THE ARINKIN-GAITSGORY TEMPEREDNESS CONJECTURE

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Abstract. Arinkin and Gaitsgory defined a category of *tempered* D-modules on Bun_G that is conjecturally equivalent to the category of quasi-coherent (not ind-coherent!) sheaves on $\mathrm{LocSys}_{\check{G}}$. However, their definition depends on the auxiliary data of a point of the curve; they conjectured that their definition is independent of this choice. Beraldo has outlined a proof of this conjecture that depends on some technology that is not currently available. Here we provide a short, unconditional proof of the Arinkin-Gaitsgory conjecture.

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1. Introduction

1.1. Statement of the main theorem.

1.1.1. Let X be a geometrically connected, smooth, and projective curve over a field k of characteristic 0. Let G be a split reductive group over k. Let Bun_G denote the moduli stack of G-bundles on X, and let $D(\mathrm{Bun}_G)$ denote the DG category of D-modules on Bun_G.

Let $\check{G}$ denote the Langlands dual group to G, and let $\mathrm{LocSys}_{\check{G}}$ denote the moduli stack of $\check{G}$-bundles on X with connection.

1.1.2. Let us begin by recalling some context from geometric Langlands.

Recall the geometric Langlands conjecture:

$$D(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \tag{1.1.1}$$

which was given in this form by [AG], following Beilinson-Drinfeld.

The right hand side has a subcategory $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$, and the left hand side should have a parallel such subcategory. Following [AG], we refer to this putative subcategory of $D(\mathrm{Bun}_G)$ as the subcategory of *tempered* D-modules on Bun_G.

There are various (not obviously equivalent) proposals for the tempered subcategory. One was given in [AG] §12, using derived geometric Satake. It is dependent on a choice of point $x \in X(k)$; we denote the resulting subcategory as $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$. As in [AG], a geometric Langlands equivalence (1.1.1) that is compatible with derived Satake at x will necessarily match $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ with $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$.[1]

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[1]For more discussion of temperedness, see [Ber6] and [Ber5].

1.1.3. We can now state our main theorem.

Theorem 1.1.3.1. *The subcategory $D(\mathrm{Bun}_G)^{x-\mathrm{temp}} \subseteq D(\mathrm{Bun}_G)$ is independent of the choice of point x.*

This result was proposed in [AG] Conjecture 12.7.5.

1.2. **Relation to work of Beraldo.**

1.2.1. A strategy of proof for Theorem 1.1.3.1 was outlined by Dario Beraldo already in 2015, yielding deeper results. We describe the ingredients for his approach below.

1.2.2. Roughly speaking, Beraldo's approach proceeds as follows.

Beraldo has explained that a Ran space (or *factorizable*) version of derived Satake would provide additional symmetries of $D(\mathrm{Bun}_G)$, refining Gaitsgory's spectral action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$. Specifically, in [Ber4], he constructed a certain monoidal category $\mathbb{H}(\mathrm{LocSys}_{\check{G}})$ receiving a monoidal functor from $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$, and has conjectured that the action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ on $D(\mathrm{Bun}_G)$ extends to $\mathbb{H}(\mathrm{LocSys}_{\check{G}})$. He has further observed that such an extension would yield Theorem 1.1.3.1, and that such an extension should follow from factorizable derived Satake (see [Ber1] §1.4.2 for related discussion, and [Ber2] for a precise assertion in the Betti setting).

1.2.3. Unfortunately, the factorizable derived Satake theorem has been slow to appear. It was claimed more than a decade ago by Gaitsgory-Lurie, and again more recently by Justin Campbell and the second author, where it is currently work in progress. In particular, at the time we are writing this, a definition of the spectral side of factorizable derived Satake has not yet appeared publicly in written form. So the full derivation of the action of Beraldo's $\mathbb{H}$ has remained heuristic.

1.2.4. Our purpose here is to provide a simple, unconditional proof of Theorem 1.1.3.1, sidestepping Beraldo's category $\mathbb{H}$ and factorizable Satake.

In particular, our argument does not resolve Beraldo's deep conjecture regarding the action of $\mathbb{H}$ on $D(\mathrm{Bun}_G)$. This remains an open problem, for which Beraldo's suggestion of using factorizable Satake (once available) continues to appear to be the most plausible strategy. Our work also does not settle other[2] applications of Beraldo's conjecture.

1.3. **Outline of the argument.**

1.3.1. The main ideas of our argument proceed as following.

1.3.2. For our point x, let $\mathcal{H}_x^{\mathrm{sph}}$ denote the associated (derived) spherical Hecke category. There is a certain object $\mathfrak{AT}_x \in \mathcal{H}_x^{\mathrm{sph}}$, which we call the *anti-tempered unit* following [Ber5].

By definition, $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ is the kernel of the corresponding Hecke functor:

$$\mathfrak{AT}_x \star - : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G).$$

1.3.3. The point x can be varied in the above description.

Specifically, there is a functor:

$$\mathfrak{AT}_X : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$$

whose fiber at x is the original functor $\mathfrak{AT}_x$, and similarly for any other point.

[2]See e.g. [Ber3] for discussion of how an action of $\mathbb{H}$ in the setting of [AGKRRV1] (and particularly [AGKRRV2]) would yield (arithmetic) Arthur parameters for unramified automorphic representations.

1.3.4. Roughly speaking, our idea is that (in a suitable sense) the functor $D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$ yields objects that are locally constant along X, so the kernels of $\mathfrak{AT}_x$ and $\mathfrak{AT}_X$ coincide.

This is easier to explain in a slightly different context – that of sheaves with nilpotent singular support of [AGKRRV1]. Recall that the Nadler-Yun theorem asserts that for any $V \in \mathsf{Rep}(\check{G})$, the Hecke functor:

$$\mathrm{Hecke}_V : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Shv}(\mathrm{Bun}_G \times X)$$

maps into the subcategory:[3]

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \otimes \mathsf{qLisse}(X) \subseteq \mathsf{Shv}(\mathrm{Bun}_G \times X).$$

Here we refer to [AGKRRV1] for notation for various categories of sheaves. By universality of the anti-tempered unit, one deduced that the corresponding Hecke functor:

$$\mathfrak{AT}_X : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Shv}(\mathrm{Bun}_G \times X)$$

similarly maps into $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \otimes \mathsf{qLisse}(X)$. If e.g. we worked with complex curves, this would mean that the functors $\mathfrak{AT}_x$ and $\mathfrak{AT}_y$ are the same up to choosing a path between x and y, and the Tannakian formalism yields such an isomorphism in general (after extending k to a finite extension).[4]

In the D-module setting, we use Gaitsgory's spectral action from [Gai1] to essentially reduce to considering Hecke eigensheaves, and then proceed from there. The reduction is in a similar spirit to [AGKRRV1] §14.3-4.

Remark 1.3.4.1. With that said, this note is logically independent of [AGKRRV1]. Indeed, all of the ingredients in our argument were already available when Arinkin-Gaitsgory formulated their conjecture.

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2. Preliminary material

Below, we collect some notation and basic constructions.

We assume the reader is generally familiar with commonly used tools in de Rham geometric Langlands, referring to [Gai2] for an introduction to these ideas.

In what follows, X is a geometrically connected, smooth, projective curve over k. For $x \in X(k)$, we let $i_x : \mathrm{Spec}(k) \to X$ denote the corresponding embedding. We let $\mathrm{Ran} = \mathrm{Ran}_X$ denote the Ran space of X.

[3]Specifically, see the discussion in [AGKRRV1] §14.2.5, immediately following the statement of the cited theorem.

[4]In particular, this sketch provides a genuine argument in the $\mathsf{Shv}_{\mathcal{N}ilp}$ setting, whether constructible (as in [AGKRRV1]) or not (as in [BZN], [NY]); the Betti case may also be deduced directly from Beraldo's ideas via [Ber2]. It should also be possible to adapt [Ber2] to the constructible [AGKRRV1] setting, but this has not yet been done as far as we know.

2.1. **Hecke functors.** We recall some preliminary constructions with Hecke functors parametrized by points of X.

Below, we work over powers of the curve and Ran space. For our point $x \in X(k)$, we let $\mathfrak{L}_x^+ G$ (resp. $\mathfrak{L}G$, resp. $\mathrm{Gr}_{G,x}$) denote the arc group (resp. loop group, resp. affine Grassmannian) based at this point. For a finite set I, let $\mathfrak{L}_{X^I}^+ G$ (resp. $\mathfrak{L}_{X^I} G$, resp. Gr_{G,X^I}) denote the standard corresponding space over X^I.

2.1.1. For a finite set I, let $\mathcal{H}_{X^I}^{\mathrm{sph}} := D(\mathrm{Gr}_{G,X^I})^{\mathfrak{L}_{X^I}^+ G}$. Similarly, we let $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ denote the Ran space version of the spherical Hecke category, and $\mathcal{H}_x^{\mathrm{sph}}$ for the spherical category at a point x.

We recall that $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ is a monoidal DG category acting canonically on $D(\mathrm{Bun}_G)$. We denote the product on $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ and its action on $D(\mathrm{Bun}_G)$ by $- \star -$.

Remark 2.1.1.1. To match conventions in [Gai2] §4, the monoidal structure on $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ that we have in mind sends an object $\mathcal{F}_1 \in \mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ (resp. $\mathcal{F}_2 \in \mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$) supported on $\{x_i\} \in \mathrm{Ran}$ (resp. $\{y_j\}$) to an object $\mathcal{F}_1 \star \mathcal{F}_2$ supported on $\{x_i\} \cup \{y_j\}$; the monoidal product is what is often called *external convolution*.

There is a closely related *pointwise* convolution structure; this makes $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ into an algebra object in the symmetric monoidal category $D(\mathrm{Ran})\text{–}\mathsf{mod}$. Here we would encode Hecke functors by the action of $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}} \in \mathsf{Alg}(D(\mathrm{Ran})\text{–}\mathsf{mod})$ on $D(\mathrm{Bun}_G) \otimes D(\mathrm{Ran})$; this does not lose information by contractibility of Ran space.

The pointwise convolution structure may be recovered from the external convolution monoidal structure on $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ by a straightforward procedure, and the two pictures should be thought of as roughly equivalent.

2.1.2. Let $\mathcal{F} \in \mathcal{H}_{X^I}^{\mathrm{sph}}$ be given.

On the one hand, $\mathcal{F}$ defines an object of $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$, so a Hecke functor $\mathcal{F} \star - : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$.

