ON THE DUNDAS-GOODWILLIE-MCCARTHY THEOREM

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Abstract. We give a modern presentation of the Dundas-Goodwillie-McCarthy theorem identifying relative K-theory and topological cyclic homology for nilpotent ring extensions.

Contents

1. Introduction

1.1. This paper proves the following theorem of Dundas-Goodwillie-McCarthy, relating K-theory and topological cyclic homology TC:

Theorem 1.1.1 ([DGM] Theorem 7.2.2.1)**.** *For $B \to A$ a morphism of connective $\mathcal{E}_1$-ring spectra such that $\pi_0(B) \to \pi_0(A)$ is surjective with kernel a nilpotent ideal, the* cyclotomic trace *map $K \to \mathrm{TC}$ induces an equivalence of spectra:*

$$\mathrm{Ker}(K(B) \to K(A)) \to \mathrm{Ker}(\mathrm{TC}(B) \to \mathrm{TC}(A)).$$

Here we use $\mathcal{E}_1$ to refer to the appropriate notion of associative algebra in the homotopical setting; A_∞ and *highly structured* are common synonyms. We refer to [NS] for an introduction to topological cyclic homology.

As we discuss at greater length below in §1.5, one of the main purposes of this note is to simplify the original arguments of Dundas-Goodwillie-McCarthy through systematic use of Lurie's ∞-categorical methods (c.f. [Lur1] and [Lur2]) and using the recent ideas of Nikolaus-Scholze.

Remark 1.1.2. Theorem [DGM] has a long history. To the extent that cyclic homology is an avatar for de Rham (or crystalline) cohomology, its origin is in [Blo].

Goodwillie proved a rationalized version of Theorem 1.1.1 in [Goo1]. The connection between K-theory and topological Hochschild homology was first proved in [DM1] and [DM2]; see also [SSW]. The cyclotomic trace[1] was constructed in [BHM] (though see [BGT] for a simpler construction). A p-adic version of Theorem 1.1.1 was proved in [McC]. Lindenstrauss-McCarthy proved the theorem for split square-zero extensions (and usual rings) in [LM].

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[1] We remind that this is a natural map $K(A) \to \mathrm{TC}(A)$ for any $\mathcal{E}_1$-algebra A. More generally, this map is defined for any essentially small stable category $\mathcal{C}$ (and in particular is Morita invariant).

For more recent developments, see [CMM] and its application in [BMS] §7.

Remark 1.1.3. It is not immediately clear how to deduce Goodwillie's rationalized version [Goo1] of Theorem 1.1.1 from Theorem 1.1.1 (because TC does not generally commute with tensor products). Our Theorem 5.15.1 provides the relevant comparison.

Similarly, Beilinson [Bei] proved a rationalized version of Theorem 1.1.1 in a p-adic setting: we refer to *loc. cit.* for the formulation. Again, Beilinson's theorem does not obviously follow from Dundas-Goodwillie-McCarthy. Forthcoming work of Ben Antieau, Akhil Mathew, and Thomas Nikolaus explains the deduction of Beilinson's theorem from Theorem 1.1.1.

1.2. **Outline of the argument.** The proof of Theorem 1.1.1 is quite striking: one shows that the cyclotomic trace induces an isomorphism on *Goodwillie derivatives*, and then formally deduces Theorem 1.1.1 from *structural properties* of the functors K-theory and TC.

Let us explain what we mean here in more detail. Fix a connective $\mathcal{E}_1$-algebra A. Then for any A-bimodule M, we can form the split square-zero extension $A \oplus M$ and take its K-theory. The induced functor $A\text{–}\mathsf{bimod} \to \mathsf{Sp}$ is not additive for a stupid reason: it does not map 0 to 0. But even the less naive functor $M \mapsto \mathrm{Ker}(K(A \oplus M) \to K(A))$ is not additive. Goodwillie's derivative construction stabilizes this latter construction to produce a new functor that does commute with direct sums, and in fact, all colimits. This Goodwillie derivative of K-theory is often called *stable K-theory*, and was first studied by Waldhausen in [Wal].

The theorem [DM1] of Dundas-McCarthy in fact identifies stable K-theory up to a cohomological shift with the functor $\mathrm{THH}(A, -) : A\text{–}\mathsf{bimod} \to \mathsf{Sp}$. Here THH denotes *topological Hochschild homology* (which is possible meaning of Hochschild homology in the setting of spectra).

The same constructions may be applied to TC. One again shows that "stable TC" coincides with THH up to shift, and that this identification is compatible with the cyclotomic trace; in particular, the cyclotomic trace induces an isomorphism on Goodwillie derivatives. (The former result is due to Hesselholt [Hes1], while the compatibility with the cyclotomic trace map seems to be due to [LM] §11.)

The *structural properties* we referred to are results about K-theory and TC commuting with certain colimits and limits and having some cohomological boundedness properties. The key point is that these structural properties are features of the two functors considered separately.

To reiterate: we are proving a theorem about the *fiber* of the cyclotomic trace map $K \to \mathrm{TC}$, which looks like a quite subtle object, it suffices to know its Goodwillie derivative and certain structural properties of K-theory and TC independently.

1.3. The method for reducing Theorem 1.1.1 to its stabilized form is called *Goodwillie calculus*. The interested reader may refer to [Goo2] for an overview of this subject, and [Lur2] §6 for a thorough treatment of the subject; however, the present note is self-contained in terms of what we use from Goodwillie calculus.

1.4. **What do we need to know about K-theory and TC?** There is a remarkable asymmetry in how K-theory and TC are treated.

In the proof of Theorem 1.1.1, we essentially never calculate anything about K-theory. The argument that its derivative is THH up to shift is quite soft (see the proof of Theorem 3.10.1). Establishing sufficient structural properties of K-theory to obtain Theorem 1.1.1 for split square-zero extensions is also not terribly difficult (and is contained in §3). For the general form of Theorem 1.1.1, some more subtle methods are needed to establish the necessary structural properties (see Theorem 5.11.2).

In contrast, our knowledge of TC is based almost entirely around an explicit formula for TC in the case of split square-zero extensions: see Theorem 4.10.1.

This disparity indicates the well-established strength of Theorem 1.1.1 for K-theory calculations.

1.5. **Comparison with [DGM].** The exposition here closely follows [DGM] at some points, but departs from it in several notable respects.

First, our treatment of topological cyclic homology follows the recent work of [NS], which clarifies the construction of TC. These notes assume some substantial familiarity with their approach to TC, and we take Nikolaus and Scholze's constructions as definitions. This allows us to circumvent equivariant homotopy theory here (although we sometimes draw notation from that subject).

We also give a somewhat streamlined approach to Goodwillie's calculus of functors in §2. In particular, we avoid (or at least mask) the analysis of cubes that appears in [DGM] §7.2.1.2.

In addition, we use Lurie's higher methods ([Lur1] and [Lur2]), which provide simpler homotopical foundations than those used in [DGM]. In particular, Lurie's approach makes it routine to treat $\mathcal{E}_1$-algebras more directly than in [DGM], where the authors avoid certain homotopy coherence questions by reducing to simplicial algebras with some cost to the conceptual clarity. We treat general connective $\mathcal{E}_1$-algebras on equal footing with classical (alias: discrete) rings, and have avoided reducing problems to (non-topological) Hochschild homology for rings.

In particular, we use higher categorical methods to directly apply Goodwillie's calculus to functors from A-bimodules to spectra.

As a final difference with [DGM], which may be more neutral than an improvement, we have also chosen to work in an abstract categorical setting where possible. That is, where possible we work with (suitable) categories $\mathcal{C}$ that would be A–mod in cases of interest. This is not because it is so important for applications to work in this generality, but because the author personally finds this setting to be clarifying, and for the sake of diversifying the literature somewhat. Other expositions tend to work directly with algebras and their modules, and the reader may readily find arguments in that more familiar language in the literature.

1.6. **Structure of these notes.** In §2-4, we treat the split square-zero case of Theorem 1.1.1. In §2, we explain how to reduce the theorem in this case to the comparison of derivatives and structural properties of these functors. In §3 and 4, we prove the corresponding facts about K-theory and TC.

Finally, in §5, we axiomatize the additional structural facts about K-theory and TC needed for the general form of Theorem 1.1.1, and then we establish these features.

1.7. **Categorical notions.** We systematically use higher category theory and higher algebra in our treatment, following [Lur1] and [Lur2]. We find it convenient to avoid "higher" terminology everywhere, so all terminology should be understood in its homotopical form: *category* means $(\infty, 1)$-*category*, *colimit* means *homotopy colimit*, and so on. In this spirit, we refer to the stable ∞-categories of [Lur2] simply as *stable categories*.

We let Gpd denote the category of (higher) groupoids, i.e., "spaces" in more standard homotopy-theoretic language.

We casually use an equals sign between two objects in a category to indicate the presence of a (hopefully) clear isomorphism.

1.8. For a category $\mathcal{C}$ and $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \in \mathsf{Gpd}$ denote the groupoid of maps from $\mathcal{F}$ to $\mathcal{G}$.

For categories $\mathcal{C}$ and $\mathcal{D}$, we let $\mathsf{Hom}(\mathcal{C}, \mathcal{D})$ denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$. More generally, use Hom to indicate the category of functors in a 2-category (meaning $(\infty, 2)$-category).

1.9. For $\mathcal{C}$ stable and $\mathcal{F} \in \mathcal{C}$, we use the notations $\mathcal{F}[1]$ and $\Sigma\mathcal{F}$ (resp. $\mathcal{F}[-1]$ and $\Omega\mathcal{F}$) interchangeably, capriciously choosing between the two.

Following the above conventions, for $f : \mathcal{F} \to \mathcal{G} \in \mathcal{C}$, we let $\mathrm{Ker}(f)$ denote the *homotopy* kernel (or *fiber*) of f, and let $\mathrm{Coker}(f)$ denote the *homotopy* cokernel (or *cofiber*, or *cone*) of f. Recall that $\mathrm{Ker}(f)[1] = \mathrm{Coker}(f)$.

For a t-structure on $\mathcal{C}$, we use cohomological notation: $\mathcal{C}^{\leqslant 0} \subseteq \mathcal{C}$ are the connective objects, and $\mathcal{C}^{\geqslant 0}$ are the coconnective objects. We let $\tau^{\leqslant n}$ and $\tau^{\geqslant n}$ denote the corresponding truncation functors.

1.10. Let StCat_{cont} denote the category of cocomplete[2] stable categories under continuous[3] exact functors (i.e., functors commuting with all colimits).

We recall from [Lur2] §4.8 that StCat_{cont} has a symmetric monoidal structure $\otimes$. The basic property is that a functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E} \in \mathsf{StCat}_{cont}$ is equivalent to a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that commutes with colimits in each variable separately. We denote the canonical (non-exact!) functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ by $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \boxtimes \mathcal{G}$.

We use this tensor product quite substantially. We refer the reader to [GR] Chapter I.1 for the relevant background material.

1.11. We let $\mathsf{Sp} \in \mathsf{StCat}_{cont}$ denote the category of spectra, which we recall is the unit for the symmetric monoidal structure on StCat_{cont}. We let $\otimes : \mathsf{Sp} \times \mathsf{Sp} \to \mathsf{Sp}$ denote the tensor (alias: smash) product, and we let $\mathbb{S} \in \mathsf{Sp}$ denote the sphere spectrum. We have the standard adjunction $\Sigma^\infty : \mathsf{Gpd} \rightleftarrows \mathsf{Sp} : \Omega^\infty$.

For $\mathcal{C}$ a stable category and $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, we let $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ denote the spectrum of maps from $\mathcal{F}$ to $\mathcal{G}$. We use $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ to mean the groupoid of maps in the category $\mathcal{C}$, forgetting it was stable. In other words, $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = \Omega^\infty \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$.

1.12. We say $\mathcal{C} \in \mathsf{StCat}_{cont}$ is *dualizable* if it is dualizable with respect to the above symmetric monoidal structure. We let the dual category be denoted $\mathcal{C}^\vee$.

In this case, for any $\mathcal{D} \in \mathsf{StCat}_{cont}$ we have $\mathcal{C}^\vee \otimes \mathcal{D} \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{StCat}_{cont}}(\mathcal{C}, \mathcal{D})$. In particular, $\mathcal{C}^\vee = \mathsf{Hom}_{\mathsf{StCat}_{cont}}(\mathcal{C}, \mathsf{Sp})$. Moreover, we see that there is an evaluation map $\mathsf{End}_{\mathsf{StCat}_{cont}}(\mathcal{C}) = \mathcal{C} \otimes \mathcal{C}^\vee \to \mathsf{Sp} \in \mathsf{StCat}_{cont}$, which we denote by $\mathrm{tr}_{\mathcal{C}}$ and refer to as the *trace*.

For $\mathcal{F} \in \mathcal{C}$ a *compact* object, the functor (by fiat) $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -) : \mathcal{C} \to \mathsf{Sp}$ commutes with colimits, so defines an object of $\mathcal{C}^\vee$. We denote this object by $\mathbb{D}\mathcal{F}$.

Example 1.12.1. If $\mathcal{C}$ is compactly generated, then $\mathcal{C}$ is dualizable. Explicitly, if $\mathcal{C}^c \subseteq \mathcal{C}$ is the (essentially small) subcategory of compact objects in $\mathcal{C}$, then $\mathsf{Ind}(\mathcal{C}^{c,op}) = \mathcal{C}^\vee$ (for Ind denoting the ind-category), where the underlying functor $\mathcal{C}^{c,op} \to \mathcal{C}^\vee$ is the map $\mathcal{F} \mapsto \mathbb{D}\mathcal{F}$ above. (See [GR] Chapter I.1 for more details.)

Example 1.12.2. For $\mathcal{C} = A\text{–}\mathsf{mod}$ in the above, note that $A\text{–}\mathsf{mod}$ is compactly generated by perfect A-modules, so $A\text{–}\mathsf{mod}$ is dualizable. Moreover, duality gives a contravariant equivalence between left and right A-modules, so $A\text{–}\mathsf{mod}^\vee = A^{op}\text{–}\mathsf{mod}$ (modules over A with the opposite multiplication).

One then obtains a standard (Morita-style) identification $\mathsf{End}_{\mathsf{StCat}_{cont}}(A\text{–}\mathsf{mod}) = A\text{–}\mathsf{mod} \otimes A^{op}\text{–}\mathsf{mod} = A\text{–}\mathsf{bimod}$. The trace map constructed above then corresponds to (topological) Hochschild homology.

[2] We omit set-theoretic considerations. So *cocomplete* should be taken to mean *presentable*. Similarly, all functors between accessible categories will themselves be assumed accessible (i.e., wherever this hypothesis is reasonable we assume it). When we refer to commutation with all colimits, we mean small colimits.

[3] We say a functor is *continuous* if it commutes with filtered colimits.

1.13. We frequently reference *sifted* colimits, which may not be familiar to all readers. We review the theory briefly here, referring to [Lur1] §5.5.8 for proofs.

A category I is *sifted* if it is non-empty and $I \to I \times I$ is cofinal. A functor commutes with sifted colimits if and only if it is continuous and commutes with geometric realizations (i.e., colimits of simplicial diagrams). A functor F commutes with all colimits if and only if it commutes with sifted colimits and finite coproducts.

For $\mathcal{C} \in \mathsf{StCat}_{cont}$, the functors $\mathcal{C} \xrightarrow{\mathcal{F} \mapsto \mathcal{F}^{\boxtimes n}} \mathcal{C}^{\otimes n}$ are typical examples of functors that commute with sifted colimits but not (outside obvious exceptional cases) general colimits.

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I also would like to thank Sasha Beilinson, Lars Hesselholt, Akhil Mathew, and Thomas Nikolaus for helpful discussions on these subjects, for encouragement to write these notes, and for their comments and corrections.

2. Goodwillie calculus

2.1. This goal for this section is to setup the proof of Theorem 1.1.1 in the split square-zero case. We end the section by formulating Theorems 2.12.1 and 2.12.2, which are about K-theory and TC respectively; although the proofs of these theorems are deferred to later sections, we deduce the split square-zero case of Theorem 1.1.1 from them in §2.12.

However, most of this section is devoted to some formal vanishing results, allowing us to carry out the strategy indicated in §1.2. Our main results here are Theorem 2.5.1 and Corollary 2.11.7; the former is a toy model for the latter. The arguments are essentially the same in the two cases, but Theorem 2.5.1 is technically simpler to formulate, and its proof essentially leads to the formulation of the more technical Corollary 2.11.7.

Remark 2.1.1. We have sought a minimalist approach to this material and have omitted many lovely aspects of Goodwillie's theory here: most notably, higher derivatives, the Goodwillie tower, and the analogy with calculus. This makes our treatment somewhat non-standard, and we refer the reader to the extensive literature in this subject (for example, [Goo2] and [Lur2] §6) for a more thorough approach.

Remark 2.1.2. The next comments are intended for the reader who wishes to tighten the connection between this section and Goodwillie's theory. Theorems 2.6.1 and 2.11.6, which provide hypotheses for a functor to vanish, can be deduced from Goodwillie's theory by standard methods. Indeed, one would prove that the Goodwillie tower converges for functors satisfying these hypotheses (c.f. Remark 2.11.3), and that the hypotheses imply that all higher Goodwillie derivatives vanish.

The methods we use below in proving these theorems are not so different from standard ones in Goodwillie's theory. However, the Goodwillie tower and higher derivatives take some work to set up, so we prefer to circumvent these constructions.

2.2. **A convention.** The reader may safely skip the present discussion.

Throughout this section, we generally consider functors between stable categories (or their subcategories) commuting with sifted colimits. This choice is not at all because this assumption is essential. These assumptions can be significantly relaxed, and we refer to [Lur2] §6 for an approach with minimal hypotheses. Note, for example, that the discussion of §2.3 carries through as is if we work with functors commuting with $\mathbb{Z}^{\geqslant 0}$-indexed colimits.

We include this hypothesis because the commutation with geometric realizations is essential in our approach; the additional commutation with all filtered colimits is fairly minor; and the commutation with sifted colimits motivates the discussion of §2.7.

2.3. **The Goodwillie derivative.** Suppose $\psi : \mathcal{C} \to \mathcal{D}$ is a functor between cocomplete stable categories commuting with sifted colimits. The *Goodwillie derivative* $\partial\psi : \mathcal{C} \to \mathcal{D}$ of ψ is initial among continuous exact functors receiving a natural transformation from ψ.

Remark 2.3.1. By [Lur2] Proposition 1.4.2.13, $\partial\psi$ is calculated as follows.

First, observe that for any $\mathcal{F} \in \mathcal{C}$, the morphisms $0 \to \mathcal{F} \to 0$ give a functorial direct sum decomposition $\psi(\mathcal{F}) = \psi_{red}(\mathcal{F}) \oplus \psi(0)$. Then ψ_{red} is *reduced*, i.e., it takes $0 \in \mathcal{C}$ to $0 \in \mathcal{D}$.

Then there is a canonical natural transformation $\Sigma \circ \psi_{red} \to \psi_{red} \circ \Sigma$, or equivalently, $\psi_{red} \to \Omega\psi_{red}\Sigma$. Finally, we have:

$$\partial\psi = \operatorname{colim}\Big(\psi_{red} \to \Omega\psi_{red}\Sigma \to \Omega^2\psi_{red}\Sigma^2 \to \dots\Big)$$

We will actually use this construction in slightly more generality.

Variant 2.3.2. Suppose $\mathcal{C}$ is equipped with a t-structure compatible with filtered colimits and we are given $\psi : \mathcal{C}^{\leqslant 0} \to \mathcal{D}$ commuting with sifted colimits. Then there is again a functor $\partial\psi : \mathcal{C} \to \mathcal{D} \in \mathsf{StCat}_{cont}$ initial among functors commuting with colimits and receiving a natural transformation $\psi \to (\partial\psi)|_{\mathcal{C}^{\leqslant 0}}$.

To construct $\partial\psi$ in this setup, note that ψ_{red} makes sense as before, and then one has:

$$\partial\psi(\mathcal{F}) = \operatorname*{colim}_{m} \operatorname*{colim}_{n \geqslant m} \Omega^n\psi_{red}(\Sigma^n\tau^{\leqslant m}\mathcal{F})$$

If the t-structure on $\mathcal{C}$ is right complete and $\psi : \mathcal{C} \to \mathcal{D}$ commutes with sifted colimits, then $\partial\psi$ in the previous sense coincides with $\partial(\psi|_{\mathcal{C}^{\leqslant 0}})$.

2.4. **Notation.** We let Alg denote the category of $\mathcal{E}_1$-algebras. We let $\mathsf{Alg}_{conn} \subseteq \mathsf{Alg}$ denote the subcategory of connective $\mathcal{E}_1$-algebras.

Recall that for $A \in \mathsf{Alg}$ and M an A-bimodule, we can form the *split square-zero extension* $\mathrm{SqZero}(A, M) \in \mathsf{Alg}$, whose underlying spectrum is $A \oplus M$.

2.5. The following result is a first approximation to the main result of this section.

Theorem 2.5.1. *Let $\Psi : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ be a functor.*

Suppose that for every $A \in \mathsf{Alg}_{conn}$, the functor:

$$\begin{gathered}\Psi_A : A\text{–}\mathsf{bimod}^{\leqslant 0} \to \mathsf{Sp} \\ M \mapsto \Psi(\mathrm{SqZero}(A, M))\end{gathered}$$

commutes with sifted colimits and has vanishing Goodwillie derivative. Suppose moreover that its underlying reduced functor $\Psi_{A,red}$ maps A–$\mathsf{bimod}^{\leqslant 0}$ to $\mathsf{Sp}^{\leqslant 0}$.

Then for every $A \in \mathsf{Alg}_{conn}$ and $M \in A$–$\mathsf{bimod}^{\leqslant -1}$, the map $\Psi(\mathrm{SqZero}(A, M)) \to \Psi(A)$ is an isomorphism.

Remark 2.5.2. In this result, Sp may be replaced by any $\mathcal{D} \in \mathsf{StCat}_{cont}$ with a left separated t-structure.

2.6. **A categorical variant.** We will deduce Theorem 2.5.1 from the following result.

Theorem 2.6.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be cocomplete stable categories equipped with t-structures compatible with filtered colimits, and suppose the t-structure on $\mathcal{D}$ is left separated. Let $\psi : \mathcal{C}^{\leqslant 0} \to \mathcal{D}^{\leqslant 0}$ be a reduced functor commuting with sifted colimits. Suppose that*[4] $\partial(\psi(\mathcal{F} \oplus -)) = 0$ *for every* $\mathcal{F} \in \mathcal{C}^{\leqslant -1}$.

