AFFINE BEILINSON-BERNSTEIN LOCALIZATION AT THE CRITICAL LEVEL FOR GL_2

SAM RASKIN

Abstract. We prove the rank 1 case of a conjecture of Frenkel-Gaitsgory: critical level Kac-Moody representations with regular central characters localize onto the affine Grassmannian. The method uses an analogue in local geometric Langlands of the existence of Whittaker models for most representations of GL_2 over a non-Archimedean field.

Contents

1. Introduction

1.1. More than a decade ago, Frenkel and Gaitsgory initiated an ambitious program to relate geometric representation theory of (untwisted) affine Kac-Moody algebras at critical level to geometric Langlands, following Beilinson-Drinfeld [BD1] and [BD2] and Feigin-Frenkel, e.g., [FF].

We refer the reader to [FG2] for an introduction to this circle of ideas. The introduction to [FG5] and the work [Gai2] may be helpful supplements.

While Frenkel-Gaitsgory were extraordinarily successful in developing representation theory at critical level (highlights include [FG1], [FG2], [FG3], [FG6], [FG5], and [FG4]), their ambitious program left many open problems. Most of these problems are dreams that are not easy to formulate precisely.

In contrast, their conjecture on critical level localization for the affine Grassmannian is a concrete representation theoretic problem. It remains the major such problem left open by their work. In this paper, we prove the Frenkel-Gaitsgory localization conjecture for rank 1 groups.

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Below, we recall the context for and statement of the Frenkel-Gaitsgory conjecture, the progress that they made on it, and outline the argument used in the present paper for GL_2.

1.2. **Notation.** In what follows, G denotes a split reductive group over a field k of characteristic 0. We fix $B \subseteq G$ a Borel subgroup with unipotent radical N and Cartan $T = B/N$. We let $\check{G}$ denote the Langlands dual group to G, and similarly $\check{B}$ and so on.

We let e.g. $G(K)$ denote the algebraic loop group of G, which is a group indscheme of ind-infinite type. We let $G(O) \subseteq G(K)$ denote its arc subgroup and $\mathrm{Gr}_G := G(K)/G(O)$ the affine Grassmannian. We refer to [BD1] for further discussion of these spaces and [Ras2] for definitions of D-modules in this context.

We follow the notational convention that all categories are assumed derived; e.g., A–mod denotes the DG (derived) category of A-modules. For $\mathcal{C}$ a DG category with a given t-structure, we let $\mathcal{C}^\heartsuit$ denote the corresponding abelian category.

1.3. **Affine Kac-Moody algebras.** Before recalling the Frenkel-Gaitsgory conjecture, we need to review the representation theory of affine Kac-Moody algebras at critical level.

1.4. Recall that for a *level* κ, by which we mean an Ad-invariant symmetric bilinear form on $\mathfrak{g}$, there is an associated central extension:

$$0 \to k \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) := \mathfrak{g} \underset{k}{\otimes} k((t)) \to 0.$$

This extension is defined by a standard 2-cocycle that vanishes on $\mathfrak{g}[[t]] := \mathfrak{g} \otimes_k k[[t]]$; in particular, the embedding $\mathfrak{g}[[t]] \hookrightarrow \mathfrak{g}((t))$ canonically lifts to an embedding $\mathfrak{g}[[t]] \hookrightarrow \widehat{\mathfrak{g}}_\kappa$.

1.5. By a *representation* of $\widehat{\mathfrak{g}}_\kappa$ on a (classical) vector space $V \in \mathsf{Vect}^\heartsuit$, we mean an action of the Lie algebra $\widehat{\mathfrak{g}}_\kappa$ such that every $v \in V$ is annihilated by $t^N\mathfrak{g}[[t]]$ for $N \gg 0$ and such that $1 \in k \subseteq \widehat{\mathfrak{g}}_\kappa$ acts by the identity.

For instance, the *vacuum* module $\mathbb{V}_\kappa := \mathrm{ind}_{\mathfrak{g}[[t]]}^{\widehat{\mathfrak{g}}_\kappa}(k)$ is such a representation. Here ind denotes induction, and we are abusing notation somewhat: we really mean to induce from $k \oplus \mathfrak{g}[[t]]$ the module k on which k acts by the identity and $\mathfrak{g}[[t]]$ acts trivially; since we only consider representations on which $k \subseteq \widehat{\mathfrak{g}}_\kappa$ acts by the identity, we expect this does not cause confusion.

We denote the abelian category of representations of $\widehat{\mathfrak{g}}_\kappa$ by $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}^\heartsuit$. The appropriate DG category $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}$ was defined in [FG4] §23; see [Gai5], [Ras5] Appendix A, or [Ras6] for other expositions.

We recall the pitfall that the forgetful functor $\mathrm{Oblv} : \widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod} \to \mathsf{Vect}$ is not conservative, i.e., it sends non-zero objects to zero.[1]

But one key advantage of $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}$ over other possible "derived categories" of $\widehat{\mathfrak{g}}_\kappa$-modules is that it admits a *level* κ *action of* $G(K)$: see [Ras6] §11 for the construction and definitions.

1.6. We let $U(\widehat{\mathfrak{g}}_\kappa)$ denote the (twisted) topological enveloping algebra of $\widehat{\mathfrak{g}}_\kappa$ (with the central element $1 \in \widehat{\mathfrak{g}}_\kappa$ set to the identity). For our purposes, $U(\widehat{\mathfrak{g}}_\kappa)$ is the pro-representation of $\widehat{\mathfrak{g}}_\kappa$:

$$\lim_n \mathrm{ind}_{t^n\mathfrak{g}[[t]]}^{\widehat{\mathfrak{g}}_\kappa}(k) \in \mathsf{Pro}(\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^\heartsuit).$$

The underlying pro-vector space is naturally an $\overrightarrow{\otimes}$-algebra algebra in the sense of [Ras6] §3 by construction, its *discrete* modules (in $\mathsf{Vect}^\heartsuit$) are the same as (classical) representations of $\widehat{\mathfrak{g}}_\kappa$.

[1] See [Ras5] §1.18 for some discussion of this point.

1.7. Let $D_\kappa(\mathrm{Gr}_G)$ denote the DG category of κ-twisted D-modules on Gr_G. There is a global sections functor:

$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -) : D_\kappa(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}.$$

This functor is a morphism of categories acted on by $G(K)$ and sends the skyscraper D-module $\delta_1 \in D_\kappa(\mathrm{Gr}_G)$ to the vacuum module $\mathbb{V}_\kappa$.

1.8. **Affine Beilinson-Bernstein localization?** Recall the finite-dimensional *Beilinson-Bernstein* localization theorem:

Theorem 1.8.1 ([BB])**.** *The functor:*

$$\Gamma(G/B, -) : D(G/B) \to \mathfrak{g}\text{–}\mathsf{mod}_0$$

is a t-exact equivalence of categories. Here $D(G/B)$ is the DG category of D-modules and $\Gamma(G/B,-)$ is the left D-module global sections functor; $\mathfrak{g}$–mod_0 is the DG category of modules over $U(\mathfrak{g})\otimes_{Z(\mathfrak{g})} k$ for $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$ and $Z(\mathfrak{g}) \to k$ the restriction of the augmentation $U(\mathfrak{g}) \to k$.

Almost as soon as Beilinson and Bernstein proved their localization theorem, there was a desire for an affine analogue that would apply for Gr_G or the affine flag variety. Results soon emerged in work of Kashiwara-Tanisaki, beginning with [KT] for so-called *negative* levels κ.

The results of Kashiwara-Tanisaki suffice for applications to Kazhdan-Lusztig problems. However, their theorems are less satisfying than Theorem 1.8.1: they do not provide an equivalence of categories, but only a fully faithful functor. Conceptually, this is necessarily the case because for negative κ, the center of $U(\widehat{\mathfrak{g}}_\kappa)$ consists only of scalars, so it is not possible to define an analogue of the category $\mathfrak{g}$–mod_0.[2]

As observed by Frenkel-Gaitsgory, this objection does not apply at *critical* level, as we recall below.

1.9. **Critical level representation theory.** For the so-called *critical* value of κ, the representation theory of the Kac-Moody algebra behaves quite differently from other levels. For completeness, we recall that critical level is $\frac{-1}{2}$ times the Killing form. We let *crit* denote the corresponding symmetric bilinear form; in particular, we use $\widehat{\mathfrak{g}}_{crit}$ (resp. $\mathbb{V}_{crit}$) in place of $\widehat{\mathfrak{g}}_\kappa$ (resp. $\mathbb{V}_\kappa$).

Theorem 1.9.1 (Feigin-Frenkel)**.** *(1) The (non-derived) center $\mathfrak{Z}$ of $U(\widehat{\mathfrak{g}}_{crit})$ is canonically isomorphic to the commutative pro-algebra of functions on the ind-scheme $\mathrm{Op}_{\check{G}}$ of* opers (on the punctured disc) *for the Langlands dual group $\check{G}$:*

$$\mathrm{Op}_{\check{G}} := \left(\check{f} + \check{\mathfrak{b}}((t))\right)dt/\check{N}(K)$$

where $\check{N}(K) \subseteq \check{G}(K)$ acts on $\check{\mathfrak{g}}((t))dt$ by gauge transformations and $\check{f}$ is a principal nilpotent element with $[\check{\rho}, \check{f}] = -\check{f}$.

We recall that, as for the Kostant slice, $\mathrm{Op}_{\check{G}}$ is (somewhat non-canonically) isomorphic to an affine space that is infinite-dimensional in both ind and pro senses (like the affine space corresponding to the k-vector space $k((t))$).

(2) The natural map:

$$\mathfrak{Z} \to \mathfrak{z} := \mathrm{End}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit}(\mathbb{V}_{crit})$$

[2]However, see [Bei] for some speculations; the suggestion is that $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}$ should be considered not as decomposing over the spectrum of its center but over a moduli of local systems on the punctured disc.

is surjective and fits into a commutative diagram:

$$\begin{array}{ccc} \mathfrak{Z} & \longrightarrow & \mathfrak{z} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Fun}(\mathrm{Op}_{\check{G}}) & \longrightarrow & \mathrm{Fun}(\mathrm{Op}_{\check{G}}^{reg}). \end{array}$$

Here $\mathrm{Op}_{\check{G}}^{reg} := (\check{f} + \check{\mathfrak{b}}[[t]])dt/\check{N}(O)$ *is the scheme of* regular *opers, on the (non-punctured) disc; we recall that the natural map* $\mathrm{Op}_{\check{G}}^{reg} \to \mathrm{Op}_{\check{G}}$ *is a closed embedding.*

We refer to [FF] and [Fre] for proofs of most of these statements; the only exception is that the map $\mathrm{Fun}(\mathrm{Op}_{\check{G}}) \hookrightarrow \mathfrak{Z}$ constructed using [FF] is an isomorphism, which is shown as [BD1] Theorem 3.7.7.[3]

We refer to [FG2] §1 and [BD1] §3 for an introduction to opers. We highlight, as in *loc. cit.*, that $\mathrm{Op}_{\check{G}}$ (resp. $\mathrm{Op}_{\check{G}}^{reg}$) is a moduli space of de Rham $\check{G}$-local systems on the punctured (resp. non-punctured disc) with extra structure.