There is also a closely related functor:

$$\mathrm{Hecke}_{\mathcal{F}} : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X^I)$$

constructed as follows. We have a standard Hecke action functor:

$$\mathcal{H}_{X^I}^{\mathrm{sph}} \otimes D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G).$$

Considering the left hand side as a $(D(X^I), \overset{!}{\otimes})$-module (via the action on the first functor), this action lifts uniquely:

$$\begin{tikzcd}
\mathcal{H}_{X^I}^{\mathrm{sph}} \otimes D(\mathrm{Bun}_G) \arrow[r, dotted] \arrow[dr] & D(\mathrm{Bun}_G) \otimes D(X^I) \simeq D(\mathrm{Bun}_G \times X^I) \arrow[d, "\mathrm{id} \otimes C^\bullet_{\mathrm{dR}}(X^I, -)"] \\
& D(\mathrm{Bun}_G).
\end{tikzcd}$$

of $D(X^I)$-module categories.[5] Finally, inserting $\mathcal{F}$ on the first tensor factor (in the dotted arrow above) gives the desired functor $\mathrm{Hecke}_{\mathcal{F}}$.

We explicitly note that composing $\mathrm{Hecke}_{\mathcal{F}}$ with de Rham cohomology along X^I gives $\mathcal{F} \star -$.

[5] Indeed, this follows from Verdier self-duality of $D(X^I)$, noting that the unit $\mathsf{Vect} \xrightarrow{\omega_{X^I} \otimes -} D(X^I)$ is dual to de Rham cohomology $C_{\mathrm{dR}}(X^I, -) : D(X^I) \to \mathsf{Vect}$, which is therefore the counit for the (cocommutative) coalgebra structure on $D(X^I)$ dual to the (commutative) algebra structure $- \overset{!}{\otimes} -$.

2.1.3. We remind the category $\mathsf{Rep}(\check{G})_{X^I}$ from [Ras1] §6, and the construction of the *naive Satake functor*:

$$\mathcal{S}_{X^I} : \mathsf{Rep}(\check{G})_{X^I} \to \mathcal{H}^{\mathrm{sph}}_{X^I}.$$

We remind from *loc. cit.* that $\mathcal{S}_{X^I}$ is t-exact and induces an equivalence on the hearts of the t-structures.

Similarly, we let:

$$\mathcal{S}_{\mathrm{Ran}} : \mathsf{Rep}(\check{G})_{\mathrm{Ran}} \to \mathcal{H}^{\mathrm{sph}}_{\mathrm{Ran}}$$

denote the Ran space version, constructed out of the above functors.

2.2. **Universal local acyclicity (and some variants).** We remind some details about objects in $\mathsf{Rep}(\check{G})_{X^I}$ being universally locally acyclic (ULA), and introduce some variants on this notion (almost ULA, quasi-ULA) in the spherical category $\mathcal{H}^{\mathrm{sph}}_{X^I}$.

We will only apply these notions when I is a singleton set, in which case they become much less abstract. However, we feel the natural generality for the development of these ideas is over powers of the curve, so we develop it there.

2.2.1. Here is an informal overview. First, ULAness refers to local constancy over some base, which for us is X^I. The definition includes a "finiteness" condition: ULA objects are automatically compact.

Recall that the constant sheaf on a classifying stack is typically non-compact, but only mildly so; in technical jargon, it is *almost compact*, see below. Therefore, in the spherical Hecke category, *almost* ULAness is more suitable than ULAness.

Finally, for *quasi-ULAness* we relax the finiteness condition; quasi-ULA objects are those that are colimits of almost ULA objects.

For simplicity, we provide somewhat ad hoc definitions in our setting that would be equivalent to more abstract definitions (uniting the definition of ULAness from [Ras1] Appendix A and almost compactness).

2.2.2. We now recall the notion of objects of $\mathsf{Rep}(\check{G})_{X^I}$ being *ULA over* X^I. In general, we refer to [Ras1] Appendix A and §6 for a detailed discussion of ULA objects in this setting.

However, briefly, we remind some points of the theory, to give the reader a sense for what ULA objects are. This material is not explicitly needed in what follows.

There is a canonical forgetful functor $\mathrm{Oblv}_{X^I} : \mathsf{Rep}(\check{G})_{X^I} \to D(X^I)$, arising from the (symmetric monoidal) forgetful functor $\mathrm{Oblv} : \mathsf{Rep}(\check{G}) \to \mathsf{Vect}$. Then an object $\mathcal{V}$ of $\mathsf{Rep}(\check{G})_{X^I}$ is ULA over X^I if and only if $\mathrm{Oblv}_{X^I}(\mathcal{V}) \in D(X^I)$ is so, where the latter notion translates to being a compact object of $D(X^I)$ (i.e., bounded with locally finitely generated cohomologies) whose cohomologies are vector bundles as quasi-coherent sheaves (i.e., *compact lisse D-modules* on X^I).[6]

In addition, there is a natural functor $\mathsf{Rep}(\check{G})^{\otimes I} \to \mathsf{Rep}(\check{G})_{X^I}$; it sends compact objects in the source to ULA objects in the target. In practice, this is a quite convenient construction of ULA objects. (See [Ras1] Proposition 6.16.1 for more discussion.)

[6] This assertion is not explicitly stated in [Ras1]. We indicate the argument here.

That Oblv_{X^I} preserves ULAness is [Ras1] Propositions 6.22.1 and B.4.4. That an object of $D(X^I)$ is ULA if and only if it is cohomologically bounded with (finite rank) vector bundle cohomologies follows from the definition of ULAness and the fact that any coherent sheaf with a connection on X^I is a vector bundle.

The converse follows immediately from the definitions, and the standard fact that for J a finite set and $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$ arbitrary, an object $\mathcal{F}$ of $\mathsf{Rep}(\check{G})^{\otimes J} \otimes \mathcal{C}$ mapping by the forgetful functor $\mathrm{Oblv} \otimes \mathrm{id}_{\mathcal{C}}$ to a compact object of $\mathcal{C}$ is itself compact. (This assertion is true for any affine algebraic group in place of $\check{G}$, but is especially obvious in this case by semi-simplicity of $\mathsf{Rep}(\check{G})^{\otimes J}$.)

Remark 2.2.2.1. Abstractly, one can imagine ULAness as analogous to compactness, but we work with a $D(X^I)$-module category (here $\mathsf{Rep}(\check{G})_{X^I}$) rather than an abstract DG category. The definition should be thought of as additionally encoding some local constancy along X^I.

Example 2.2.2.2. For us, the main example is when I is a singleton, so $\mathsf{Rep}(\check{G})_X = \mathsf{Rep}(\check{G}) \otimes D(X)$; then for $V \in \mathsf{Rep}(\check{G})$ compact, the object $V \boxtimes \omega_X \in \mathsf{Rep}(\check{G}) \otimes D(X) = \mathsf{Rep}(\check{G})_X$ is compact.

2.2.3. We will need the following technical notion in what follows.

Definition 2.2.3.1. The subcategory $\mathcal{H}^{\mathrm{sph,aULA}}_{X^I} \subseteq \mathcal{H}^{\mathrm{sph}}_{X^I}$ of *almost ULA* objects the full (non-cocomplete) subcategory generated under finite colimits and direct summands by applying $\mathcal{S}_{X^I}$ to objects of $\mathsf{Rep}(\check{G})_{X^I}$ ULA over X^I. The subcategory $\mathcal{H}^{\mathrm{sph,qULA}}_{X^I} \subseteq \mathcal{H}^{\mathrm{sph}}_{X^I}$ of *quasi-ULA* objects is the full subcategory generated under colimits by almost ULA objects.

The following remark explains the baroque terminology. However, the subtle finiteness issues at play are not salient to our purposes, even though we have accounted for them in the terminology. Therefore, the ready can safely ignore the remark.

Remark 2.2.3.2. Recall that e.g., the skyscraper sheaf $\delta_1 \in \mathcal{H}^{\mathrm{sph}}_x$ at the origin $1 \in \mathrm{Gr}_{G,x}$ is not compact; rather, it is *almost compact* in the technical sense.[7]

For similar reasons, the standard spherical sheaves over X^I are not literally ULA over X^I; we use the term *almost ULA* in parallel with *almost compact*.

2.2.4. The following result characterizes almost ULA objects more explicitly.

Lemma 2.2.4.1. *An object $\mathcal{F} \in \mathcal{H}^{\mathrm{sph}}_{X^I}$ is almost ULA if and only if it is bounded in the t-structure and each cohomology:*

$$H^i(\mathcal{F}) \in \mathcal{H}^{\mathrm{sph},\heartsuit}_{X^I}$$

maps to a vector bundle with connection on X^I along the map:

$$\mathcal{H}^{\mathrm{sph},\heartsuit}_{X^I} \xrightarrow{(\mathcal{S}^{\heartsuit}_{X^I})^{-1}} \mathsf{Rep}(\check{G})^{\heartsuit}_{X^I} \xrightarrow{\mathrm{Oblv}^{\heartsuit}_{X^I}} D(X^I)^{\heartsuit}.$$

Proof. Let $\mathcal{C} \subseteq \mathcal{H}^{\mathrm{sph}}_{X^I}$ be the subcategory of objects $\mathcal{F}$ satisfying the stated conditions.

Note that $\mathcal{C}$ is clearly closed under finite colimits. Moreover, the functor $\mathcal{S}_{X^I}$ maps ULA objects of $\mathsf{Rep}(\check{G})_{X^I}$ into $\mathcal{C}$ by design. Therefore, $\mathcal{H}^{\mathrm{sph,aULA}}_{X^I} \subseteq \mathcal{C}$.

Conversely, if $\mathcal{F} \in \mathcal{C}$, each of its cohomologies is almost ULA by assumption, so $\mathcal{F}$ is as well by definition and boundedness.

□

Corollary 2.2.4.2. *The property of an object $\mathcal{F} \in \mathcal{H}^{\mathrm{sph}}_{X^I}$ being almost ULA over X is étale local on X.*

Remark 2.2.4.3. Probably Corollary 2.2.4.2 is also true for quasi-ULAness, but we do not need this. Because the t-structure on $\mathcal{H}^{\mathrm{sph}}_{X^I}$ is not left separated, we do not have an analogue of Lemma 2.2.4.1 for quasi-ULA objects.

[7] We remind that for $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$ equipped with a t-structure, an object $\mathcal{F} \in \mathcal{C}$ is *almost compact* if $\tau^{\geqslant -n}\mathcal{F} \in \mathcal{C}^{\geqslant -n}$ is compact for every n, or equivalently, the functor:

$$\mathcal{C}^{\geqslant -n} \hookrightarrow \mathcal{C} \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F},-)} \mathsf{Vect}$$

commutes with filtered colimits for every n. For instance, coherent complexes on finite type schemes are almost compact, but only perfect complexes are actually compact.

The more relevant example in the present context is that the constant D-module on $\mathbb{B}G$ (for G reductive) is almost compact in $D(\mathbb{B}G)$ but not compact (unless G is the trivial group); see [DG1] Example 7.1.4 for more discussion. It follows a posteriori that the unit object in the spherical category is not compact.