Then $\psi(\mathcal{F}) = 0$ *for every* $\mathcal{F} \in \mathcal{C}^{\leqslant -1}$.

Proof that Theorem 2.6.1 implies Theorem 2.5.1. Fix $A \in \mathsf{Alg}_{conn}$ and define $\psi : A\text{–}\mathsf{bimod}^{\leqslant 0} \to \mathsf{Sp}$ as $\Psi_{A,red}$. We claim that the hypotheses of Theorem 2.6.1 are satisfied.

There are explicit assumptions in Theorem 2.5.1 that ψ commutes with sifted colimits and maps into $\mathsf{Sp}^{\leqslant 0}$.

We need to show that for $M \in A\text{–}\mathsf{bimod}^{\leqslant -1}$, the Goodwillie derivative of $\psi(M \oplus -)$ is zero; in fact, we will show this for $M \in A\text{–}\mathsf{bimod}^{\leqslant 0}$. For $M = 0$, this is an assumption. In general, we have:

$$\psi(M \oplus -) \oplus \Psi(A) = \Psi(\mathrm{SqZero}(A, M \oplus -)) = \Psi(\mathrm{SqZero}(\mathrm{SqZero}(A, M), -)).$$

The latter functor has vanishing derivative by the hypothesis that the Goodwillie derivative of $\Psi_{\mathrm{SqZero}(A,M)}$ vanishes.

□

Remark 2.6.2. Akhil Mathew communicated the following example to us, showing that Theorem 2.6.1 is sharp. Let $\mathcal{C} = \mathcal{D} = \mathbb{F}_p\text{–}\mathsf{mod}$ be the categories of $\mathbb{F}_p$-vector spaces for p some prime. We will construct a non-zero functor $\mathbb{F}_p\text{–}\mathsf{mod}^{\leqslant 0} \to \mathbb{F}_p\text{–}\mathsf{mod}^{\leqslant 0}$ which satisfies the hypotheses of Theorem 2.6.1.

For $V \in \mathbb{F}_p\text{–}\mathsf{mod}^{\heartsuit}$, let $\mathrm{Sym}^{\mathrm{perf}}(V)$ denote the perfection of the symmetric algebra on V, i.e., $\mathrm{colim}_n \mathrm{Sym}(V)$ with Frobenius as structure maps. By [Lur2] Theorem 1.3.3.8, there is a unique functor $\mathbb{L}\mathrm{Sym}^{\mathrm{perf}}(-) : \mathbb{F}_p\text{–}\mathsf{mod}^{\leqslant 0} \to \mathbb{F}_p\text{–}\mathsf{mod}^{\leqslant 0}$ commuting with sifted colimits and whose restriction to $\mathbb{F}_p\text{–}\mathsf{mod}^{\heartsuit}$ is $\mathrm{Sym}^{\mathrm{perf}}(-)$.

By [BS] Proposition 11.6, $\mathbb{L}\mathrm{Sym}^{\mathrm{perf}}(-)$ vanishes on $\mathbb{F}_p\text{–}\mathsf{mod}^{\leqslant -1}$; in particular, its Goodwillie derivative vanishes. Then the evident formula:

$$\mathbb{L}\mathrm{Sym}^{\mathrm{perf}}(V \oplus W) = \mathbb{L}\mathrm{Sym}^{\mathrm{perf}}(V) \otimes \mathbb{L}\mathrm{Sym}^{\mathrm{perf}}(W)$$

implies that for every $V \in \mathbb{F}_p\text{–}\mathsf{mod}^{\leqslant 0}$, the functor $\big(W \mapsto \mathbb{L}\mathrm{Sym}^{\mathrm{perf}}(V \oplus W)\big)$ also has vanishing Goodwillie derivative.

2.7. **The bilinear obstruction to linearity.** To prove Theorem 2.6.1, it is convenient to use the following construction.

In the notation of *loc. cit.*, note that ψ commutes with all colimits if and only if it commutes with pairwise direct sums (because ψ is reduced and commutes with sifted colimits). Therefore, B_ψ may be understood as the obstruction to ψ commuting with all colimits.

For $\mathcal{F}, \mathcal{G} \in \mathcal{C}^{\leqslant 0}$, define:

$$B_\psi(\mathcal{F}, \mathcal{G}) = \mathrm{Coker}(\psi(\mathcal{F}) \oplus \psi(\mathcal{G}) \to \psi(\mathcal{F} \oplus \mathcal{G})) \in \mathcal{D}.$$

Note that:

$$\psi(\mathcal{F} \oplus \mathcal{G}) = \psi(\mathcal{F}) \oplus \psi(\mathcal{G}) \oplus B_\psi(\mathcal{F}, \mathcal{G})$$

because the composition:

[4] For clarity: the notation indicates the Goodwillie derivative of the functor $\mathcal{G} \mapsto \psi(\mathcal{F} \oplus \mathcal{G})$.

$$\psi(\mathcal{F}) \oplus \psi(\mathcal{G}) \to \psi(\mathcal{F} \oplus \mathcal{G}) = \psi(\mathcal{F} \times \mathcal{G}) \to \psi(\mathcal{F}) \times \psi(\mathcal{G}) = \psi(\mathcal{F}) \oplus \psi(\mathcal{G})$$

is the identity.

Therefore, ψ commutes with colimits if and only if $B_\psi = 0$.

2.8. **Simplicial review.** We will prove Theorem 2.6.1 using some standard simplicial methods.

Suppose $\mathcal{F}_\bullet$ is a simplicial object in $\mathcal{C}$. We let $|\mathcal{F}_\bullet|$ denote the *geometric realization* of this simplicial diagram, i.e., the colimit.

Similarly, let $|\mathcal{F}_\bullet|_{\leqslant n}$ denote the partial geometric realization $\operatorname{colim}_{\mathbf{\Delta}_{\leqslant n}^{op}} \mathcal{F}_\bullet$. Here $\mathbf{\Delta}_{\leqslant n} \subseteq \mathbf{\Delta}$ is the full subcategory of simplices of order $\leqslant n$. Note that:

$$|\mathcal{F}_\bullet| = \operatorname{colim}_n |\mathcal{F}_\bullet|_{\leqslant n}.$$

Finally, we recall (c.f. to the proof of the Dold-Kan theorem):

Lemma 2.8.1. *For $n \geqslant 0$, $\operatorname{Coker}(|\mathcal{F}_\bullet|_{\leqslant n} \to |\mathcal{F}_\bullet|_{\leqslant n+1})$ is a direct summand of $\mathcal{F}_{n+1}[n+1]$.*

2.9. Our main technique is the following.

Lemma 2.9.1. *Suppose $\psi : \mathcal{C}^{\leqslant 0} \to \mathcal{D}$ commutes with sifted colimits. For every $\mathcal{F} \in \mathcal{C}$, $\psi(\Sigma\mathcal{F})$ admits an increasing filtration*[5] $\operatorname{fil}_\bullet \psi(\Sigma\mathcal{F})$ *such that:*

- $\operatorname{fil}_i \psi(\Sigma\mathcal{F}) = 0$ *for $i < 0$.*
- *For $i \geqslant 0$, $\operatorname{gr}_i \psi(\Sigma\mathcal{F})$ is a direct summand of $\psi(\mathcal{F}^{\oplus i})[i]$.*
- *More precisely, $\operatorname{gr}_0 \psi(\Sigma\mathcal{F}) = \psi(0)$, $\operatorname{gr}_1(\psi(\Sigma\mathcal{F})) = \psi_{red}(\mathcal{F})[1]$, and $\operatorname{gr}_2(\psi(\Sigma\mathcal{F})) = B_{\psi_{red}}(\mathcal{F},\mathcal{F})[2]$.*

Proof. There is a canonical simplicial diagram:

$$\ldots \mathcal{F} \oplus \mathcal{F} \substack{\to\\\to\\\to} \mathcal{F} \rightrightarrows 0$$

with geometric realization $\Sigma\mathcal{F}$. (For example, this simplicial diagram is the Cech construction for $0 \to \Sigma\mathcal{F}$.)

Because ψ commutes with geometric realizations, we have:

$$\psi(\Sigma\mathcal{F}) = |\psi(\mathcal{F}^{\oplus\bullet})|.$$

We then set $\operatorname{fil}_i \psi(\Sigma\mathcal{F}) = |\psi(\mathcal{F}^{\oplus\bullet})|_{\leqslant i}$. This filtration tautologically satisfies the first property, and it satisfies the second property by Lemma 2.8.1. The third property follows by refining Lemma 2.8.1 to identify exactly which summand occurs, which we omit here (see e.g. [Lur2] Lemma 1.2.4.17). □

Remark 2.9.2. In the terminology of Goodwillie calculus, we generally have:

$$\operatorname{gr}_i \psi(\Sigma\mathcal{F}) = \operatorname{cr}_i \psi(\mathcal{F}, \ldots, \mathcal{F})[i]$$

where $\operatorname{cr}_i \psi$ is the *ith cross-product* of ψ. (In particular, B_ψ is non-standard notation for $\operatorname{cr}_2 \psi$.) However, we will not explicitly need the higher cross products.

As a first consequence, we obtain:

Corollary 2.9.3. *Suppose that $\psi : \mathcal{C}^{\leqslant 0} \to \mathcal{D}^{\leqslant 0}$ is reduced and commutes with sifted colimits. Then for every $n \geqslant 0$, $\psi(\mathcal{C}^{\leqslant -n}) \subseteq \mathcal{D}^{\leqslant -n}$.*

[5]For us, all filtrations are assumed exhaustive.

Proof. By induction, it suffices to show $\psi(\mathcal{C}^{\leqslant -1}) \subseteq \mathcal{D}^{\leqslant -1}$. Suppose $\mathcal{F} \in \mathcal{C}^{\leqslant 0}$; we need to show $\psi(\Sigma\mathcal{F}) \in \mathcal{D}^{\leqslant -1}$.

We use the filtration of Lemma 2.9.1. It suffices to show $\mathrm{gr}_i\,\psi(\Sigma\mathcal{F}) \in \mathcal{D}^{\leqslant -1}$. Then $\mathrm{gr}_0\,\psi(\Sigma\mathcal{F}) = 0$ as ψ is reduced; and for $i > 0$, $\mathrm{gr}_i\,\psi(\Sigma\mathcal{F})$ is a summand of $\psi(\mathcal{F}^i)[i]$, which is clearly in degrees $\leqslant -i < 0$. ∎

2.10. We now prove Theorem 2.6.1.

Proof of Theorem 2.6.1. We will show by induction on n that these hypotheses on ψ force $\psi(\mathcal{C}^{\leqslant -1}) \subseteq \mathcal{D}^{\leqslant -n}$. The case $n = 1$ is given by Corollary 2.9.3. In what follows, we assume the inductive hypothesis for n and deduce it for $n + 1$.

Step 1. First, we claim that for $\mathcal{F} \in \mathcal{C}^{\leqslant -1}$, the functor:

$$B_\psi(\mathcal{F}, -)[-1] : \mathcal{C}^{\leqslant 0} \to \mathcal{D}$$

satisfies the hypotheses of Theorem 2.6.1.

Clearly this functor is reduced and commutes with sifted colimits.

Let us show that this functor maps $\mathcal{C}^{\leqslant 0}$ into $\mathcal{D}^{\leqslant 0}$. Fix $\mathcal{G} \in \mathcal{C}^{\leqslant 0}$. The reduced functor $B_\psi(-, \mathcal{G})$ commutes with sifted colimits and maps $\mathcal{C}^{\leqslant 0}$ into $\mathcal{D}^{\leqslant 0}$. By Corollary 2.9.3, $B_\psi(\mathcal{F}, \mathcal{G}) \in \mathcal{D}^{\leqslant -1}$, so $B_\psi(\mathcal{F}, \mathcal{G})[-1] \in \mathcal{D}^{\leqslant 0}$ as desired.

Finally, note that for any $\mathcal{F}' \in \mathcal{C}^{\leqslant -1}$, the functor:

$$\Omega B_\psi(\mathcal{F}, \mathcal{F}' \oplus -) : \mathcal{C}^{\leqslant 0} \to \mathcal{D}$$

has vanishing Goodwillie derivative, as it is a summand of the functor $\psi(\mathcal{F} \oplus \mathcal{F}' \oplus -)[-1]$.

Therefore, we may apply the inductive hypothesis to this functor. We obtain that $B_\psi(\mathcal{F}, -)$ maps $\mathcal{C}^{\leqslant -1}$ to $\mathcal{D}^{\leqslant -n-1}$. In particular, $B_\psi(\mathcal{F}, \mathcal{F}) \in \mathcal{D}^{\leqslant -n-1}$.

Step 2. Next, we claim that:

$$\mathrm{Coker}(\Sigma\psi(\mathcal{F}) \to \psi(\Sigma\mathcal{F})) \in \mathcal{D}^{\leqslant -n-3}.$$

Note that by the construction of Lemma 2.9.1, the map:

$$\Sigma\psi(\mathcal{F}) = \mathrm{gr}_1\,\psi(\Sigma\mathcal{F}) = \mathrm{fil}_1\,\psi(\Sigma\mathcal{F}) \to \psi(\Sigma\mathcal{F})$$

is the canonical map used in the definition of the Goodwillie derivative. Therefore, it suffices to show that $\mathrm{gr}_i\,\psi(\Sigma\mathcal{F}) \in \mathcal{D}^{\leqslant -n-3}$ for $i \geqslant 2$.

By induction, $\psi(\mathcal{F}^i) \in \mathcal{D}^{\leqslant -n}$ by induction. Therefore, $\mathrm{gr}_i\,\psi(\Sigma\mathcal{F}) \in \mathcal{D}^{\leqslant -n-i}$. This gives the claim for $i \geqslant 3$.

If $i = 2$, then $\mathrm{gr}_i\,\psi(\Sigma\mathcal{F}) = B_\psi(\mathcal{F}, \mathcal{F})[2]$, and by the previous step $B_\psi(\mathcal{F}, \mathcal{F}) \in \mathcal{D}^{\leqslant -n-1}$ as needed.

Step 3. Finally, the previous step implies that for $\mathcal{F} \in \mathcal{C}^{\leqslant -1}$, the map:

$$\psi(\mathcal{F}) \to \Omega\psi(\Sigma\mathcal{F})$$

is an isomorphism on H^{-n} (where this notation denotes the cohomology functor for the t-structure on $\mathcal{D}$).

More generally, for any $m \geqslant 0$, the functor $\Omega^m\psi\Sigma^m : \mathcal{C}^{\leqslant 0} \to \mathcal{D}^{\leqslant 0}$ satisfies our hypotheses, so we find that $\Omega^m\psi\Sigma^m(\mathcal{F}) \to \Omega^{m+1}\psi(\Sigma^{m+1}\mathcal{F})$ is an isomorphism on H^{-n}.

Finally, we obtain $H^{-n}(\psi(\mathcal{F})) \xrightarrow{\simeq} H^{-n}(\partial\psi(\mathcal{F}))$ is an isomorphism. But of course, $\partial\psi = 0$, so we obtain $\psi(\mathcal{F}) \in \mathcal{D}^{\leqslant -n-1}$, providing the inductive step.

□

2.11. **Full vanishing results.** We now wish to extend the above results to give vanishing for connective objects. Throughout this section, $\mathcal{C}, \mathcal{D} \in \mathsf{StCat}_{cont}$ are equipped with t-structures compatible with filtered colimits.

The following definition is obviously quite natural in this context.

Definition 2.11.1. A functor $\psi : \mathcal{C}^{\leq 0} \to \mathcal{D}^{\leq 0}$ is *extensible* if there exists $\widetilde{\psi} : \mathcal{C}^{\leq 1} \to \mathcal{D}^{\leq 1}$ commuting with sifted colimits with $\widetilde{\psi}|_{\mathcal{C}^{\leq 0}} = \psi$.

It is convenient to introduce the following notion as well.

Definition 2.11.2. A functor $\psi : \mathcal{C}^{\leq 0} \to \mathcal{D}^{\leq 0}$ is *pseudo-extensible* if:

- ψ is reduced and commutes with sifted colimits.
- Every φ in $\mathcal{S}_\psi$ maps $\mathcal{C}^{\leq 0} \to \mathcal{D}^{\leq 0}$, where $\mathcal{S}_\psi \subseteq \mathsf{Hom}(\mathcal{C}, \mathcal{D})$ is the minimal subgroupoid such that $\psi \in \mathcal{S}_\psi$ and such that for every $\mathcal{F} \in \mathcal{C}^{\leq 0}$ and $\varphi \in \mathcal{S}_\psi$, $B_\varphi(\mathcal{F}, -)[-1] \in \mathcal{S}_\psi$.

Remark 2.11.3. In standard terminology from Goodwillie calculus, one can show that if ψ commutes with sifted colimits, then ψ is *(-1)-analytic* if and only if $\psi[n]$ is pseudo-extensible for some n.

Lemma 2.11.4. *If $\psi : \mathcal{C}^{\leq 0} \to \mathcal{D}^{\leq 0}$ is extensible, then it is pseudo-extensible.*

Proof. Clearly any $\varphi \in \mathcal{S}_\psi$ is extensible. Therefore, by induction we are reduced to showing that for $\mathcal{F} \in \mathcal{C}^{\leq 0}$, $B_\psi(\mathcal{F}, -)[-1]$ maps $\mathcal{C}^{\leq 0}$ to $\mathcal{D}^{\leq 0}$.

Let $\widetilde{\psi} : \mathcal{C}^{\leq 1} \to \mathcal{D}^{\leq 1}$ be as in the definition of extensibility. Clearly $B_{\widetilde{\psi}}$ maps $\mathcal{C}^{\leq 1} \times \mathcal{C}^{\leq 1}$ to $\mathcal{D}^{\leq 1}$, so applying Corollary 2.9.3 once in each variable, we find $B_{\widetilde{\psi}}$ maps $\mathcal{C}^{\leq 0} \times \mathcal{C}^{\leq 0}$ to $\mathcal{D}^{\leq -1}$. Here the functor coincides with B_ψ, and incorporating the shift we get the claim.

□

The flexibility the next result affords is ultimately the reason we consider pseudo-extensible functors here.

Lemma 2.11.5. *In the above setting, suppose the t-structure on $\mathcal{D}$ is left complete. Let $\mathcal{I}$ be a filtered category, and suppose we are given a diagram $\mathcal{I}^{op} \xrightarrow{i \mapsto \psi_i} \mathsf{Hom}(\mathcal{C}^{\leq 0}, \mathcal{D}^{\leq 0})$ of pseudo-extensible functors. Suppose that for every n there exists $i \in \mathcal{I}$ such that $\tau^{\geq -n}\psi_j \xrightarrow{\simeq} \tau^{\geq -n}\psi_i$ for all $i \to j \in \mathcal{I}$.*

Then the value-wise limit of functors $\psi = \lim_{i \in \mathcal{I}^{op}} \psi_i$ is pseudo-extensible.

Proof. For $\mathcal{F}, \mathcal{G} \in \mathcal{C}^{\leq 0}$, we have:

$$B_\psi(\mathcal{F}, \mathcal{G}) = \lim_i B_{\psi_i}(\mathcal{F}, \mathcal{G})$$

by definition of B_-. Therefore, so we are reduced (by induction, say) to showing that $B_\psi(\mathcal{F}, \mathcal{G}) \in \mathcal{D}^{\leq -1}$.

Recall that left completeness and our stabilization hypotheses imply $\tau^{\geq -n}\psi = \tau^{\geq -n}\psi_i$ for i in $\mathcal{I}$ sufficiently large (depending on n). Therefore, $\tau^{\geq -n}\psi$ is reduced and commutes with sifted colimits for every n, which implies the same for the functor ψ (by left completeness of the t-structure on $\mathcal{D}$).

Similarly, we have $\tau^{\geq -n} \lim_i B_{\psi_i}(\mathcal{F}, \mathcal{G}) = \tau^{\geq -n} B_{\psi_i}(\mathcal{F}, \mathcal{G})$ for i sufficiently large. So clearly $B_\psi(\mathcal{F}, \mathcal{G}) \in \mathcal{D}^{\leq -1}$, since this is true for each ψ_i by pseudo-extensibility.

□

We now have the following result, which in the extensible case is just a rephrasing of Theorem 2.6.1.

Theorem 2.11.6. *In the setting of Theorem 2.6.1, suppose ψ is pseudo-extensible and $\partial(\psi(\mathcal{F} \oplus -)) = 0$ for every $\mathcal{F} \in \mathcal{C}^{\leqslant 0}$. Then ψ maps $\mathcal{C}^{\leqslant 0}$ to $\cap \mathcal{D}^{\leqslant -n}$.*

Proof. First, we claim $\psi(\mathcal{F}) \in \mathcal{D}^{\leqslant -1}$ for all $\mathcal{F} \in \mathcal{C}^{\leqslant 0}$. As in the proof of Theorem 2.6.1, it suffices to show $\mathrm{gr}_i\, \psi(\Sigma\mathcal{F}) \in \mathcal{D}^{\leqslant -3}$ for all i, and this is automatic for $i \geqslant 3$. For $i = 2$, we have $\mathrm{gr}_2\, \psi(\Sigma\mathcal{F}) = B_\psi(\mathcal{F}, \mathcal{F})[2]$, and $B_\psi(\mathcal{F}, \mathcal{F}) \in \mathcal{D}^{\leqslant -1}$ by pseudo-extensibility.

Next, observe that any $\varphi \in \mathcal{S}_\psi$ is pseudo-extensible, and by induction the Goodwillie derivatives of the functors $\varphi(\mathcal{F} \oplus -)$ vanish for any $\mathcal{F} \in \mathcal{C}^{\leqslant 0}$. Therefore, by the above argument, every $\varphi \in \mathcal{S}_\psi$ maps $\mathcal{C}^{\leqslant 0}$ into $\mathcal{D}^{\leqslant -1}$.

Finally, we see that $\psi[-1]$ is pseudo-extensible, so by induction we obtain the result. □

We immediately deduce the following.

Corollary 2.11.7. *In the setting of Theorem 2.5.1, suppose that the functors $\Psi_{A,red} : A\text{–}\mathsf{bimod}^{\leqslant 0} \to \mathsf{Sp}$ are pseudo-extensible.*

Then $\Psi(A \oplus M) \xrightarrow{\simeq} \Psi(A)$ for any $M \in A\text{–}\mathsf{bimod}^{\leqslant 0}$.