Remark 1.9.2. The definition of opers here is slightly different from the original one used by Beilinson-Drinfeld and rather follows the definition advocated by Gaitsgory. In this definition, an isogeny of reductive groups induces an isomorphism on spaces of opers, unlike in [BD1]. For $\check{G}$ semisimple, the definition here coincides with the definition in [BD1] for the associated adjoint group. We refer to [Bar] for a more geometric discussion.

1.10. **Localization at critical level.** The functor:

$$\Gamma^{\mathsf{IndCoh}} : D_{crit}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$$

fails to be an equivalence for two related reasons.

First, recall that $\Gamma^{\mathsf{IndCoh}}(\delta_1) = \mathbb{V}_{crit}$. As for any skyscraper D-module, $\mathrm{End}(\delta_1) = k$, while by Theorem 1.9.1, $\mathbb{V}_{crit}$ has a large endomorphism algebra. Worse still, $\mathbb{V}_{crit}$ has large self-Exts by [FT] and [FG2] §8.

Moreover, there are central character restrictions on the essential image of Γ^{IndCoh}. Say $M \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}$ is *regular* if $I := \mathrm{Ker}(\mathfrak{Z} \to \mathfrak{z})$ acts on M trivially, and let $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{reg} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}$ denote the corresponding subcategory (which is not closed under extensions). Then for any $\mathcal{F} \in D_{crit}(\mathrm{Gr}_G)$, the cohomology groups of $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$ will be regular, for the same reason as the analogous statement in the finite-dimensional setting.

1.11. In [FG2], Frenkel and Gaitsgory in effect proposed that these are the only obstructions. We recall their conjecture now.

First, in [FG4] §23, an appropriate DG category $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$ character was constructed: we review the construction in §6. There is a canonical action of the symmetric monoidal DG category $\mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{reg})$ on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$ commuting with the critical level $G(K)$-action.[4]

Next, recall that geometric Satake [MV1] gives an action of $\mathsf{Rep}(\check{G}) = \mathsf{QCoh}(\mathbb{B}\check{G})$ on $D_{crit}(\mathrm{Gr}_G)$ by convolution.

[3] In fact, the mere existence of this map (and its good properties) is all we really need. That the map is an isomorphism is nice, but not strictly necessary.

[4] There are actually important technical issues involving this $G(K)$-action that should probably be overlooked at the level of an introduction; we refer to §1.22 and §6.10 for further discussion.

Moreover, $\mathrm{Op}^{reg}_{\check{G}}$ has a canonical $\check{G}$-bundle; indeed, $\mathrm{Op}^{reg}_{\check{G}}$ is the moduli of $\check{G}$-local systems on the formal disc $\mathcal{D} = \mathrm{Spec}(k[[t]])$ with extra structure, giving a map:

$$\mathrm{Op}^{reg}_{\check{G}} \to \mathrm{LocSys}_{\check{G}}(\mathcal{D}) = \mathbb{B}\check{G}.$$

In particular, there is a canonical symmetric monoidal functor:

$$\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}).$$

According to Beilinson-Drinfeld's *birth of opers* theorem from [BD1], Γ^{IndCoh} is a canonically morphism of $(G(K), \mathsf{Rep}(\check{G}))$-bimodule categories (c.f. §7).

Conjecture 1.11.1 (Frenkel-Gaitsgory, [FG2] Main conjecture 8.5.2)**.** *The induced functor:*

$$\Gamma^{\mathrm{Hecke}} : D_{crit}(\mathrm{Gr}_G) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$$

is a t-exact equivalence of DG categories.

We can now state:

Main Theorem (Thm. 7.14.1)**.** *Conjecture 1.11.1 is true for G of semisimple rank* 1*.*

Corollary 1.11.1. *For $\chi \in \mathrm{Op}^{reg}_{\check{G}}(k)$ a regular oper (defined over k), let $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{\chi}$ denote the abelian category of $\widehat{\mathfrak{g}}_{crit}$-modules on which $\mathfrak{Z}$ acts through its quotient $\mathfrak{Z} \twoheadrightarrow \mathfrak{z} \xrightarrow{\chi} k$, and let $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\chi}$ denote the appropriate DG category.*

Then for $G = GL_2$, the functor:

$$D_{crit}(\mathrm{Gr}_G) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{Vect} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\chi}$$

induced by global sections is a t-exact equivalence, where Vect is a $\mathsf{Rep}(\check{G})$-module category via the map $\mathrm{Spec}(k) \xrightarrow{\chi} \mathrm{Op}^{reg}_{\check{G}} \to \mathbb{B}\check{G}$.

Corollary 1.11.2. *Let $G = GL_2$ and let $\chi_1, \chi_2 \in \mathrm{Op}^{reg}_{\check{G}}(k)$ be two regular opers (defined over k). Then any isomorphism of the underlying $\check{G}$-local systems of χ_1 and χ_2 gives rise to an equivalence of abelian categories:*

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{\chi_1} \simeq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{\chi_2}.$$

Remark 1.11.3. We highlight a wrong perspective on Corollary 1.11.2; this remark may safely be skipped.

For $G = PGL_2$, one can show that the group scheme Aut of automorphisms of the formal disc acts transitively on $\mathrm{Op}^{reg}_{\check{G}}$, giving rise to an elementary construction of equivalences of categories as in Corollary 1.11.2 in this case.

However, these are not the equivalences produced by Corollary 1.11.2. First, at the level of DG categories, the equivalences using the action of Aut are not $G(K)$-equivariant: the $G(K)$-actions differ via the action of Aut on $G(K)$. In contrast, the equivalences produced using Corollary 1.11.1 are manifestly $G(K)$-equivariant.

Concretely, this implies that for a k-point $g \in G(K)$, if $g \cdot \mathbb{V}_{crit} := \mathrm{ind}^{\widehat{\mathfrak{g}}_{crit}}_{\mathrm{Ad}_g(\mathfrak{g}[[t]])}(k)$ and $g \cdot \mathbb{V}_{crit,\chi} := (g \cdot \mathbb{V}_{crit}) \otimes_{\mathfrak{z},\chi} k$, then Corollary 1.11.2 maps $g \cdot \mathbb{V}_{crit,\chi_1}$ to $g \cdot \mathbb{V}_{crit,\chi_2}$. For $\gamma \in \mathrm{Aut}$ and $\chi_2 = \gamma \cdot \chi_1$, the resulting isomorphism produced *using* γ (not Corollary 1.11.2) rather sends $g \cdot \mathbb{V}_{crit,\chi_1}$ to $\gamma(g) \cdot \mathbb{V}_{crit,\chi_2}$.

In addition, one can see that the equivalences produced using the Aut action depend on isomorphisms of underlying $\check{G}^{\wedge}_{\check{B}}$-bundles of regular opers (in this PGL_2 case), not merely the underlying $\check{G}$-bundles.

1.12. **Viewpoints.** We refer to the introduction and §2 of [FG6] for a discussion of Conjecture 1.11.1 and its consequences. We highlight some ways of thinking about it here.

- For the representation theorist, Theorem 7.14.1 provides an affine analogue of Beilinson-Bernstein similar to their original result, c.f. the discussion in §1.8. The equivalences of Corollary 1.11.2 provide analogues of translation functors at critical level. By Theorem 1.13.1, the content of Theorem 7.14.1 amounts to a structure theorem for regular $\widehat{\mathfrak{g}}_{crit}$-modules (for $\mathfrak{g} = \mathfrak{sl}_2$).
- For the number theorist, Theorem 7.14.1 provides a first non-trivial test of Frenkel-Gaitsgory's proposal [FG2] for local geometric Langlands beyond Iwahori invariants.

  Roughly, Frenkel-Gaitsgory propose that for σ a de Rham $\check{G}$-local system on the punctured disc, there should be an associated DG category $\mathcal{C}_\sigma$ with an action of $G(K)$.[5] This construction should mirror the usual local Langlands correspondence, leading to many expected properties of this assignment, c.f. [Gai2].

  A striking part of their proposal does not have an arithmetic counterpart. For χ an oper with underlying local system σ, Frenkel-Gaitsgory propose $\mathcal{C}_\sigma = \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_\chi \in G(K)\text{–}\mathsf{mod}_{crit}$, where we use similar notation to Corollary 1.11.1. We remark that Frenkel-Zhu [FZ] and Arinkin [Ari] have shown that any such σ admits an oper structure (assuming, to simplify the discussion, that σ is a field-valued point).

  In particular, one expects equivalences as in Corollary 1.11.2, at least on the corresponding derived categories.

  Our results provide the first verification beyond Iwahori invariants of their ideas.

Remark 1.12.1. We have nothing to offer to the combinatorics of representations. The previous results of Frenkel-Gaitsgory suffice[6] to treat problems of Kazhdan-Lusztig nature, c.f. [AF].

1.13. **Previously known results.** Frenkel-Gaitsgory were able to show the following results, valid for any reductive G.

We let $I = G(O) \times_G B$ be the Iwahori subgroup of $G(O)$ and $\mathring{I} = G(O) \times_G N$ its prounipotent radical.

Theorem 1.13.1. *The functor* Γ^{Hecke} *is fully faithful, preserves compact objects, and is an equivalence on* $\mathring{I}$*-equivariant categories. Moreover, the restriction of* Γ^{Hecke} *to the* $\mathring{I}$*-equivariant category* $D_{crit}(\mathrm{Gr}_G)^{\mathring{I}} \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$ *is* t*-exact.*

Remark 1.13.2. The fully faithfulness is [FG2] Theorem 8.7.1; we give a simpler proof of this result in Appendix B. The existence of the continuous right adjoint $\mathrm{Loc}^{\mathrm{Hecke}}$ is proved as in [FG4] §23.5-6. The equivalence on $\mathring{I}$-equivariant categories and t-exactness of the functor is Theorem 1.7 of [FG6].[7]

[5] Most invariantly, this action should have critical level, which is (slightly non-canonically) equivalent to level 0.