2.2.5. *The case where* $|I| = 1$. Suppose I is a singleton set. We now make the above notions more explicit in this case.

First, $\mathsf{Rep}(\check{G})_X \simeq \mathsf{Rep}(\check{G}) \otimes D(X)$.

Even more explicitly, if $V^{\check{\lambda}} \in \mathsf{Rep}(\check{G})^{\heartsuit}$ is an irreducible representation with highest weight $\check{\lambda} \in \check{\Lambda}^+$, then we obtain:

$$\mathsf{Rep}(\check{G}) \otimes D(X) \simeq \bigoplus_{\check{\lambda} \in \check{\Lambda}^+} D(X)$$

as $\mathsf{Rep}(\check{G}) \simeq \oplus_{\check{\lambda} \in \check{\Lambda}^+} \mathsf{Vect}$. Therefore, we may canonically write any $\mathcal{F} \in \mathsf{Rep}(\check{G})_X$ as:

$$\mathcal{F} = \bigoplus_{\check{\lambda} \in \check{\Lambda}^+} V^{\check{\lambda}} \boxtimes \mathcal{F}_{\check{\lambda}}$$

for $\mathcal{F}_{\check{\lambda}} \in D(X)$.

In this case, it follows immediately from the definitions that $\mathcal{F}$ is ULA if and only if:

- Each $\mathcal{F}_{\check{\lambda}}$ is compact lisse D-module (i.e., it is bounded and each cohomology is a vector bundle with connection).
- $\mathcal{F}_{\check{\lambda}} = 0$ for all but finitely many $\check{\lambda} \in \check{\Lambda}^+$.

2.3. **Intermediate results.** We now formulate two intermediate results, from which we easily deduce Theorem 1.1.3.1.

2.3.1. *Local constancy.* Let $\mathcal{F} \in \mathcal{H}_X^{\mathrm{sph}}$ be given. For $x \in X(k)$, let $\mathcal{F}_x \in \mathcal{H}_x^{\mathrm{sph}}$ denote the !-fiber of $\mathcal{F}$ at x.

We let:

$$\mathrm{Hecke}_{\mathcal{F}} : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$$

denote the following functor.

By construction, the composition:

$$D(\mathrm{Bun}_G) \xrightarrow{\mathrm{Hecke}_{\mathcal{F}}} D(\mathrm{Bun}_G \times X) \xrightarrow{(\mathrm{id} \times i_x)^!} D(\mathrm{Bun}_G)$$

is the usual Hecke functor:

$$\mathcal{F}_x \star - : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$$

defined by $\mathcal{F}_x$.

2.3.2. With the above preliminary constructions out of the way, we can state:

Theorem 2.3.2.1. *Suppose $\mathcal{F} \in \mathcal{H}_X^{\mathrm{sph}}$ is quasi-ULA. Then* $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}}) = \mathrm{Ker}(\mathcal{F}_x \star -)$.

This is the main technical result of the present paper; its proof is given in §4.

2.3.3. *Projectors.* We follow terminology from [Ber5].

Define the *tempered unit (at x)* $\mathbb{1}_x^{\tau} \in \mathcal{H}_x^{\mathrm{sph}}$ as follows. We recall the *derived Satake theorem* of [BF], which asserts:[8,9]

$$\mathcal{H}_x^{\mathrm{sph}} \simeq \mathsf{IndCoh}_{\mathrm{Nilp}}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \subseteq \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}).$$

There are adjoint functors:

$$\Xi : \mathsf{QCoh}(\mathbb{B}\check{G})^{\mathbb{S}^2}) \rightleftarrows \mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2}) : \Psi.$$

[8] See [AG] §12 for this precise statement.

[9] Here $\mathbb{S}^2 \in \mathsf{Gpd}$ is the 2-sphere and for a stack $\mathcal{Y}$, $\mathcal{Y}^{\mathbb{S}^2}$ indicates the prestack of maps from $\mathbb{S}^2$ to $\mathcal{Y}$. In our case, it is straightforward to see that $(\mathbb{B}\check{G})^{\mathbb{S}^2} \simeq (\Omega_0\check{\mathfrak{g}})/\check{G}$, where $\Omega_0\check{\mathfrak{g}} \simeq \mathrm{Spec}(k) \times_{\check{\mathfrak{g}}} \mathrm{Spec}(k)$.

Moreover, the unit object in $\mathcal{H}_x^{\mathrm{sph}}$ corresponds to the trivial representation $\mathsf{triv} \in \mathsf{Rep}(\check{G})^{\heartsuit} = \mathsf{IndCoh}_{\mathrm{Nilp}}((\mathbb{B}\check{G})^{\mathbb{S}^2})^{\heartsuit}$. We then take $\mathbb{1}_x^{\tau}$ to correspond to $\Xi\Psi(\mathsf{triv})$.

2.3.4. By definition, there is a canonical map:

$$\mathbb{1}_x^{\tau} \to \delta_1 \in \mathcal{H}_x^{\mathrm{sph}}.$$

We then define the *anti-tempered unit (at x)* as:

$$\mathfrak{AT}_x := \mathrm{Ker}(\mathbb{1}_x^{\tau} \to \delta_1).$$

By definition, an object $\mathcal{G} \in D(\mathrm{Bun}_G)$ lies in $D(\mathrm{Bun}_G)^{x-\mathrm{temp}}$ if and only if $\mathfrak{AT}_x \star \mathcal{G} = 0$.

2.3.5. We now have the following basic observation.

Lemma 2.3.5.1. *There is a canonical object $\mathfrak{AT} \in \mathcal{H}_X^{\mathrm{sph,qULA}}$ (not depending on the choice of point $x \in X(k)$) with $!$-fiber $\mathfrak{AT}_x \in \mathcal{H}_x^{\mathrm{sph}}$ at x.*

The idea is that $\mathfrak{AT}_x$ does not depend seriously on the choice of point $x \in X$, so extends by universality to an object locally constant along the curve. So in this sense, Lemma 2.3.5.1 should be thought of as obvious.

Making this idea precise requires a slight bit of work, in part because [BF] is written with reference to a marked point. We provide a proof via Whittaker categories in §3; it is overkill and arguably loses sight of the basic geometric idea, but it accomplishes what it needs to.

2.4. **Gaitsgory's spectral action.** We now review the main results of [Gai1]; see also [Gai2] §4.3-4.5 and §11.1.

Roughly speaking, Theorem 2.4.1.1 below says that the Hecke action on $D(\mathrm{Bun}_G)$ satisfies a certain local constancy property along the curve. Indeed, it may be compared to corresponding results in [NY] and [AGKRRV1] in the setting of nilpotent singular support; there the Nadler-Yun theorem, which is a precise local constancy property of Hecke functors, is shown to be equivalent to the existence of a spectral action of QCoh of a suitable space of local systems.

Ultimately, the application of this result is to use this "local constancy" property (and Lemma 2.3.5.1) to relate anti-tempered projectors at different points. Of course, making this idea precise will be the content of §4.

2.4.1. First, there is a canonical symmetric monoidal functor:

$$\mathrm{Loc} : \mathsf{Rep}(\check{G})_{\mathrm{Ran}} \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

from *loc. cit.* It admits a fully faithful continuous right adjoint (cf. *loc. cit.*); therefore, the restriction functor:

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})\text{–}\mathsf{mod} \to \mathsf{Rep}(\check{G})_{\mathrm{Ran}}\text{–}\mathsf{mod}$$

is fully faithful. (Here modules are taken in the symmetric monoidal category $\mathsf{DGCat}_{\mathrm{cont}}$ of cocomplete DG categories).

On the other hand, there is an action of $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$ on $D(\mathrm{Bun}_G)$ that is constructed as:

$$\mathsf{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathcal{S}_{\mathrm{Ran}}} \mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}} \curvearrowright D(\mathrm{Bun}_G).$$

Theorem 2.4.1.1 (Gaitsgory, [Gai1], [Gai2] Theorem 4.5.2)**.** *The above action of $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$ on $D(\mathrm{Bun}_G)$ factors through a (necessarily unique) action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ via the localization functors.*

Remark 2.4.1.2. Related results in other contexts have also recently been obtained: see [NY], [AGKRRV1], [FS]. In these other contexts, the proofs are more conceptual.

We again use $- \star -$ to denote the action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ on $D(\mathrm{Bun}_G)$.

2.4.2. *The coaction functor.* The following construction derived from the spectral action plays an important role in our analysis.

We have an action functor:

$$\mathrm{act} : \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G).$$

As the first factor is canonically self-dual, we obtain a functor:

$$\mathrm{coact} : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}).$$

Explicitly, the functor coact is characterized by the commutative diagram:

$$\begin{tikzcd}
\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes D(\mathrm{Bun}_G) \arrow[r, "\mathrm{id}\otimes\mathrm{coact}"] \arrow[dd, "\mathrm{act}"] & \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[d, "s_{23}"] \\
& \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes D(\mathrm{Bun}_G) \arrow[d] \\
D(\mathrm{Bun}_G) & \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes D(\mathrm{Bun}_G). \arrow[l, "\Gamma(\mathrm{LocSys}_{\check{G}}, -)\otimes\mathrm{id}_{D(\mathrm{Bun}_G)}"']
\end{tikzcd}$$

Here s_{23} swaps the second and third tensor factors, the lower arrow on the right is obtained from the functor:

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

tensoring quasi-coherent sheaves together.

Remark 2.4.2.1. The following heuristic may be helpful for tracking constructions with coact. Informally, for $\mathcal{F} \in D(\mathrm{Bun}_G)$, we can imagine $\mathrm{coact}(\mathcal{F}) \in D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ as a family of D-modules on Bun_G indexed by local systems $\sigma \in \mathrm{LocSys}_{\check{G}}$. The fiber $\mathrm{coact}(\mathcal{F})_\sigma \in D(\mathrm{Bun}_G)$ at a fixed local system is by construction the Beilinson spectral projector applied to $\mathcal{F}$, i.e., it is obtained by universally producing a Hecke eigensheaf with eigenvalue σ from $\mathcal{F}$.

2.4.3. *A technical observation.* We now record a technical result that will play a role in §4; the reader may safely skip this material for now and refer back to it as needed.

The following observation is convenient for relating the spectral action on $D(\mathrm{Bun}_G)$ to the setting of Theorem 2.3.2.1.

Lemma 2.4.3.1. *Let $\mathcal{F} \in \mathcal{H}_X^{\mathrm{sph}}$ be arbitrary (i.e., we do not assume it to be quasi-ULA). Then* $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})$ *is a* $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$*-submodule category of* $D(\mathrm{Bun}_G)$.