2.12. **The split square-zero case of Theorem 1.1.1.** The following two results will be shown in §3 and §4 respectively.

Theorem 2.12.1. *(1) For $A \in \mathsf{Alg}_{conn}$, the functor:*

$$A\text{–}\mathsf{bimod}^{\leqslant 0} \to \mathsf{Sp}^{\leqslant 0}$$
$$M \mapsto K(\mathrm{SqZero}(A, M))$$

commutes with sifted colimits. Moreover, the underlying reduced functor is extensible in the sense of §2.11.

(2) For $A \in \mathsf{Alg}_{conn}$, the Goodwillie derivative of the functor:

$$A\text{–}\mathsf{bimod}^{\leqslant 0} \to \mathsf{Sp}$$
$$M \mapsto K(A \oplus M)$$

is canonically isomorphic to the functor $M \mapsto \mathrm{THH}(A, M)[1]$.

Theorem 2.12.2. *(1) For $A \in \mathsf{Alg}_{conn}$, the functor:*

$$A\text{–}\mathsf{bimod}^{\leqslant 0} \to \mathsf{Sp}$$
$$M \mapsto \mathrm{TC}(\mathrm{SqZero}(A, M))$$

is pseudo-extensible in the sense of §2.11.

(2) For A as above and $M \in A\text{–}\mathsf{bimod}^{\leqslant 0}$:

$$\mathrm{TC}_{red}(\mathrm{SqZero}(A, M)) := \mathrm{Ker}(\mathrm{TC}(\mathrm{SqZero}(A, M)) \to \mathrm{TC}(A)) \in \mathsf{Sp}^{\leqslant -1}.$$

(3) The Goodwillie derivative of the above functor is canonically isomorphic to $\mathrm{THH}(A, -)[1]$. Moreover, this isomorphism is compatible with the cyclotomic trace and the isomorphism of Theorem 2.12.1.

Remark 2.12.3. Throughout these notes, K denotes *connective* K-theory. Adapting [Bei] Lemma 2.3 to the setting of connective ring spectra justifies omitting negative K-groups.

2.13. Let us assume the above theorems for now and deduce Theorem 1.1.1 in the split square-zero case.

Define the functor:

$$\Psi^{DGM} : \mathsf{Alg}_{conn} \to \mathsf{Sp}$$
$$A \mapsto \mathrm{Coker}(K(A) \to \mathrm{TC}(A))$$

where the map $K(A) \to \mathrm{TC}(A)$ is the cyclotomic trace map. It suffices to show Ψ^{DGM} satisfies the hypotheses of Theorem 2.5.1 and Corollary 2.11.7.

$\Psi^{DGM}_{red}(\mathrm{SqZero}(A, -)) : A\text{–}\mathsf{bimod}^{\leqslant 0} \to \mathsf{Sp}$ is pseudo-extensible by Theorem 2.12.1 (1) and Theorem 2.12.2 (1) as this property is preserved under cokernels. Moreover, this functor maps into $\mathsf{Sp}^{\leqslant 0}$ by Theorem 2.12.2 (2) (c.f. Remark 2.12.3). Finally, its Goodwillie derivative vanishes by Theorem 2.12.1 (2) and Theorem 2.12.2 (3).

3. K-theory

3.1. In this section, we prove Theorem 2.12.1. It is convenient in working with K-theory to generalize to a categorical setting, and we do so in what follows.

3.2. **Split square-zero extensions categorically.** First, we interpret the theory of split square-zero extensions in the categorical setting.

Suppose $\mathcal{C} \in \mathsf{StCat}_{cont}$ and $T : \mathcal{C} \to \mathcal{C} \in \mathsf{StCat}_{cont}$ is a (continuous, exact) endomorphism.

Definition 3.2.1. $\mathrm{SqZero}(\mathcal{C}, T)$ is the category of pairs $\mathcal{F} \in \mathcal{C}$ and $\eta : \mathcal{F} \to T(\mathcal{F})$ a *locally nilpotent* endomorphism, i.e., the colimit of the diagram:

$$\mathcal{F} \xrightarrow{\eta} T(\mathcal{F}) \xrightarrow{T(\eta)} T(\mathcal{F}) \xrightarrow{T^2(\eta)} \ldots$$

is zero.

Proposition 3.2.2. *Suppose* $A \in \mathsf{Alg}$ *and* $M \in A\text{–}\mathsf{bimod}$*. Let* $T_M := (M[1] \otimes_A -) : A\text{–}\mathsf{mod} \to A\text{–}\mathsf{mod}$*. Then there is a canonical equivalence:*

$$\mathrm{SqZero}(A, M)\text{–}\mathsf{mod} \simeq \mathrm{SqZero}(A\text{–}\mathsf{mod}, T_M)$$

such that the diagram:

$$\begin{array}{ccc} \mathrm{SqZero}(A, M)\text{–}\mathsf{mod} & \longrightarrow & \mathrm{SqZero}(A\text{–}\mathsf{mod}, T_M) \\ & {\scriptstyle A \underset{\mathrm{SqZero}(A,M)}{\otimes} -} \searrow \quad \swarrow {\scriptstyle (\mathcal{F},\eta) \mapsto \mathcal{F}} & \\ & A\text{–}\mathsf{mod} & \end{array}$$

commutes.

Proof. We construct the functor:

$$F : \mathrm{SqZero}(A, M)\text{–}\mathsf{mod} \to \mathrm{SqZero}(A\text{–}\mathsf{mod}, T_M)$$

as follows. Suppose $N \in \mathrm{SqZero}(A, M)\text{–}\mathsf{mod}$. We must have $F(N) = A \otimes_{\mathrm{SqZero}(A,M)} N$ as objects of $A\text{–}\mathsf{mod}$; it remains to define the map η (in the above notation). This map is the boundary for the obvious exact triangle:

$$T_M(F(N))[-1] = M \underset{A}{\otimes} F(N) \to N \to F(N) \xrightarrow{+1}$$

where N is regarded as an A-module through the section $A \to \mathrm{SqZero}(A, M)$. In the case $N = \mathrm{SqZero}(A, M)$, one sees that this triangle is split, so η is 0; this implies in general that η is locally nilpotent.

Now observe that the diagram:

$$\begin{array}{ccc} \mathrm{SqZero}(A, M)\text{–}\mathsf{mod} & \xrightarrow{F} & \mathrm{SqZero}(A\text{–}\mathsf{mod}, T_M) \\ & \searrow \quad \swarrow_{(\mathcal{F},\eta)\mapsto \mathrm{Ker}(\eta)} & \\ & A\text{–}\mathsf{mod} & \end{array}$$

tautologically commutes, where the left arrow is restriction along the morphism $A \to \mathrm{SqZero}(A, M)$.

To show that F is an equivalence, it suffices to show that each of the above functors to A–mod is monadic and the induced functor of monads is an isomorphism. The left functor is tautologically monadic. The right functor admits the left adjoint $\mathcal{F} \mapsto (\mathcal{F}, \eta = 0)$; it obviously commutes with colimits and is conservative by local nilpotence of η. Recall that to check the induced map of monads is an isomorphism, it is enough to see that F is intertwined by the left adjoints to the vertical arrows, which is evident.

Moreover, this equivalence also clearly makes the diagram from the proposition commute.

□

We also use the following observation.

Proposition 3.2.3. *In the above setting, suppose that* $\mathcal{C}$ *is compactly generated. Then* $\mathrm{SqZero}(\mathcal{C}, T)$ *is compactly generated. An object of* $\mathrm{SqZero}(\mathcal{C}, T)$ *is compact if and only if the underlying object of* $\mathcal{C}$ *is.*

Proof. Suppose $(\mathcal{F}, \eta_{\mathcal{F}}), (\mathcal{G}, \eta_{\mathcal{G}}) \in \mathrm{SqZero}(\mathcal{C}, T)$. Note that:

$$\underline{\mathrm{Hom}}_{\mathrm{SqZero}(\mathcal{C},T)}\big((\mathcal{F}, \eta_{\mathcal{F}}), (\mathcal{G}, \eta_{\mathcal{G}})\big) = \mathrm{Eq}\left(\underline{\mathrm{Hom}}_{\mathrm{SqZero}(\mathcal{C},T)}(\mathcal{F}, \mathcal{G}) \rightrightarrows \underline{\mathrm{Hom}}_{\mathrm{SqZero}(\mathcal{C},T)}(\mathcal{F}, \mathcal{G})\right)$$

where the two maps in the equalizer are composition with $\eta_{\mathcal{F}}$ and $\eta_{\mathcal{G}}$ respectively. This implies that if $\mathcal{F}$ is compact in $\mathcal{C}$, then $(\mathcal{F}, \eta_{\mathcal{F}})$ is compact in $\mathrm{SqZero}(\mathcal{C}, T)$ (as colimits commutes with finite limits in $\mathcal{C}$).

We claim that $\mathrm{SqZero}(\mathcal{C}, T)$ is compactly generated by objects $(\mathcal{F}, \eta = 0)$ for $\mathcal{F}$ compact in $\mathcal{C}$. Indeed, suppose $(\mathcal{G}, \eta_{\mathcal{G}}) \in \mathrm{SqZero}(\mathcal{C}, T)$ receives only the zero map from such objects. This implies $\mathrm{Ker}(\eta_{\mathcal{G}}) = 0$, as:

$$\underline{\mathrm{Hom}}_{\mathrm{SqZero}(\mathcal{C},T)}\big((\mathcal{F}, 0), (\mathcal{G}, \eta_{\mathcal{G}})\big) = \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathrm{Ker}(\eta_{\mathcal{G}})).$$

Then local nilpotence of $\eta_{\mathcal{G}}$ implies $\mathcal{G} = 0$, as desired.

□

3.3. **Variant.** Here is a sort of alternative to the square-zero extension construction above, which is more convenient for our purposes.

Let $\mathcal{C}$ be a compactly generated stable category and let $T : \mathcal{C} \to \mathcal{C} \in \mathsf{StCat}_{cont}$ be an endomorphism. We define $\mathcal{C}^T$ as $\mathsf{Ind}(\mathcal{C}^{T,c})$ where $\mathcal{C}^{T,c}$ is the category of pairs $(\mathcal{F}, \eta)$ with $\mathcal{F} \in \mathcal{C}^c$ and $\eta : \mathcal{F} \to T(\mathcal{F})$. We remark that there is no local nilpotence hypothesis here.

Example 3.3.1. For k a field and $\mathcal{C} = k$–mod and $T = \mathrm{id}$, there are natural identifications:

$$\mathrm{SqZero}(\mathcal{C}, T) = \mathrm{Ker}(k[t]\text{–}\mathsf{mod} \xrightarrow{k(t) \underset{k[t]}{\otimes} -} k(t)\text{–}\mathsf{mod})$$
$$\mathcal{C}^T = \mathrm{Ker}(k[t]\text{–}\mathsf{mod} \xrightarrow{k[t,t^{-1}] \underset{k[t]}{\otimes} -} k[t,t^{-1}]\text{–}\mathsf{mod}).$$

3.4. We have the following compatibility in the above setting. Note that there is always a fully-faithful functor $\mathrm{SqZero}(\mathcal{C}, T) \to \mathcal{C}^T$ preserving compact objects.

Lemma 3.4.1. *For connective A and $M \in A\text{–}\mathsf{bimod}^{\leqslant 0}$, the natural functor:*

$$\mathrm{SqZero}(A\text{–}\mathsf{mod}, T_M) \to A\text{–}\mathsf{mod}^{T_M}$$

is an equivalence of categories.

In particular, $A\text{–}\mathsf{mod}^{T_M}$ is canonically equivalent to $\mathrm{SqZero}(A, M)\text{–}\mathsf{mod}$.

Proof. It suffices to observe that for any $\mathcal{F} \in A\text{–}\mathsf{mod}^c$, any map $\eta : \mathcal{F} \to T_M(\mathcal{F})$ is automatically locally nilpotent as $\mathrm{colim}_n\, T_M^n(\mathcal{F}) = 0$ (since $\mathcal{F}$ is bounded above and $T_M = M \otimes_A -[1]$ lowers cohomological degrees by 1).

The second point follows from Proposition 3.2.2.

□

Remark 3.4.2. Note that there is no contradiction with Example 3.3.1: there $\mathrm{id}_{\mathcal{C}} = T_{k[-1]}$ and $k[-1]$ is not connective.

3.5. **Structural features of K-theory.** We now prove the first point of Theorem 2.12.1.

Proof of Theorem 2.12.1 (1)*.* To prove the extensibility, define:

$$\widetilde{K}_A : A\text{–}\mathsf{bimod}^{\leqslant 1} \to \mathsf{Sp}^{\leqslant 0} \subseteq \mathsf{Sp}$$
$$\widetilde{K}_A(M) := K(A\text{–}\mathsf{mod}^{T_M,c}).$$

For $M \in A\text{–}\mathsf{bimod}^{\leqslant 0}$, note that $\widetilde{K}_A(M) = K(\mathrm{SqZero}(A, M))$ by Lemma 3.4.1. Therefore, we need only to show that $\widetilde{K}_A$ commutes with sifted colimits.

Let $\mathsf{Proj}(A) \subseteq A\text{–}\mathsf{mod}^c$ denote the full subcategory of *(finitely-generated) projective* A-modules, i.e., the full subcategory of $A\text{–}\mathsf{mod}^c$ consisting of summands of $A^{\oplus n}$ for some $n \in \mathbb{Z}^{\geqslant 0}$. Let $\mathsf{Proj}(A)^{T_M} \subseteq A\text{–}\mathsf{mod}^{T_M,c}$ be the full subcategory of pairs $(\mathcal{F}, \eta)$ with $\mathcal{F} \in \mathsf{Proj}(A)$.

Note that $\mathsf{Proj}(A)^{T_M}$ is an exact[6] category. Standard[7] arguments show that $K(\mathsf{Proj}(A)^{T_M}) \xrightarrow{\simeq} \widetilde{K}_A(M)$, where the left hand side indicates Waldhausen K-theory of this exact category. (It is essential that T_M is right t-exact here.)

Because $\Omega^\infty : \mathsf{Sp}^{\leqslant 0} \to \mathsf{Gpd}$ commutes with sifted colimits (as follows from thinking of connective spectra as group-like $\mathcal{E}_\infty$-monoids), it suffices to show that the functor:

$$\Omega^\infty \widetilde{K}_A : A\text{–}\mathsf{bimod}^{\leqslant 1} \to \mathsf{Gpd}$$

commutes with sifted colimits, or just as well, that $\Omega^{\infty-1}\widetilde{K}_A$ does. The latter is the geometric realization of Waldhausen's $S_\bullet$ construction, so it suffices to show the individual terms of the $S_\bullet$ construction commute with sifted colimits in M here.

[6]In the higher categorical sense: see [Bar] for an introduction in this setup.

[7]See [Fon] for a general format for such problems. In particular, the main theorem of *loc. cit.* implies our present claim. We thank Thomas Nikolaus for directing us to this reference.

To simplify the notation, we explain why $M \mapsto S_2(\mathsf{Proj}(A)^{T_M}) \in \mathsf{Gpd}$ commutes with sifted colimits: the general case is the same with more notation.

First, recall that $S_2(\mathsf{Proj}(A))$ is the groupoid of data $\mathcal{F}_1, \mathcal{F}_2 \in \mathsf{Proj}(A)$ and a map $f : \mathcal{F}_1 \to \mathcal{F}_2$ with $\mathrm{Coker}(f) \in \mathsf{Proj}(A) \subseteq A\text{–}\mathsf{mod}$.

Then we similarly have:

$$S_2(\mathsf{Proj}(A)^{T_M}) = \underset{(f:\mathcal{F}_1 \to \mathcal{F}_2)\in S_2(\mathsf{Proj}(A))}{\mathrm{colim}} \mathrm{Hom}_{A\text{–}\mathsf{mod}}(\mathcal{F}_1, T_M(\mathcal{F}_1)) \underset{\mathrm{Hom}_{A\text{–}\mathsf{mod}}(\mathcal{F}_1, T_M(\mathcal{F}_2))}{\times} \mathrm{Hom}_{A\text{–}\mathsf{mod}}(\mathcal{F}_2, T_M(\mathcal{F}_2)).$$

Therefore, it suffices to show that for every point $(f : \mathcal{F}_1 \to \mathcal{F}_2) \in S_2(\mathsf{Proj}(A))$, the above expression commutes with sifted colimits in M.

Note that:

$$\mathrm{Hom}_{A\text{–}\mathsf{mod}}(\mathcal{F}_1, T_M(\mathcal{F}_1)) \underset{\mathrm{Hom}_{A\text{–}\mathsf{mod}}(\mathcal{F}_1, T_M(\mathcal{F}_2))}{\times} \mathrm{Hom}_{A\text{–}\mathsf{mod}}(\mathcal{F}_2, T_M(\mathcal{F}_2)) =$$
$$\Omega^\infty\Big(\underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}}(\mathcal{F}_1, T_M(\mathcal{F}_1)) \underset{\underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}}(\mathcal{F}_1, T_M(\mathcal{F}_2))}{\times} \underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}}(\mathcal{F}_2, T_M(\mathcal{F}_2))\Big).$$

Before passing to Ω^∞, this expression clearly commutes with all colimits in M: this follows from compactness of the $\mathcal{F}_i$ and the fact that A–mod is stable.

As $\Omega^\infty : \mathsf{Sp}^{\leqslant 0} \to \mathsf{Gpd}$ is conservative and commutes with sifted colimits, it suffices to show that this fiber product lies in $\mathsf{Sp}^{\leqslant 0}$. Each term in the fiber product is connective because $\mathcal{F}_i \in \mathsf{Proj}(A)$ and $T_M(\mathcal{F}_i) \in A\text{–}\mathsf{mod}^{\leqslant 0}$. Then the fiber product is connective because $\underline{\mathrm{Hom}}(\mathcal{F}_2, T(\mathcal{F}_2)) \to \underline{\mathrm{Hom}}(\mathcal{F}_1, T(\mathcal{F}_2))$ is surjective on H^0 (because $\mathrm{Coker}(f) \in \mathsf{Proj}(A)$).

□

3.6. **Derivative of K-theory.** We now give the calculation of Goodwillie derivatives in an appropriate categorical setup.

3.7. For $(\mathcal{C}, T)$ as in §3.3, let $K_{\mathcal{C}}(T) \in \mathsf{Sp}$ be the (connective) K-theory of $\mathcal{C}^{T,c}$.

We have the following easy result.

Lemma 3.7.1. *For every compactly generated $\mathcal{C}$, the functor $K_{\mathcal{C}} : \mathsf{End}_{\mathsf{StCat}_{cont}}(\mathcal{C}) \to \mathsf{Sp}$ commutes with filtered colimits.*

Proof. Clearly the functor $T \mapsto \mathcal{C}^{T,c}$ (as a functor to essentially small stable categories, say) commutes with filtered colimits, so the result follows from the commutation of K-theory and filtered colimits.

□

3.8. Note that $K_{\mathcal{C}}$ has a little more functoriality.

Let $\mathsf{StCat}_{cg,endo}$ be the following 1-category.[8] Objects are pairs $(\mathcal{C}, T)$ with $\mathcal{C}$ a compactly generated stable category and $T : \mathcal{C} \to \mathcal{C}$ a functor commuting with colimits. Morphisms are lax commuting diagrams:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ \downarrow F & \overset{\varepsilon}{\swarrow} & \downarrow F \\ \mathcal{D} & \xrightarrow{T'} & \mathcal{D} \end{array}$$

[8]Meaning $(\infty, 1)$-category, of course; we are distinguishing here from an $(\infty, 2)$-category.

(so $\varepsilon : FT \to T'F$ is a natural transformation) with F preserving compact objects.[9]

Then $(\mathcal{C}, T) \mapsto K_{\mathcal{C}}(T)$ upgrades to a functor out of $\mathsf{StCat}_{cg,endo}$. For a diagram as above, the map $K_{\mathcal{C}}(T) \to K_{\mathcal{D}}(T')$ is induced by the functor:

$$\mathcal{C}^{T,c} \to \mathcal{D}^{T',c}$$

$$(\mathcal{F}, \eta : \mathcal{F} \to T(\mathcal{F})) \mapsto \big(F(\mathcal{F}), F(\mathcal{F}) \xrightarrow{F(\eta)} FT(\mathcal{F}) \xrightarrow{\varepsilon} T'F(\mathcal{F})\big).$$

3.9. In the setting of §3.3, let $\partial K_{\mathcal{C}}$ denote the Goodwillie derivative of the functor $K_{\mathcal{C}} : \mathsf{End}_{\mathsf{StCat}_{cont}}(\mathcal{C}) \to \mathsf{Sp}$; although $K_{\mathcal{C}}$ does not commute with sifted colimits, the definition and construction of the Goodwillie derivative still apply (c.f. §2.2). Moreover, $\partial K_{\mathcal{C}}$ commutes with arbitrary colimits by Lemma 3.7.1.

3.10. By Lemma 3.4.1, the following result is a generalization of Theorem 2.12.1 (2).

Theorem 3.10.1. *The functor ∂K is canonically isomorphic to the trace functor $\mathrm{tr}_{\mathcal{C}} : \mathrm{End}(\mathcal{C}) \to \mathsf{Sp}$.*

The proof of this result occupies the remainder of this section.

3.11. We will prove Theorem 3.10.1 using the following convenient characterization of ∂K.

First, note that $K(\mathsf{Sp})$ has a canonical base-point, i.e., there's a canonical map $\mathbb{S} \to K(\mathsf{Sp}) \in \mathsf{Sp}$ corresponding to the point of $\Omega^\infty K(\mathsf{Sp})$ which is the class of the sphere spectrum. Similarly, we have a canonical point of $K_{\mathsf{Sp}}(\mathrm{id}_{\mathsf{Sp}})$ defined by $(\mathbb{S}, \mathrm{id}_{\mathbb{S}}) \in \mathsf{Sp}^{\mathrm{id}_{\mathsf{Sp}},c}$.

We recall that the notation $\mathsf{StCat}_{cg,endo}$ introduced in §3.8.

