcal{C} \otimes \mathsf{Lisse}_{\mathcal{Y}}(X) \to \mathcal{C} \otimes \mathsf{QCoh}(\mathcal{Y})$$

is conservative.

Proof.

Step 1. First, we note that if $F : \mathcal{D}_1 \to \mathcal{D}_2 \in \mathsf{DGCat}_{cont}$ is conservative and $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is dualizable, then $\mathrm{id}_{\mathcal{C}} \otimes F : \mathcal{C} \otimes \mathcal{D}_1 \to \mathcal{C} \otimes \mathcal{D}_2$ is conservative. Indeed, we can rewrite this functor as:

$$\mathcal{C} \otimes \mathcal{D}_1 = \mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathcal{C}^\vee, \mathcal{D}_1) \xrightarrow{\varphi \mapsto F\varphi} \mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathcal{C}^\vee, \mathcal{D}_2) = \mathcal{C} \otimes \mathcal{D}_2$$

in which form it is manifestly conservative. Therefore, we are reduced to considering $\mathcal{C} = \mathsf{Vect}$ in the assertion.

Step 2. Next, suppose S is an eventually coconnective scheme locally almost of finite type. Let $|S|$ denote the set of points of its underlying topological space; for $s \in |S|$, we write $\kappa(s)$ for the residue field at this point, s for $\mathrm{Spec}(\kappa(s))$, and $i_s : s \to S$ for the structural morphism.

We then note that the functor:

$$\mathsf{QCoh}(S) \xrightarrow{\{i_s^*\}_{s\in|S|}} \prod_{s\in|S|} \mathsf{QCoh}(s)$$

is conservative. Indeed, this follows from [Lur] Lemma 2.6.1.3 and the conservativeness of the restriction $S^{cl} \hookrightarrow S$ (which is easy from eventual coconnectivity of S).

In our setting, let $\pi : S \to \mathcal{Y}$ be a flat cover. We find that the restriction functor:

$$\mathsf{QCoh}(\mathcal{Y}) \to \prod_{s\in|S|} \mathsf{QCoh}(s)$$

is conservative. By the same reasoning as before, for any dualizable DG category $\mathcal{D}$, the functor:

$$\mathcal{D} \otimes \mathsf{QCoh}(\mathcal{Y}) \to \mathcal{D} \otimes \prod_{s\in|S|} \mathsf{QCoh}(s) \xrightarrow{\simeq} \prod_{s\in|S|} \mathcal{D} \otimes \mathsf{QCoh}(s)$$

is conservative. In particular, this applies for $\mathcal{D} = D(X)$.

Step 3. By the above, we have a commutative diagram:

$$\begin{array}{ccccc} \mathsf{Lisse}_{\mathcal{Y}}(X) & \hookrightarrow & D(X) \otimes \mathsf{QCoh}(\mathcal{Y}) & \xrightarrow{i_x^! \otimes \mathrm{id}} & \mathsf{QCoh}(\mathcal{Y}) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{s\in|S|} \mathsf{Lisse}_s(X) & \hookrightarrow & \prod_{s\in|S|} D(X) \otimes \mathsf{QCoh}(s) & \xrightarrow{i_x^! \otimes \mathrm{id}} & \prod_{s\in|S|} \mathsf{QCoh}(s). \end{array}$$

The middle and right vertical arrows are conservative, so the same is true of the left vertical arrow. Therefore, to see that the top line is conservative, it suffices to show that for each $s \in |S|$, the functor:

$$i_x^! : \mathsf{Lisse}_s(X) \to \mathsf{QCoh}(s)$$

is conservative.

Therefore, we are reduced to the case where $S = \mathrm{Spec}(\kappa)$ for some field κ/k.

Step 4. Let $X_\kappa := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\kappa)$. Note that $D(X) \otimes \mathsf{Vect}_\kappa = D_{/\kappa}(X_\kappa)$, where we regard X_κ as a scheme over the field κ) and write $D_{/\kappa}$ to emphasize this (reminding that implicitly, the category of D-modules depends on the structural map to Spec of a field). Moreover, $X_{dR} \times \mathrm{Spec}(\kappa) = X_{\kappa,dR/\mathrm{Spec}(\kappa)}$, so $\mathsf{Lisse}_{\mathrm{Spec}(\kappa)}(X) \subseteq D_{/\kappa}(X_\kappa)$ is the subcategory of ($\mathrm{Spec}(\kappa)$-families of) lisse D-modules on X_κ, considering the latter as a scheme over $\mathrm{Spec}(\kappa)$.

This is all to say that we are reduced to the case where $\kappa = k$, as the only difference is notational.