[6] Up to mild central character restrictions coming from [FG4]. These restrictions are understood among experts to be inessential.

[7] The results we cite here are not formulated in exactly the given form in the cited works. For the purposes of the introduction, we ignore this issue and address these gaps in the body of the paper.

1.14. **Methods.** Below, we outline the proof of Theorem 7.14.1. However, to motivate this, we highlight a methodological point.

Across their works at critical level, Frenkel and Gaitsgory use remarkably little about actual critical level representations. Indeed, they rely primarily on Feigin and Frenkel's early results, some basic properties of Wakimoto modules, and the Kac-Kazhdan theorem.

But using the action of $G(K)$ on $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}$ and constructions/results from geometric Langlands, Frenkel and Gaitsgory were able to prove deep results about representations at critical level; see e.g. [FG4].

In other words, their works highlight an important methodological point: the theory of group actions on categories provides a bridge:

$$\left\{\begin{array}{l}\text{Geometry and higher}\\ \text{representation theory of}\\ \text{groups}\end{array}\right\} \overset{\text{Group actions on categories}}{\rightsquigarrow} \left\{\begin{array}{l}\text{Representation theory}\\ \text{of Lie algebras}\end{array}\right\}$$

For loop groups in particular, a great deal was known at the time about $G(O)$ and Iwahori invariants: see e.g. [MV1], [AB], [ABG], and [ABB$^+$].

More recently, Whittaker invariants have been added to the list: see [Ras5]. These can be used to simplify many arguments from Frenkel-Gaitsgory, as e.g. in Appendix B.

1.15. As we outline below, our methods are in keeping with the above. The main new idea and starting point of the present paper, Theorem 5.1.1, is exactly about the higher representation theory of $PGL_2(K)$.

1.16. Group actions on categories inherently involve derived categories. Therefore, one has the striking fact that although Corollary 1.11.2 is about abelian categories (of modules!), the proof we give involves sophisticated homological methods and careful analysis of objects in degree $-\infty$ in various DG categories.

1.17. **Sketch of the proof.** We now give the Platonic ideal of the proof of the main theorem.

1.18. First, one readily reduces to proving Conjecture 1.11.1 for any G of semisimple rank 1; for us, it is convenient to focus on $G = PGL_2$.

1.19. The following result is one of the key new ideas of this paper:

Theorem (Thm. 5.1.1)**.** *Let $G = PGL_2$ and let $\mathcal{C}$ be acted on by $G(K)$ (perhaps with level κ).*

Then $\mathcal{C}$ is generated under the action of $G(K)$ by its Whittaker category $\mathsf{Whit}(\mathcal{C}) := \mathcal{C}^{N(K),\psi}$ and its $\mathring{I}$-equivariant category $\mathcal{C}^{\mathring{I}}$.

The relation to the equivalence part of the Frenkel-Gaitsgory conjecture is immediate: By fully faithfulness of Γ^{Hecke} (Theorem 1.13.1), Theorem 7.14.1 is reduced to showing essential surjectivity. Applying Theorem 5.1.1 to the essential image of Γ^{Hecke}, one immediately obtains Theorem 7.14.1 from Theorem 1.13.1 and:

Theorem (Thm. 8.3.1)**.** *For any reductive G, the functor Γ^{Hecke} induces an equivalence on Whittaker categories.*

The latter result is an essentially immediate consequence of the affine Skryabin theorem from [Ras5] and the classical work [FGV].

1.20. Theorem 5.1.1 warrants some further discussion.

First, this result mirrors the fact that for PGL_2 over a local, non-Archmidean field, irreducible representations admit Whittaker models, or else are one of the two 1-dimensional characters trivial on the image of SL_2.

We now give an intentionally informal heuristic for Theorem 5.1.1 that may safely be skipped.

For general reductive G and $\mathcal{C} \in G(K)\text{–}\mathsf{mod}_{crit}$, let $\mathcal{C}' \subseteq \mathcal{C}$ be the subcategory generated under the $G(K)$-action by $\mathsf{Whit}(\mathcal{C})$.

Assuming some form of local geometric Langlands, one expects that the local Langlands parameters of $\mathcal{C}/\mathcal{C}'$ to consist only of those $\sigma \in \mathrm{LocSys}_{\check{G}}(\mathring{\mathcal{D}})$ that lift to a point of $\mathrm{LocSys}_{\check{P}}(\mathring{\mathcal{D}})$ at which the map $\mathrm{LocSys}_{\check{P}}(\mathring{\mathcal{D}}) \to \mathrm{LocSys}_{\check{G}}(\mathring{\mathcal{D}})$ is singular; here $\check{P}$ is some parabolic subgroup of $\check{G}$ and $\mathring{\mathcal{D}} = \mathrm{Spec}(k((t)))$ is the formal punctured disc.

For $\check{G} = SL_2$, the only parabolic we need to consider is the Borel $\check{B}$. Then $\sigma \in \mathrm{LocSys}_{\check{B}}$ is the data of an extension:

$$0 \to (\mathcal{L}, \nabla) \to (\mathcal{E}, \nabla) \to (\mathcal{L}^\vee, \nabla) \to 0$$

$(\mathcal{L}, \nabla)$ a line bundle with connection on the punctured disc (and $\mathcal{L}^\vee$ equipped with the dual connection to that of $\mathcal{L}$). At such a point, the cokernel of the map of tangent spaces induced by $\mathrm{LocSys}_{\check{B}}(\mathring{\mathcal{D}}) \to \mathrm{LocSys}_{\check{G}}(\mathring{\mathcal{D}})$ is:

$$H^1_{dR}(\mathring{\mathcal{D}}, (\check{\mathfrak{g}}/\check{\mathfrak{b}})_\sigma) = H^1_{dR}(\mathring{\mathcal{D}}, (\mathcal{L}^\vee, \nabla)^{\otimes 2}).$$

This group will vanish unless $(\mathcal{L}, \nabla)^{\otimes 2}$ is trivial, i.e., unless $(\mathcal{E}, \nabla) \in \mathrm{LocSys}_{\check{G}}(\mathring{\mathcal{D}})$ or its quadratic twist has unipotent monodromy.

It is expected that for $\mathcal{D} \in G(K)\text{–}\mathsf{mod}_{crit}$ with local Langlands parameters having unipotent monodromy (resp. up to twist by a 1-dimensional character) is generated under the $G(K)$-action by its Iwahori invariants (resp. its Iwahori invariants twisted by a suitable character of I trivial on $\mathring{I}$).

This justifies that for $G = PGL_2$, one should expect $\mathcal{C}/\mathcal{C}'$ above to be generated by its $\mathring{I}$-invariants. (And in fact, following the above reasoning, one can refine Theorem 5.1.1 to show that $\mathcal{C}$ is even generated by $\mathsf{Whit}(\mathcal{C})$ and its invariants with respect to the Iwahori subgroup of $SL_2(K)$.)

1.21. The argument we provide for Theorem 5.1.1 is novel. Its decategorified version gives a new proof of the corresponding result in usual harmonic analysis.

We use the perspective of [Ras5] on Whittaker categories, which allows us to study the Whittaker construction via (finite-dimensional!) algebraic groups. (We summarize the most relevant parts of [Ras5] in §5.2.)

In §3, we introduce a new technique in the finite-dimensional settings suggested by [Ras5], which we call *Whittaker inflation*. In that context, Theorem 3.4.1 shows that subcategories generated under group actions by Whittaker invariants are large in a suitable sense. These ideas apply for a general reductive group G, and have counterparts in the decategorified setting.

In §5, we introduce a method of *descent* that is specific for PGL_2. Combined with the results of §3, descent immediately gives Theorem 5.1.1.

1.22. For clarity, we highlight that there is one technical issue in the above argument: there is not an a priori $G(K)$-action on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$, so the above argument does not apply as is. Instead, there is a closely related but inequivalent category, $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$, with an evident $G(K)$-action (coming from [Ras6]). We refer to §6.10 for a more technical discussion of this point.

This distinction makes the second half of the paper more technical, and requires finer analysis than was suggested in §1.19.

1.23. The t-exactness in Theorem 7.14.1 is proved by another instance of the descent argument highlighted above. The details are in §10, with some auxiliary support in §11.

1.24. Finally, we highlight that the vast majority of the intermediate results in this paper apply to general reductive groups G. In particular, this includes the results of §3, which are a key ingredient in the proof of Theorem 7.14.1.

The descent arguments discussed above are where we use that $G = PGL_2$; here the key input is that every element in the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ is either regular or 0. The situation strongly suggests that there should be some (more complicated) generalization of the descent method that applies for higher rank groups as well.

1.25. **Structure of this paper.** The first part of the paper is purely geometric, primarily involving monoidal categories of D-modules on algebraic groups.

In §3, we introduce the *inflation* method discussed above. In §4, we provide some refinements of these ideas that are needed later in the paper; this section includes some results on Whittaker models for the finite-dimensional group G that are of independent interest.

In §5, we prove our theorem on the existence of Whittaker models for most categorical representations of $PGL_2(K)$ and introduce the *descent* argument discussed above.

1.26. The second part of the paper applies the above material to critical level Kac-Moody representations.

In §6, we introduce the DG category $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg}$ following [FG4]. To study this DG category using group actions, we import the main results from [Ras6] here.

In §7, we recall in detail the key constructions from the formulation of Conjecture 1.11.1. We formulate three lemmas from which we deduce our main result, Theorem 7.14.1.

The proofs of these lemmas occupy §9-11. Roughly, §9 is devoted to showing that the functor Γ^{Hecke} is essentially surjective, while §10 is devoted to showing that it is t-exact. The final section, §11, provides additional technical support related to the distinction between $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg}$ and $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg,naive}$.

Finally, §8 collects results on the behavior of Γ^{Hecke} on Iwahori and Whittaker equivariant categories; the former results are due to Frenkel-Gaitsgory [FG6], while the latter are original.

1.27. There are two appendices.

In Appendix A, we compare our construction of the global sections functor to the more classical one used by Kashiwara-Tanisaki, Beilinson-Drinfeld and Frenkel-Gaitsgory.

In Appendix B, we reprove the Frenkel-Gaitsgory theorem that Γ^{Hecke} is fully faithful.

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2. Preliminary material

2.1. In this section, we collect some notation and constructions that will be used throughout the paper.

2.2. As in §1.2, we always work over a field k of characteristic 0.

2.3. **Reductive groups.** Throughout the paper, G denotes a split reductive group, B denotes a fixed Borel with unipotent radical N and Cartan $T = B/N$.