Lemma 3.11.1. *Suppose that we are given a functor $\Phi : \mathsf{StCat}_{cg,endo} \to \mathsf{Sp}$, which we denote by $(\mathcal{C}, T) \mapsto \Phi_{\mathcal{C}}(T)$. Suppose that we are given a base-point $x : \mathbb{S} \to \Phi_{\mathsf{Sp}}(\mathrm{id}_{\mathsf{Sp}}) \in \mathsf{Sp}$.*

Suppose moreover that:

- *For every $\mathcal{C}$, the functor $\Phi_{\mathcal{C}}(-)$ commutes with colimits.*
- *The functor Φ is* additive *in the following sense. Abuse notation in writing T for the endofunctor of $\mathsf{Hom}(\Delta^1, \mathcal{C}) = \{\mathcal{F} \to \mathcal{G} \in \mathcal{C}\}$ sending $\mathcal{F} \to \mathcal{G}$ to $T(\mathcal{F}) \to T(\mathcal{G})$. Then we suppose (using the notation of §3.3) that the map:*

$$\mathsf{Hom}(\Delta^1, \mathcal{C})^T \to \mathcal{C} \times \mathcal{C}$$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{G} \\ \downarrow{\scriptstyle \eta_{\mathcal{F}}} & & \downarrow{\scriptstyle \eta_{\mathcal{G}}} \\ T(\mathcal{F}) & \longrightarrow & T(\mathcal{G}) \end{array} \mapsto \big((\mathcal{F}, \eta_{\mathcal{F}}), (\mathrm{Coker}(f), \eta_{\mathrm{Coker}(f)})\big)$$

induces an isomorphism $\Phi_{\mathsf{Hom}(\Delta^1,\mathcal{C})}(T) \xrightarrow{\simeq} \Phi_{\mathcal{C}}(T) \times \Phi_{\mathcal{C}}(T)$.

Then there is a unique natural transformation:

$$\partial K_{\mathcal{C}}(T) \to \Phi_{\mathcal{C}}(T)$$

of functors $\mathsf{StCat}_{cg,endo} \to \mathsf{Sp}$ equipped with a structure of based map when evaluated on $(\mathsf{Sp}, \mathrm{id}_{\mathsf{Sp}})$. In particular, $(\mathcal{C}, T) \mapsto \partial K_{\mathcal{C}}(T)$ is initial with respect to the above data.

[9] This does not completely define a structure of category, of course: we have not written compositions, never mind higher data. But all of this data is implicit in the standard 2-categorical structure on StCat_{cont}; we refer to [GR] for the appropriate formalism, including how to properly define this category.

Proof. Suppose $(\mathcal{C}, T) \in \mathsf{StCat}_{cg,endo}$. By functoriality of Φ and using its base-point, we have a canonical map:

$$\mathrm{Hom}_{\mathsf{StCat}_{cg,endo}}((\mathsf{Sp}, \mathrm{id}_{\mathsf{Sp}}), (\mathcal{C}, T)) = \mathcal{C}^{T,c,\simeq} \to \mathrm{Hom}_{\mathsf{Sp}}(\Phi_{\mathsf{Sp}}(\mathrm{id}_{\mathsf{Sp}}), \Phi_{\mathcal{C}}(T)) \to \Omega^{\infty}\Phi_{\mathcal{C}}(T) \in \mathsf{Gpd}.$$

By additivity of Φ and the Waldhausen construction of K-theory, this map factors through a canonical map from $\Omega^{\infty}K_{\mathcal{C}}(T)$, and by Lemma 3.11.2 this canonically upgrades to a map of spectra $K_{\mathcal{C}}(T) \to \Phi_{\mathcal{C}}(T)$. Finally, by definition of the Goodwillie derivative, this natural transformation factors through $\partial K_{\mathcal{C}}(T) \to \Phi_{\mathcal{C}}(T)$. Clearly this construction is natural in $(\mathcal{C}, T)$, giving the claim. □

We used the following result in the course of the proof, which we explicitly record for clarity.

Lemma 3.11.2. *For $\mathcal{D}$ stable and $F, G : \mathcal{D} \to \mathsf{Sp}$ exact functors, natural transformations between F and G are the same as natural transformations between the functors $\Omega^{\infty}F, \Omega^{\infty}G : \mathcal{D} \to \mathsf{Gpd}$. That is, the natural map:*

$$\mathrm{Hom}_{\mathsf{Hom}(\mathcal{D},\mathsf{Sp})}(F, G) \to \mathrm{Hom}_{\mathsf{Hom}(\mathcal{D},\mathsf{Gpd})}(\Omega^{\infty}F, \Omega^{\infty}G)$$

is an isomorphism.

Proof. We have:

$$\mathrm{Hom}_{\mathsf{Hom}(\mathcal{D},\mathsf{Sp})}(F, G) = \lim_{n} \mathrm{Hom}_{\mathsf{Hom}(\mathcal{D},\mathsf{Gpd})}\left(\Omega^{\infty}(F[n]), \Omega^{\infty}(G[n])\right).$$

Now observe that each of the structural maps in this limit is an isomorphism (as suspension is an equivalence for both $\mathcal{C}$ and $\mathcal{D}$). □

3.12. We now prove Theorem 3.10.1.

Proof of Theorem 3.10.1. We verify that the functor $(\mathcal{C}, T) \mapsto \mathrm{tr}_{\mathcal{C}}(T)$ satisfies the same universal property as in Lemma 3.11.1.

Step 1. First, note that there actually is a canonical such a functor out of $\mathsf{StCat}_{cg,endo}$: this follows from the functoriality of traces discussed in §4 (and almost established in [KP] §1).

Clearly this functor commutes with colimits in T. It is straightforward to check additivity; we omit the verification here.

Moreover, $\mathrm{tr}_{\mathsf{Sp}}(\mathrm{id}_{\mathsf{Sp}}) = \mathbb{S} \in \mathsf{Sp}$, so there is a tautological base-point here.

It remains to show universality of the trace. So suppose that we are given Φ as in Lemma 3.11.1.

Step 2. We now make some preliminary constructions.

Fix $(\mathcal{C}, T) \in \mathsf{StCat}_{cg,endo}$ and suppose $\mathcal{F} \in \mathcal{C}^c$ compact. Then there is a canonical map:

$$\varepsilon_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, T(\mathcal{F})) \to \mathcal{C}^{T,c,\simeq} \to \Omega^{\infty}\Phi_{\mathcal{C}}(T) \in \mathsf{Gpd}$$

with the first map being obvious and the second map coming from Lemma 3.11.1.

Moreover, suppose that for some $n \geqslant 0$, we are given a diagram:

$$\mathcal{F}_0 \xrightarrow{\alpha_0} \ldots \xrightarrow{\alpha_{n-1}} \mathcal{F}_n \xrightarrow{\alpha_n} T(\mathcal{F}_0) \xrightarrow{T(\alpha_0)} \ldots \xrightarrow{T(\alpha_{n-1})} T(\mathcal{F}_n) \in \mathcal{C}$$

with each $\mathcal{F}_i$ compact. For each i, we have an induced map $\mathcal{F}_i \to T(\mathcal{F}_i)$, and we claim that the induced point of $\Omega^{\infty}\Phi_{\mathcal{C}}(T)$ is canonically independent of i. More precisely, we have a simplicial groupoid sending $[n]$ to the groupoid of diagrams as above, and we claim there is a natural transformation to the constant simplicial groupoid with value $\Omega^\infty\Phi_{\mathcal{C}}(T)$ that coincides with the construction $\varepsilon_{\mathcal{F}}$ for $n = 0$. (The additivity of Φ is essential here.)

First, observe that for any $\mathcal{F} \in \mathcal{C}^c$, $\varepsilon_{\mathcal{F}}$ is pointed (and even upgrades to a map of spectra) by exactness of $\Phi_{\mathcal{C}}(-)$. Under the above hypotheses, we may regard $\mathcal{F}_n$ as a filtered object of $\mathcal{C}$, and the given data as a filtered map $\mathcal{F}_n \to T(\mathcal{F}_n)$. Clearly on associated graded, the induced maps:

$$\mathrm{Coker}(\mathcal{F}_i \to \mathcal{F}_{i+1}) \to \mathrm{Coker}(T(\mathcal{F}_i) \to T(\mathcal{F}_{i+1}))$$

are zero for all $i \geqslant 0$. Additivity (and the pointedness noted above) then implies that:

$$\varepsilon_{\mathcal{F}_n}(T(\alpha_{n-1} \dots T(\alpha_0)\alpha_n)) = \varepsilon_{\mathcal{F}_0}(\alpha_n \dots \alpha_1\alpha_0).$$

This argument immediately upgrades to give the desired natural transformation of simplicial groupoids; we omit the details. Moreover, we note that these constructions are natural in $(\mathcal{C}, T) \in \mathsf{StCat}_{cg,endo}$ in the obvious sense.

Step 3. Now fix $\mathcal{C}$ compactly generated and stable. Recall that $\mathcal{C}$ is dualizable in StCat_{cont}, so:

$$\mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\simeq} \mathsf{End}_{\mathsf{StCat}_{cont}}(\mathcal{C}).$$

Therefore, it is enough to give the natural transformation of the induced functors:

$$\mathcal{C} \times \mathcal{C}^\vee \to \mathsf{Sp}.$$

The left hand side is $\mathsf{Ind}(\mathcal{C}^c \times \mathcal{C}^{c,op})$, so it is enough to construct our natural transformation when restricted to $\mathcal{C}^c \times \mathcal{C}^{c,op}$, i.e., for functors of the form $\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}$ for $\mathcal{F}, \mathcal{G} \in \mathcal{C}^c$.

Step 4. We have $\mathrm{tr}_{\mathcal{C}}(\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}) = \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$. By Lemma 3.11.2, it suffices to construct natural maps:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \to \Phi_{\mathcal{C}}(\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}) \in \mathsf{Gpd}.$$

We have map of spectra:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F}) \otimes \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\mathcal{F}, \underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F}) \otimes \mathcal{G}) = \underline{\mathrm{Hom}}(\mathcal{F}, (\mathcal{G} \boxtimes \mathbb{D}\mathcal{F})(\mathcal{F})).$$

Applying Ω^∞ and using the construction from Step 2, we obtain a map:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \to \Omega^\infty\Phi_{\mathcal{C}}(\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}).$$

This map is clearly natural in the variable $\mathcal{G}$, only natural with respect to isomorphisms in the variable $\mathcal{F}$. I.e., we have constructed a natural transformation of functors $\mathcal{C}^c \times \mathcal{C}^{c,op,\simeq} \to \mathsf{Gpd}$.

Step 5. It remains to upgrade the above construction to a natural transformation of functors defined on all of $\mathcal{C}^c \times \mathcal{C}^{c,op}$.

First, at a homotopically naive level, suppose we are given $f : \mathcal{F}_0 \to \mathcal{F}_1$ and $g : \mathcal{F}_1 \to \mathcal{G}$. We a priori obtain two points of $\Omega^\infty\Phi_{\mathcal{C}}(\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}_0)$:

$$\begin{aligned} gf \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_0, \mathcal{G}), &\quad \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_0, \mathcal{G}) \to \Omega^\infty\Phi(\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}_0) \\ f \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_1, \mathcal{G}), &\quad \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_1, \mathcal{G}) \to \Omega^\infty\Phi(\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}_1)\Omega^\infty\Phi(\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}_0). \end{aligned} \tag{3.12.1}$$

We claim that they are canonically identified.

We prove this by identifying both points with a third: we have a canonical map:

$$\mathcal{F}_1 = \mathbb{S} \otimes \mathcal{F}_1 \xrightarrow{f \otimes g} \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{F}_1) \otimes \mathcal{G} = (\mathcal{G} \boxtimes \mathbb{D}\mathcal{F}_0)(\mathcal{F}_1)$$

which (by Step 2) gives a point of $\Omega^\infty\Omega^\infty\Phi(\mathcal{G}\boxtimes\mathbb{D}\mathcal{F}_0)$.

It is tautological that this map coincides with the second map in (3.12.1). To identify it with the first, we use the diagram:

$$\mathcal{F}_0 \xrightarrow{f} \mathcal{F}_1 \xrightarrow{\mathrm{id}_{\mathcal{F}_0}\otimes g} (\mathcal{G}\boxtimes\mathbb{D}\mathcal{F}_0)(\mathcal{F}_0) \to (\mathcal{G}\boxtimes\mathbb{D}\mathcal{F}_0)(\mathcal{F}_1)$$

and Step 2.

To upgrade this map to a homotopically correct one, one shows that we have a morphism of the complete Segal spaces defined by $\mathcal{C}^c \times \mathcal{C}^{c,op}$ and Gpd respectively; obviously this uses the full simplicial construction from Step 2. We leave the details to the reader.

□

4. Topological cyclic homology

4.1. In this section, we prove Theorem 2.12.2. We will deduce this result from an explicit calculation of TC for split square-zero extensions, see Theorem 4.10.1 below.

Remark 4.1.1. Throughout this section, if not otherwise mentioned, stable categories lie in StCat_{cont} and functors between stable categories are morphisms there (i.e., continuous exact functors).

4.2. **Mea culpa and references.** Throughout this section, we need various functoriality properties of traces. Unfortunately, these are not so well documented at the moment.

In §4.3, §4.4 and §4.7, we indicate what functoriality we require. This material (especially first two of these sections), is well-known folklore that does not seem to quite have a convenient reference.

We do not feel so much guilt on this point for three reasons. First, some of this functoriality is established in [KP] §1. Moreover, the constructions (especially Proposition 1.2.9) from *loc. cit.* can be readily be generalized to provide the desired functoriality using Segal spaces.

Second, Thomas Nikolaus has forthcoming work [Nik] completely establishing the functoriality we postulate here.

Finally, if we had chosen to work with algebras instead of categories (as is all we need in practice), then one could make do with the methods of [NS]. But we have not used this approach here because we find it to be not as well-suited as the categorical approach for the problems at hand.

4.3. **Review of traces.** Let $\mathcal{C}$ be dualizable in StCat_{cont}. Then the trace functor:

$$\mathrm{tr}_{\mathcal{C}} : \mathsf{End}_{\mathsf{StCat}_{cont}}(\mathcal{C}) \to \mathsf{Sp}$$

satisfies:

$$\mathrm{tr}_{\mathcal{C}}(TS) = \mathrm{tr}_{\mathcal{C}}(ST)$$

for $S, T \in \mathsf{End}_{\mathsf{StCat}_{cont}}(\mathcal{C})$, and more generally:

$$\mathrm{tr}_{\mathcal{C}}(T_1 \dots T_n) = \mathrm{tr}_{\mathcal{C}}(T_2 \dots T_n T_1)$$

for $T_1, \dots T_n \in \mathsf{End}_{\mathsf{StCat}_{cont}}(\mathcal{C})$. Here we are lazily writing an equals sign for an existence of canonical isomorphism; and more functorially, we should work with a series of functors indexed by a cyclic set.

In particular, there is a $\mathbb{Z}/n$-action on[10] $\mathrm{tr}_{\mathcal{C}}(T^n)$ for any T. As a variant, we have a cyclic functor with constant value $\mathrm{tr}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}})$ and whose underlying simplicial functor is constant; this recovers the usual $\mathbb{BZ}$-action on THH.

[10]For clarity: throughout this section, e.g. T^n denotes the n-fold composition of T with itself (and not, say, the n-fold product of it with itself).

Remark 4.3.1. Here we are using somewhat non-standard notation: we use $\mathbb{BZ}$ rather than S^1 or $\mathbb{T}$ to emphasize that the story is not at all transcendental. Note that in this perspective, the usual homomorphism $\mathbb{Z}/n \to S^1 = \mathbb{BZ}$ corresponds to the extension $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n$ of abelian groups.

4.4. Next, recall the following additional functoriality of traces. Note that we have used some of this material already in the proof of Theorem 2.12.1.

Suppose $T \in \mathsf{End}(\mathcal{C})$ and $S \in \mathsf{End}(\mathcal{D})$. Suppose moreover that $\psi : \mathcal{C} \to \mathcal{D} \in \mathsf{StCat}_{cont}$ is an exact functor between dualizable stable categories that admits a continuous right adjoint, and that we are given a natural transformation:

$$\psi T \to S\psi.$$

Then there is an induced map:

$$\mathrm{tr}_{\mathcal{C}}(T) \to \mathrm{tr}_{\mathcal{D}}(S)$$

satisfying expected compatibilities.

Note that in such a case, for $n \in \mathbb{Z}^{>0}$ we also obtain a natural transformation:

$$\psi T^n \to S\psi T^{n-1} \to \ldots \to S^{n-1}\psi T \to S^n\psi$$

and so a map:

$$\mathrm{tr}_{\mathcal{C}}(T^n) \to \mathrm{tr}_{\mathcal{D}}(S^n).$$

By construction, this map is $\mathbb{Z}/n$-equivariant.

Example 4.4.1. Suppose $\mathcal{C} = \mathsf{Sp}$ and T is the identity functor. Then a functor ψ as above is equivalent to a compact object $\mathcal{F} \in \mathcal{D}$, and a natural transformation as above is equivalent to a map $\eta : \mathcal{F} \to S(\mathcal{F})$. From this datum, the above constructs a canonical map:

$$\mathrm{tr}_{\mathsf{Sp}}(\mathrm{id}_{\mathsf{Sp}}) = \mathbb{S} \to \mathrm{tr}_{\mathcal{D}}(S)$$

i.e., it gives a point of $\Omega^\infty \,\mathrm{tr}_{\mathcal{D}}(S)$. For later use, we denote this point $\mathrm{tr}_{\mathcal{F}}(\eta)$.

We remark that we have already seen this construction in Example 4.4.1.

4.5. **Calculation of** THH. Suppose that $\mathcal{C} \in \mathsf{StCat}_{cont}$ is dualizable and equipped with a continuous endofunctor T.

Proposition 4.5.1. *There is a canonical $\mathbb{BZ}$-equivariant isomorphism:*

$$\mathrm{THH}(\mathcal{C}) \bigoplus \oplus_{n>0} \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{SqZero}(\mathcal{C}, T)).$$

Here $\mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}}$ is the right adjoint induction functor from spectra with (naive) $\mathbb{Z}/n$-actions to spectra with $\mathbb{BZ}$-actions.

Proof. Here is the method.

Suppose $\mathcal{D}$ a dualizable stable category and we wish to calculate its Hochschild homology. Note that $\mathcal{D} \otimes \mathcal{D}^\vee \xrightarrow{\simeq} \mathsf{End}(\mathcal{D})$, and the trace map on the right hand side corresponds to the canonical pairing on the left hand side. Then we might try to calculate this tensor product in some explicit terms, calculate what corresponds to the identity functor for $\mathcal{D}$, and then apply the evaluation functor. This will be our approach for $\mathcal{D} = \mathrm{SqZero}(\mathcal{C}, T)$.

We remark that some of the manipulations below may also be understood in terms of usual square-zero extensions (using Proposition 3.2.2), and we encourage the reader to do the exercise of translating.

Step 1. First, we claim that for any $\mathcal{D} \in \mathsf{StCat}_{cont}$, the functor:

$$\mathrm{SqZero}(\mathcal{C}, T) \otimes \mathcal{D} \xrightarrow{\simeq} \mathrm{SqZero}(\mathcal{C} \otimes \mathcal{D}, T \otimes \mathrm{id}_{\mathcal{D}}) \tag{4.5.1}$$

is an equivalence. We will show both sides map to $\mathcal{C} \otimes \mathcal{D}$ monadically, and the induced morphism of monads is an isomorphism.

Note that the functor $\mathrm{SqZero}(\mathcal{C}, T) \xrightarrow{(\mathcal{F},\eta)\mapsto \mathrm{Ker}(\eta)} \mathcal{C}$ admits a left adjoint equipping an object of $\mathcal{C}$ with the zero map to T of itself. Moreover, the local nilpotence condition in the definition of $\mathrm{SqZero}(\mathcal{C}, T)$ implies that this functor is conservative, and therefore (being continuous) monadic. Note that the underlying monad on $\mathcal{C}$ sends $\mathcal{F} \in \mathcal{C}$ to $\mathcal{F} \oplus T(\mathcal{F})[-1] = \mathrm{Ker}(0 : \mathcal{F} \to T(\mathcal{F}))$.

Then we recall that monadicity is preserved under tensor products in StCat_{cont}, so the left hand side of (4.5.1) maps monadically to $\mathcal{C} \otimes \mathcal{D}$. Moreover, applying the above to $\mathrm{SqZero}(\mathcal{C} \otimes \mathcal{D}, T \otimes \mathrm{id}_{\mathcal{D}})$, we obtain that this category maps monadically to $\mathcal{C} \otimes \mathcal{D}$. Then it is immediate to verify that the functor in (4.5.1) intertwines these monadic functors and induces an equivalence of monads on $\mathcal{C} \otimes \mathcal{D}$, and therefore is an equivalence.

Step 2. Next, we claim that $\mathrm{SqZero}(\mathcal{C}, T)$ is dualizable with dual $\mathrm{SqZero}(\mathcal{C}^\vee, T^\vee)$. Here we recall that a functor $T : \mathcal{C} \to \mathcal{C}$ induces a dual functor $T^\vee : \mathcal{C}^\vee \to \mathcal{C}^\vee$; explicitly, for $\lambda \in \mathcal{C}^\vee = \mathsf{Hom}(\mathcal{C}, \mathsf{Sp})$, $T^\vee(\lambda) = \lambda \circ T$.

First, we construct the evaluation map:

$$\mathrm{SqZero}(\mathcal{C}, T) \otimes \mathrm{SqZero}(\mathcal{C}^\vee, T^\vee) \to \mathsf{Sp}.$$

It is equivalent to construct a functor:

$$\mathrm{SqZero}(\mathcal{C}, T) \times \mathrm{SqZero}(\mathcal{C}^\vee, T^\vee) \to \mathsf{Sp}$$

commuting with colimits in each variable separately. This pairing sends:

$$\big((\mathcal{F}, \eta : \mathcal{F} \to T(\mathcal{F})), (\lambda, \mu : \lambda \to \lambda T)\big)$$

to:

$$\mathrm{Eq}\left(\lambda(\mathcal{F}) \underset{\mu}{\overset{\lambda(\eta)}{\rightrightarrows}} \lambda T(\mathcal{F})\right).$$

Next, we define the coevaluation map:

$$\mathsf{Sp} \to \mathrm{SqZero}(\mathcal{C}, T) \otimes \mathrm{SqZero}(\mathcal{C}^\vee, T^\vee).$$

For this, it is helpful to realize the right hand side more explicitly using the previous step. Iteratively applying the previous step, we obtain:

$$\mathrm{SqZero}(\mathcal{C}, T) \otimes \mathrm{SqZero}(\mathcal{C}^\vee, T^\vee) = \mathrm{SqZero}(\mathcal{C} \otimes \mathrm{SqZero}(\mathcal{C}^\vee, T^\vee), T \otimes \mathrm{id}) =$$
$$\mathrm{SqZero}\left(\mathcal{C} \otimes \mathcal{C}^\vee, (T \otimes \mathrm{id}_{\mathcal{C}}^\vee) \times (\mathrm{id}_{\mathcal{C}} \otimes T^\vee)\right).$$

Noting that $\mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\simeq} \mathsf{End}(\mathcal{C})$ by duality, we obtain that objects of the above tensor product are the same as data:[11]

[11] A posteriori, we have $\mathrm{SqZero}(\mathcal{C}, T) \otimes \mathrm{SqZero}(\mathcal{C}^\vee, T^\vee) \xrightarrow{\simeq} \mathsf{End}(\mathrm{SqZero}(\mathcal{C}, T))$.