Remark 2.4.3.2. Very naively, the idea is that Hecke operators commute. Unfortunately, this does not quite apply, as we will explain now. First, the derived Hecke category $\mathcal{H}_X^{\mathrm{sph}}$; at least if we elide the difference with the pointwise version of the category, it is a consequence of derived Satake that $\mathcal{H}_X^{\mathrm{sph}}$ is actually symmetric monoidal, so satisfies a commutativity property of the above type. Second, the action of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ comes from an action of $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$, which again is a symmetric monoidal category. However, to unite the two, it is natural to work with $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$; an unpublished result of Gaitsgory shows that $\mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ cannot be made symmetric monoidal. Therefore, we have to be slightly more careful.

Remark 2.4.3.3. We will only substantively apply Lemma 2.4.3.1 when $\mathcal{F}$ is supported at a single point, in which case the argument below could be streamlined in an evident fashion. However, as in Remark 4.1.1.1, the general assertion serves as a sanity check at a certain point in §4.

Proof of Lemma 2.4.3.1.

Step 1. By construction of the spectral action, it suffices to show that for any $\mathcal{V} \in \mathsf{Rep}(\check{G})_{\mathrm{Ran}}$, $\mathcal{V} \star -$ preserves $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})$.

It suffices to check this for a collection of objects $\mathcal{V}$ generating $\mathsf{Rep}(\check{G})$ under the monoidal product and convolutions. In this way, we can assume $\mathcal{V} \in \mathsf{Rep}(\check{G})_X \subseteq \mathsf{Rep}(\check{G})_{\mathrm{Ran}}$.[10]

Let $\mathcal{H} \in \mathcal{H}_X^{\mathrm{sph}}$ denote $\mathcal{S}_X(\mathcal{V})$, i.e., the image of $\mathcal{V}$ under geometric Satake. By definition, it is equivalent to show that the Hecke functor $\mathcal{H} \star - : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)$ preserves $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})$.

Step 2. As in Remark 2.1.1.1, $\mathcal{H}_X^{\mathrm{sph}}$ has a natural structure of algebra in $D(X)$–mod, which encodes the pointwise convolution. As such, it acts on $D(\mathrm{Bun}_G) \otimes D(X) \in D(X)$–$\mathsf{mod}$.

From this perspective, the action functor:

$$\mathcal{H}_X^{\mathrm{sph}} \otimes D(\mathrm{Bun}_G) = \mathcal{H}_X^{\mathrm{sph}} \underset{D(X)}{\otimes} \big(D(\mathrm{Bun}_G) \otimes D(X)\big) \xrightarrow{\mathrm{act}} D(\mathrm{Bun}_G) \otimes D(X) = D(\mathrm{Bun}_G \times X)$$

sends $\mathcal{F} \boxtimes \mathcal{G}$ to $\mathrm{Hecke}_{\mathcal{F}}(\mathcal{G})$.

On the other hand, we remind that the action of $\mathcal{H} \in \mathcal{H}_X^{\mathrm{sph}} \subseteq \mathcal{H}_{\mathrm{Ran}}^{\mathrm{sph}}$ (considered with its *external* convolution structure) on $D(\mathrm{Bun}_G)$ is computed by applying $\mathrm{Hecke}_{\mathcal{H}}$ and then pushing forward along the projection $\mathrm{Bun}_G \times X \to \mathrm{Bun}_G$.

Therefore, we see that the problem is solved whenever $\mathcal{F}$ and $\mathcal{H}$ *commute* in $\mathcal{H}_X^{\mathrm{sph}}$, i.e., $\mathcal{F} \boxtimes \mathcal{H}$ and $\mathcal{H} \boxtimes \mathcal{F}$ map to isomorphic objects under:

$$\mathcal{H}_X^{\mathrm{sph}} \otimes \mathcal{H}_X^{\mathrm{sph}} \to \mathcal{H}_X^{\mathrm{sph}} \underset{D(X)}{\otimes} \mathcal{H}_X^{\mathrm{sph}} \to \mathcal{H}_X^{\mathrm{sph}}$$

the assertion is clear.

Below, we will reduce to this situation using the symmetric monoidal structure on $\mathcal{H}_x^{\mathrm{sph}}$ constructed in [BF]. Their symmetric monoidal structure surely extends to one on $\mathcal{H}_X^{\mathrm{sph}} \in \mathsf{Alg}(D(X)\text{–}\mathsf{mod})$, but we do not address this here; we instead reduce to their claim by working locally on X.

Step 3. Let $j : U \to X$ be an open subscheme. Observe that $\mathrm{Hecke}_{j_* j^* \mathcal{F}}$ is computed as the composition:

$$D(\mathrm{Bun}_G) \xrightarrow{\mathrm{Hecke}_{\mathcal{F}}} D(\mathrm{Bun}_G \times X) \xrightarrow{(\mathrm{id} \times j)^*} D(\mathrm{Bun}_G \times U) \xrightarrow{(\mathrm{id} \times j)_*} D(\mathrm{Bun}_G \times X).$$

Therefore, we can assume $\mathcal{F}$ is pushed forward from some open U that admits an étale map to $\mathbb{A}^1$; indeed, such U's form an open cover of X. In this case, we claim that $\mathcal{F}$ automatically commutes with $\mathcal{H} \in \mathcal{H}_X^{\mathrm{sph}}$, which would conclude the argument.

---

[10]We explain the reduction in more detail. First, by definition, $\mathsf{Rep}(\check{G})_{\mathrm{Ran}}$ is generated under colimits by objects in the images of the functors $\mathsf{Rep}(\check{G})_{X^I} \to \mathsf{Rep}(\check{G})_{\mathrm{Ran}}$ as I varies over all finite sets. Second, by construction, we have a commutative diagram:

$$\begin{tikzcd} \mathsf{Rep}(\check{G})_X^{\otimes I} \arrow[r] \arrow[d] & \mathsf{Rep}(\check{G})_{X^I} \arrow[d] \\ \mathsf{Rep}(\check{G})_{\mathrm{Ran}}^{\otimes I} \arrow[r] & \mathsf{Rep}(\check{G})_{\mathrm{Ran}} \end{tikzcd}$$

where the lower right map is the monoidal product. That the upper right arrow generates its target under colimits follows from [Ras1] Theorem 6.17.1 (or rather, its proof, specifically, the explicit construction of ULA generators used there).

Indeed, we then have a commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_X^{\mathrm{sph}} \otimes \mathcal{H}_U^{\mathrm{sph}} \arrow[r, "\mathrm{id}\otimes j_*"] \arrow[dd, "j^*\otimes \mathrm{id}"] & \mathcal{H}_X^{\mathrm{sph}} \otimes \mathcal{H}_X^{\mathrm{sph}} \arrow[r] & \mathcal{H}_X^{\mathrm{sph}} \underset{D(X)}{\otimes} \mathcal{H}_X^{\mathrm{sph}} \arrow[dr] & \\
 & & & \mathcal{H}_X^{\mathrm{sph}} \\
\mathcal{H}_U^{\mathrm{sph}} \otimes \mathcal{H}_U^{\mathrm{sph}} \arrow[r] & \mathcal{H}_U^{\mathrm{sph}} \underset{D(X)}{\otimes} \mathcal{H}_U^{\mathrm{sph}} \arrow[r] & \mathcal{H}_U^{\mathrm{sph}} \arrow[ur, "j_*"'] &
\end{tikzcd}$$

and similarly starting with $\mathcal{H}_U^{\mathrm{sph}} \otimes \mathcal{H}_X^{\mathrm{sph}}$. Therefore, it suffices to observe that $\mathcal{H}_U^{\mathrm{sph}}$ upgrades to an object of $\mathsf{ComAlg}(D(U)\text{–}\mathsf{mod})$, i.e., is symmetric monoidal as a DG category over U_{dR}.

This follows from the observation:

$$\mathcal{H}_U^{\mathrm{sph}} \simeq \mathcal{H}_0^{\mathrm{sph}} \otimes D(U) \in \mathsf{Alg}(D(U)\text{–}\mathsf{mod})$$

via (3.2.1), noting that the derived Satake isomorphism of [BF] shows that $\mathcal{H}_0^{\mathrm{sph}}$ admits a symmetric monoidal structure enhancing its convolution monoidal structure.

□

3. Proof of Lemma 2.3.5.1

Here we prove the existence of a version of a version $\mathfrak{AT}$ of the anti-tempered unit defined over the curve X and quasi-ULA over it.

We accomplish this by reinterpreting temperedness via Whittaker categories. This material is well-known and overlaps greatly with [Ber5] §2.5. However, we were unable to find a proof of Lemma 3.1.1.1 in the existing literature, so we have provided details below, however digressive they may be.

More specifically, we write a formula for $\mathfrak{AT}_x$ using Whittaker averaging. This formula is purely on the geometric side, so makes sense as we vary the point, yielding the object $\mathfrak{AT}$. We check that $\mathfrak{AT}$ is quasi-ULA by reduction to the case $X = \mathbb{A}^1$; our various loop groups over X then split as products, making the local constancy along the curve evident.

3.1. **Derived Satake and Whittaker averaging.**

3.1.1. We record the following folklore result.

In what follows, we consider the monoidal functor:

$$\mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \to \mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \simeq \mathcal{H}_x^{\mathrm{sph}}$$

where the equivalence is the derived Satake equivalence of [BF] discussed earlier.

Finally, we let π denote the natural map $(\mathbb{B}\check{G})^{\mathbb{S}^2} \to \mathbb{B}\check{G}$ (coming from the base point in $\mathbb{S}^2$).

Lemma 3.1.1.1. *The following diagram in* $\mathsf{DGCat}_{\mathrm{cont}}$ *commutes:*

$$\begin{tikzcd}
\mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \arrow[r] \arrow[d, "\pi_*^{\mathsf{IndCoh}}"] & \mathcal{H}_x^{\mathrm{sph}} \arrow[r, equal] & D(\mathrm{Gr}_{G,x})^{\mathfrak{L}_x^+ G} \arrow[d, "\mathrm{Av}_!^{\psi}"] \\
\mathsf{IndCoh}(\mathbb{B}\check{G}) \arrow[r, equal] & \mathsf{Rep}(\check{G}) \arrow[r, "\simeq"] & \mathsf{Whit}(\mathrm{Gr}_{G,x}).
\end{tikzcd}$$

Here $\mathsf{Whit}(\mathrm{Gr}_{G,x}) := D(\mathrm{Gr}_{G,x})^{\mathfrak{L}_x N,\psi}$*, and we remind that the composition:*

$$\mathsf{Rep}(\check{G}) \xrightarrow{\mathcal{S}_x} \mathcal{H}_x^{\mathrm{sph}} \xrightarrow{\mathrm{Av}_!^{\psi}} \mathsf{Whit}(\mathrm{Gr}_{G,x})$$

is an equivalence by the geometric Casselman-Shalika formula of [FGV].