We let $\Lambda = \mathrm{Hom}(T, \mathbb{G}_m)$ be the lattice of weights of T and $\mathrm{Hom}(\mathbb{G}_m, T)$ the lattice of coweights. Let $\rho \in \Lambda \otimes \mathbb{Q}$ be the half sum of positive roots and $\check{\rho} \in \check{\Lambda} \otimes \mathbb{Q}$ be the half sum of positive coroots. We denote the pairing between Λ and $\check{\Lambda}$ by $(-,-)$.

We let e.g. $\mathfrak{g}$ denote the Lie algebra of G, $\mathfrak{b}$ the Lie algebra of B, and so on.

We let $\check{G}$ denote the Langlands dual group to G, considered as an algebraic group over k. It naturally comes equipped with a choice Borel $\check{B}$ with radical $\check{N}$ and Cartan $\check{T} = \check{B}/\check{N}$.

2.4. **Higher categories.** Following standard conventions in the area, we freely use Lurie's theory [Lur1] [Lur2] of higher category theory. To simplify the terminology, we use *category* to mean $(\infty, 1)$-*category*.

2.5. **DG categories.** We let DGCat_{cont} denote the symmetric monoidal category of presentable (in particular, cocomplete) DG categories, referring to [GR3] Chapter I for more details. As in *loc. cit.*, the binary product underlying this symmetric monoidal structure is denoted $\otimes$. We recall that $\mathsf{Vect} \in \mathsf{DGCat}_{cont}$ is the unit for this tensor product.

2.6. For $\mathcal{A} \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ an algebra in this symmetric monoidal category, we typically write $\mathcal{A}$–mod for $\mathcal{A}$–$\mathsf{mod}(\mathsf{DGCat}_{cont})$, i.e., the category of modules for $\mathcal{A}$ in DGCat_{cont}.

2.7. For $\mathcal{C}$ a DG category and $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, we use the notation $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ to denote the corresponding object of Vect, as distinguished from the corresponding ∞-groupoid $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = \Omega^\infty \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$.

2.8. For $\mathcal{C}, \mathcal{D}$ objects of a 2-category (i.e., (meaning: $(\infty, 2)$-category) C, we use the notation $\mathsf{Hom}_{\mathsf{C}}(\mathcal{C}, \mathcal{D}) \in \mathsf{Cat}$ to denote the corresponding category of maps.

When C is enriched over DGCat_{cont}, we use the same notation for the DG category of maps. E.g., this applies for $\mathsf{C} = \mathsf{DGCat}_{cont}$ or $\mathsf{C} = \mathcal{A}$–mod for $\mathcal{A}$ as above.

2.9. We use the notation $(-)^\vee$ to denote duals of dualizable objects in symmetric monoidal categories. In particular, for $\mathcal{C} \in \mathsf{DGCat}_{cont}$ dualizable in the sense of [GR3], we let $\mathcal{C}^\vee \in \mathsf{DGCat}_{cont}$ denote the corresponding dual category.

2.10. For a DG category $\mathcal{C}$ with t-structure, we use cohomological notation: $\mathcal{C}^{\leqslant 0}$ denotes the connective objects and $\mathcal{C}^{\geqslant 0}$ denotes the coconnective objects. We let $\mathcal{C}^\heartsuit = \mathcal{C}^{\leqslant 0} \cap \mathcal{C}^{\geqslant 0}$ denote the heart of the t-structure.

2.11. **Classical objects.** Where we wish to say that an object lives in some traditional $(1,1)$-category, we often refer to it as *classical*. So e.g., a *classical vector space* refers to an object of $\mathsf{Vect}^\heartsuit$, while a *classical (ind)scheme* is being distinguished from a DG (ind)scheme.

2.12. D**-modules.** For an indscheme S of ind-finite type, we let $D(S)$ denote the DG category of D-modules on S as defined in [GR3]. For a map $f : S \to T$, we let $f^!$ and $f_{*,dR}$ denote the corresponding D-module functors.

We recall that for S an indscheme of possibly infinite type, there are two categories of D-modules, denoted $D^*(S)$ and $D^!(S)$. We refer to [Ras2] for the definitions in this setting.

2.13. **Group actions on categories.** We briefly recall some constructions from the theory of group actions on categories.

2.14. Suppose H is a Tate group indscheme in the sense of [Ras6] §7, i.e., H is a group indscheme that admits a group subscheme $K \subseteq H$ such that H/K is an indscheme of ind-finite type.

We recall from [Ras2] the category $D^*(H)$ is canonically monoidal. By definition, we let H–mod denote the category $D^*(H)$–mod and refer to objects of this category as categories with a *strong H-action*. We typically omit the adjective *strong*; where we refer only to an H-action, we mean a strong H-action.

For $\mathcal{C} \in H$–mod and $\mathcal{F} \in D^*(H)$, we let $\mathcal{F} \star - : \mathcal{C} \to \mathcal{C}$ denote the (*convolution*) functor defined by the action.

2.15. For $\mathcal{C} \in H$–mod, we have the *invariants category* and *coinvariants* categories:

$$\mathcal{C}^H := \underline{\mathrm{Hom}}_{H\text{–}\mathsf{mod}}(\mathsf{Vect}, \mathcal{C}), \qquad \mathcal{C}_H := \mathsf{Vect} \underset{D^*(H)}{\otimes} \mathcal{C}.$$

Here Vect is given the *trivial* H-action.

We let $\mathrm{Oblv} : \mathcal{C}^H \to \mathcal{C}$ denote the forgetful functor. Recall from [Ber] §2 and §4 that if H is a group scheme with prounipotent tail, then $\mathrm{Oblv} : \mathcal{C}^H \to \mathcal{C}$ admits a continuous right adjoint $\mathrm{Av}_* = \mathrm{Av}_*^H$ that is functorial in $\mathcal{C}$. The composition $\mathrm{Oblv}\,\mathrm{Av}_* : \mathcal{C} \to \mathcal{C}$ is given by convolution with the constant D-module $k_H \in D^*(H)$.

More generally, as in [Ber] §2.5.4, for any character $\psi : H \to \mathbb{G}_a$, we may form the twisted invariants and coinvariants categories:

$$\mathcal{C}^{H,\psi}, \mathcal{C}_{H,\psi}.$$

We use similar notation to the above, though (for H a group scheme) we often write $\mathrm{Av}_*^{H,\psi} = \mathrm{Av}_*^{\psi}$ to emphasize the character.

2.16. For $\mathcal{C}$ with a right H-action and $\mathcal{D}$ with a left H-action, we let $\mathcal{C} \overset{H}{\otimes} \mathcal{D}$ denote the H-invariants for the induced diagonal action on $\mathcal{C} \otimes \mathcal{D}$.

2.17. Given a central extension $\widehat{H}$ of H by a torus T and an element $\lambda \in \mathfrak{t}^\vee$, we have a category H–mod_λ of categories acted on by H with *level* λ, and such that for $\lambda = 0$ we have H–$\mathsf{mod}_0 = H$–mod. We refer to [Ras6] §11.3 and [Ras5] §1.30 for definitions.

For $H = G(K)$ the loop group, ad-invariant symmetric bilinear forms $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to k$ define the above data, c.f. *loc. cit.* In particular, we obtain $G(K)$–mod_κ for any κ.

In the presence of a level, we can form invariants and coinvariants for group indschemes H' equipped with a map $H' \to H$ and a trivialization[8] of the corresponding central extension of H'.

For instance, for $H = G(K)$, this applies to $N(K)$ and $G(O)$, or any subgroup of either. Indeed, the Kac-Moody extension is canonically trivialized over each of these subgroups.

Where the level is obviously implied, we sometimes allow ourselves simply to refer to *H-actions*, *H-equivariant functors*, and so on.

2.18. We recall from [Ras6] that for H as above, there is a canonical category $\mathfrak{h}$–mod of modules for the Lie algebra $\mathfrak{h}$ of H and a canonical action of H on $\mathfrak{h}$–mod. We remind that if H is not of finite type, the forgetful functor $\mathfrak{h}$–$\mathsf{mod} \to \mathsf{Vect}$ is not conservative.

One has similar reasoning in the presence of a level. For instance, we have a canonical object $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod} \in G(K)$–$\mathsf{mod}_\kappa$. We refer to [Ras6] §11 for further discussion.

[8]One can do better: the important thing is to have a specified action of H' on Vect with the given level.

2.19. We will sometimes reference the theory of weak actions of Tate group indschemes. We let H–mod_{weak} denote the category of DG categories with weak H-actions, defined as in [Ras6] §7. We use the notation $\mathcal{C} \mapsto \mathcal{C}^{H,w}, \mathcal{C}_{H,w}$ to denote weak invariants and coinvariants functors.

2.20. We will frequently reference compatibilities between t-structures and group actions. We refer to [Ras5] Appendix B and [Ras6] §10 for definitions and basic results.

2.21. Finally, we end with informal remarks.

The theory of loop group actions on DG categories, especially weak actions, is somewhat involved to set up, c.f. [Ras6]. With that said, as a black box, the theory is fairly intuitive to use and provides quite useful insights.

Therefore, we hope that the sometimes frequent references to [Ras6] and the more formal parts (e.g., Appendix B) of [Ras5] will not cause the reader too much indigestion.

3. Whittaker inflation

3.1. The main result of this section is Theorem 3.4.1, which is one of the key innovations of this paper. For higher jet groups G_n (see below) of a reductive group G, this result precisely measures how much information is lost by the corresponding analogues of the Whittaker model.

The proof uses some constructions with Heisenberg group actions on categories, which we recall here. This material is a categorical version of the usual representation theory of Heisenberg groups over finite fields. Similar ideas were used in [Gai3], though the application was of different nature there.

3.2. For H an algebraic group and $n \geqslant 1$, we let H_n denote the algebraic group of maps from $\mathrm{Spec}(k[[t]]/t^n)$ to H. In particular, $H_1 = H$.

Let $\{e_i \in \mathfrak{n}\}_{i \in \mathcal{I}_G}$ be Chevalley generators of $\mathfrak{n}$ indexed by $\mathcal{I}_G$ the set of simple roots. Let $\psi : N_n \to \mathbb{G}_a$ be defined as the composition:

$$N_n \to N_n/[N,N]_n = \prod_{i \in \mathcal{I}_G} (\mathbb{G}_a)_n \cdot e_i \xrightarrow{\mathrm{sum}} (\mathbb{G}_a)_n = (\mathbb{G}_a) \underset{k}{\otimes} k[[t]]/t^n \to \mathbb{G}_a$$

where the last map is induced by the functional:

$$\begin{aligned} k[[t]]/t^n &\to k \\ \sum a_i t^i &\mapsto a_{n-1}. \end{aligned}$$

For the remainder of this section, we assume that n is at least 2. The main result of this section answers the question: for $\mathcal{C} \in G_n$–mod, how much information do the invariants $\mathcal{C}^{N_n,\psi}$ remember about $\mathcal{C}$?