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & ST \\ {\scriptstyle\beta}\downarrow & & \\ TS & & \end{array} \tag{4.5.2}$$

with:

$$\operatorname{colim}\left(S \xrightarrow{\alpha} ST \xrightarrow{(-\circ T)(\alpha)} ST^2 \ldots\right) = 0$$
$$\operatorname{colim}\left(S \xrightarrow{\beta} TS \xrightarrow{(T\circ -)(\beta)} T^2S \ldots\right) = 0.$$

Now our coevaluation map is specified by an object of the above tensor product (since it is a continuous exact functor out of spectra). In the graphical display of (4.5.2), this object is:

$$\begin{array}{ccc} \oplus_{n\geqslant 0} T^n & \xrightarrow{\pi} & \oplus_{n\geqslant 1} T^n \\ {\scriptstyle\pi}\downarrow & & \\ \oplus_{n\geqslant 1} T^n & & \end{array}$$

where π denotes the natural projection.

To verify that this actually defines a duality datum, we should show that the composition:

$$\operatorname{SqZero}(\mathcal{C}, T) \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} \operatorname{SqZero}(\mathcal{C}, T) \otimes \operatorname{SqZero}(\mathcal{C}^\vee, T^\vee) \otimes \operatorname{SqZero}(\mathcal{C}, T) \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} \operatorname{SqZero}(\mathcal{C}, T)$$

is isomorphic to the identity functor (by the symmetry of $\mathcal{C}$ and $\mathcal{C}^\vee$ here, this suffices).

This composition sends $(\mathcal{F}, \eta) \in \operatorname{SqZero}(\mathcal{C}, T)$ to:

$$\mathcal{F}' = \operatorname{Eq}(\oplus_{n\geqslant 0} T^n(\mathcal{F}) \underset{\pi}{\overset{\oplus_n T^n(\eta)}{\rightrightarrows}} \oplus_{n\geqslant 1} T^n(\mathcal{F}))$$

equipped with the map $\eta' : \mathcal{F}' \to T(\mathcal{F}')$ induced by the natural projection from $\mathcal{F}'$ to:

$$T(\mathcal{F}') = \operatorname{Eq}(\oplus_{n\geqslant 1} T^n(\mathcal{F}) \rightrightarrows \oplus_{n\geqslant 2} T^n(\mathcal{F})).$$

We wish to construct a functorial isomorphism $(\mathcal{F}, \eta) \simeq (\mathcal{F}', \eta')$.

First, we observe:[12]

$$\operatorname{Eq}(\oplus_{n\geqslant 0} T^n(\mathcal{F}) \underset{\mathrm{id}}{\rightrightarrows} \oplus_{n\geqslant 0} T^n(\mathcal{F})) = \operatorname{Coeq}(\oplus_{n\geqslant 0} T^n(\mathcal{F}) \rightrightarrows \oplus_{n\geqslant 0} T^n(\mathcal{F}))[-1] =$$
$$\operatorname{colim}_{n\geqslant 0} T^n(\mathcal{F})[-1] = 0$$

Explicitly, the endofunctor of $\operatorname{SqZero}(\mathcal{C}, T)$ corresponding to the above data sends $(\mathcal{F}, \eta)$ to:

$$\mathcal{F}' := \operatorname{Eq}\left(S(\mathcal{F}) \underset{\alpha}{\overset{S(\eta)}{\rightrightarrows}} ST(\mathcal{F})\right)$$

with $\eta' : \mathcal{F}' \to T(\mathcal{F}')$ induced by taking equalizers along rows in the (appropriately commuting) diagram:

$$\begin{array}{ccc} S(\mathcal{F}) & \underset{\alpha}{\overset{S(\eta)}{\rightrightarrows}} & ST(\mathcal{F}) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ TS(\mathcal{F}) & \underset{T(\alpha)}{\overset{TS(\eta)}{\rightrightarrows}} & TST(\mathcal{F}). \end{array}$$

[12]Note the difference from the equalizer we aim to calculate: it is in indexing the second term in the equalizer.

by local nilpotence of η. Therefore, the equalizer we are trying to calculate is:

$$\operatorname{Coker}\left(\operatorname{Eq}(0 \rightrightarrows \mathcal{F}) \to 0\right) = \operatorname{Coker}(\mathcal{F}[-1] \to 0) = \mathcal{F}.$$

The additional compatibility between η and η' is readily seen.

Step 3. We now obtain a formula for Hochschild homology as a bare spectrum by composing the evaulation and coevaluation maps.

First, note that the evaluation map sends an object (4.5.2) to:

$$\operatorname{Eq}(\operatorname{tr}_{\mathcal{C}}(S) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \operatorname{tr}_{\mathcal{C}}(ST) \simeq \operatorname{tr}_{\mathcal{C}}(TS)).$$

Indeed, this follows by identifying the two on "pure tensors" $(\mathcal{F}, \eta) \boxtimes (\lambda, \mu)$ as in the construction of the evaluation map.

We thus obtain:

$$\operatorname{THH}(\operatorname{SqZero}(\mathcal{C}, T)) \xrightarrow{\simeq} \bigoplus_{n \geqslant 0} \operatorname{Eq}\left(\operatorname{tr}_{\mathcal{C}}(T^n) \underset{\sigma_n}{\overset{\mathrm{id}}{\rightrightarrows}} \operatorname{tr}_{\mathcal{C}}(T^n)\right) \tag{4.5.3}$$

where $\sigma_n : \operatorname{tr}_{\mathcal{C}}(T^n) \xrightarrow{\simeq} \operatorname{tr}_{\mathcal{C}}(T^n)$ is the action of the generator of $\mathbb{Z}/n$ on this trace (c.f. §4.3).

We observe that the summand:

$$\operatorname{Eq}\left(\operatorname{tr}_{\mathcal{C}}(T^n) \underset{\sigma_n}{\overset{\mathrm{id}}{\rightrightarrows}} \operatorname{tr}_{\mathcal{C}}(T^n)\right)$$

is isomorphic as a spectrum to $\operatorname{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \operatorname{tr}_{\mathcal{C}}(T^n)$.

Therefore, the left and right hand sides of (4.5.3) have natural $\mathbb{BZ}$-actions. It remains to show that isomorphism of (4.5.3) upgrades to a $\mathbb{BZ}$-equivariant one.

Step 4. First, as a (slightly[13]) toy version of the problem, fix $n > 0$. We claim that the composition:

$$\operatorname{THH}(\operatorname{SqZero}(\mathcal{C}, T)) \xrightarrow{(4.5.3)} \operatorname{Eq}\left(\operatorname{tr}_{\mathcal{C}}(T^n) \underset{\sigma_n}{\overset{\mathrm{id}}{\rightrightarrows}} \operatorname{tr}_{\mathcal{C}}(T^n)\right) \to \operatorname{tr}_{\mathcal{C}}(T^n)$$

is $\mathbb{Z}/n$-equivariant, and that the induced map:

$$\operatorname{THH}(\operatorname{SqZero}(\mathcal{C}, T)) \to \operatorname{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \operatorname{tr}_{\mathcal{C}}(T^n) = \operatorname{Eq}\left(\operatorname{tr}_{\mathcal{C}}(T^n) \underset{\sigma_n}{\overset{\mathrm{id}}{\rightrightarrows}} \operatorname{tr}_{\mathcal{C}}(T^n)\right)$$

is the natural projection arising from (4.5.3).

Note that we have a natural functor:

$$\operatorname{Oblv} : \operatorname{SqZero}(\mathcal{C}, T) \to \mathcal{C}$$
$$(\mathcal{F}, \eta) \mapsto \mathcal{F}$$

that admits a continuous right adjoint.[14] Moreover, there is a canonical natural transformation:

$$\operatorname{Oblv} \to T \circ \operatorname{Oblv} \tag{4.5.4}$$

that evaluates on $(\mathcal{F}, \eta)$ as the map η.

By §4.4, we obtain a natural $\mathbb{Z}/n$-equivariant map:

[13]If the right hand side were a product instead of a sum, what we explain here would be adequate. And in fact, for our application we may assume $\operatorname{tr}_{\mathcal{C}}(T^n) \in \mathsf{Sp}^{\leqslant -n}$, which forces the direct sum and direct product to coincide.

[14]Explicitly, this right adjoint sends $\mathcal{G} \in \mathcal{C}$ to $\oplus_{n \geqslant 0} T^n(\mathcal{G})$ equipped with the projection map $\oplus_{n \geqslant 0} T^n(\mathcal{G}) \to T(\oplus_{n \geqslant 0} T^n(\mathcal{G})) = \oplus_{n \geqslant 1} T^n(\mathcal{G})$.

$$\mathrm{THH}(\mathrm{SqZero}(\mathcal{C},T)) = \mathrm{tr}_{\mathrm{SqZero}(\mathcal{C},T)}(\mathrm{id}_{\mathrm{SqZero}(\mathcal{C},T)}) \to \mathrm{tr}_{\mathcal{C}}(T^n).$$

It is straightforward to see that this map has the desired compatibilities with (4.5.3).

Step 5. Finally, we explain how to complete the argument. We do this using a general format for THH (and traces more generally) to have gradings.[15]

Let $\mathsf{Rep}(\mathbb{G}_m)$ denote the symmetric monoidal category of $\mathbb{Z}$-graded spectra with the convolution[16] monoidal structure.[17] A *grading* on $\mathcal{D} \in \mathsf{StCat}_{cont}$ is the datum of a category $\mathcal{D}^{gr} \in \mathsf{StCat}_{cont}$ (of "graded objects in $\mathcal{D}$") equipped with $\mathsf{Rep}(\mathbb{G}_m)$-module category structure and an isomorphism:

$$\mathcal{D}^{gr} \underset{\mathsf{Rep}(\mathbb{G}_m)}{\otimes} \mathsf{Sp} \xrightarrow{\simeq} \mathcal{D}.$$

Here we are using the symmetric monoidal functor $\mathsf{Rep}(\mathbb{G}_m) \to \mathsf{Sp}$ of forgetting the grading.[18]

We claim that if $\mathcal{D}$ is dualizable and equipped with a grading, then $\mathrm{THH}(\mathcal{D})$ is naturally graded spectrum.

Indeed, one can show[19] that $\mathcal{D}^{gr}$ is automatically dualizable as a $\mathsf{Rep}(\mathbb{G}_m)$-module category. Therefore, we can take its THH (the trace of the identity) in this category to obtain an object:

$$\mathrm{THH}_{/\mathsf{Rep}(\mathbb{G}_m)}(\mathcal{D}^{gr}) \in \mathsf{Rep}(\mathbb{G}_m).$$

Then by functoriality, this object maps to $\mathrm{THH}(\mathcal{D})$ under the forgetful functor $\mathsf{Rep}(\mathbb{G}_m) \to \mathsf{Sp}$, i.e., it induces a grading on $\mathrm{THH}(\mathcal{D})$. Moreover, this construction makes manifest functoriality of traces in the graded setting, similar to §4.3; we do not spell out the details here.

We apply this to $\mathcal{D} = \mathrm{SqZero}(\mathcal{C},T)$. We set $\mathrm{SqZero}(\mathcal{C},T)^{gr}$ to be the category whose objects are collections $\mathcal{F}_n \in \mathcal{C}$ for each $n \in \mathbb{Z}$ and equipped with maps $\eta_n : \mathcal{F}_n \to T(\mathcal{F}_{n+1})$ that are locally nilpotent in the sense that $\mathrm{colim}_n T^n(\mathcal{F}_{n+m}) = 0$ for any m.

This category has an obvious action of $\mathsf{Rep}(\mathbb{G}_m)$, and the functor:

$$\mathrm{SqZero}(\mathcal{C},T)^{gr} \to \mathrm{SqZero}(\mathcal{C})$$
$$\big((\mathcal{F}_n)_{n\in\mathbb{Z}}, (\eta_n)_{n\in\mathbb{Z}}\big) \mapsto (\oplus_n \mathcal{F}_n, \oplus_n \eta_n)$$

naturally upgrades to an equivalence:

---

[15] The present discussion admits some natural extensions, which we highlight here.

First, there is a notion of *graded cyclotomic spectrum*, which is discussed in §5.18. (The key point is that the Frobenius at p multiplies degrees by p.) In particular, up to adapting [AMGR] to the graded setting, the present discussion shows that THH of a graded object of StCat_{cont} is a graded cyclotomic spectrum.

Moreover, similar ideas may developed in the filtered setting.

[16] I.e., if $\mathcal{F}$ and $\mathcal{G}$ are spectra, then $\mathcal{F}(n) \otimes \mathcal{G}(m) = (\mathcal{F} \otimes \mathcal{G})(n+m)$, where e.g. $\mathcal{F}(n)$ indicates we consider $\mathcal{F}$ as graded purely in degree n, on the left hand side $\otimes$ indicates our convolution monoidal structure, and on the right hand side it indicates the usual tensor product.

[17] This category is readily seen to in fact be comodules over the (bi-$\mathcal{E}_\infty$) Hopf algebra $\Sigma^\infty\mathbb{Z}$.

[18] Here is another language for categorical gradings, which the reader may safely skip. We use some terminology and notation that we do not wish to explain here.

Note that [Gai] Theorem 2.2.2 is true for spectra in the special case $G = \mathbb{G}_m$: the proof from *loc. cit.* §7.2 works in this setup. (This is closely related to $\mathbb{G}_m$ being linearly reductive in any characteristic.)

The upshot is that a grading on $\mathcal{D}$ is equivalent to a weak $\mathbb{G}_m$-action on $\mathcal{D}$, where one recovers $\mathcal{D}^{gr}$ as $\mathcal{D}^{\mathbb{G}_m,w}$.

For example, in slightly imprecise terms, the relevant weak $\mathbb{G}_m$-action on $\mathrm{SqZero}(\mathcal{C},T)$ that we use below scales the map η.

[19] E.g., using the previous footnote and standard techniques.

$$\mathrm{SqZero}(\mathcal{C},T)^{gr} \underset{\mathsf{Rep}(\mathbb{G}_m)}{\otimes} \mathsf{Sp} \xrightarrow{\simeq} \mathrm{SqZero}(\mathcal{C},T).$$

Then it is straightforward to verify that the grading on $\mathrm{THH}(\mathrm{SqZero}(\mathcal{C},T))$ coming from the above coincides with the grading appearing in (4.5.3).

Now note that the natural transformation (4.5.4) is *graded of degree 1* in the natural sense. Therefore, we can apply the method from the previous step to see that the degree n part of $\mathrm{THH}(\mathrm{SqZero}(\mathcal{C},T))$ is $\mathbb{BZ}$-equivariantly isomorphic to $\mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n)$, completing the argument.

□

4.6. **Cyclotomic structure.** In Proposition 4.5.1, for $\mathcal{C}$ dualizable and $T : \mathcal{C} \to \mathcal{C}$ continuous and exact, we calculated $\mathrm{THH}(\mathrm{SqZero}(\mathcal{C},T))$ with its $\mathbb{BZ}$-action. We now wish to describe its cyclotomic structure.

4.7. First, we need some additional functoriality for traces. Fix p a prime. Then we claim that there is a *Tate diagonal* map:

$$\Delta_p : \mathrm{tr}_{\mathcal{C}}(T) \to \mathrm{tr}_{\mathcal{C}}(T^p)^{t\mathbb{Z}/p}$$

functorial in T (actually, satisfying functoriality as in §4.4, but we do not need this).

First, note that $\mathsf{End}(\mathcal{C}) \xrightarrow{T \mapsto \mathrm{tr}_{\mathcal{C}}(T^p)^{t\mathbb{Z}/p}} \mathsf{Sp}$ is exact. The argument[20] is standard, c.f. [NS] Proposition III.1.1.

By Lemma 3.11.2, it suffices to construct the natural transformation of functors to Gpd obtained by applying Ω^∞. Moreover, as $T \mapsto \mathrm{tr}_{\mathcal{C}}(T)$ commutes with all colimits, it suffices to define the restriction our natural transformation when restricted along:

$$\mathcal{C} \times \mathcal{C}^\vee \to \mathcal{C} \otimes \mathcal{C}^\vee = \mathsf{End}(\mathcal{C}).$$

Now for $(\mathcal{F},\lambda) \in \mathcal{C} \times \mathcal{C}^\vee$, the trace of the corresponding functor is $\lambda(\mathcal{F})$, while the trace of its p-fold composition is $\lambda(\mathcal{F})^{\otimes p}$. Then we use the natural map:

$$\Omega^\infty \lambda(\mathcal{F}) \xrightarrow{\simeq} \Big(\big(\Omega^\infty \lambda(\mathcal{F})\big)^p\Big)^{h\mathbb{Z}/p} \to \big(\Omega^\infty \lambda(\mathcal{F})^{\otimes p}\big)^{h\mathbb{Z}/p} =$$
$$\Omega^\infty \lambda(\mathcal{F})^{\otimes p, h\mathbb{Z}/p} \to \Omega^\infty \lambda(\mathcal{F})^{\otimes p, t\mathbb{Z}/p}.$$

Example 4.7.1. For $\mathcal{C} = \mathsf{Sp}$, a functor T of this type is necessarily the tensor product with some spectrum. In this case, the Tate diagonal construction above recovers that of [NS] §III.1.

Variant 4.7.2. Generalizing [NS] §III.3, there is some additional functoriality. For example, for $n > 0$, the Tate diagonal map:

$$\Delta_p : \mathrm{tr}_{\mathcal{C}}(T^n) \to \mathrm{tr}_{\mathcal{C}}(T^{np})^{t\mathbb{Z}/p}$$

is naturally $\mathbb{Z}/n$-equivariant, where we use the natural $(\mathbb{Z}/np)/(\mathbb{Z}/p) = \mathbb{Z}/n$-action on the right hand side.

[20] We remark that this argument is a variant of the standard combinatorial proof of Fermat's little theorem; see Wikipedia for example.

4.8. We now construct a cyclotomic structure on the $\mathbb{BZ}$-spectrum:

$$\bigoplus_{n\geqslant 1} \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n).$$

We remark that a version of this construction appears in [LM] in a related context (but using the formalism of equivariant homotopy theory).

For simplicity, we assume $\mathrm{tr}_{\mathcal{C}}(T^n) \in \mathsf{Sp}^{\leqslant 0}$ for all n so we are in the setting of [NS] (the general non-connective setting can be treated following [AMGR]). So for every prime p, we need to construct a suitable "Frobenius" map.

For any integer n, we have the $\mathbb{Z}/n$-equivariant Tate diagonal:

$$\mathrm{tr}_{\mathcal{C}}(T^n) \to \mathrm{tr}_{\mathcal{C}}(T^{np})^{t\mathbb{Z}/p}.$$

We then induce to $\mathbb{BZ}$-representations:

$$\mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n) \to \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \left(\mathrm{tr}_{\mathcal{C}}(T^{np})^{t\mathbb{Z}/p}\right)$$

and observe[21] that there is a natural map:

$$\mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \left(\mathrm{tr}_{\mathcal{C}}(T^{np})^{t\mathbb{Z}/p}\right) \to \left(\mathrm{Ind}_{\mathbb{Z}/np}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^{np})\right)^{t\mathbb{Z}/p}$$

that is equivariant for the multiplication by p-map $\mathbb{BZ} \to \mathbb{BZ}$.

Composing the above morphisms and taking the direct sum over n, we obtain:

$$\varphi_p : \bigoplus_{n\geqslant 1} \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n) \to \bigoplus_{n\geqslant 1} \left(\mathrm{Ind}_{\mathbb{Z}/np}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^{np})\right)^{t\mathbb{Z}/p}$$

which is again equivariant against $p : \mathbb{BZ} \to \mathbb{BZ}$. By [NS], these maps over all p define a cyclotomic structure assuming connectivity. (And again, refining this construction somewhat gives a cyclotomic structure in general, following [AMGR].)

Tracing the constructions, we have:

---

[21] To see this, suppose in generality that we are given:

$$\begin{array}{ccccc} K & \longrightarrow & H_1 & \longrightarrow & G_1 \\ \| & & \downarrow & & \downarrow \\ K & \longrightarrow & H_2 & \longrightarrow & G_2 \end{array}$$

where the rows are fiber sequences of groups, the maps $H_i \to G_i$ are epimorphisms (i.e., surjective on π_0), and K is a finite (discrete) group.

Then for V a spectrum with a (naive) H_1-action, there is an obvious commuting diagram of spectra with G_2-actions:

$$\begin{array}{ccc} \mathsf{Ind}_{H_1}^{H_2}(V)_{hK} & \longrightarrow & \mathsf{Ind}_{G_1}^{G_2}(V_{hK}) \\ \downarrow & & \downarrow \\ \mathsf{Ind}_{H_1}^{H_2}(V)^{hK} & \longleftarrow & \mathsf{Ind}_{H_1}^{H_2}(V^{hK}) \end{array}$$

where the vertical maps come from norm maps for K.

We take $K = \mathbb{Z}/p \to H_1 = \mathbb{Z}/np \to G_1 = \mathbb{Z}/p$ and $K = \mathbb{Z}/p \to H_2 = \mathbb{BZ} \xrightarrow{p} G_2 = \mathbb{BZ}$ as our rows, with the natural maps relating them. Then the morphism in the top row of our diagram above is an isomorphism, since the functor $\mathsf{Ind}_{\mathbb{Z}/m}^{\mathbb{BZ}}$ commutes with colimits for any m. Passing to kernels along the vertical arrows then gives the desired map.

Lemma 4.8.1. *This cyclotomic structure is the canonical one on:*

$$\mathrm{THH}_{red}(\mathrm{SqZero}(\mathcal{C}, T)) := \mathrm{Ker}\left(\mathrm{THH}(\mathrm{SqZero}(\mathcal{C}, T)) \to \mathrm{THH}(\mathcal{C})\right)$$

under the isomorphism of Proposition 4.5.1.

4.9. **Genuine fixed points.** It is convenient to introduce the following notation.