Step 5. We are now essentially done: the functor $i_x^! : \mathsf{Lisse}(X) := \mathsf{Lisse}_{\mathrm{Spec}(k)}(X) \to \mathsf{Vect}$ is t-exact up to shift and is obviously conservative on the heart of the t-structure, so is conservative (as the t-structure on $\mathsf{Lisse}(X)$ is left separated).

□

3.3.2. We now observe the following.

Lemma 3.3.2.1. *For any quasi-ULA $\mathcal{F}$, the composition:*

$$D(\mathrm{Bun}_G) \xrightarrow{\text{coact}} D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id}} D(\mathrm{Bun}_G \times X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) = D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

maps into the subcategory:

$$D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X).$$

Proof. First, note that:

$$D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X) \to D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

is indeed fully faithful: e.g., the embedding $\mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X) \hookrightarrow D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ admits a continuous right adjoint by definition, so tensoring with it preserves fully faithfulness.

Now, by definition of quasi-ULAness, we are immediately reduced to considering the case where $\mathcal{F}$ is almost ULA. Such an object is cohomologically bounded, so we are reduced to the case where $\mathcal{F}$ is concentrated in degree zero.

In this case, $\mathcal{F}$ necessarily is a direct sum of terms of the form $\mathcal{S}_X(V \otimes \sigma)$ where $\sigma \in D(X)^\heartsuit$ is a finite rank local system, $V \in \mathsf{Rep}(\check{G})^\heartsuit$ is finite dimensional, we consider $V \otimes \sigma$ as an object of $\mathsf{Rep}(\check{G})_X$, and we remind that $\mathcal{S}_X$ denotes the geometric Satake functor (cf. [Ras] §6, especially Proposition 6.22.1 and Lemma 6.23.1).

Now observe that $\mathrm{Hecke}_{\mathcal{S}_X(V \otimes \sigma)}$ differs from $\mathrm{Hecke}_{\mathcal{S}_X(V \otimes \omega_X)}$ by applying $\mathrm{id}_{D(\mathrm{Bun}_G)} \otimes (\sigma \overset{!}{\otimes} -) \otimes \mathrm{id}_{\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})}$. Clearly this operation preserves the subcategory $D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X)$, so we may take $\mathcal{F} = \mathcal{S}_X(V \otimes \omega_X)$ instead. We simplify the notation by writing $\mathcal{F} = \mathcal{S}_X(V)$.

Next, recall that $V \in \mathsf{Rep}(\check{G})$ defines a canonical vector bundle $\mathcal{E}_V$ on $X_{dR} \otimes \mathrm{LocSys}_{\check{G}}$. We then observe that the compositions:

$$D(\mathrm{Bun}_G) \xrightarrow{\text{coact}} D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\mathrm{Hecke}_{\mathcal{S}(V)} \otimes \mathrm{id}} D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

and:

$$D(\mathrm{Bun}_G) \xrightarrow{\text{coact}} D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\mathrm{id} \otimes \mathcal{E}_V \otimes \mathrm{id}}$$
$$D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes (- \otimes -)}$$
$$D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

coincide (by construction[3] of Loc). The latter clearly maps into $D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X)$, as desired.

□

[3]Specifically, we use the following fact, which is tautological from the construction of Loc. Suppose $\mathcal{M} \in D(X)$. We obtain an object $V \otimes \mathcal{M} \in \mathsf{Rep}(\check{G})_X$. Let $\lambda_{\mathcal{M}} : D(X) \to \mathsf{Vect}$ be the functor Verdier dual to $\mathcal{M}$, i.e., the functor $C^\bullet_{dR}(X, \mathcal{M} \overset{!}{\otimes} -)$. Then $\mathrm{Loc}(V \otimes \mathcal{M}) \in \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ is (functorially in $\mathcal{M}$) calculated as the image of $\mathcal{E}_V$ under $\lambda_{\mathcal{M}} \otimes \mathrm{id} : D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$.

3.3.3. By Lemma 3.3.2.1, we have:

$$(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\, \mathrm{coact}(\mathcal{G}) \in D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X).$$

Moreover, by (3.1.1), this object vanishes when we apply $(\mathrm{id} \otimes i_x^!)$ to it. Therefore, by Lemma 3.2.1.1, we have:

$$(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\, \mathrm{coact}(\mathcal{G}) = 0.$$

Here we observe that $D(\mathrm{Bun}_G)$ is dualizable by [DG1], and that $\mathrm{LocSys}_{\check{G}}$ is eventually coconnective e.g. by [AG] §10. This concludes the proof of (3.1.2), hence of Theorem 2.2.2.1.

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