3.3. As $n \geqslant 2$, we have a homomorphism:

$$\begin{aligned} \mathfrak{g} \otimes \mathbb{G}_a &\to G_n \\ (\xi \in \mathfrak{g}) &\mapsto \exp(t^{n-1}\xi). \end{aligned} \tag{3.3.1}$$

This map realizes $\mathfrak{g} \otimes \mathbb{G}_a$ as a normal subgroup of G_n. Note that the adjoint action of G_n on this normal subgroup is given by:

$$G_n \xrightarrow{\mathrm{ev}} G \overset{\mathrm{adjoint}}{\curvearrowright} \mathfrak{g}.$$

If $\mathcal{C}$ is acted on by G_n, it is thus acted on by $\mathfrak{g} \otimes \mathbb{G}_a$ by restriction, or using Fourier transform, by $D(\mathfrak{g}^\vee)$ equipped with the $\overset{!}{\otimes}$-tensor product. (We omit the tensoring with $\mathbb{G}_a$ because we are not concerned with the additive structure on $\mathfrak{g}^\vee$ here.)

Fix a symmetric, linear G-equivariant identification $\kappa : \mathfrak{g} \simeq \mathfrak{g}^\vee$ for the remainder of this section. Therefore, $\mathcal{C}$ is acted on by $D(\mathfrak{g})$ with its $\overset{!}{\otimes}$-monoidal structure. In particular, for S a scheme mapping to $\mathfrak{g}$, we may form $\mathcal{C}|_S := \mathcal{C} \otimes_{D(\mathfrak{g})} D(S)$.

Define $\mathcal{C}_{reg}$ as $\mathcal{C}|_{\mathfrak{g}_{reg}}$ where $\mathfrak{g}_{reg} \subseteq \mathfrak{g}$ is the subset of regular elements. We have adjoint functors:

$$j^! : \mathcal{C} \rightleftarrows \mathcal{C}_{reg} : j_{*,dR}$$

with the right adjoint $j_{*,dR}$ being fully faithful: indeed, these properties are inherited from the corresponding situation $j^! : D(\mathfrak{g}) \rightleftarrows D(\mathfrak{g}_{reg}) : j_{*,dR}$ for $j : \mathfrak{g}_{reg} \hookrightarrow \mathfrak{g}$ the embedding.

Because $\mathfrak{g}_{reg} \subseteq \mathfrak{g}$ is closed under the adjoint action of G, and since G_n acts on $\mathfrak{g}^\vee \simeq \mathfrak{g}$ through the adjoint action of G, it follows that $\mathcal{C}_{reg}$ is acted on by G_n so that the comparison functors with $\mathcal{C}$ are G_n-equivariant.

3.4. **Main theorem.** We have $\mathfrak{g} \otimes \mathbb{G}_a \cap N_n = \mathfrak{n} \otimes \mathbb{G}_a$, and under the Fourier transform picture above, we have:

$$\mathcal{C}^{\mathfrak{n} \otimes \mathbb{G}_a, \psi|_{\mathfrak{n} \otimes \mathbb{G}_a}} \simeq \mathcal{C}|_{f+\mathfrak{b}}.$$

Here f a principal nilpotent whose image in $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{n}^\vee$ is $\psi|_{\mathfrak{n} \otimes \mathbb{G}_a}$.

In particular, because $f + \mathfrak{b} \subseteq \mathfrak{g}_{reg}$, it follows that $\mathcal{C}^{N_n,\psi} \simeq \mathcal{C}_{reg}^{N_n,\psi}$. The following result states that this is the only loss in (N_n, ψ)-invariants.

Theorem 3.4.1. *The functor:*

$$G_n\text{–}\mathsf{mod}_{reg} \xrightarrow{\mathcal{C} \mapsto \mathcal{C}^{N_n,\psi}} \mathsf{DGCat}_{cont}$$

is conservative, where G_n–$\mathsf{mod}_{reg} \subseteq G$–$\mathsf{mod}$ is the full subcategory consisting of $\mathcal{C}$ with $\mathcal{C}_{reg} = \mathcal{C}$.

Here are some consequences.

Corollary 3.4.2. *For every $\mathcal{C} \in G_n$–mod, the convolution functor:*

$$D(G_n)^{N_n,-\psi} \underset{\mathcal{H}_{N_n,\psi}}{\otimes} \mathcal{C}^{N_n,\psi} \to \mathcal{C}$$

is fully faithful with essential image $\mathcal{C}_{reg}$. Here $\mathcal{H}_{N_n,\psi} = D(G_n)^{N_n \times N_n, (\psi, -\psi)}$ is the appropriate Hecke category for the pair $(G_n, (N_n, \psi))$.

Proof. Note that this functor is G_n-equivariant and that its essential image factors through G_n (by the above analysis). Therefore, by Theorem 3.4.1, it suffices to show that it is an equivalence on (N_n, ψ)-invariants, which is clear.

□

Corollary 3.4.3. *Observe that $D(G_n)_{reg}$ admits a unique monoidal structure such that the localization functor $D(G_n) \to D(G_n)_{reg}$ is monoidal.*

Then $D(G_n)_{reg}$ and $\mathcal{H}_{N_n,\psi}$ (as defined in the previous corollary) are Morita equivalent, with bimodule $D(G_n)^{N_n,\psi}$ defining this equivalence.

The remainder of this section is devoted to the proof of Theorem 3.4.1.

3.5. **Example: $n = 2$ case.** First, we prove Theorem 3.4.1 in the $n = 2$ case. This case is simpler than the general case, and contains one of the main ideas in the proof of the general case.

Note that by Fourier transform along $\mathfrak{g} \otimes \mathbb{G}_a \subseteq G_2$, an action of G_2 on $\mathcal{C}$ is equivalent to the datum of G on $\mathcal{C}$, and an action of $(D(\mathfrak{g}), \overset{!}{\otimes})$ on $\mathcal{C}$ as an object of G–mod (where G acts on $D(\mathfrak{g})$ by the adjoint action). In the sheaf of categories language [Gai6], we obtain:

$$G_2\text{–}\mathsf{mod} \simeq \mathsf{ShvCat}_{/(\mathfrak{g}/G)_{dR}}.$$

The functor of (N_2, ψ)-invariants then corresponds to global sections of the sheaf of categories over $(f + \mathfrak{b}/N)_{dR}$, i.e., the de Rham space of the Kostant slice. Recall that the Kostant slice $f + \mathfrak{b}/N$ is an affine scheme and maps smoothly to $\mathfrak{g}/G$ with image $\mathfrak{g}_{reg}/G$.

As the Kostant slice is a scheme (not a stack), [Gai6] Theorem 2.6.3 implies that $(f + \mathfrak{b}/N)_{dR}$ is 1-affine. In particular, its global sections functor is conservative.

Therefore, it suffices to note that pullback of sheaves of categories along the map $(f + \mathfrak{b}/N)_{dR} \to (\mathfrak{g}_{reg}/G)_{dR}$ is conservative. However, in the diagram:

$$\begin{array}{ccc} f + \mathfrak{b}/N & \longrightarrow & \mathfrak{g}_{reg}/G \\ \downarrow & & \downarrow \\ (f + \mathfrak{b}/N)_{dR} & \longrightarrow & (\mathfrak{g}_{reg}/G)_{dR} \end{array}$$

pullback for sheaves of categories along the vertical maps is conservative for formal reasons (e.g., write de Rham as the quotient by the infinitesimal groupoid), and conservativeness of pullback along the upper arrow follows from descent of sheaves of categories along smooth (or more generally fppf) covers, c.f. [Gai6] Theorem 1.5.2. This implies that pullback along the bottom arrow is conservative as well.

Remark 3.5.1. It follows from the above analysis that the Hecke algebra $\mathcal{H}_2$ (in the notation of Corollary 3.4.2) is equivalent to D-modules on the group scheme of regular centralizers.

3.6. **Heisenberg groups.** We will deduce the general case of Theorem 3.4.1 from the representation theory of Heisenberg groups, which we digress to discuss now.

Let V be a finite-dimensional vector space. In the following discussion, we do not distinguish between V and the additive group scheme $V \otimes_k \mathbb{G}_a$.

Let $H = H(V)$ denote the corresponding *Heisenberg group*; by definition, H is the semidirect product:

$$V \ltimes (V^\vee \times \mathbb{G}_a)$$

where V acts on $V^\vee \times \mathbb{G}_a$ via:

$$v \cdot (\lambda, c) = (\lambda, c + \lambda(v)), \quad (v, \lambda, c) \in V \times V^\vee \times \mathbb{G}_a.$$

Remark 3.6.1. Note that H only depends on the symplectic vector space $W = V \times V^\vee$, not on the choice of polarization $V \subseteq W$. But the above presentation is convenient for our purposes.

3.7. Observe that $\mathbb{G}_a \subseteq H$ is central. In particular, $D(\mathbb{A}^1) \overset{\text{Fourier}}{\simeq} D(\mathbb{G}_a)$ maps centrally to H, where we use $D(\mathbb{A}^1)$ to indicate that we consider the $\overset{!}{\otimes}$-monoidal structure and $D(\mathbb{G}_a)$ to indicate the convolution monoidal structure.

Let H–$\mathsf{mod}_{reg} \subseteq H$–$\mathsf{mod}$ denote the subcategory where $D(\mathbb{A}^1)$ acts through its localization $D(\mathbb{A}^1 \backslash 0)$, i.e., where all Fourier coefficients are non-zero.

Theorem 3.7.1. *The functor:*

$$H\text{–}\mathsf{mod}_{reg} \xrightarrow{\mathcal{C}\mapsto\mathcal{C}^V} D(\mathbb{A}^1\backslash 0)\text{–}\mathsf{mod}$$

is an equivalence.

Corollary 3.7.2. *The functor of V-invariants is conservative on H–mod_{reg}.*

Proof of Theorem 3.7.1. Note that by duality, V acts on $V \times \mathbb{A}^1$; explicitly, this is given by the formula:

$$v \cdot (w, c) := (w - c \cdot v, c)$$

By Fourier transform along $V^\vee \times \mathbb{G}_a \subseteq H$, we see that an H-action on $\mathcal{C}$ is equivalent to giving a V-action on $\mathcal{C}$ (where V is given its natural additive structure), and an additional $(D(V \times \mathbb{A}^1), \overset{!}{\otimes})$-action on $\mathcal{C}$ in the category V–mod.