For p a prime, let $\mathrm{tr}_{\mathcal{C}}(T^p)^{\mathbb{Z}/p} \in \mathsf{Sp}$ denote the *genuine fixed points*, which by definition is the fiber product:

$$\begin{array}{ccc}
\mathrm{tr}_{\mathcal{C}}(T^p)^{\mathbb{Z}/p} & \longrightarrow & \mathrm{tr}_{\mathcal{C}}(T^p)^{h\mathbb{Z}/p} \\
\downarrow & & \downarrow \\
\mathrm{tr}_{\mathcal{C}}(T) & \xrightarrow{\Delta_p} & \mathrm{tr}_{\mathcal{C}}(T^p)^{t\mathbb{Z}/p}.
\end{array}$$

More generally, for $n \geqslant 1$, we construct the genuine fixed points $\mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n}$ as the iterated fiber product:

$$\begin{aligned}
\mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n} := {} & \\
\mathrm{tr}_{\mathcal{C}}(T) & \underset{\substack{\prod_{\substack{p|n \\ p \text{ prime}}} \mathrm{tr}_{\mathcal{C}}(T^p)^{t\mathbb{Z}/p}}}{\times} \prod_{\substack{p|n \\ p \text{ prime}}} \mathrm{tr}_{\mathcal{C}}(T^p)^{h\mathbb{Z}/p} \ldots \underset{\substack{\prod_{\substack{pd|n \\ \varpi(d)=k \\ p \text{ prime}}} \mathrm{tr}_{\mathcal{C}}(T^{pd})^{t\mathbb{Z}/p, h\mathbb{Z}/d}}}{\times} \prod_{\substack{d|n \\ \varpi(d)=k+1}} \mathrm{tr}_{\mathcal{C}}(T^d)^{h\mathbb{Z}/d} \ldots \\
& \ldots \underset{\substack{\prod_{\substack{pd|n \\ \varpi(d)=\varpi(n)-1 \\ p \text{ prime}}} \mathrm{tr}_{\mathcal{C}}(T^{pd})^{t\mathbb{Z}/p, h\mathbb{Z}/d}}}{\times} \mathrm{tr}_{\mathcal{C}}(T^n)^{h\mathbb{Z}/n}.
\end{aligned}$$

Here for $d \in \mathbb{Z}^{>0}$, we let $\varpi(d) = \sum_{p \text{ prime}} v_p(d)$. The structure maps in the above iterated fiber product are constructed as follows. Going right we use the Tate diagonal maps:

$$\mathrm{tr}_{\mathcal{C}}(T^d)^{h\mathbb{Z}/d} \to \mathrm{tr}_{\mathcal{C}}(T^{pd})^{t\mathbb{Z}/p, h\mathbb{Z}/d}$$

coming from Variant 4.7.2. And going left we simply use the canonical projection from invariants to the Tate construction.

Note that for any $d \mid n$, there is a canonical restriction map:

$$\mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n} \to \mathrm{tr}_{\mathcal{C}}(T^d)^{\mathbb{Z}/d}.$$

Lemma 4.9.1. *For any n there is a canonical isomorphism:*

$$\mathrm{tr}_{\mathcal{C}}(T^n)_{h\mathbb{Z}/n} \xrightarrow{\simeq} \mathrm{Ker}\left(\mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n} \to \lim_{d|n} \mathrm{tr}_{\mathcal{C}}(T^d)^{\mathbb{Z}/d}\right).$$

Proof. The limit on the right is calculated as an iterated fiber product as in the definition of $\mathrm{tr}_{\mathcal{C}}(T^n)$, but where we omit the last fiber product in its definition. Therefore, the kernel we are trying to compute coincides with:

$$\mathrm{Ker}\Big(\mathrm{tr}_{\mathcal{C}}(T^n)^{h\mathbb{Z}/n} \to \prod_{\substack{pd|n \\ \varpi(d)=\varpi(n)-1 \\ p \text{ prime}}} \mathrm{tr}_{\mathcal{C}}(T^{pd})^{t\mathbb{Z}/p, h\mathbb{Z}/d}\Big)$$

Note that on the right, pd is necessarily equal to n, so we can rewrite this expression as:

$$\mathrm{Ker}\Big(\mathrm{tr}_{\mathcal{C}}(T^n)^{h\mathbb{Z}/n} \to \prod_{\substack{p|n \\ p \text{ prime}}} \mathrm{tr}_{\mathcal{C}}(T^n)^{t\mathbb{Z}/p, h\mathbb{Z}/(n/p)}\Big).$$

Now we observe that for any $V \in \mathsf{Sp}^{\leqslant 0}$ with a (naive) $\mathbb{Z}/n$-action, the map:

$$V^{t\mathbb{Z}/n} \to \prod_{\substack{p|n \\ p \text{ prime}}} V^{t\mathbb{Z}/p, h\mathbb{Z}/(n/p)}$$

is an isomorphism. Indeed, it is easy to see[22] that:

$$V^{t\mathbb{Z}/n} \xrightarrow{\simeq} \prod_{p \text{ prime}} V^{t\mathbb{Z}/p^{v_p(n)}, h\mathbb{Z}/(n/p^{v_p(n)})}$$

for arbitrary V, and in the connective case we can further apply [NS] Lemma II.4.1, which is a version of the Tate orbit lemma.

Therefore, by our connectivity assumption on traces of powers of T, we need to calculate:

$$\mathrm{Ker}\Big(\mathrm{tr}_{\mathcal{C}}(T^n)^{h\mathbb{Z}/n} \to \mathrm{tr}_{\mathcal{C}}(T^n)^{t\mathbb{Z}/n}\Big)$$

which is certainly $\mathrm{tr}_{\mathcal{C}}(T^n)_{h\mathbb{Z}/n}$.

□

4.10. **Calculation of** TC.

We use the above as follows.

Theorem 4.10.1. *Suppose that for every $n > 0$, $\mathrm{tr}_{\mathcal{C}}(T^n) \in \mathsf{Sp}^{\leqslant -n}$.*

Then there is a natural isomorphism:

$$\mathrm{TC}_{red}(\mathrm{SqZero}(\mathcal{C}, T)) := \mathrm{Ker}\big(\mathrm{TC}(\mathrm{SqZero}(\mathcal{C}, T)) \to \mathrm{TC}(\mathcal{C})\big) \xrightarrow{\simeq} \lim_n \mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n}$$

where the limit is over positive integers ordered under divisibility.

In particular, $\mathrm{TC}_{red}(\mathrm{SqZero}(\mathcal{C}, T))$ *has a complete decreasing filtration indexed by positive integers under divisibility, and there is a canonical isomorphism:*

$$\mathrm{gr}_n \mathrm{TC}_{red}(\mathrm{SqZero}(\mathcal{C}, T)) \xrightarrow{\simeq} \mathrm{tr}_{\mathcal{C}}(T^n)_{h\mathbb{Z}/n}.$$

Remark 4.10.2. This result is implicit in [LM].

Proof of Theorem 4.10.1. By the Nikolaus-Scholze formula [NS] Corollary 1.5 for TC (using the connectivity assumption of §4.8), we have:

$$\mathrm{TC}_{red}(\mathrm{SqZero}(\mathcal{C}, T)) = \mathrm{Eq}\big(\mathrm{THH}_{red}(\mathrm{SqZero}(\mathcal{C}, T))^{h\mathbb{B}\mathbb{Z}} \rightrightarrows \prod_{p \text{ prime}} \mathrm{THH}_{red}(\mathrm{SqZero}(\mathcal{C}, T))^{t\mathbb{Z}/p, h\mathbb{B}\mathbb{Z}}\big).$$

[22] E.g., one notes that $V^{t\mathbb{Z}/n}$ is n-adically complete and shows that $V^{t\mathbb{Z}/p^{v_p(n)}, h\mathbb{Z}/(n/p^{v_p(n)})}$ is its p-adic completion.

Here we recall that on the right hand side, we are taking the residual $\mathbb{BZ}$-action using the natural isomorphism $\mathbb{BZ} = \mathrm{Coker}(\mathbb{Z}/p \to \mathbb{BZ})$.[23] We further recall that one map in the equalizer uses the cyclotomic Frobenius, and the other uses the tautological projection from homotopy $\mathbb{Z}/p$-invariants to the Tate construction.

Now under our assumption of increasing connectivity on the traces of powers of T, we have:

$$\mathrm{THH}_{red}(\mathrm{SqZero}(\mathcal{C}, T) \xrightarrow[\simeq]{\text{Prop. 4.5.1}} \bigoplus_n \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n) = \prod_n \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n). \tag{4.10.1}$$

Therefore, we have:

$$\mathrm{THH}_{red}(\mathrm{SqZero}(\mathcal{C}, T))^{h\mathbb{BZ}} = \big(\prod_n \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n)\big)^{h\mathbb{BZ}} = \prod_n \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n)^{h\mathbb{BZ}} =$$
$$\prod_n \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n)^{h\mathbb{BZ}} = \prod_n \mathrm{tr}_{\mathcal{C}}(T^n)^{h\mathbb{Z}/n}.$$

For p prime, we obtain:

$$\mathrm{THH}_{red}(\mathrm{SqZero}(\mathcal{C}, T))^{t\mathbb{Z}/p, h\mathbb{BZ}} = \prod_n \big(\mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n)\big)^{t\mathbb{Z}/p, h\mathbb{BZ}}.$$

First, note that the factor n vanishes if p does not divide n. Indeed, in this case we have:

$$\big(\mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n)\big)^{t\mathbb{Z}/p, h\mathbb{BZ}} = \big(\mathrm{Ind}_{\mathbb{Z}/pn}^{\mathbb{BZ}} \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{Z}/pn} \mathrm{tr}_{\mathcal{C}}(T^n)\big)^{t\mathbb{Z}/p, h\mathbb{BZ}} = \Big(\mathrm{Ind}_{\mathbb{Z}/pn}^{\mathbb{BZ}} \big(\mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{Z}/pn} \mathrm{tr}_{\mathcal{C}}(T^n)\big)^{t\mathbb{Z}/p}\Big)^{h\mathbb{BZ}} = 0.$$

And if p does divide n, then we have:

$$\mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n)\big)^{t\mathbb{Z}/p, h\mathbb{BZ}} = \mathrm{tr}_{\mathcal{C}}(T^n)^{t\mathbb{Z}/p, h\mathbb{Z}/(n/p)}.$$

By Lemma 4.8.1, the Frobenius map at p is given by the product over n of the maps:

$$\mathrm{tr}_{\mathcal{C}}(T^n)^{h\mathbb{Z}/n} \to \mathrm{tr}_{\mathcal{C}}(T^{pn})^{t\mathbb{Z}/p, h\mathbb{Z}/n}.$$

Now the fact that the equalizer above is the limit of genuine fixed points is formal from the definition of genuine fixed points. Moreover, the associated graded term was already calculated in Lemma 4.9.1.

□

Remark 4.10.3. The mild connectivity assumption that $\mathrm{tr}_{\mathcal{C}}(T^n) \in \mathsf{Sp}^{\leqslant 0}$ for all n played an inessential role in our calculation of the cyclotomic structure on $\mathrm{THH}(\mathrm{SqZero}(\mathcal{C}, T))$ and in the definition of genuine fixed points $\mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n}$: it is straightforward to generalize to the non-connective setting here. In other words, we were merely lazy there (the cost being our use of the Tate orbit lemma).

However, in the proof of the above theorem, the harsher assumption that the connectivity of $\mathrm{tr}_{\mathcal{C}}(T^n)$ tends to ∞ with n played an essential role above. Indeed, it was crucially used in (4.10.1), which for example allowed us to compute the homotopy $\mathbb{BZ}$-invariants termwise.

[23]Here we are taking the cokernel in the category of $\mathcal{E}_\infty$-groups. In a less commutative setting, it would be better to note that there is a fiber sequence $\mathbb{Z}/p \to \mathbb{BZ} \xrightarrow{p} \mathbb{BZ}$ with the right map surjective on π_0; this is the appropriate notion of "group quotient" in the homotopical setting.

4.11. **Proof of the main theorem.** We can now prove the main result of this section.

Proof of Theorem 2.12.2. For (essentially notational) convenience, we begin by ignoring the compatibility with K-theory but treating the general categorical setup, using Proposition 3.2.2.

As t-structures are used in Theorem 2.12.2, we define a t-structure on $\mathsf{End}(\mathcal{C})$ by setting $T \in \mathsf{End}(\mathcal{C})^{\leqslant 0}$ if $\mathrm{tr}_{\mathcal{C}}(T^n) \in \mathsf{Sp}^{\leqslant -n}$ for every $n > 0$. Note that for A a connective $\mathcal{E}_1$-algebra and $M \in A\text{–}\mathsf{bimod}^{\leqslant 0}$, we have:[24]

$$T_M \in \mathsf{End}(A\text{–}\mathsf{mod})^{\leqslant 0}$$

as $T_M := M \otimes_A -[1]$ gives so $\mathrm{tr}_{A\text{–}\mathsf{mod}}(T_M^n) = \mathrm{THH}(A, M^{\otimes n}[n])$.

Now first observe that the functor:

$$\mathsf{End}(\mathcal{C}) \to \mathsf{Sp}$$
$$T \mapsto \mathrm{tr}_{\mathcal{C}}(T^n)_{h\mathbb{Z}/n}$$

commutes with sifted colimits. Indeed, $T \mapsto T^n$ commutes with sifted colimits (as composition of functors commutes with colimits in each variable), and traces and coinvariants both commute with all colimits.

Moreover, for $n \neq 1$, we claim that the Goodwillie derivative of this functor vanishes. Again, by exactness of traces and coinvariants, it suffices to show this for the functor sending T to its n-fold self-composition T^n. Here it is straightforward[25] to see that the natural map:

$$T^n \to \Omega((\Sigma T)^n) = \Omega(\Sigma^n(T^n)) = \Sigma^{n-1}(T^n)$$

is nullhomotopic (and naturally so in T).

Therefore, by Lemma 4.9.1 and induction, the functor:[26]

$$\mathsf{End}(\mathcal{C}) \to \mathsf{Sp}$$
$$T \mapsto \mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n}$$

commutes with sifted colimits, the canonical projection $\mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n} \to \mathrm{tr}_{\mathcal{C}}(T)$ realizes the right hand side as the Goodwillie derivative of the left hand side (as functors of T).

Next, observe that for $T \in \mathsf{End}(\mathcal{C})^{\leqslant 0}$, we have:

$$\mathrm{Ker}\left(\mathrm{TC}_{red}(\mathrm{SqZero}(\mathcal{C}, T)) \to \mathrm{tr}_{\mathcal{C}}(T^{n!})^{\mathbb{Z}/n!}\right) \in \mathsf{Sp}^{<-n}$$

by Theorem 4.10.1, as $\mathrm{gr}_m \mathrm{TC}_{red}(\mathrm{SqZero}(\mathcal{C}, T)) = \mathrm{tr}_{\mathcal{C}}(T^m)_{h\mathbb{Z}/m} \in \mathsf{Sp}^{<-n}$ for $m > n$. From here the results are formal: by left completeness of the t-structure on Sp, we obtain Theorem 2.12.2 (2), and similarly the identification of the Goodwillie derivative from Theorem 2.12.2 (3). Moreover, Lemma 2.11.5 and the observation that each $\mathsf{End}(\mathcal{C})^{\leqslant 0} \xrightarrow{T \mapsto \mathrm{tr}_{\mathcal{C}}(T^n)^{\mathbb{Z}/n}} \mathsf{Sp}$ is manifestly extensible imply pseudo-extensibility of TC_{red}, i.e., Theorem 2.12.2 (1).

It now remains to show the compatibility with the cyclotomic trace from K-theory and the identifications of Goodwillie derivatives. For this, we suppose that $\mathcal{C}$ is compactly generated. Recall from §3.3 that in this case we have $\mathrm{SqZero}(\mathcal{C}, T) \subseteq \mathcal{C}^T$, with this inclusion preserving compact objects. We will show that the diagram:

[24] Here we are using the t-structure we just constructed. So in other words, there are more connective objects in this t-structure than the usual one on $\mathsf{End}(A\text{–}\mathsf{mod}) = A\text{–}\mathsf{bimod}$.

[25] See [Lur2] Proposition 6.1.3.4 for a statement in the general setting of Goodwillie calculus.

[26] Here we ask the reader to believe that Lemma 4.9.1 is true in the non-connective setting for appropriate definition of genuine fixed points, or to read $\mathsf{End}(\mathcal{C})$ as $\mathsf{End}(\mathcal{C})^{\leqslant 1}$.

$$\begin{array}{ccc} K(\mathrm{SqZero}(\mathcal{C},T)^c) & \longrightarrow & K(\mathcal{C}^{T,c}) \\ \downarrow & & \downarrow \\ \mathrm{TC}(\mathrm{SqZero}(\mathcal{C},T)) & \longrightarrow & \mathrm{tr}_{\mathcal{C}}(T) \end{array}$$

commutes, where the left vertical arrow is the cyclotomic trace map, and the bottom and right arrows are maps to Goodwillie derivatives (using the above for TC and Theorem 3.10.1 for K-theory). This suffices by Lemma 3.4.1.

Now observe that (by construction) the map $\mathrm{TC}(\mathrm{SqZero}(\mathcal{C},T)) \to \mathrm{tr}_{\mathcal{C}}(T)$ factors through $\mathrm{THH}(\mathrm{SqZero}(\mathcal{C},T)$. Hence, we can show the commutation of the above diagram with THH in place of TC, and the Dennis trace replacing the cyclotomic trace.

By Lemma 3.11.2, it suffices to show the commutation of the above diagram functorially in T after applying Ω^∞. Moreover, by Lemma 3.11.1, it suffices to show the commutation of the diagram:

$$\begin{array}{ccc} \mathrm{SqZero}(\mathcal{C},T)^{c,\simeq} & \longrightarrow & \Omega^\infty K(\mathcal{C}^{T,c}) \\ \downarrow & & \downarrow \\ \Omega^\infty \mathrm{THH}(\mathrm{SqZero}(\mathcal{C},T)) & \longrightarrow & \Omega^\infty \mathrm{tr}_{\mathcal{C}}(T). \end{array}$$

Now suppose $(\mathcal{F},\eta) \in \mathrm{SqZero}(\mathcal{C},T)^c$, i.e., $\mathcal{F} \in \mathcal{C}^c$ and $\eta : \mathcal{F} \to T(\mathcal{F})$ is locally nilpotent. Recall that Example 4.4.1 produced a point $\mathrm{tr}_{\mathcal{F}}(\eta) \in \Omega^\infty \mathrm{tr}_{\mathcal{C}}(T)$, i.e., *loc. cit.* gave a map $\mathrm{SqZero}(\mathcal{C},T)^{c,\simeq} \to \Omega^\infty \mathrm{tr}_{\mathcal{C}}(T)$. We claim that each leg of the above diagram identifies with this map.

For the upper leg of the diagram, this is tautological from the proof of Lemma 3.11.1.

We now treat the lower leg. The Dennis trace map applied to $(\mathcal{F},\eta)$ produces a point of $\Omega^\infty \mathrm{THH}(\mathrm{SqZero}(\mathcal{C},T))$, which (by definition of the Dennis trace) is $\mathrm{tr}_{(\mathcal{F},\eta)}(\mathrm{id}_{\mathcal{F},\eta})$ in the notation of Example 4.4.1. The proof of Proposition 4.5.1 shows that its image under Ω^∞ of the map:[27]

$$\mathrm{THH}(\mathrm{SqZero}(\mathcal{C},T)) = \mathrm{THH}(\mathcal{C}) \bigoplus \oplus_{n>0} \mathrm{Ind}_{\mathbb{Z}/n}^{\mathbb{BZ}} \mathrm{tr}_{\mathcal{C}}(T^n) \to \oplus_{n\geqslant 0} \mathrm{tr}_{\mathcal{C}}(T^n)$$

is $(\mathrm{tr}_{\mathcal{F}}(\mathrm{id}_{\mathcal{F}}), \mathrm{tr}_{\mathcal{F}}(\eta), \mathrm{tr}_{\mathcal{F}}(T(\eta)\eta), \ldots)$ (which genuinely gives a point of the direct sum by local nilpotence). Clearly the projection of this point to $\mathrm{tr}_{\mathcal{C}}(T)$ is $\mathrm{tr}_{\mathcal{F}}(\eta)$.

□

5. Reduction to the split square-zero case

5.1. In this section, we prove the general form of Theorem 1.1.1.

5.2. We return to the general format of Theorem 2.5.1 and Corollary 2.11.7, letting $\Psi : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ be a functor. Our present goal is to axiomatize when Ψ is constant along general nilpotent extensions.

5.3. **Square-zero extensions.** We briefly review the theory of square-zero in the homotopical setting.

Let $A \in \mathsf{Alg}$ be fixed, and let $m : A \otimes A \to A$ denote the multiplication. For $I \in A\text{–}\mathsf{bimod}$, a *square-zero extension* of A by I is by definition a morphism $\delta : \mathrm{Ker}(m) \to I[1] \in A\text{–}\mathsf{bimod}$.

To obtain an algebra from such a datum, recall that δ is equivalent by adjunction to a morphism $A \to \mathrm{SqZero}(A, I[1]) \in \mathsf{Alg}$. We then set the underlying algebra of (A, I, δ) to be the fiber product:

[27]Note that we use a direct sum indexed by $n \geqslant 0$ in the last term; the 0-fold composition of T with itself is the identity functor, so the leading term here is $\mathrm{THH}(\mathcal{C})$.

$$A \underset{\mathrm{SqZero}(A, I[1])}{\times} A$$

where one of the structure morphisms is induced by δ and the other by the zero map $0 \to I[1]$ (noting $A = \mathrm{SqZero}(A, 0)$). For example, for $\delta = 0$ we recover the split square-zero extension $\mathrm{SqZero}(A, I)$.

This construction enhances to define a category $\mathsf{Alg}^{\mathrm{SqZero}}$ whose objects are data (A, I, δ), and where morphisms $(A_1, I_1, \delta_1) \to (A_2, I_2, \delta_2)$ consist of a morphism $f : A_1 \to A_2$ of algebras and a commutative diagram of A_1-bimodules:

$$\begin{array}{ccc} \mathrm{Ker}(m_1) & \xrightarrow{f} & \mathrm{Ker}(m_2) \\ \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} \\ I_1[1] & \longrightarrow & I_2[1]. \end{array}$$

We let $\mathsf{Alg}^{\mathrm{SqZero}}_{conn}$ be the full subcategory of $\mathsf{Alg}^{\mathrm{SqZero}}$ consisting of objects (A, I, δ) where $A \in \mathsf{Alg}_{conn}$ and $I \in A\text{–}\mathsf{bimod}^{\leqslant 0}$.