Remark 3.1.1.2. This compatibility is standard, and in fact a defining feature of the forthcoming approach to factorizable derived Satake referenced in §1.2.3. It has been used without proof elsewhere in the literature, but is not so obvious from the methods of [BF].[11] To fill this gap, we provide a self-contained proof of the commutativity of this diagram here; from the perspective of our paper, this is digressive.

Proof of Lemma 3.1.1.1. Let:

$$F : \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \to \mathsf{IndCoh}(\mathbb{B}\check{G}) = \mathsf{QCoh}(\mathbb{B}\check{G})$$

denote the functor defined by the diagram:

$$\begin{tikzcd}
\mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \arrow[r] & \mathcal{H}_x^{\mathrm{sph}} \arrow[r, equal] \arrow[d] & D(\mathrm{Gr}_{G,x})^{\mathfrak{L}_x^+ G} \arrow[d, "\mathrm{Av}_!^{\psi}"] \\
\mathsf{IndCoh}(\mathbb{B}\check{G}) \arrow[r, equal] & \mathsf{Rep}(\check{G}) & \mathsf{Whit}(\mathrm{Gr}_{G,x}). \arrow[l, "\simeq"']
\end{tikzcd}$$

Our task is to identify F with π_*^{IndCoh}.

Let $\iota : \mathbb{B}\check{G} \to (\mathbb{B}\check{G})^{\mathbb{S}^2}$ denote the natural map. Recall that $\iota_*^{\mathsf{IndCoh}} : \mathsf{Rep}(\check{G}) = \mathsf{IndCoh}(\mathbb{B}\check{G}) \to \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2})$ is monoidal for the convolution monoidal structure on $(\mathbb{B}\check{G})^{\mathbb{S}^2}$ used in derived Satake. Moreover, the diagram:

$$\begin{tikzcd}
\mathsf{Rep}(\check{G}) \arrow[d, "\iota_*^{\mathsf{IndCoh}}"] \arrow[dr, "\mathcal{S}_x"] & \\
\mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \arrow[r] & \mathcal{H}_x^{\mathrm{sph}}
\end{tikzcd}$$

commutes by design, cf. [BF]. It follows (by construction of the equivalence $\mathsf{Rep}(\check{G}) \simeq \mathsf{Whit}(\mathrm{Gr}_{G,x})$) that we have a commutative diagram:

$$\begin{tikzcd}
\mathsf{Rep}(\check{G}) \arrow[d, "\iota_*^{\mathsf{IndCoh}}"] \arrow[dr, "\mathrm{id}"] & \\
\mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \arrow[r, "F"] & \mathsf{Rep}(\check{G})
\end{tikzcd}$$

Therefore, the functor F is naturally $\mathsf{Rep}(\check{G})$-linear and satisfies:

$$F \circ \iota_*^{\mathsf{IndCoh}} \simeq \mathrm{id} \in \mathsf{End}_{\mathsf{Rep}(\check{G})\text{–}\mathsf{mod}}(\mathsf{Rep}(\check{G})). \tag{3.1.1}$$

We will show that any such functor is isomorphic to π_*^{IndCoh}.

We have:

$$\mathsf{Hom}_{\mathsf{Rep}(\check{G})\text{–}\mathsf{mod}}(\mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}), \mathsf{Rep}(\check{G})) \simeq \mathsf{Hom}_{\mathsf{DGCat}_{\mathrm{cont}}}(\mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}), \mathsf{Vect}) \simeq \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}). \tag{3.1.2}$$

[11] Although [BF] has the word *Whittaker* in its title, its usage refers to the appearance of the Kostant section on the *spectral* side of the equivalence; here we use it on the geometric side.

Here the first isomorphism is composition with the functor of invariants (or global sections on $\mathbb{B}\check{G}$) $\mathsf{Rep}(\check{G}) \to \mathsf{Vect}$. The second functor is Serre duality.

We let $\mathcal{K} \in \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2})$ be the object corresponding to F under (3.1.2). Explicitly, this means that for $\mathcal{F} \in \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2})$, we have:

$$F(\mathcal{F}) \simeq \Gamma^{\mathsf{IndCoh}}((\mathbb{B}\check{G})^{\mathbb{S}^2}, \mathcal{F} \overset{!}{\otimes} \mathcal{K})$$

functorially in $\mathcal{F}$; here $- \overset{!}{\otimes} -$ is the natural (symmetric) monoidal structure on IndCoh for which ω is the unit (explicitly: !-pullback the external product along the diagonal map). We see that our goal is to construct an isomorphism $\mathcal{K} \simeq \omega_{(\mathbb{B}\check{G})^{\mathbb{S}^2}}$.

By the projection formula, (3.1.1) translates to the assertion:

$$\iota^!(\mathcal{K}) \simeq \omega_{\mathbb{B}\check{G}} \in \mathsf{IndCoh}(\mathbb{B}\check{G}). \tag{3.1.3}$$

By a standard argument (e.g., by Koszul duality), it follows that $\mathcal{K}$ lies in the essential image of $\Xi : \mathsf{QCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \hookrightarrow \mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2})$. As $(\mathbb{B}\check{G})^{\mathbb{S}^2}$ is Gorenstein, we can therefore write:[12]

$$\mathcal{K} \simeq \Xi(\mathcal{K}_0) \otimes \omega_{(\mathbb{B}\check{G})^{\mathbb{S}^2}}.$$

for some $\mathcal{K}_0 \in \mathsf{QCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2})$; now our goal is to show $\mathcal{K}_0 \simeq \mathcal{O}_{(\mathbb{B}\check{G})^{\mathbb{S}^2}}$.

Now (3.1.3) translates to the assertion:

$$\iota^*(\mathcal{K}_0) \simeq \mathcal{O}_{\mathbb{B}\check{G}} \in \mathsf{QCoh}(\mathbb{B}\check{G}). \tag{3.1.4}$$

As $\mathbb{B}\check{G} \hookrightarrow (\mathbb{B}\check{G})^{\mathbb{S}^2}$ is a nil-isomorphism and the target is eventually coconnective, it follows that $\mathcal{K}_0$ is connective. In addition, we obtain a canonical map:

$$\alpha : \mathcal{O}_{(\mathbb{B}\check{G})^{\mathbb{S}^2}} \to \iota_*\iota^*(\mathcal{K}_0). \tag{3.1.5}$$

Because $\mathcal{K}_0$ is connective and $(\mathbb{B}\check{G})^{\mathbb{S}^2}$ equals $\mathbb{B}\check{G}$ at the classical level, the map $\mathcal{K}_0 \to \iota_*\iota^*\mathcal{K}_0$ induces an isomorphism on H^0 (top cohomology for the t-structure). In addition, because $\Gamma : \mathsf{QCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \to \mathsf{Vect}$ is t-exact (because $(\mathbb{B}\check{G})^{\mathbb{S}^2}$ is the quotient of an affine scheme by a reductive group), the map $\mathcal{K}_0 \to \iota_*\iota^*\mathcal{K}_0$ induces an isomorphism on $H^0\Gamma((\mathbb{B}\check{G})^{\mathbb{S}^2}, -)$. It therefore follows that (3.1.5) lifts to a map:

$$\beta : \mathcal{O}_{(\mathbb{B}\check{G})^{\mathbb{S}^2}} \to \mathcal{K}_0.$$

fitting into a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{(\mathbb{B}\check{G})^{\mathbb{S}^2}} \arrow[d, "\beta"] \arrow[dr, "\alpha"] & \\
\mathcal{K}_0 \arrow[r] & \iota_*\iota^*(\mathcal{K}_0).
\end{tikzcd}$$

Therefore, $\iota^*(\beta)$ is the isomorphism (3.1.4); as ι^* is conservative, β is itself an isomorphism, concluding the argument.

□

[12]We are slightly abusing notation here: we have previously used the notation Ξ in the setting of nilpotent (coherent) singular support; here we are using it for the full IndCoh category.

Remark 3.1.1.3. Commutation of a diagram in a higher category is a property, not a structure. The above produces a *non-canonical* commutation structure on the relevant diagram. This is the cost of our cheap argument, which did not use much about the [BF] construction.

Fortunately, we will not need anything more than the existence of a commutative diagram in our applications.

3.1.2. We now deduce some consequences of Lemma 3.1.1.1.

Corollary 3.1.2.1. *The functor:*

$$\mathrm{Av}_!^{\psi} : \mathcal{H}_x^{\mathrm{sph}} \to \mathsf{Whit}(\mathrm{Gr}_{G,x})$$

admits a left adjoint, denoted $\mathrm{Av}_!^{\psi,L}$.

Proof. By Lemma 3.1.1.1, this follows from the corresponding property of the functor π_*^{IndCoh} : $\mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \to \mathsf{IndCoh}(\mathbb{B}\check{G})$, which in turn follows from the map π being eventually coconnective.

□

Remark 3.1.2.2. Note that $\mathcal{H}_x^{\mathrm{sph}}$ has a natural t-structure. The same is true (but less obvious) of $\mathsf{Whit}(\mathrm{Gr}_{G,x})$, cf. [Ras2]. The equivalence $\mathsf{Whit}(\mathrm{Gr}_{G,x}) \simeq \mathsf{Rep}(\check{G})$ is t-exact for the natural t-structure; for our present purposes, we may even take this as a definition of the t-structure on $\mathsf{Whit}(\mathrm{Gr}_{G,x})$. We remark that $\mathrm{Av}_!^{\psi}$ is t-exact and an equivalence on hearts while its left adjoint $\mathrm{Av}_!^{\psi,L}$ has bounded amplitude; indeed, this follows from the corresponding evident facts on the spectral side.

Remark 3.1.2.3. For $|I| > 1$, the functor $\mathrm{Av}_!^{\psi} : \mathcal{H}_{X^I}^{\mathrm{sph}} \to \mathsf{Whit}(\mathrm{Gr}_{G,X^I})$ can be seen *not* to admit a left adjoint (although $*$-averaging provides a right adjoint). This reflects the fact that we do not have a good geometric argument for its existence; we need derived Satake (over a point, with Lemma 3.1.1.1) to see it.