Using the sheaf of categories language [Gai6], this is equivalent to the data of a sheaf of categories on $(V_{dR} \times \mathbb{A}^1_{dR})/V_{dR}$, where we are quotienting using the above action. The corresponding object of H–mod lies in H–mod_{reg} if and only if the sheaf of categories is pushed forward from:

$$(V_{dR} \times \mathbb{A}^1_{dR}\backslash 0)/V_{dR} = \mathbb{A}^1_{dR}\backslash 0.$$

Therefore, we obtain an equivalence of the above type. Geometrically, this equivalence is given by taking global sections of a sheaf of categories, which for $(V_{dR} \times \mathbb{A}^1_{dR})/V_{dR}$ corresponds to taking (strong) V-invariants for the corresponding H-module category.

□

3.8. **Proof of Theorem 3.4.1.** We now return to the setting of Theorem 3.4.1. The remainder of this section is devoted to the proof of this result.

In what follows, for $\mathfrak{h}$ a nilpotent Lie algebra, we let $\exp(\mathfrak{h})$ denote the corresponding unipotent algebraic group.

Let $N_n^m = \exp(t^{n-m}\mathfrak{n}[[t]]/t^n\mathfrak{n}[[t]]) \subseteq N_n$ for $1 \leqslant m \leqslant n$. For example, for $m = 1$ we recover the group $\mathfrak{n} \otimes \mathbb{G}_a \subseteq G_n$.

We will show by induction on m that the functor of (N_n^m, ψ)-invariants is conservative on G_n–mod_{reg}.

3.9. As a base case, we first show the claim for $m = 1$.

Here the assertion follows by the argument of §3.5. Indeed, we have a homomorphism $G_2 \to G_n$ which identifies $G \subseteq G_2, G_n$ and $\mathfrak{g} \otimes \mathbb{G}_a \subseteq G_2, G_n$. Restricting along this homomorphism, we obtain that ψ-invariants for $N \cdot N_n^1 \subseteq N_n$ is conservative, and a fortiori, (N_n^1, ψ)-invariants is as well.

3.10. We[9] now observe that the above argument extends to treat any $m \leqslant \frac{n}{2}$.

In this case, the subalgebra $t^{n-m}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] \subseteq \mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] = \mathrm{Lie}(G_n)$ is abelian. Clearly this subalgebra is normal; the adjoint action of G_n on it is given via the representation:

$$G_n \to G_m \curvearrowright \mathrm{Lie}(G_m) = \mathfrak{g}[[t]]/t^m\mathfrak{g}[[t]] \overset{t^m \cdot -}{\simeq} t^{n-m}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]].$$

[9]The arguments in §3.10 and §3.14 are not needed in the case $\mathfrak{g} = \mathfrak{sl}_2$, which is what we use for our application to the localization theorem. Indeed, for $\mathfrak{g} = \mathfrak{sl}_2$, in the argument in §3.13, one only needs to consider (in the notation of *loc. cit.*) $r = 1$, in which case $\mathfrak{g}_{1-r} = \mathfrak{t}$ is abelian, hence the last equation in (3.13.1) holds for trivial reasons. Given that equality, the rest of the argument goes through for $m \geqslant 2$.

In other words, the reader who is only interested in Theorem 7.14.1 can safely skip §3.10 and §3.14.

We therefore have a homomorphism:

$$G_n \ltimes t^{n-m}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] \otimes \mathbb{G}_a \to G_n$$

whose restriction to G_n is the identity and whose restriction to $t^{n-m}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] \otimes \mathbb{G}_a$ is the exponential of the embedding $t^{n-m}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] \hookrightarrow \mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]]$.

Considering $\mathcal{C}$ as a category acted on by $G_n \ltimes t^{n-m}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]]$ via the above map and Fourier transforming as in Example 3.5, we can view this action as the data of making $\mathcal{C}$ into a sheaf of categories on $(\mathfrak{g}[[t]]/t^m\mathfrak{g}[[t]])_{dR}/G_{n,dR}$. Here we have identified the dual of $(t^{n-m}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]])$ with $\mathfrak{g}[[t]]/t^m\mathfrak{g}[[t]]$ via the pairing $(\xi_1, \xi_2) \mapsto \mathrm{Res}(t^{-n}\kappa(\xi_1, \xi_2)dt)$ (for κ as above).

Define:

$$(\mathfrak{g}[[t]]/t^m\mathfrak{g}[[t]])_{reg} = \mathfrak{g}[[t]]/t^m\mathfrak{g}[[t]] \times_{\mathfrak{g}} \mathfrak{g}_{reg}.$$

By the regularity assumption on $\mathcal{C}$, the above sheaf of categories is pushed-forward from:

$$(\mathfrak{g}[[t]]/t^m\mathfrak{g}[[t]])_{reg,dR}/G_{n,dR}.$$

Then (N_n^m, ψ)-invariants correspond to global sections of $(f + \mathfrak{b}[[t]]/t^m\mathfrak{b}[[t]])_{dR}$ with coefficients in the above sheaf of categories. As the map:

$$(f + \mathfrak{b}[[t]]/t^m\mathfrak{b}[[t]]) \to (\mathfrak{g}[[t]]/t^m\mathfrak{g}[[t]])_{reg}/G_m$$

is a smooth cover (as it is obtained by applying jets to a smooth cover), the same is true of:

$$(f + \mathfrak{b}[[t]]/t^m\mathfrak{b}[[t]]) \to (\mathfrak{g}[[t]]/t^m\mathfrak{g}[[t]])_{reg}/G_n.$$

As $f + \mathfrak{b}[[t]]/t^m\mathfrak{b}[[t]]$ is a scheme, the reasoning of §3.5 gives us the desired result.

3.11. In §3.14, we will give a separate argument to treat the case $n = 2m - 1$; of course, this is only possible for n odd. The argument is not complicated, but a little involved to set up, so we postpone the argument for the moment.

Combined with §3.10, this gives the result for all $m \leqslant \frac{n+1}{2}$.

3.12. We now perform the induction; we assume the conservativeness for $m - 1$ and show it for our given $m \leqslant n$. By the inductive hypothesis as established above (though postponed in one case to §3.14), we may assume $m \geqslant \frac{n+2}{2}$.

We will give the argument here by another inductive argument. As above, let $\mathfrak{g} = \oplus_s \mathfrak{g}_s$ be the principal grading defined by the coweight $\check{\rho} : \mathbb{G}_m \to G^{ad}$ of the adjoint group G^{ad} of G. So for example, $e_i \in \mathfrak{g}_1$ and $\mathfrak{n} = \oplus_{s \geqslant 1}\mathfrak{g}_s$. For $r \geqslant 1$, let $\mathfrak{n}_{\geqslant r} := \oplus_{s \geqslant r}\mathfrak{g}_s$.

Now define:

$$N_n^{m,r} := \exp\left(t^{n-m+1}\mathfrak{n}[[t]] + t^{n-m}\mathfrak{n}_{\geqslant r}[[t]]/t^n\mathfrak{n}[[t]]\right) \subseteq N_n^m \subseteq N_n.$$

We will show by descending induction on $r \geqslant 1$ that $(N_n^{m,r}, \psi)$-invariants is conservative. Note that this result is clear from our hypothesis on m for $r \gg 0$, since then $\mathfrak{n}_{\geqslant r} = 0$ and $N_n^{m,r} = N_n^{m-1}$. Moreover, a proof for all r implies the next step in the induction with respect to m, since $N_n^{m,1} = N_n^m$, so would complete the proof of Theorem 3.4.1.

3.13. For $r \geqslant 1$, assume the conservativeness (in the regular setting) of $(N_n^{m,r+1}, \psi)$-invariants; we will deduce it for $N_n^{m,r}$. The idea is to make a Heisenberg group act on $(N_n^{m,r+1}, \psi)$-invariants so that invariants with respect to a Lagrangian gives $(N_n^{m,r}, \psi)$-invariants.

Step 1. Define $\mathfrak{h}_0 \subseteq \mathrm{Lie}(G_n) = \mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]]$ as:

$$t^{m-1}\mathfrak{g}_{1-r} \oplus \mathrm{Lie}(N_n^{m,r}).$$

Observe that $\mathfrak{h}_0$ is a Lie subalgebra. Indeed:

$$\begin{aligned} [t^{m-1}\mathfrak{g}_{1-r}, t^{n-m+1}\mathfrak{g}[[t]]] &\subseteq t^n\mathfrak{g}[[t]], \\ [t^{m-1}\mathfrak{g}_{1-r}, t^{n-m}\mathfrak{n}_{\geqslant r}] &\subseteq t^{n-1}\mathfrak{n}, \text{ and} \\ [t^{m-1}\mathfrak{g}_{1-r}, t^{m-1}\mathfrak{g}_{1-r}] &\subseteq t^{2m-2}\mathfrak{g}[[t]] \subseteq t^n\mathfrak{g}[[t]] \end{aligned} \tag{3.13.1}$$

where the last embedding uses the assumption $m \geqslant \frac{n+2}{2}$.

In the same way, we see that[10] $\mathrm{Lie}(N_n^{m,r+1})$ is a normal Lie subalgebra of $\mathfrak{h}_0$, and that for $\xi \in \mathfrak{h}_0$ and $\varphi \in \mathrm{Lie}(N_n^{m,r+1})$, $\psi([\xi, \varphi]) = 0$.

Moreover, $\mathfrak{h}_0$ is nilpotent, so exponentiates to a group $H_0 \subseteq G_n$.[11] Combining this with the above, we see that H_0 acts on $(N_n^{m,r+1}, \psi)$-invariants for any category with an action of G_n.

Step 2. Let $\mathfrak{g}'_{1-r} \subseteq \mathfrak{g}_{1-r}$ denote $\mathrm{Ad}_f^{2r-1}(\mathfrak{g}_r)$. Observe that the pairing:

$$\psi([-,-]) : \mathfrak{g}_r \otimes \mathfrak{g}_{1-r} \to k \tag{3.13.2}$$

induces a perfect pairing between $\mathfrak{g}_r$ and $\mathfrak{g}'_{1-r}$. Indeed, the diagram:

$$\begin{array}{ccccc} \mathfrak{g}_r \otimes \mathfrak{g}_r & \xrightarrow{\mathrm{id}\otimes \mathrm{Ad}_f^{2r-1}} & \mathfrak{g}_r \otimes \mathfrak{g}_{1-r} & \xrightarrow{\mathrm{id}\otimes \mathrm{Ad}_f} & \mathfrak{g}_r \otimes \mathfrak{g}_{-r} \\ & & & \searrow^{\psi([-,-])} & \downarrow{\scriptstyle -\kappa(-,-)} \\ & & & & k \end{array}$$

commutes[12], and $\mathrm{Ad}_f^{2r} : \mathfrak{g}_r \to \mathfrak{g}_{-r}$ is an isomorphism by $\mathfrak{sl}_2$-representation theory.