5.4. **Notation.** Throughout this section, for $\Psi : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ and $B \to A$ a square-zero extension, we will use the notation $\Psi(B/A)$ for $\mathrm{Ker}(\Psi(B) \to \Psi(A))$.

5.5. We now introduce the following hypotheses on our functor $\Psi : \mathsf{Alg}_{conn} \to \mathsf{Sp}$.

Definition 5.5.1. Ψ is *convergent* if for any $A \in \mathsf{Alg}_{conn}$, the natural morphism:

$$\Psi(A) \to \lim_n \Psi(\tau^{\geqslant -n} A)$$

is an isomorphism.

Definition 5.5.2. Ψ *infinitesimally commutes with sifted colimits* if the functor:

$$\mathsf{Alg}^{\mathrm{SqZero}}_{conn} \to \mathsf{Sp}$$
$$(B \to A) \mapsto \Psi(B/A)$$

commutes with sifted colimits.

Proposition 5.5.3. *Suppose that Ψ is convergent and infinitesimally commutes with sifted colimits. Suppose moreover that Ψ is constant on split square-zero extensions, i.e., for every $A \in \mathsf{Alg}_{conn}$ and $M \in A\text{–}\mathsf{bimod}^{\leqslant 0}$, the morphism $\Psi(\mathrm{SqZero}(A, M)/A) = 0$.*

Then Ψ is infinitesimally constant*, that is, for every $f : B \to A \in \mathsf{Alg}_{conn}$ with $H^0(B) \to H^0(A)$ surjective with nilpotent kernel, the map $\Psi(B) \to \Psi(A)$ is an isomorphism.*

Proof. First, suppose $B \to A$ is a square-zero extension.

Note that if $A = T(V)$ is a *free* $\mathcal{E}_1$-algebra on $V = \mathbb{S}^{\oplus J}$ for some set J, then the then this square-zero extension is necessarily a split square-zero extension. Indeed, $\mathrm{Ker}(m)$ is then canonically isomorphic to $T(V) \otimes V \otimes T(V)$ as a $T(V)$-bimodule; in particular, it is free on V. It follows that for $I \in T(V)\text{–}\mathsf{bimod}^{-1}$, any morphism $\mathrm{Ker}(m) \to I[1]$ is necessarily nullhomotopic. In particular, our hypothesis implies $\Psi(B) \xrightarrow{\simeq} \Psi(A)$ for A of this form.

In general, $A = |F_\bullet|$ can be written as a geometric realization of free $\mathcal{E}_1$-algebras F_n as above.[28]

[28]This follows from the formalism of [Lur1] §5.5.8, especially Lemma 5.5.8.14.

Then $B = |F_\bullet \times_A B|$, and each $F_n \times_A B \to F_n$ is a square-zero extension, and this is a simplicial object of $\mathsf{Alg}_{conn}^{\mathrm{SqZero}}$. Therefore, if Ψ infinitesimally commutes with sifted colimits, we have:

$$\Psi(B/A) = |\Psi(F_\bullet \times_A B/F_\bullet)| = |0| = 0.$$

This shows the result for general square-zero extensions.

Next, fix A. Recall (c.f. [Lur2] Corollary 7.4.1.28) that each morphism $\tau^{\geqslant -n-1}A \to \tau^{\geqslant -n}A$ admits a structure of square-zero extension, so by the above is an isomorphism on Ψ. By convergence, we then have:

$$\Psi(A) \xrightarrow{\simeq} \lim_n \Psi(\tau^{\geqslant -n}A) \xrightarrow{\simeq} \Psi(H^0(A)).$$

Then for $f : B \to A$ surjective with nilpotent kernel, we have $\Psi(B) = \Psi(H^0(B))$ and $\Psi(A) = \Psi(H^0(A))$. Clearly $H^0(B) \to H^0(A)$ is a composition of square-zero extensions, so we obtain the result.

□

5.6. **Application to Dundas-Goodwillie-McCarthy.** Proposition 5.5.3 and the split square-zero case of Theorem 1.1.1 reduce the general case of Theorem 1.1.1 to the following to results.

Theorem 5.6.1. *The functors* $K, \mathrm{TC} : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ *are convergent and infinitesimally commute with sifted colimits.*

We prove this result separately for K-theory and TC below.

5.7. **K-theory.** First, convergence of K-theory is simple: the description of K-theory as a group completion makes it clear that $\tau^{\geqslant -n-1}K(A) = \tau^{\geqslant -n-1}K(\tau^{\geqslant -n}A)$.

We will show K-theory infinitesimally commutes with sifted colimits essentially following [DGM]. The proof uses *Volodin's construction*, which is an avatar of Milnor's definition of K_2 that we review below.[29]

5.8. **Volodin theory.** Fix $A \in \mathsf{Alg}_{conn}$ in what follows. Recall that $\mathsf{Proj}(A)$ denotes the category of projective A-modules, i.e., summands of $A^{\oplus n}$. Let $\mathsf{Proj}(A)^{\simeq} \in \mathsf{Gpd}$ denote the underlying groupoid of this category. Direct sums make $\mathsf{Proj}(A)^{\simeq}$ into an $\mathcal{E}_\infty$-space, and we recall that $\Omega^\infty K(A)$ is its group completion.

Note that if once we invert $A \in \mathsf{Proj}(A)^{\simeq}$, we obtain a group. This motivates considering the canonical map:

$$\chi = \chi_A : \operatorname*{colim}_{n \geqslant 0} \mathsf{Proj}(A)^{\simeq} \to \Omega^\infty K(A) \in \mathsf{Gpd}$$

where the structure maps in this colimit are given by adding $A \in \mathsf{Proj}(A)$. Note that the left hand side does not necessarily admit an $\mathcal{E}_\infty$-structure, and this map is not at all an equivalence. However, it is an isomorphism on π_0.

Volodin's construction will provide a convenient expression for $\mathrm{fib}(\chi)$ (the fiber over $0 \in \Omega^\infty K(A)$).

[29]I do not feel like I understand this method so well. Perhaps there is a more conceptual approach.

5.9. For $n \geqslant 0$, let $GL_n(A)$ denote the group[30] of automorphisms of $A^{\oplus n}$.

Consider $A^{\oplus n}$ as a filtered A-module in the standard way $0 \to A \to A^{\oplus 2} \to \ldots \to A^{\oplus n}$. Let $B_n(A)$ be the group of automorphisms of $A^{\oplus n}$ as a filtered A-module and let $U_n(A)$ denote the kernel of the "symbol" map:[31]

$$B_n(A) \to \prod_{i=1}^{n} GL_1(A).$$

5.10. We will need some care about the functoriality of the above constructions.

Let fSet^{inj} denote the category of finite sets and injective maps between them. Then for any $I \in \mathsf{fSet}^{inj}$, we have $GL_I(A) := \mathrm{Aut}(A^{\oplus I})$; and for $I \hookrightarrow J$, we have a morphism $GL_I(A) \to GL_J(A)$ induced by the direct sum decomposition $A^{\oplus J} = A^{\oplus I} \oplus A^{\oplus J \setminus I}$.

Relatedly, we have a functor from $\mathsf{fSet}^{inj} \to \mathsf{Gpd}$ sending I to $\mathsf{Proj}(A)^{\simeq}$ for every I, and sending a morphism $I \hookrightarrow J$ to the map $\mathsf{Proj}(A)^{\simeq} \xrightarrow{-\oplus A^{J\setminus I}} \mathsf{Proj}(A)^{\simeq}$. Note that the source of the map χ is obtained by taking the colimit of this construction along the map $\mathbb{Z}^{\geqslant 0} \xrightarrow{n \mapsto \{1,\ldots,n\}} \mathsf{fSet}^{inj}$. (And one can show that the target is the colimit along fSet^{inj}, although we will not need this fact.)

The construction of U_n depends also on a linear ordering, i.e., for $I \in \mathbf{\Delta}_{aug}^{inj}$ a finite set with a linear ordering, we have the group $U_I(A)$ functorial for injective order preserving maps, and mapping naturally to mapping to $GL_I(A)$.

Moreover, the natural map:

$$\mathbb{B}U_I(A) \to \mathbb{B}GL_I(A) \to \mathsf{Proj}(A)^{\simeq} \to \Omega^{\infty}K(A)$$

is canonically constant with constant value $A^{\oplus I}$. Indeed, this is clear from the Waldhausen realization of $K(A)$.

5.11. Putting the functoriality above together, let $\mathcal{J}$ denote the category (even poset) of pairs $I \in \mathbf{\Delta}_{aug}^{inj}$ and an isomorphism $\alpha : I \simeq \{1, \ldots, n\}$, where morphisms $(I, \alpha) \to (J, \beta)$ are order-preserving maps $f : I \to J$ making the diagram:

$$\begin{array}{ccc} I & \xrightarrow{f} & J \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \{1, \ldots, n\} & \hookrightarrow & \{1, \ldots, m\} \end{array}$$

commute, where the bottom arrow is the standard embedding.

Definition 5.11.1. The *Volodin space* $\mathcal{X}(A)$ is the groupoid $\mathrm{colim}_{(I,\alpha)\in\mathcal{J}}\, \mathbb{B}U_I(A)$.

We claim the observations from §5.10 in effect equip $\mathcal{X}(A)$ with a canonical map to $\mathrm{fib}(\chi)$.

Indeed, note that $\mathcal{J}$ maps to $\mathbb{Z}^{\geqslant 0}$ (using the isomorphism α from the pair (I, α)), inducing a map to $\mathbb{B}U_I(A) \to \mathbb{B}GL_{|I|}(A) \to \mathsf{Proj}(A)^{\simeq}$. On colimits, we obtain a map to $\mathrm{colim}_{n \geqslant 0} \mathsf{Proj}(A)$, i.e., the source of the map χ. Then the Waldhausen realization of K-theory shows that the composite map to $\Omega^{\infty}K(A)$ is constant with value the base-point.

Theorem 5.11.2. *The map* $\mathcal{X}(A) \to \mathrm{fib}(\chi)$ *is an equivalence.*

Remark 5.11.3. The reader willing to take this result on faith may safely skip ahead to §5.12.

[30]Meaning group-like $\mathcal{E}_1$-groupoid.

[31]I.e., the ith factor takes the induced automorphism of $\mathrm{gr}_i\, A^{\oplus n} = A$.

Proof of Theorem 5.11.2 (sketch). This result is shown in [FOV] Proposition 2.6. We give some indications of the ideas that go into it here.

Roughly, the construction works as follows. Note that $\mathcal{X}(A)$ is connected (since $\mathcal{J}$ is contractible, admitting an initial object). Let $\mathrm{St}^{der}(A) := \Omega\mathcal{X}(A)$ be the *derived Steinberg group* of A, so $\mathcal{X}(A) = \mathrm{colim}_{(I,\alpha)\in\mathcal{J}} U_I(A)$, the colimit being taken in the category Gp of group-like $\mathcal{E}_1$-groupoids. (It is straightforward to see from this description that $\pi_0(\mathrm{St}^{der}(A))$ is the usual Steinberg group of $\pi_0(A)$, which is the reason we use this terminology.)

In essence, the argument relating $\mathrm{St}^{der}(A)$ and $K(A)$ imitates the classical argument relating the Steinberg group and K_2 of usual ring.

Step 1. First, suppose $G \in \mathsf{Gp}$ is a group (in the homotopy-theoretic sense).

By definition, a *central extension* of G by an $\mathcal{E}_2$-group[32] A is the data of a pointed map $\mathbb{B}G \to \mathbb{B}^2 A$. We frequently let E denote the fiber of the map $G \to \mathbb{B}A$, which clearly fits into a fiber sequence of groups:

$$A \to E \to G.$$

We sometimes say $A \to E \to G$ is a central extension to mean it arises by this procedure.

Step 2. We say G is *perfect* if $\pi_0(G)$ is perfect in the usual sense, i.e., its (non-derived) abelianization is trivial. We claim that perfect G admits a *universal central extension*, i.e., an initial central extension.

Indeed, in this case Quillen's plus construction[33] implies that there is simply-connected $Y \in \mathsf{Gpd}$ with a map from $\mathbb{B}G$ realizing Y as the initial simply-connected space with a map from $\mathbb{B}G$ (indeed: $Y = (\mathbb{B}G)^+$). Clearly Y is initial for pointed maps as well. Then looping implies G has a universal central extension by $\Omega^2 Y$.

We denote the universal central extension by $G^{univ} \to G$ in this case.

Step 3. Next, we recall that there is a simple recognition principle for a central extension $A \to E \to G$ to be the universal one.

Namely, this is the case if and only if E is an *acyclic group*, meaning that $\Sigma^\infty(\mathbb{B}E) = 0 \in \mathsf{Sp}$ (or equivalently, the group homology of E is trivial).

Indeed, this is a slight rephrasing of a well-known feature of Quillen's plus construction (see [DGM] Theorem 3.1.1.7 for example).

Step 4. Now suppose G is perfect and we are given a central extension:

$$A \to H \to G$$

with H perfect as well. Then we claim the natural map $H^{univ} \to G^{univ}$ is an isomorphism.

Indeed, let A_G^{univ} denote $\mathrm{Ker}(G^{univ} \to G)$. By universality, we have a canonical (pointed) map $\mathbb{B}^2 A_G^{univ} \to \mathbb{B}^2 A$. Let F denote its fiber. Note that F receives a canonical map from $\mathbb{B}H$ with fiber $\mathbb{B}G^{univ}$.

We claim F is simply-connected. Indeed, its π_1 is a quotient of $\pi_0(A)$, so abelian, but also a quotient of the perfect group $\pi_0(H)$, so trivial.

Therefore, F defines a central extension:

$$\Omega^2 F \to G^{univ} \to H$$

[32] Meaning a group-like $\mathcal{E}_2$-groupoid, i.e., a double loop space.

[33] See [Hoy] for a discussion in the higher categorical setting.

which is the universal one by the recognition principle: G^{univ} is an acyclic group, being the universal central extension for G.

Step 5. We now weaken the recognition principle given above (c.f. [DGM] Lemma 3.1.1.13).

Suppose we are given $f : E \to G$ surjective on π_0 with G perfect and E an acyclic group, but without assuming centrality of the extension. Observe that acyclicity of E implies E is perfect with $E^{univ} \xrightarrow{\simeq} E$. Therefore, we obtain a map $E^{univ} \to G^{univ}$, and we can ask when this map is an isomorphism.

We claim that this is the case if we assume that for every $g \in E$, the induced (conjugation) automorphism of $\mathrm{Ker}(f)$ is trivial on $\pi_*(\mathrm{Ker}(f))$.

Indeed, we have the fiber sequence:

$$\mathbb{B}\,\mathrm{Ker}(f) \to \mathbb{B}E \to \mathbb{B}G$$

with $\pi_1(\mathbb{B}E) = \pi_0(E)$ acting trivially on $\pi_*(\mathbb{B}\,\mathrm{Ker}(f)) = \pi_{*+1}(\mathrm{Ker}(f))$. A well-known obstruction theoretic argument then shows that for every $n \geqslant 0$, the map $\tau_{\leqslant n}E \to \tau_{\leqslant n}G$ factors as a composition of central extensions:

$$\tau_{\leqslant n}E = H_0 \to H_1 \to \ldots \to H_r \to H_{r+1} = \tau_{\leqslant n}G \in \mathsf{Gp}.$$

The maps $H_i \to H_{i+1}$ are surjective on π_0 (being extensions), so each group H_i is perfect. Therefore, the previous step implies that each such map induces an isomorphism on universal central extensions. We then obtain $(\tau_{\leqslant n}E)^{univ} \xrightarrow{\simeq} (\tau_{\leqslant n}G)^{univ}$.

Finally, it is clear that $G^{univ} = \lim_n(\tau_{\leqslant n}G)^{univ}$ (and similarly for E), giving the claim.

Step 6. We now apply these methods to study $\mathrm{St}^{der}(A)$, leaving details to references.

First, an argument of Suslin [Sus] shows that $\mathrm{St}^{der}(A)$ is an acyclic group. More precisely, he shows that for any $(I, \alpha) \in \mathcal{J}$ and $r \geqslant 0$, there is a map $(I, \alpha) \to (J, \beta) \in \mathcal{J}$ such that $\mathbb{B}U_I(A) \to \mathbb{B}U_J(A)$ is zero on homology groups $H_{\leqslant r}$ with coefficients in some field. This immediately implies the vanishing of such homology groups for $\mathcal{X}(A) = \mathbb{B}\,\mathrm{St}^{der}(A)$, which implies the vanishing for integral coefficients, which gives our desired acyclicity.

(We remark that although Suslin formally treats classical algebras, his argument readily adapts to the connective $\mathcal{E}_1$-setting.)

Step 7. Next, we would like to realize $\mathrm{St}^{der}(A)$ as the universal central extension of something.

Let $GL_\infty(A) := \mathrm{colim}_n\, GL_n(A) = \Omega(\mathrm{colim}_n\, \mathsf{Proj}(A))$. Although $GL_\infty(A)$ is not perfect, it is not so far: by Whitehead's lemma, $\pi_0(GL_\infty(A))$ has perfect derived group an (non-derived) abelianization $K_1(A)(= K_1(\pi_0(A)))$. Therefore, the kernel of the composition:

$$GL_\infty(A) \to \Omega^{\infty+1}K(A) \xrightarrow{\Omega\chi} \pi_0(\Omega^{\infty+1}K(A)) = K_1(A)$$

is perfect. Let us denote[34] this kernel by G.

There is a natural map $f : \mathrm{St}^{der}(A) \to G$ induced by our map from $\mathcal{X}(A)$ to the fiber of χ. We claim that this realizes $\mathrm{St}^{der}(A)$ as the universal central extension of G.

Indeed, by acyclicity of $\mathrm{St}^{der}(A)$, it suffices to show that any $g \in \mathrm{St}^{der}(A)$ acts trivially on $\pi_*(\mathrm{Ker}(f))$. This is a variant of the classical argument that the usual Steinberg group is a central extension [Mil] Theorem 5.1, which we outline below (see also [FOV] §4 and [DGM] Lemma 3.1.3.4).

Let $\mathcal{J}_{\leqslant n} \subseteq \mathcal{J}$ be the full subcategory of pairs (I, α) with $|I| \leqslant n$. Let $\mathrm{St}_n^{der}(A)$ be $\mathrm{colim}_{\mathcal{J}_{\leqslant n}} \mathrm{St}_n^{der} \in \mathsf{Gp}$.

[34]More standard notation would be $E(A)$, but we have been using E above for extensions.

Note that $\mathrm{St}_n^{der}(A)$ has a canonical map to $GL_n(A)$, so acts on the spectrum $A^{\oplus n}$. Moreover, the map $\mathrm{St}_n^{der}(A) \to \mathrm{St}_{n+1}(A)$ clearly factors through $\mathrm{St}_n^{der}(A) \rhd \Omega^\infty A^{\oplus n}$.

Let K'_n denote the kernel of the map $\mathrm{St}_n^{der}(A) \to GL_n(A)$, and let $K'_{n+\frac{1}{2}}$ denote $\mathrm{Ker}(\mathrm{St}_n^{der}(A) \rhd \Omega^\infty A^{\oplus n} \to GL_{n+1}(A))$. Clearly the natural map $K'_n \to K'_{n+1}$ factors through $K'_{n+\frac{1}{2}}$.

It is immediate to see that for $x \in \Omega^\infty A^{\oplus n}$, the natural conjugation action on $K'_{n+\frac{1}{2}}$ fixes the image of K'_n, i.e., if we compose $\varphi : K'_n \to K'_{n+\frac{1}{2}}$ with this conjugation map, we obtain a map canonically homotopic to φ. This implies that the image of x in $\pi_0(\mathrm{St}^{der}(A))$ acts by the identity on the image of $\pi_*(K'_n)$ in $\mathrm{colim}_m\, \pi_*(K'_m) = \pi_*(\mathrm{Ker}(\mathrm{St}^{der}(A) \to GL_\infty(A)))$.

The same arguments apply if we replace e.g. K'_n by $K_n := \mathrm{Ker}(\mathrm{St}_n(A) \to GL_n(A) \times_{GL_\infty(A)} G)$. Then $\mathrm{colim}_n\, K_n$ is the kernel of the map $\mathrm{St}(A) \to GL_\infty(A)$ (by filteredness of the colimit here).

Then observe that $\pi_0(\mathrm{St}^{der}(A))$ is the usual Steinberg group of $\pi_0(A)$, and the above shows that in the standard notation, $x_{ij}(\alpha)$ acts trivially on the image of $\pi_*(K_n)$ in $\pi_*(\mathrm{St}^{der}(A))$ for $i < j$ and $j > n$. A similar argument shows the claim for $x_{ji}(\alpha)$ for such i and j. Such elements then generate the (usual) Steinberg group, completing the argument.

Step 8. Let G be as above. By Quillen's plus construction of K-theory, the plus construction of $\mathbb{B}G$ is $\Omega^\infty \tau^{\leqslant -2} K(A)$.

Recall that the classifying space of the universal central extension of $\mathbb{B}G$ is the fiber of the canonical map to the plus construction. Therefore, we have a diagram:

$$\begin{array}{ccccc}
\mathcal{X}(A) = \mathbb{B}\,\mathrm{St}^{der}(A) & \longrightarrow & \mathbb{B}G & \longrightarrow & \Omega^\infty \tau^{\leqslant -2} K(A) \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{B}GL_\infty(A) & \longrightarrow & \Omega^\infty \tau^{\leqslant -1} K(A) \\
 & & \downarrow & & \downarrow \\
 & & \underset{n}{\mathrm{colim}}\,\mathsf{Proj}(A) & \longrightarrow & \Omega^\infty K(A)
\end{array}$$

where the squares are all Cartesian and the top row is a fiber sequence, implying the theorem. (In particular, the universal central extension $\mathrm{St}^{der}(A)$ of G is by $\Omega^{\infty+2} K(A)$, in analogy with the classical definition of K_2.)

□

5.12. **Consequences.** We now deduce the following.

Corollary 5.12.1. *The functor:*

$$\mathsf{Alg}_{conn} \to \mathsf{Gpd}$$
$$A \mapsto \mathrm{fib}(\chi_A)$$

commutes with sifted colimits.