3.1.3. Next, we define the endofunctor $\Theta_x : \mathcal{H}_x^{\mathrm{sph}} \to \mathcal{H}_x^{\mathrm{sph}}$ as:

$$\Theta_x(\mathcal{F}) := |(\mathrm{Av}_!^{\psi,L}\,\mathrm{Av}_!^{\psi})^{\bullet+1}(\mathcal{F})|. \tag{3.1.6}$$

In other words, we form the simplicial diagram:

$$\ldots \substack{\rightarrow\\ \rightarrow\\ \rightarrow} (\mathrm{Av}_!^{\psi,L}\,\mathrm{Av}_!^{\psi}\,\mathrm{Av}_!^{\psi,L}\,\mathrm{Av}_!^{\psi})(\mathcal{F}) \rightrightarrows (\mathrm{Av}_!^{\psi,L}\,\mathrm{Av}_!^{\psi})(\mathcal{F})$$

and form its geometric realization (i.e., colimit). As the simplicial diagram above is naturally augmented, there is a canonical natural transformation $\Theta_x \to \mathrm{id}$.

Corollary 3.1.3.1. *Under derived Satake, the endofunctor* Θ_x *and its natural map to the identity correspond to the endofunctor* $\Xi\Psi : \mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \to \mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2})$ *along with its natural map to the identity.*

This is immediate from Lemma 3.1.1.1.

3.2. **Working over the curve.** We now apply the above to prove Lemma 2.3.5.1.

3.2.1. Observe that we have a natural $D(X)$-module category $\mathsf{Whit}(\mathrm{Gr}_{G,X})$ consisting of D-modules on $\mathrm{Gr}_{G,X}$ satisfying equivariance for $\mathfrak{L}_X N$ against a non-degenerate character. We again have a natural functor $\mathcal{H}_X^{\mathrm{sph}} \to \mathsf{Whit}(\mathrm{Gr}_{G,X})$, which we again denote $\mathrm{Av}_!^{\psi}$ by abuse.

Lemma 3.2.1.1. *The functor* $\mathrm{Av}_! : \mathcal{H}_X^{\mathrm{sph}} \to \mathsf{Whit}(\mathrm{Gr}_{G,X})$ *admits a left adjoint* $\mathrm{Av}_!^{\psi,L}$ *that is* $D(X)$*-linear.*

Proof. Let $\mathcal{Y}$ be a prestack and let $F : \mathsf{C} \to \mathsf{D}$ be a morphism of sheaves on categories on $\mathcal{Y}$; see [Gai3] for the definitions. To check whether F admits a left adjoint in the 2-category $\mathsf{ShvCat}_{/\mathcal{Y}}$, we may check this locally in the fppf topology on $\mathcal{Y}$; indeed, this follows immediately from descent for sheaves of categories (cf. [Gai3] Appendix A).

The above problem may be put into this format for $\mathcal{Y} = X_{\mathrm{dR}}$ by [Gai3] Theorem 2.6.3. In other words, we see that the assertion we are trying to prove is flat local on X_{dR}, and in particular, étale local. Therefore, we are reduced to the case $X = \mathbb{A}^1$.

In this case, we have:

$$\begin{gathered}\mathcal{H}_{\mathbb{A}^1}^{\mathrm{sph}} = \mathcal{H}_0^{\mathrm{sph}} \otimes D(\mathbb{A}^1) \\ \mathsf{Whit}(\mathrm{Gr}_{G,\mathbb{A}^1}) = \mathsf{Whit}(\mathrm{Gr}_{G,0}) \otimes D(\mathbb{A}^1)\end{gathered} \tag{3.2.1}$$

for $x = 0 \in \mathbb{A}^1$. These identifications are compatible with the functors $\mathrm{Av}_!^{\psi}$, i.e., the version over X is the pointwise version tensored with $\mathrm{id}_{D(\mathbb{A}^1)}$. Therefore, we obtain the result from Corollary 3.1.2.1.

□

3.2.2. We now define the object $\mathfrak{AT}$ from Lemma 2.3.5.1.

First, observe that there is a functor:

$$\Theta_X : \mathcal{H}_X^{\mathrm{sph}} \to \mathcal{H}_X^{\mathrm{sph}}$$

defined by the same formula as (3.1.6), but understanding the symbols as being for their versions over X instead of the pointwise versions. There is again a canonical natural transformation:

$$\Theta_X \to \mathrm{id}\,.$$

Clearly applying $i_x^!$ to this data recovers the pointwise version.

Let $\delta_{1,X} \in \mathcal{H}_X^{\mathrm{sph}}$ denote the unit for the monoidal structure, i.e., the pushforward of ω_X along the unit map $X \to \mathrm{Gr}_{G,X}$. We then set:

$$\mathfrak{AT} := \mathrm{Ker}\left(\Theta_X(\delta_{1,X}) \to \delta_{1,X}\right) \in \mathcal{H}_X^{\mathrm{sph}}.$$

By Corollary 3.1.3.1, $i_x^! \mathfrak{AT} = \mathfrak{AT}_x$.

3.2.3. It remains to show $\mathfrak{AT}_X$ is quasi-ULA.

Lemma 3.2.3.1. *The endofunctor:*

$$\mathrm{Av}_!^{\psi,L}\,\mathrm{Av}_!^{\psi} : \mathcal{H}_X^{\mathrm{sph}} \to \mathcal{H}_X^{\mathrm{sph}}$$

preserves almost ULA objects.

Proof. By Corollary 2.2.4.2, this assertion is Zariski local on X. Therefore, we may assume X admits an étale map to $\mathbb{A}^1$. Such an isomorphism induces equivalences (3.2.1) (with X replacing $\mathbb{A}^1$ in those formulae).

In particular, we obtain:

$$\mathcal{H}_X^{\mathrm{sph}} \simeq \mathcal{H}_0^{\mathrm{sph}} \otimes D(X) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2}) \otimes D(X)$$

from derived Satake at a point. The endofunctor $\mathrm{Av}_!^{\psi,L}\mathrm{Av}_!^{\psi}$ corresponds to $\mathrm{Av}_!^{\psi,L}\mathrm{Av}_!^{\psi}\otimes\mathrm{id}_{D(X)}$ under the first isomorphism, and then to $\pi^{*,\mathsf{IndCoh}}\pi_*^{\mathsf{IndCoh}}\otimes\mathrm{id}_{D(X)}$ by Lemma 3.1.1.1.

By definition and the discussion of §2.2.5, almost ULA objects of $\mathcal{H}_X^{\mathrm{sph}}$ correspond to the subcategory of $\mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2})\otimes D(X)$ generated under finite colimits by objects of the form:

$$\mathcal{F}_1\boxtimes\mathcal{F}_2$$

for:[13]

$$\mathcal{F}_1\in\mathsf{Coh}((\mathbb{B}\check{G})^{\mathbb{S}^2})\subseteq\mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2})$$

and $\mathcal{F}_2\in D(X)$ a compact lisse D-module. Such an object maps to:

$$\pi^{*,\mathsf{IndCoh}}\pi_*^{\mathsf{IndCoh}}(\mathcal{F}_1)\boxtimes\mathcal{F}_2$$

which is of the same form (as π is finite and eventually coconnective), yielding the claim.

□

Remark 3.2.3.2. Note that in the above argument, $\pi^{*,\mathsf{IndCoh}}\pi_*^{\mathsf{IndCoh}}(\mathcal{F}_1)$ actually lies in $\mathsf{Perf}((\mathbb{B}\check{G})^{\mathbb{S}^2})\subseteq\mathsf{Coh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2})$, i.e., it is honestly compact in $\mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2})$. Correspondingly, one can refine the statement of Lemma 3.2.3.1 to say that the monad $\mathrm{Av}_!^{\psi,L}\mathrm{Av}_!^{\psi}$ maps almost ULA objects to *honestly* ULA objects (as defined for any $D(X)$-module category as in [Ras1]).

Finally, we have:

Proof of Lemma 2.3.5.1. By definition of $\mathfrak{AT}$, it suffices to show that $\delta_{1,X}$ and $\Theta_X(\delta_{1,X})$ are quasi-ULA. Clearly $\delta_{1,X}$ is so; in fact, it is almost ULA.

By definition, $\Theta_X(\delta_{1,X})$ is a colimit of terms $(\mathrm{Av}_!^{\psi,L}\mathrm{Av}_!^{\psi})^n(\delta_{1,X})$, so it suffices to show each such term is almost ULA. This follows from Lemma 3.2.3.1.

□

4. Proof of Theorem 2.3.2.1

Throughout this section, we fix $\mathcal{F}\in\mathcal{H}_X^{\mathrm{sph}}$ quasi-ULA.

It is clear that $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})\subseteq\mathrm{Ker}(\mathcal{F}_x\star-)$. Therefore, it remains to show the converse. To this end, we fix an object $\mathcal{G}\in\mathrm{Ker}(\mathcal{F}_x\star-)$; our objective is to show that $\mathcal{G}\in\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})$.

4.1. **Setup.**

We now begin with some reductions in proof of Theorem 2.3.2.1.

4.1.1. First, observe that it suffices to show:

$$\mathrm{coact}(\mathcal{G})\in\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})\otimes\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})\subseteq D(\mathrm{Bun}_G)\otimes\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}).\tag{4.1.1}$$

Indeed, we have:

$$\big(\mathrm{id}\otimes\Gamma(\mathrm{LocSys}_{\check{G}},-)\big)\circ\mathrm{coact}=\mathrm{id}_{D(\mathrm{Bun}_G)}$$

so that the above assertion would imply that:

$$\mathcal{G}=\big(\mathrm{id}\otimes\Gamma(\mathrm{LocSys}_{\check{G}},-)\big)\,\mathrm{coact}(\mathcal{G})$$

lies in $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})$.

Remark 4.1.1.1. By Lemma 2.4.3.1, coact necessarily sends $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})$ to $\mathrm{Ker}(\mathrm{Hecke}_{\mathcal{F}})\otimes\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$. Therefore, we are not attempting to prove an unreasonable assertion.

[13]The displayed containment is via the natural quotient functor $\mathsf{IndCoh}((\mathbb{B}\check{G})^{\mathbb{S}^2})\to\mathsf{IndCoh}_{\mathcal{N}ilp}((\mathbb{B}\check{G})^{\mathbb{S}^2}))$, which we remind is fully faithful on eventually coconnective objects.

4.1.2. We now form the following commutative diagram, whose analysis is central to the argument.

$$\begin{tikzcd}
\mathrm{Ker}(\mathcal{F}_x \star -) \arrow[r, "\mathrm{coact}"] \arrow[d, hook] \arrow[ddr, bend right=60, "(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id}) \circ \mathrm{coact}"'] & \mathrm{Ker}(\mathcal{F}_x \star -) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[d, hook] \arrow[ddr, bend left, "0"] & \\
D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coact}"] & D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[d, "\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id}"] \arrow[dr, "(\mathcal{F}_x \star -) \otimes \mathrm{id}"] & \\
& D(\mathrm{Bun}_G \times X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[r, "(\mathrm{id} \times i_x)^! \otimes \mathrm{id}"] & D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})
\end{tikzcd}$$

Here the commutative upper left square exists by Lemma 2.4.3.1 (applied to $\mathcal{F}_x \in \mathcal{H}_x^{\mathrm{sph}} \subseteq \mathcal{H}_X^{\mathrm{sph}}$).