Define $\mathfrak{h}'_0 \subseteq \mathfrak{h}_0$ as:

$$t^{m-1}\mathfrak{g}'_{1-r} \oplus \mathrm{Lie}(N_n^{m,r}).$$

Again, $\mathfrak{h}'_0$ integrates to a group H'_0.

Step 3. Finally, recall that the adjoint action of H_0 fixes $N_n^{m,r+1} \subseteq H_0$ and preserves its character ψ to $\mathbb{G}_a$. Let $K \subseteq N_n^{m,r+1}$ be the kernel of ψ: clearly K is normal in H_0.

One immediately observes that $H := H'_0/K$ is a Heisenberg group. The central $\mathbb{G}_a$ is induced by the map:

$$\mathbb{G}_a = N_n^{m,r+1}/K \to H'_0/K = H.$$

The vector space defining the Heisenberg group is $t^{n-m}\mathfrak{g}_r$, and its dual is embedded as $t^{m-1}\mathfrak{g}'_{1-r} = H'_0/K$.

[10] The same is true for r instead of $r+1$, but the statement with the character is not.

[11] The embedding exponentiates because $\mathfrak{h}_0 \subseteq \mathfrak{n} + t\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]]$, i.e., the Lie algebra of a unipotent subgroup of G_n. Here we use that $m \geqslant 2$.

[12] Proof: write $\psi(-)$ as $\kappa(f, -)$ and use Ad-invariance of κ.

Now observe that our Heisenberg group H acts on $\mathcal{C}^{N_n^{m,r+1},\psi}$ for any $\mathcal{C}$ acted on by G_n, with its central $\mathbb{G}_a$ acting through the exponential character. Now the result follows from Corollary 3.7.2.

3.14. As above, it remains to show the result in the special case that $n = 2m-1$ for some $m \geqslant 2$. We do so below.

Step 1. We need some auxiliary constructions.

Let $\xi \in \mathfrak{g}_{reg}$ be a k-point (i.e., a regular element of $\mathfrak{g}$ in the usual sense). Let $\mathfrak{z}_\xi \subseteq \mathfrak{g}$ denote the centralizer of ξ.

Then $\mathfrak{g}/\mathfrak{z}_\xi$ carries an alternating form:

$$(\varphi_1, \varphi_2)_\xi := \kappa(\xi, [\varphi_1, \varphi_2]) = \kappa([\xi, \varphi_1], \varphi_2).$$

The second equality holds as κ is G-invariant, and shows that $(-,-)_\xi$ descends to $\mathfrak{g}/\mathfrak{z}_\xi$. Moreover, as κ is non-degenerate, we see from the last expression that $(-,-)_\xi$ is non-degenerate on $\mathfrak{g}/\mathfrak{z}_\xi$, hence symplectic.

Step 2. In the above setting, suppose that ξ lies in the Kostant slice $f + \mathfrak{b}$.

In this case, we claim that the composition $\mathfrak{n} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{z}_\xi$ is injective, and that $\mathfrak{n} \subseteq \mathfrak{g}/\mathfrak{z}_\xi$ is Lagrangian with respect to the symplectic form $(-,-)_\xi$.

Indeed, it is standard that $\mathfrak{z}_\xi \cap \mathfrak{n} = 0$ (this is the infinitesimal version of the freeness of the action of N on $f + \mathfrak{b}$), giving the injectivity.

We now claim that $\mathfrak{n}$ is isotropic for the above form. For $\varphi_1, \varphi_2 \in \mathfrak{n}$, we have:

$$(\varphi_1, \varphi_2)_\xi = \kappa(\xi, [\varphi_1, \varphi_2])$$

by definition; we claim this inner product is zero. Let $\mathfrak{g} = \oplus_s \mathfrak{g}_s$ be the *principal grading* of $\mathfrak{g}$, i.e., the grading defined by the coweight $\check{\rho} : \mathbb{G}_m \to G^{ad}$. Then $[\varphi_1, \varphi_2] \in [\mathfrak{n}, \mathfrak{n}] = \oplus_{s \geqslant 2} \mathfrak{g}_s$, while $\xi \in f + \mathfrak{b} \subseteq \oplus_{s \geqslant -1} \mathfrak{g}_s$. By invariance of κ, for $\widetilde{\xi} \in \mathfrak{g}_s, \widetilde{\varphi} \in \mathfrak{g}_r$, we have $\kappa(\widetilde{\xi}, \widetilde{\varphi}) = 0$ unless $r + s = 0$, giving the claim.

Finally, $2\dim(\mathfrak{n}) + \dim(\mathfrak{z}_\xi) = \dim(\mathfrak{n}) + \dim(\mathfrak{n}^-) + \dim(\mathfrak{t}) = \dim(\mathfrak{g})$, so $\mathfrak{n} \subseteq \mathfrak{g}/\mathfrak{z}_\xi$ is in fact Lagrangian.

Step 3. Next, we observe that the above generalizes to the scheme-theoretic situation in which we allow ξ to vary.

More precisely, let $\widetilde{\mathcal{W}} = \mathfrak{g} \otimes \mathcal{O}_{\mathfrak{g}_{reg}}$ be the constant vector bundle on $\mathfrak{g}_{reg}$ with fiber $\mathfrak{g}$. This bundle carries a subbundle $\mathfrak{z} \subseteq \widetilde{W}$ of regular centralizers; e.g., the fiber of $\mathfrak{z}$ at $\xi \in \mathfrak{g}(k)$ is $\mathfrak{z}_\xi$.

The quotient:

$$\mathcal{W} := \widetilde{\mathcal{W}}/\mathfrak{z}$$

is a vector bundle on $\mathfrak{g}_{reg}$. Our earlier construction defines a symplectic form on $\mathcal{W}$. Moreover, after pulling back along the embedding $i : f + \mathfrak{b} \hookrightarrow \mathfrak{g}_{reg}$, the constant bundle with fiber $\mathfrak{n}$ defines a Lagrangian subbundle of the vector bundle $i^*(\mathcal{W})$.

Step 4. We now record some general results in the above setting.

Let S be a scheme of finite type and let $\mathcal{W}$ be a symplectic vector bundle on S. We denote the total space of $\mathcal{W}$ by the same notation.

Define the *Heisenberg group scheme* $\mathcal{H} = \mathcal{H}(\mathcal{W})$ over S as the extension:

$$0 \to \mathbb{G}_{a,S} \to \mathcal{H} \to \mathcal{W} \to 0$$

where $\mathcal{H} = \mathcal{W} \times_S \mathbb{G}_{a,S}$ as a scheme, and the group law is given by the formula:

$$(w_1, \lambda_1) \cdot (w_2, \lambda_2) = (w_1 + w_2, \lambda_1 + \lambda_2 + \frac{1}{2}(w_1, w_2)), \quad (w_i, \lambda_i) \in \mathcal{W} \times_S \mathbb{G}_{a,S}$$

where the term (w_1, w_2) denotes the symplectic pairing.

Example 3.14.1. For example, if $S = \mathrm{Spec}(k)$ and $\mathcal{W} = W = V \times V^\vee$ with the evident symplectic form, then the above recovers the Heisenberg group denoted $H(V)$ earlier.

In the general setting above, let $\mathbb{B}_S\mathcal{H} = S/\mathcal{H}$ denote classifying space of the group scheme $\mathcal{H}$. By a (strong) action of $\mathcal{H}(\mathcal{W})$ on a category, we mean a sheaf of categories on $(\mathbb{B}_S\mathcal{H})_{dR}$; by 1-affineness of S_{dR} and of the morphism $\mathcal{H}_{dR} \to S_{dR}$ ([Gai6] Theorem 2.6.3), this data is equivalent to that of a module category for $D(\mathcal{H}) \in \mathsf{Alg}(D(S)\text{–}\mathsf{mod})$ with its natural convolution monoidal structure. We denote the corresponding 2-category by $\mathcal{H}\text{–}\mathsf{mod}$.

As when working over a point, we have a subcategory $\mathcal{H}\text{–}\mathsf{mod}_{reg} \subseteq \mathcal{H}\text{–}\mathsf{mod}$: Fourier transform for the central $\mathbb{G}_{a,S} \subseteq \mathcal{H}$ makes any object of $\mathcal{H}\text{–}\mathsf{mod}$ into a $(D(S \times \mathbb{A}^1), \overset{!}{\otimes})$ module category, and we ask that this action factors through $D(S \times (\mathbb{A}^1 \backslash 0))$.

Lemma 3.14.2. *Suppose $\mathcal{N} \subseteq \mathcal{W}$ is a Lagrangian subbundle. Then the functor of strong $\mathcal{N}$-invariants defines an equivalence:*

$$\mathcal{H}\text{–}\mathsf{mod}_{reg} \xrightarrow{\simeq} D(S \times (\mathbb{A}^1 \backslash 0))\text{–}\mathsf{mod}.$$

Proof. In the case where $\mathcal{W}$ admits a Lagrangian splitting $\mathcal{W} = \mathcal{N} \times \mathcal{N}^\vee$, the same argument as over a point applies.

Étale locally, such a splitting exists: indeed, étale locally, $\mathcal{W}$ admits Darboux coordinates (as a torsor for a smooth group scheme is étale locally trivial), and then by the Bruhat decomposition for the Lagrangian Grassmannian, $\mathcal{N}$ admits a complement after a further Zariski localization.

Therefore, we obtain the result by étale descent for sheaves of categories on S_{dR}, see [Gai6] Corollary 1.5.4.

□

We also need a mild extension of the above.

Suppose we are given a vector bundle $\widetilde{\mathcal{W}}$ on S equipped with an epimorphism $\pi : \widetilde{\mathcal{W}} \twoheadrightarrow \mathcal{W}$. We form the group scheme $\widetilde{\mathcal{H}} := \mathcal{H} \times_{\mathcal{W}} \widetilde{\mathcal{W}}$, i.e., the pullback of the extension $\mathcal{H}$ of $\mathcal{W}$ to $\widetilde{\mathcal{W}}$. We can again speak of (strong) $\widetilde{\mathcal{H}}$-actions; we define regularity as for $\mathcal{H}$, i.e., with respect to the central $\mathbb{G}_a$.