Proof. By Theorem 5.11.2, it suffices to show this for $A \mapsto \mathcal{X}(A)$ instead. We are reduced to showing $A \mapsto \mathbb{B}U_n(A)$ commutes with sifted colimits for every n, and then to $A \mapsto U_n(A) \in \mathsf{Gpd}$. But $U_n(A)$ is functorially isomorphic to $A^{\oplus\binom{n-1}{2}}$, which clearly commutes with sifted colimits in A.

□

5.13. We now show K-theory infinitesimally commutes with sifted colimits.

For convenience, we use some notation from the proof of Theorem 5.11.2 and let $\mathrm{St}^{der}(A) = \Omega\mathfrak{X}(A) \in \mathsf{Gp}$. By Theorem 5.11.2, for any $A \in \mathsf{Alg}_{conn}$ we have a canonical homomorphism $\mathrm{St}^{der}(A) \to GL_\infty(A)$ with:

$$GL_\infty(A)/\mathrm{St}^{der}(A) = \Omega^{\infty+1}K(A)$$

(where the quotient on the left is the usual geometric realization of the bar construction).

Similarly, we have:

Lemma 5.13.1. *Let $B \to A$ is a square-zero extension.*

As in §5.4, we let $\mathrm{St}^{der}(B/A) := \mathrm{Ker}(\mathrm{St}^{der}(B) \to \mathrm{St}^{der}(A))$, and similarly for $K(B/A)$ and $GL_\infty(B/A)$.

Then:

$$GL_\infty(B/A)/\mathrm{St}^{der}(B/A) = \Omega^{\infty+1}K(B/A).$$

Proof. We have the commutative diagram:

$$\begin{array}{ccccc}
GL_\infty(B/A)/\mathrm{St}^{der}(B/A) & \longrightarrow & \mathbb{B}\,\mathrm{St}^{der}(B/A) & \longrightarrow & \mathbb{B}GL_\infty(B/A) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\infty+1}K(B) & \longrightarrow & \mathbb{B}\,\mathrm{St}^{der}(B) & \longrightarrow & \mathbb{B}GL_\infty(B) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{\infty+1}K(A) & \longrightarrow & \mathbb{B}\,\mathrm{St}^{der}(A) & \longrightarrow & \mathbb{B}GL_\infty(A)
\end{array}$$

with rows being fiber sequences. Moreover, because $\pi_0(\mathrm{St}^{der}(B)) \twoheadrightarrow \pi_0(\mathrm{St}^{der}(A))$ and $\pi_0(GL_\infty(B)) \twoheadrightarrow \pi_0(GL_\infty(A))$ (by the square-zero condition), the right two columns are also fiber sequences. This gives the claim.

□

Now observe that the functor:

$$\begin{aligned}
\mathsf{Alg}_{conn}^{\mathrm{SqZero}} &\to \mathsf{Gpd} \\
(B \to A) &\mapsto GL_\infty(B/A)
\end{aligned}$$

commutes with sifted colimits. Indeed, the right hand side is isomorphic to infinite matrices with coefficients in $\Omega^\infty \mathrm{Ker}(B \to A)$, so is a direct sum of infinitely many copies of this functor. Therefore, the claim follows from noting that $(B \to A) \mapsto \Omega^\infty \mathrm{Ker}(B \to A)$ commutes with sifted colimits.

Mapping to $\mathrm{St}^{der}(B/A)$ obviously commutes with sifted colimits, so we obtain $(B \to A) \mapsto \Omega^{\infty+1}K(B/A)$ infinitesimally commutes with sifted colimits. Then the claim follows because $\pi_0(\Omega^\infty K(B/A)) = 0$ and $\Omega^{\infty+1} : \mathsf{Sp}^{\leqslant -1} \to \mathsf{Gpd}$ commutes with sifted colimits.

5.14. **TC.** We now show that TC is convergent and infinitesimally commutes with sifted colimits.

Notation 5.14.1. In what follows, for a spectrum V we let $V_{\mathbb{Q}}$ denote $V \otimes \mathbb{Q}$.

5.15. The following result, which also serves as a key consistency check with Goodwillie's original work on the subject [Goo1], is crucial.

Theorem 5.15.1. *For $B \to A \in \mathsf{Alg}_{conn}$ a square-zero extension, the natural map:*

$$\mathrm{TC}(B/A)_{\mathbb{Q}} \to \mathrm{TC}(B_{\mathbb{Q}}/A_{\mathbb{Q}})$$

is an isomorphism.

We will prove this result in what follows.

5.16. **Notation.** Let V be a cyclotomic spectrum.

Let $\mathrm{TC}^{-}(V)$ denote $V^{h\mathbb{BZ}}$. Similarly, let $\mathrm{TP}(V) = V^{t\mathbb{BZ}}$ be the *periodic topological cyclic homology* of V, where this notation indicates the Tate construction on the circle[35] (see e.g. [NS] §I.4).

Let $\mathrm{TP}^\wedge(V)$ denote the profinite completion of $\mathrm{TP}(V)$, that is:

$$\mathrm{TP}^\wedge(V) := \mathrm{Coker}\left(\underline{\mathrm{Hom}}_{\mathsf{Sp}}(\mathbb{Q}, \mathrm{TP}(V)) \to \mathrm{TP}(V)\right).$$

Recall that we have:

$$\mathrm{TC}(V) = \mathrm{Eq}(\mathrm{TC}^{-}(V) \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}^\wedge(V)) \tag{5.16.1}$$

by [NS] II.4.3. Here can is the tautological map (lifting to TP itself), and φ is the *cyclotomic Frobenius* map (which amalgamates Frobenius maps at all primes).

5.17. **Construction of the meromorphic Frobenius.** We have the following observation.

Lemma 5.17.1. *Let V be a spectrum with a (naive) $\mathbb{BZ}$-action. Suppose $(V_{\mathbb{Q}})^{t\mathbb{BZ}} = 0$. Then there is a canonical isomorphism:*

$$(V^{h\mathbb{BZ}})_{\mathbb{Q}} \xrightarrow{\simeq} (V^{t\mathbb{BZ}})_{\mathbb{Q}} \times (V_{\mathbb{Q}})^{h\mathbb{BZ}}.$$

Proof. By definition of the Tate construction for $\mathbb{BZ}$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccc} (V_{h\mathbb{BZ}})_{\mathbb{Q}}[1] & \longrightarrow & (V^{h\mathbb{BZ}})_{\mathbb{Q}} & \longrightarrow & (V^{t\mathbb{BZ}})_{\mathbb{Q}} \\ \downarrow & & \downarrow & & \downarrow \\ (V_{\mathbb{Q}})_{h\mathbb{BZ}}[1] & \longrightarrow & (V_{\mathbb{Q}})^{h\mathbb{BZ}} & \longrightarrow & (V_{\mathbb{Q}})^{t\mathbb{BZ}}. \end{array}$$

Clearly the left vertical map is an isomorphism, so the right square is Cartesian. Now our vanishing hypothesis gives the result.

□

We now make the following definition following [Hes2].

Definition 5.17.2. A cyclotomic spectrum V *admits a meromorphic Frobenius* if $\mathrm{TP}(V)$ is profinite[36] and $\mathrm{TP}(V_{\mathbb{Q}})$ is profinite.

Remark 5.17.3. Note that $\mathrm{TP}(V_{\mathbb{Q}})$ being profinite is equivalent to vanishing, since a profinite and rational spectrum is necessarily zero.

---

[35] We remind the reader of our notational convention from Remark 4.3.1.

[36] I.e., $\underline{\mathrm{Hom}}(\mathbb{Q}, \mathrm{TP}(V)) = 0$, or equivalently, $\mathrm{TP}(V) \xrightarrow{\simeq} \mathrm{TP}^\wedge(V)$.

Suppose V admits a meromorphic Frobenius. We then have the canonical map:

$$\varphi^{mer} : \mathrm{TP}(V)_{\mathbb{Q}} \to \mathrm{TP}(V)_{\mathbb{Q}}$$

given as the composition:

$$\mathrm{TP}(V)_{\mathbb{Q}} \xrightarrow{\text{Lem. 5.17.1}} \mathrm{TC}^{-}(V)_{\mathbb{Q}} \xrightarrow{\varphi\otimes\mathbb{Q}} \mathrm{TP}^{\wedge}(V)_{\mathbb{Q}} = \mathrm{TP}(V)_{\mathbb{Q}}.$$

Lemma 5.17.4. *Suppose V is a cyclotomic spectrum that admits a meromorphic Frobenius. Then* $\mathrm{TC}(V)_{\mathbb{Q}} \xrightarrow{\simeq} \mathrm{TC}(V_{\mathbb{Q}})$ *if and only if* $\mathrm{id} - \varphi^{mer} : \mathrm{TP}^{\wedge}(V)_{\mathbb{Q}} \to \mathrm{TP}^{\wedge}(V)_{\mathbb{Q}}$ *is an isomorphism.*

Proof. We clearly have:

$$\mathrm{Ker}(\mathrm{id} - \varphi^{mer}) \xrightarrow{\simeq} \mathrm{Ker}\left(\mathrm{TC}(V)_{\mathbb{Q}} \to \mathrm{TC}(V_{\mathbb{Q}})\right)$$

using Lemma 5.17.1.

□

Remark 5.17.5. $\mathrm{THH}(\mathbb{F}_p)$ admits a meromorphic Frobenius with non-invertible $\mathrm{id} - \varphi^{mer}$: see [NS] §IV.4.

5.18. **Filtrations.** We now give some sufficient hypotheses to test the invertibility of $\mathrm{id} - \varphi^{mer}$ in the above setting.

Definition 5.18.1. A *connectively filtered cyclotomic spectrum* is a connective spectrum V with a complete[37] filtration $\mathrm{fil}_\bullet V$ by connective spectra, an action of $\mathbb{BZ}$ as a filtered spectrum, and $\mathbb{BZ}$-equivariant filtered *Frobenius* maps:

$$\varphi_p : \mathrm{fil}_\bullet \to \mathrm{fil}_{p\cdot\bullet} V^{t\mathbb{Z}/p}$$

for every prime p. (As is standard, we are considering the residual[38] $\mathbb{BZ} = (\mathbb{BZ})/(\mathbb{Z}/p)$-action on the right hand side.)

Remark 5.18.2. In the above, note that V is a cyclotomic spectrum. Moreover, the notation indicates that φ_p maps $\mathrm{fil}_n V$ to $\mathrm{fil}_{pn} V^{t\mathbb{Z}/p}$ for every p and n. Similarly, $\mathrm{gr}_\bullet V = \oplus_n \mathrm{gr}_n V$ is also a *graded* cyclotomic spectrum, which (similarly to the proof of Proposition 4.5.1) means $\varphi_p : \mathrm{gr}_n V \to \mathrm{gr}_{pn} V^{t\mathbb{Z}/p}$.

Remark 5.18.3. The non-connective version of the above notion may readily be extracted from [AMGR].

Remark 5.18.4. A (non-connective) version of this notion appeared in [Bru] using the language of equivariant homotopy theory.

Example 5.18.5. If A is a filtered $\mathcal{E}_1$-algebra with $\mathrm{fil}_n A$ connective for all n, then the standard filtration on $\mathrm{THH}(A)$ upgrades to a structure of connectively filtered cyclotomic spectrum. Indeed, this is a ready adaptation of the Nikolaus-Scholze construction (or follows from the functoriality of traces outlined in §4.7). We have $\mathrm{gr}_\bullet \mathrm{THH}(A) = \mathrm{THH}(\mathrm{gr}_\bullet A)$ as (graded) cyclotomic spectra.

Proposition 5.18.6. *Let V be a connectively filtered cyclotomic spectrum such that:*

- $\mathrm{fil}_{-1} V \xrightarrow{\simeq} \mathrm{fil}_0 V \xrightarrow{\simeq} \dots \xrightarrow{\simeq} V$.

---

[37] I.e., $\lim_n \mathrm{fil}_{-n} V = 0$.

[38] The quotient notation is misleading here: it refers to the existence of the fiber sequence $\mathbb{Z}/p \to \mathbb{BZ} \xrightarrow{p} \mathbb{BZ}$ (with the right map of course surjective on π_0).

- *For every* n, $\mathrm{fil}_{-m} V \in \mathsf{Sp}^{\leqslant -n}$ *for* $m \gg 0$.
- *The cyclotomic spectrum* $\mathrm{gr}_\bullet V$ *admits a meromorphic Frobenius.*

Then $\mathrm{TP}(V)$ *admits a meromorphic Frobenius with* $\mathrm{id} - \varphi^{mer} : \mathrm{TP}(V) \otimes \mathbb{Q} \to \mathrm{TP}(V) \otimes \mathbb{Q}$ *an isomorphism.*

From Lemma 5.17.4, we immediately deduce:

Corollary 5.18.7. *In the setting of Proposition 5.18.6,* $\mathrm{TC}(V) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{TC}(V_{\mathbb{Q}})$.

Proof of Proposition 5.18.6.

Step 1. First, note that $\mathbb{BZ}$-equivariance of the filtration on V, $\mathrm{TC}^-(V)$ and $V_{h\mathbb{BZ}}$ inherit filtrations as well. These filtrations are complete: for $\mathrm{TC}^-(V)$ this is automatic, and for $V_{h\mathbb{BZ}}$ this follows because the filtration on $\tau^{\geqslant -n} V_{h\mathbb{BZ}}$ is bounded from below for any n.

The norm map $V_{h\mathbb{BZ}}[1] \to \mathrm{TC}^-(V)$ is clearly filtered, so $\mathrm{TP}(V)$ also has a complete filtration. Note that $\mathrm{gr}_n \mathrm{TP}(V) = (\mathrm{gr}_n V)^{t\mathbb{BZ}}$, and similarly for the other players. In particular, $\mathrm{gr}_i \mathrm{TP}(V) = 0$ for $n \geqslant 0$.

In particular, we see that $\underline{\mathrm{Hom}}(\mathbb{Q}, \mathrm{TP}(V))$ has a complete filtration with associated graded $\underline{\mathrm{Hom}}(\mathbb{Q}, \mathrm{gr}_\bullet \mathrm{TP}(V)) = 0$. The same logic applies for $V_{\mathbb{Q}}$, noting that the filtration is complete for the same reason as for $V_{h\mathbb{BZ}}$. Therefore, we obtain $\underline{\mathrm{Hom}}(\mathbb{Q}, \mathrm{gr}_\bullet \mathrm{TP}(V_{\mathbb{Q}})) = 0$ and deduce that V admits a meromorphic Frobenius.

Step 2. Note that Lemma 5.17.1 applies just as well in the setting of (connectively) filtered cyclotomic spectra. Therefore, the meromorphic Frobenius on $\mathrm{TP}(V)_{\mathbb{Q}}$ maps $\mathrm{fil}_n \mathrm{TP}(V)_{\mathbb{Q}}$ $(= (\mathrm{fil}_n \mathrm{TP}(V))_{\mathbb{Q}})$ to $\mathrm{fil}_{pn} \mathrm{TP}(V)_{\mathbb{Q}}$. Since $\mathrm{fil}_{-1} \mathrm{TP}(V)_{\mathbb{Q}} = \mathrm{TP}(V)_{\mathbb{Q}}$, we would be done if this filtration on $\mathrm{TP}(V)_{\mathbb{Q}}$ were complete. But this is not typically true, so some additional argument is needed.

Step 3. Fix n, and assume m is large enough that $\tau^{\geqslant -n} \mathrm{fil}_{-m} V = 0$.

We claim $\varphi^{mer} : \tau^{\geqslant -n} \mathrm{TP}(V)_{\mathbb{Q}} \to \mathrm{TP}(V)_{\mathbb{Q}}$ is actually *integral* on $\mathrm{fil}_{-m} V$. That is, there are a canonical maps $f_n : \tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V) \to \tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V)$ fitting into commutative diagrams:

$$\begin{array}{ccc}
\tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V) & \xrightarrow{\quad f_n \quad} & \tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V) \\
\downarrow & & \downarrow \\
\tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V)_{\mathbb{Q}} & \xrightarrow{\quad \varphi^{mer} \quad} & \tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V)
\end{array}$$

and compatible in the natural sense as we suitably vary n and m.

Indeed, our assumptions give $\mathrm{fil}_{-m} V_{h\mathbb{BZ}} \in \mathsf{Sp}^{\leqslant -n}$, so $\mathrm{fil}_{-m} V^{h\mathbb{BZ}} \to \mathrm{fil}_{-m} V^{t\mathbb{BZ}}$ induces an isomorphism on $\tau^{\geqslant -n}$. Now our map f_n is induced by the cyclotomic Frobenius, noting that $\mathrm{TP}(V) = \mathrm{TP}^\wedge(V)$ as filtered spectra by assumption. It is immediate to see that f_n fits into a commutative diagram as above.

Step 4. Let $\mathrm{TP}_p^\wedge(V)$ be the p-adic completion of $\mathrm{TP}(V)$. Note that by construction, f_n induces a map $\tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V) \to \tau^{\geqslant -n} \mathrm{fil}_{-pm} \mathrm{TP}_p^\wedge(V)$.

In particular, f_n itself maps through $\tau^{\geqslant -n} \mathrm{fil}_{-2m} \mathrm{TP}(V)$. We deduce that $\mathrm{id} - f_n : \tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V) \to \tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V)$ is invertible (since the filtration on $\mathrm{TP}(V)$ is complete).

Because tensoring with $\mathbb{Q}$ is t-exact, the map $\mathrm{id} - \varphi^{mer} : \tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V)_{\mathbb{Q}} \to \tau^{\geqslant -n} \mathrm{fil}_{-m} \mathrm{TP}(V)_{\mathbb{Q}}$ is an isomorphism. Since $\mathrm{id} - \varphi^{mer}$ was an isomorphism on all associated graded terms, we now deduce that it is an isomorphism after applying $\tau^{\geqslant -n}$ for any n, and therefore is itself an isomorphism.

□

5.19. To prove Theorem 5.15.1, it suffices (by Corollary 5.18.7) to show the following.

Lemma 5.19.1. *For $B \to A \in \mathsf{Alg}_{conn}$ a square-zero extension,* $\mathrm{THH}(B/A)$ *admits a filtration as in Proposition 5.18.6.*

Proof. Suppose B is a square-zero extension of A by $M \in A\text{–}\mathsf{bimod}^{\leqslant 0}$. Note that B admits a two-step descending filtration with $\mathrm{gr}_{\bullet} B = \mathrm{SqZero}(A, M)$ (more precisely: $\mathrm{gr}_0 = A$ and $\mathrm{gr}_{-1} = M$).

As in Example 5.18.5, $\mathrm{THH}(B/A)$ is a connectively filtered cyclotomic spectrum with $\mathrm{gr}_{\bullet} \mathrm{THH}(B/A) = \mathrm{THH}(\mathrm{SqZero}(A, M)/A)$. By connectivity of A and B, we have $\mathrm{fil}_{-n-1} \mathrm{THH}(B/A) \in \mathsf{Sp}^{\leqslant -n}$. Moreover, it is clear that $\mathrm{gr}_i \mathrm{THH}(B/A) = 0$ for $i \geqslant 0$.

It remains to show that $\mathrm{THH}(\mathrm{SqZero}(A, M)/A)$ admits a meromorphic Frobenius. As in the proof of Theorem 4.10.1, we have:

$$\mathrm{TP}(\mathrm{SqZero}(A, M)/A) = \prod_{n \geqslant 1} \mathrm{THH}(A, M[1]^{\otimes n})^{t\mathbb{Z}/n}.$$

The $\mathbb{Z}/n$-Tate construction applied to connective spectra produces n-adically complete spectra, showing that $\mathrm{TP}(\mathrm{SqZero}(A, M)/A)$ is profinite. We then have:

$$\mathrm{TP}\left((\mathrm{SqZero}(A, M)/A)_{\mathbb{Q}}\right) = \mathrm{TP}(\mathrm{SqZero}(A_{\mathbb{Q}}, M_{\mathbb{Q}})/A_{\mathbb{Q}})$$

which is profinite by the above. □

5.20. **Sifted colimits.** We now show:

Proposition 5.20.1. TC *infinitesimally commutes with sifted colimits.*

Proof. It suffices to show that the functors $\mathrm{TC}(-)_{\mathbb{Q}}$ and $\mathrm{TC}(-)/p$ infinitesimally commute with sifted colimits, where p varies over all primes. (Indeed, the functor $\mathsf{Sp} \to \mathsf{Sp}$ of tensoring with $\mathbb{Q} \bigoplus \oplus_p \mathbb{S}/p$ is continuous and conservative.)

In fact, the functor $\mathrm{TC}(-)/p : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ (without relativization) commutes with sifted colimits by [CMM] Corollary 2.15.[39]

Now for $B \to A$ a square-zero extension, we have $\mathrm{TC}(B/A)_{\mathbb{Q}} = \mathrm{TC}(B_{\mathbb{Q}}/A_{\mathbb{Q}})$ by Theorem 5.15.1. Note that $\mathrm{TC} = \mathrm{TC}^-$ for rational ring spectra, so we have $\mathrm{TC}(B_{\mathbb{Q}}/A_{\mathbb{Q}}) = \mathrm{TC}^-(B_{\mathbb{Q}}/A_{\mathbb{Q}})$. Finally, as $\mathrm{TP}(B_{\mathbb{Q}}/A_{\mathbb{Q}}) = 0$ by Lemma 5.19.1, we have $\mathrm{TC}^-(B_{\mathbb{Q}}/A_{\mathbb{Q}}) = \mathrm{THH}(B_{\mathbb{Q}}/A_{\mathbb{Q}})_{h\mathbb{BZ}}[1]$, and this functor manifestly commutes with sifted colimits. □

5.21. **Convergence.** It remains to show that $\mathrm{TC} : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ is convergent, which is straightforward.

Note that $\mathrm{THH} : \mathsf{Alg}_{conn} \to \mathsf{Sp}$ is convergent, since $\tau^{\geqslant -n} \mathrm{THH}(A) \xrightarrow{\simeq} \tau^{\geqslant -n} \mathrm{THH}(\tau^{\geqslant -n} A)$ for every n. We formally obtain convergence of TC^-, since it is obtained as homotopy invariants from THH.

Note that $\mathrm{THH}(-)_{h\mathbb{BZ}}$ is convergent for the same reason as THH. Combined with the above, we obtain that TP is convergent. This clearly implies that its profinite completion is convergent, so we obtain the same property for TC from (5.16.1).

[39] In fact, it is clear from the proof that in Theorem 5.6.1, we only needed Ψ to infinitesimally commute with geometric realizations, and here the argument is more elementary: see [CMM] Corollary 2.6.

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