We consider $\mathcal{G}$ as an object of the top left term. To orient the reader, we remind that our objective is to show to prove (4.1.1), which is the assertion that $\mathcal{G}$ is annihilated by the curved arrow on the left.

By the diagram, we observe:

$$((\mathrm{id} \times i_x^!) \otimes \mathrm{id})(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\,\mathrm{coact}(\mathcal{G}) = 0. \tag{4.1.2}$$

4.2. **Local constancy along the curve.** The idea is that quasi-ULAness of $\mathcal{F}$ implies that $(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\,\mathrm{coact}(\mathcal{G})$ is locally constant along the curve in a suitable sense, so (4.1.2) implies its vanishing. We formulate this precisely and apply it to derive our desired conclusion below.

4.2.1. *Lisse sheaves.* Our goal is to define a category of "$\mathcal{Y}$-families of lisse sheaves on X"[14] for a (pre)stack $\mathcal{Y}$.

First, when $\mathcal{Y} = S = \mathrm{Spec}(A)$ is affine, we define the (non-cocomplete) DG category $\mathsf{Lisse}_S(X)^c \subseteq \mathsf{QCoh}(S) \otimes D(X)$ to be the full subcategory consisting of objects mapping to $\mathsf{Perf}(S \times X)$ under the forgetful functor $\mathsf{QCoh}(S) \otimes D(X) \to \mathsf{QCoh}(S \times X)$. We then define $\mathsf{Lisse}_S(X)$ as $\mathsf{Ind}(\mathsf{Lisse}_S(X)^c)$, the ind-category of $\mathsf{Lisse}_S(X)^c$.

Remark 4.2.1.1. In the terminology of [Ras1] Appendix A, we could say that $\mathsf{Lisse}_S(X)^c$ is the subcategory of ULA objects in the $D(X)$-module category $\mathsf{QCoh}(S) \otimes D(X)$.

Note that objects of $\mathsf{Lisse}_S(X)^c$ are compact in $\mathsf{QCoh}(S) \otimes D(X)$ (cf. [Ras1] Corollary B.4.2). Therefore, the natural functor $\mathsf{Lisse}_S(X) \to \mathsf{QCoh}(S) \otimes D(X)$ is fully faithful.

For a map $f : S \to T$, the pullback $f^* \otimes \mathrm{id}_{D(X)} : \mathsf{QCoh}(T) \otimes D(X) \to \mathsf{QCoh}(S) \otimes D(X)$ obviously maps $\mathsf{Lisse}_T(X)$ to $\mathsf{Lisse}_S(X)$. Therefore, we obtain a contravariant functor $\mathsf{AffSch}^{op} \xrightarrow{S \mapsto \mathsf{Lisse}_S(X)} \mathsf{DGCat}_{\mathrm{cont}}$. We extend this functor to general prestacks by right Kan extension. Explicitly, for $\mathcal{Y}$ any prestack, this means we define:[15]

$$\mathsf{Lisse}_{\mathcal{Y}}(X) := \lim_{S \to \mathcal{Y}} \mathsf{Lisse}_S(X) \subseteq \lim_{S \to \mathcal{Y}} \mathsf{QCoh}(S) \otimes D(X) = \mathsf{QCoh}(\mathcal{Y}) \otimes D(X).$$

[14]In what follows, we could assume X is an arbitrary connected smooth variety, not necessarily 1-dimensional or proper.

[15]The final equality is justified by dualizability of $D(X)$. We also remark that $\mathsf{QCoh}(\mathcal{Y}) \otimes D(X) \xrightarrow{\simeq} \mathsf{QCoh}(\mathcal{Y} \times X_{\mathrm{dR}})$ by [GR] Proposition 3.1.7.

Remark 4.2.1.2. Note that $\mathsf{Lisse}_{\mathcal{Y}}(X)$ is a $\mathsf{QCoh}(\mathcal{Y})$-submodule category of $\mathsf{QCoh}(\mathcal{Y}) \otimes D(X)$. Indeed, by definition, this reduces to the affine case. For $\mathcal{Y} = S$ affine, as $\mathsf{QCoh}(S)$ is generated under colimits and shifts by its monoidal unit, any cocomplete $\mathcal{D} \subseteq \mathcal{C}$ with $\mathcal{C} \in \mathsf{QCoh}(S)\text{–}\mathsf{mod}$ is a $\mathsf{QCoh}(S)$-submodule category. So the claim is essentially formal.

4.2.2. Let $x \in X(k)$. We abuse notation in letting $i_x^!$ denote the composition:

$$\mathsf{Lisse}_{\mathcal{Y}}(X) \hookrightarrow \mathsf{QCoh}(\mathcal{Y}) \otimes D(X) \xrightarrow{\mathrm{id} \otimes i_x^!} \mathsf{QCoh}(\mathcal{Y}) \otimes \mathsf{Vect} = \mathsf{QCoh}(\mathcal{Y}).$$

We will use the following result.

Proposition 4.2.2.1. *Suppose $\mathcal{Y}$ is an Artin stack locally almost of finite type and eventually coconnective. Then the functor $i_x^! : \mathsf{Lisse}_{\mathcal{Y}}(X) \to \mathsf{QCoh}(\mathcal{Y})$ is conservative.*

More generally, for any dualizable DG category $\mathcal{C}$, the functor:

$$\mathrm{id}_{\mathcal{C}} \otimes i_x^! : \mathcal{C} \otimes \mathsf{Lisse}_{\mathcal{Y}}(X) \to \mathcal{C} \otimes \mathsf{QCoh}(\mathcal{Y})$$

is conservative.

Proof.

Step 1. First, we note that if $F : \mathcal{D}_1 \to \mathcal{D}_2 \in \mathsf{DGCat}_{\mathrm{cont}}$ is conservative and $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$ is dualizable, then $\mathrm{id}_{\mathcal{C}} \otimes F : \mathcal{C} \otimes \mathcal{D}_1 \to \mathcal{C} \otimes \mathcal{D}_2$ is conservative. Indeed, we can rewrite this functor as:

$$\mathcal{C} \otimes \mathcal{D}_1 = \mathsf{Hom}_{\mathsf{DGCat}_{\mathrm{cont}}}(\mathcal{C}^\vee, \mathcal{D}_1) \xrightarrow{\varphi \mapsto F\varphi} \mathsf{Hom}_{\mathsf{DGCat}_{\mathrm{cont}}}(\mathcal{C}^\vee, \mathcal{D}_2) = \mathcal{C} \otimes \mathcal{D}_2$$

in which form it is manifestly conservative. Therefore, we are reduced to considering $\mathcal{C} = \mathsf{Vect}$ in the assertion.

Step 2. Next, suppose S is an eventually coconnective scheme locally almost of finite type. Let $|S|$ denote the set of points of its underlying topological space; for $s \in |S|$, we write $\kappa(s)$ for the residue field at this point, s for $\mathrm{Spec}(\kappa(s))$, and $i_s : s \to S$ for the structural morphism.

We then note that the functor:

$$\mathsf{QCoh}(S) \xrightarrow{\{i_s^*\}_{s \in |S|}} \prod_{s \in |S|} \mathsf{QCoh}(s)$$

is conservative. Indeed, this follows from [Lur] Lemma 2.6.1.3 and the conservativeness of the restriction along $\iota : S^{cl} \hookrightarrow S$; here $S^{\mathrm{cl}} \subseteq S$ is the underlying classical scheme of S. We remark that the conservativeness of ι^* uses that S is eventually coconnective.[16]

In our setting, let $\pi : S \to \mathcal{Y}$ be a flat cover. We find that the restriction functor:

$$\mathsf{QCoh}(\mathcal{Y}) \to \prod_{s \in |S|} \mathsf{QCoh}(s)$$

[16] The argument is standard, but we find it easier to supply an argument here than to find a reference. Suppose $\mathcal{Q} \in \mathsf{QCoh}(S)$ with $\iota^*(\mathcal{Q}) = 0$ is given. We wish to show $\mathcal{Q} = 0$.

Let $\mathsf{Ann}^{\otimes}(\mathcal{Q}) \subseteq \mathsf{QCoh}(S)$ be the subcategory of objects $\mathcal{P} \in \mathsf{QCoh}(S)$ with $\mathcal{P} \otimes_{\mathcal{O}_S} \mathcal{Q} = 0$. It suffices to show that $\mathcal{O}_S \in \mathsf{Ann}^{\otimes}(\mathcal{Q})$.

For $\widetilde{\mathcal{Q}} \in \mathsf{QCoh}(S^{\mathrm{cl}})$, we have:

$$\mathcal{Q} \underset{\mathcal{O}_S}{\otimes} \iota_*(\widetilde{\mathcal{Q}}) = \iota_*(\iota^*(\mathcal{Q}) \underset{\mathcal{O}_{S^{\mathrm{cl}}}}{\otimes} \widetilde{\mathcal{Q}}) = 0.$$

Therefore, $\iota_* : \mathsf{QCoh}(S^{\mathrm{cl}}) \to \mathsf{QCoh}(S)$ maps into $\mathsf{Ann}^{\otimes}(\mathcal{Q})$.

As ι_* is a t-exact functor that is an equivalence on hearts, it follows that $\mathsf{QCoh}(S)^\heartsuit \subseteq \mathsf{Ann}^{\otimes}(\mathcal{Q})$. It then follows that the minimal subcategory of $\mathsf{QCoh}(S)$ closed under shifts and colimits that contains $\mathsf{QCoh}(S)^\heartsuit$ lies in $\mathsf{Ann}^{\otimes}(\mathcal{Q})$. In particular, any object that is cohomologically bounded in $\mathsf{QCoh}(S)$ lies in this annihilator.

Because S is eventually coconnective, $\mathcal{O}_S$ is cohomologically bounded, yielding the claim.

is conservative. By the same reasoning as before, for any dualizable DG category $\mathcal{D}$, the functor:

$$\mathcal{D} \otimes \mathsf{QCoh}(\mathcal{Y}) \to \mathcal{D} \otimes \prod_{s \in |S|} \mathsf{QCoh}(s) \xrightarrow{\simeq} \prod_{s \in |S|} \mathcal{D} \otimes \mathsf{QCoh}(s)$$

is conservative. In particular, this applies for $\mathcal{D} = D(X)$.

Step 3. By the above, we have a commutative diagram:

$$\begin{tikzcd}
\mathsf{Lisse}_{\mathcal{Y}}(X) \arrow[r, hook] \arrow[d] & D(X) \otimes \mathsf{QCoh}(\mathcal{Y}) \arrow[r, "i_x^! \otimes \mathrm{id}"] \arrow[d] & \mathsf{QCoh}(\mathcal{Y}) \arrow[d] \\
\prod_{s \in |S|} \mathsf{Lisse}_s(X) \arrow[r, hook] & \prod_{s \in |S|} D(X) \otimes \mathsf{QCoh}(s) \arrow[r, "i_x^! \otimes \mathrm{id}"] & \prod_{s \in |S|} \mathsf{QCoh}(s).
\end{tikzcd}$$

The middle and right vertical arrows are conservative, so the same is true of the left vertical arrow. Therefore, to see that the top line is conservative, it suffices to show that for each $s \in |S|$, the functor:

$$i_x^! : \mathsf{Lisse}_s(X) \to \mathsf{QCoh}(s)$$

is conservative.

Therefore, we are reduced to the case where $S = \mathrm{Spec}(\kappa)$ for some field κ/k.

Step 4. Let $X_\kappa := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\kappa)$. Note that $D(X) \otimes \mathsf{Vect}_\kappa = D_{/\kappa}(X_\kappa)$, where we regard X_κ as a scheme over the field κ) and write $D_{/\kappa}$ to emphasize this (reminding that implicitly, the category of D-modules depends on the structural map to Spec of a field). Moreover, $X_{\mathrm{dR}} \times \mathrm{Spec}(\kappa) = X_{\kappa, \mathrm{dR}/\mathrm{Spec}(\kappa)}$, so $\mathsf{Lisse}_{\mathrm{Spec}(\kappa)}(X) \subseteq D_{/\kappa}(X_\kappa)$ is the subcategory of (Spec(κ)-families of) lisse D-modules on X_κ, considering the latter as a scheme over Spec(κ).

This is all to say that we are reduced to the case where $\kappa = k$, as the only difference is notational.

Step 5. We are now essentially done by standard arguments. Below, write $\mathsf{Lisse}(X)$ for $\mathsf{Lisse}_{\mathrm{Spec}(k)}(X)$, and similarly for $\mathsf{Lisse}(X)^c$.

Note that objects of $\mathsf{Lisse}(X)^c \subseteq D(X)$ are exactly the coherent (i.e., compact) D-modules with singular support in the zero section; therefore, $\mathsf{Lisse}(X)^c$ is closed under truncations for the t-structure on $D(X)$, and the heart $\mathsf{Lisse}(X)^{c,\heartsuit} \subseteq D(X)^\heartsuit$ of the resulting t-structure is closed under subobjects. As the t-structure on $D(X)$ is compatible with filtered colimits, it follows that $\mathsf{Lisse}(X) \subseteq D(X)$ is also closed under truncations, so inherits a similar t-structure.

Next, note that $i_x^! : \mathsf{Lisse}(X)^c \to \mathsf{Vect}$ is t-exact up to shift. Indeed, by construction of D-module functors, we have a commutative diagram:

$$\begin{tikzcd}
D(X) \arrow[r, "\mathrm{Oblv}"] \arrow[dr, "i_x^!"'] & \mathsf{QCoh}(X) \arrow[d, "i_x^{!,\mathsf{QCoh}}"] \\
& \mathsf{Vect}.
\end{tikzcd}$$

Here Oblv is the so-called *right* forgetful functor; we remind that it is t-exact. Moreover, the functor $i_x^{!,\mathsf{QCoh}}$ is the quasi-coherent !-pullback functor, i.e., the right adjoint to $i_{x,*} : \mathsf{Vect} \to \mathsf{QCoh}(X)$. Explicitly, for $\mathcal{Q} \in \mathsf{QCoh}(X)$, we have $i_x^{!,\mathsf{QCoh}}(\mathcal{Q}) = i_x^*(\mathcal{Q} \otimes \omega_X^{-1})$. In particular, if $\mathcal{Q}$ is a vector bundle concentrated in degree 0, then $i_x^!(\mathcal{Q})$ is concentrated in degree[17] $\dim X$. This clearly implies the t-exactness of $i_x^![\dim X]$ on $\mathsf{Lisse}(X)^c$.

It then follows formally that $i_x^! : \mathsf{Lisse}(X) \to \mathsf{Vect}$ is t-exact up to shift.

[17] Of course, $\dim X = 1$ for us. However, for the present purpose, X can be taken to be any connected smooth variety, so we write $\dim X$ instead to suggest this generality.

Finally, let $\sigma \in \mathsf{Lisse}(X)$ with $i_x^!(\sigma) = 0$; we wish to show that $\sigma = 0$. First, assume σ lies in $\mathsf{Lisse}(X)^{c,\heartsuit}$. Then by the above, up to shifts and tensoring with a line, $i_x^!(\sigma)$ is the fiber of the vector bundle underlying σ; as X is connected, this forces $\sigma = 0$. For general σ, it suffices to show that for every integer j and every subobject:

$$\widetilde{\sigma} \subseteq H^j(\sigma), \;\; \sigma \in \mathsf{Lisse}(X)^{c,\heartsuit}$$

we have $\widetilde{\sigma} = 0$. By the case we have treated, it suffices to show $i_x^!(\widetilde{\sigma}) = 0$, which follows from t-exactness of $i_x^![\dim X]$:

$$i_x^!(\widetilde{\sigma})[\dim X] \subseteq i_x^!(H^j(\sigma))[\dim X] = H^{j+\dim X}(i_x^!\sigma) = 0 \in \mathsf{Vect}^{\heartsuit}.$$

□

4.2.3. We now observe the following.

Lemma 4.2.3.1. *For any quasi-ULA $\mathcal{F}$, the composition:*

$$D(\mathrm{Bun}_G) \xrightarrow{\text{coact}} D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id}} D(\mathrm{Bun}_G \times X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) = D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

maps into the subcategory:

$$D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X).$$

Proof. First, note that:

$$D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X) \to D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$$

is indeed fully faithful: e.g., the embedding $\mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X) \hookrightarrow D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ admits a continuous right adjoint by definition, so tensoring with it preserves fully faithfulness.

Now, by definition of quasi-ULAness, we are immediately reduced to considering the case where $\mathcal{F}$ is almost ULA. By the same logic, we may assume $\mathcal{F} = \mathcal{S}_X(\mathcal{V})$ for some ULA $\mathcal{V} \in \mathsf{Rep}(\check{G})_X$, where we remind and we remind that $\mathcal{S}_X$ denotes the geometric Satake functor.

As in §2.2.5, $\mathcal{V}$ is necessarily is a direct sum of terms of the form $V \boxtimes \sigma$ where $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ is finite-dimensional and $\sigma \in D(X)$ is compact and lisse, i.e., cohomologically bounded with cohomologies that are vector bundles with flat connections.

Now observe that $\mathrm{Hecke}_{\mathcal{S}_X(V \otimes \sigma)}$ differs from $\mathrm{Hecke}_{\mathcal{S}_X(V \otimes \omega_X)}$ by applying $\mathrm{id}_{D(\mathrm{Bun}_G)} \otimes (\sigma \overset{!}{\otimes} -)$. Clearly this operation preserves the subcategory $D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X)$, so we are reduced to considering the case $\mathcal{F} = \mathcal{S}_X(V \otimes \omega_X)$ instead. We simplify the notation by writing $\mathcal{F} = \mathcal{S}_X(V)$.

Next, recall that V defines a canonical vector bundle $\mathcal{E}_V$ on $X_{\mathrm{dR}} \times \mathrm{LocSys}_{\check{G}}$. Specifically, we have a defining map:

$$X_{\mathrm{dR}} \times \mathrm{LocSys}_{\check{G}} = X_{\mathrm{dR}} \times \mathcal{M}aps(X_{\mathrm{dR}}, \mathbb{B}\check{G}) \to \mathbb{B}\check{G}.$$

We consider V as a vector bundle on $\mathbb{B}\check{G}$ by the identification $\mathsf{Rep}(\check{G}) \simeq \mathsf{QCoh}(\mathbb{B}\check{G})$, and its pullback along the above map is (by definition) the vector bundle $\mathcal{E}_V$. By definition, we can consider $\mathcal{E}_V$ as an object of $\mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X) \subseteq D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) = \mathsf{QCoh}(X_{\mathrm{dR}} \times \mathrm{LocSys}_{\check{G}})$.

We then observe that we have a commutative diagram:

$$\begin{tikzcd}
D(\mathrm{Bun}_G) \arrow[rr, "\mathrm{coact}"] \arrow[d, "\mathrm{coact}"] & & D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[dd, "\mathrm{Hecke}_{\mathcal{S}_X(V)} \otimes \mathrm{id}"] \\
D(\mathrm{Bun}_G) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[d, "\mathrm{id} \otimes \mathcal{E}_V \otimes \mathrm{id}"] & & \\
D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[rr, "\mathrm{id} \otimes \mathrm{id} \otimes m"] & & D(\mathrm{Bun}_G) \otimes D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})
\end{tikzcd}$$

by construction[18] of Loc. Here $m : \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ is the tensor product. Now the claim follows as $\mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X)$ and $\mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X) \subseteq D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ is a $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$-submodule category (cf. Remark 4.2.1.2).

□

4.2.4. *Conclusion.* At this point, we combine the above ideas.

By Lemma 4.2.3.1, we have:

$$(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\, \mathrm{coact}(\mathcal{G}) \in D(\mathrm{Bun}_G) \otimes \mathsf{Lisse}_{\mathrm{LocSys}_{\check{G}}}(X).$$

Moreover, by (4.1.2), this object vanishes when we apply $(\mathrm{id} \otimes i_x^!)$ to it. Therefore, by Proposition 4.2.2.1, we have:

$$(\mathrm{Hecke}_{\mathcal{F}} \otimes \mathrm{id})\, \mathrm{coact}(\mathcal{G}) = 0.$$

Here we observe that $D(\mathrm{Bun}_G)$ is dualizable by [DG2], and that $\mathrm{LocSys}_{\check{G}}$ is eventually coconnective e.g. by [AG] §10. This concludes the proof of (4.1.1), and hence of Theorem 2.3.2.1.

References

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[18] Specifically, we use the following fact, which is tautological from the construction of Loc. Suppose $\mathcal{M} \in D(X)$. We obtain an object $V \otimes \mathcal{M} \in \mathsf{Rep}(\check{G})_X$. Let $\lambda_{\mathcal{M}} : D(X) \to \mathsf{Vect}$ be the functor Verdier dual to $\mathcal{M}$, i.e., the functor $C^\bullet_{\mathrm{dR}}(X, \mathcal{M} \overset{!}{\otimes} -)$. Then $\mathrm{Loc}(V \otimes \mathcal{M}) \in \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ is (functorially in $\mathcal{M}$) calculated as the image of $\mathcal{E}_V$ under $\lambda_{\mathcal{M}} \otimes \mathrm{id} : D(X) \otimes \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$.

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