Lemma 3.14.3. *Suppose $\mathcal{N} \subseteq \mathcal{W}$ is a Lagrangian subbundle, and suppose we are given a lift $\mathcal{N} \hookrightarrow \widetilde{\mathcal{W}}$ of this embedding over π. In particular, we obtain an embedding of the additive group scheme $\mathcal{N}$ into $\widetilde{\mathcal{H}}$.*

Then the functor of (strong) $\mathcal{N}$-invariants is conservative on $\widetilde{\mathcal{H}}\text{–}\mathsf{mod}_{reg}$.

Proof. As in the proof of Lemma 3.14.2, we are reduced by Zariski descent to the case where S is affine.

In this case, the embedding $\mathcal{N} \hookrightarrow \widetilde{\mathcal{W}}$ extends to a map $\mathcal{W} \to \widetilde{\mathcal{W}}$ splitting the projection (because $\mathcal{W}/\mathcal{N}$ is a vector bundle). This gives a map $\mathcal{H} \to \widetilde{\mathcal{H}}$ splitting the canonical projection that is the identity on the centrally embedded $\mathbb{G}_{a,S}$, and which is compatible with embeddings from $\mathcal{N}$. Therefore, the result in this case follows from Lemma 3.14.2.

□

We remark that $\widetilde{\mathcal{W}}$ inherits an alternating form from $\mathcal{W}$, and $\widetilde{\mathcal{H}}$ may be interpreted as a degenerate version of a Heisenberg group scheme.

Step 5. We can now conclude the argument. We remind that we have assumed $n = 2m-1$ for some $m \geqslant 2$.

We have the following extension of Lie algebras, which is between abelian Lie algebras:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t^{n-m+1}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] & \longrightarrow & t^{n-m}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] & \longrightarrow & t^m\mathfrak{g}[[t]]/t^{m+1}\mathfrak{g}[[t]] & \longrightarrow & 0. \\ & & \| & & \| & & \downarrow \simeq & & \\ & & t^m\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] & & t^{m-1}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] & & \underline{\mathfrak{g}} & & \end{array}$$

Here we write $\underline{\mathfrak{g}}$ to emphasize we are considering the *abelian* Lie algebra with vector space $\mathfrak{g}$.

As an extension of vector spaces, the above has an obvious splitting $(\xi \in \underline{\mathfrak{g}}) \mapsto t^{n-m}\xi$, so we see that the corresponding Lie algebra is a Heisenberg Lie algebra for the degenerate alternating form:

$$\begin{gathered} \mathfrak{g} \otimes \mathfrak{g} \to t^{n-1}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t] \subseteq t^m\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] \\ (\xi_1, \xi_2) \mapsto [t^{m-1}\xi_1, t^{m-1}\xi_2]. \end{gathered}$$

Passing to algebraic groups, we see that an action of $\exp(t^{m-1}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]])$ on $\mathcal{C}$ amounts to the following data. First, performing a Fourier transform along the central $\exp(t^m\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]]) = t^m\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]] \otimes \mathbb{G}_a$, we obtain a sheaf of categories on:

$$((t^m\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]])^\vee)_{dR} \simeq (\mathfrak{g}[[t]]/t^{n-m}\mathfrak{g}[[t]])_{dR} = (\mathfrak{g}[[t]]/t^{m-1}\mathfrak{g}[[t]])_{dR}$$

where the $\simeq$ is constructed as in §3.10; we denote the sheaf of categories corresponding to $\mathcal{C}$ by C. The remaining data encoding the full $\exp(t^{m-1}\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]])$-action amounts to an action of a degenerate Heisenberg group $\widetilde{\mathcal{H}}$ on C. In detail: form a constant vector bundle on $(\mathfrak{g}[[t]]/t^{m-1}\mathfrak{g}[[t]])$ with fiber $\mathfrak{g}$, and equip it with the (degenerate) alternating form whose fiber at $\xi \in (\mathfrak{g}[[t]]/t^{m-1}\mathfrak{g}[[t]])$ is:

$$(\varphi_1, \varphi_2) \in \mathfrak{g} \times \mathfrak{g} \mapsto \kappa([\xi(0), \varphi_1], \varphi_2)$$

where $\xi(0)$ indicates the image of ξ in $\mathfrak{g}$ obtained by $t \mapsto 0$. The corresponding Heisenberg group scheme $\widetilde{\mathcal{H}}$ defined by this data acts strongly on C.

In these terms, $\mathcal{C}^{N_n^{m-1},\psi}$ is calculated as global sections of C on $(f + \mathfrak{b}[[t]]/t^{m-1}\mathfrak{b}[[t]])_{dR}$; by §3.10, the assignment $(\mathcal{C} \in G_n\text{–}\mathsf{mod}_{reg}) \mapsto \mathcal{C}^{N_n^{m-1},\psi}$ is conservative.

Now observe that the constant vector bundle $\mathcal{N}$ on $f + \mathfrak{b}[[t]]/t^{m-1}\mathfrak{b}[[t]]$ with fiber $\mathfrak{n}$ satisfies the assumptions of Lemma 3.14.3 by Step 3, where the notation of Step 3 matches that of Lemma 3.14.3 (up to pulling back from $\mathfrak{g}_{reg}$ or $f + \mathfrak{b}$). We obtain $\mathcal{C}^{N_n^m,\psi}$ by passing to invariants for this Lagrangian subbundle; by Lemma 3.14.3, that functor is conservative, giving the claim.

4. Convolution for finite Whittaker categories

4.1. In this section, we extend the results from §3. These extensions are given in §4.7. This material plays technical roles in §10 and §11. The reader may safely skip this section on a first read and refer back where necessary.

Key roles are played by Theorems 4.2.1 and 4.3.1. The author finds these results to be of independent interest.[13]

4.2. **Main result.** The first main result of this section is the following:

Theorem 4.2.1. *For any $n \geqslant 1$ and any $\mathcal{C} \in G_n$–mod, the convolution functor:*

$$D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi} \to \mathcal{C}$$

admits a left adjoint. Here $D(G_n)^{N_n,-\psi}$ is the equivariant category for the action of N_n on G_n on the right.

Moreover, this left adjoint is isomorphic to the composition:

$$\mathcal{C} \xrightarrow{\mathrm{coact}[-2\dim G_n]} D(G_n) \overset{N_n}{\otimes} \mathcal{C} \xrightarrow{\mathrm{Av}_*^{N_n,-\psi} \otimes \mathrm{id}_{\mathcal{C}}} D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi}.$$

(Because of the diagonal N_n-equivariance and by unipotence of N_n, the functor $\mathrm{Av}_*^{N_n,-\psi} \otimes \mathrm{id}_{\mathcal{C}}[2\dim N_n]$ may be replaced by $\mathrm{Av}_*^{N_n,-\psi} \otimes \mathrm{Av}_*^{N_n,-\psi}[2\dim N_n]$ or $\mathrm{id}_{D(G_n)} \otimes \mathrm{Av}_*^{N_n,-\psi}[2\dim N_n]$.)

The proof bifurcates into the cases $n \geqslant 2$ and $n = 1$. In the former case, the argument is quite similar to the proof of Theorem 3.4.1.

4.3. **Reformulation.** First, we begin with a somewhat more convenient formulation of Theorem 4.2.1.

Theorem 4.3.1. *Let $n \geqslant 1$ and let $\mathcal{C} \in G_n \times G_n$–mod. Then the left adjoint to:*[14]

$$\mathcal{C}^{N_n \times N_n,(\psi,-\psi)} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^{\Delta N_n} \xrightarrow{\mathrm{Av}_*^{\Delta G_n}} \mathcal{C}^{\Delta G_n}$$

is defined, where $\Delta : G_n \to G_n \times G_n$ is the diagonal embedding. For convenience, we denote this left adjoint by $\mathrm{Av}_!^{\psi,-\psi}$.

Moreover, the canonical natural transformation:

$$\mathrm{Av}_!^{\psi,-\psi} \to \mathrm{Av}_*^{\psi,-\psi}[2\dim N_n] \in \mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathcal{C}^{\Delta G}, \mathcal{C}^{N \times N,(\psi,-\psi)})$$

is an equivalence.

Remark 4.3.2. In the case $n = 1$ and $\mathcal{C} = D(G) \in G \times G$–mod, Theorem 4.3.1 is [BBM] Theorem 1.5 (2). However, even in the $n = 1$ case, the result is new e.g. for $\mathcal{C} = D(G \otimes G)$.

Remark 4.3.3. In §6, we will only need the $n > 1$ case of Theorem 4.2.1. We include the proof in the $n = 1$ case only for the sake of completeness.

Proof that Theorem 4.3.1 implies Theorem 4.2.1. Suppose $\mathcal{C} \in G_n$–mod is given. We form $D(G_n) \otimes \mathcal{C} \in G_n \times G_n$–mod. By Theorem 4.3.1 (and changing ψ by a sign), the map:

$$\big(D(G_n) \otimes \mathcal{C}\big)^{N_n \times N_n,(-\psi,\psi)} = D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi} \xrightarrow{\mathrm{Oblv}} D(G_n) \overset{N_n}{\otimes} \mathcal{C} \xrightarrow{\mathrm{Av}_*^{\Delta G_n}} D(G_n) \overset{G_n}{\otimes} \mathcal{C} \xrightarrow{\simeq} \mathcal{C}.$$

[13]For instance, using Theorem 4.3.1 and standard arguments (relying on [Ras3]), one obtains geometric proofs of [Gin] Theorem 1.6.3 (similarly, Proposition 3.1.2). In particular, these arguments show that the t-exactness from *loc. cit.* Theorem 1.6.3 applies as well in the ℓ-adic context in characteristic p (using Artin-Schreier sheaves instead of exponential D-modules, and needing no special reference to [Ras3] because "non-holonomic" objects are meaningless here).

[14]Note that $(\psi, -\psi)$ restricted to the diagonal ΔN_n is the trivial character.

By definition, the resulting functor is the convolution functor, so that convolution functor admits a left adjoint. We similarly obtain the formula for the left adjoint in Theorem 4.2.1.

□

Below we prove Theorem 4.3.1, splitting it up into different cases.

4.4. **Proof of Theorem 4.3.1 for $n = 2$.** We freely use the notation and observations from §3.5. As in *loc. cit.*, we have:

$$G_2 \times G_2\text{–}\mathsf{mod} \simeq \mathsf{ShvCat}_{/\mathfrak{g}_{dR}/G_{dR} \times \mathfrak{g}_{dR}/G_{dR}}.$$

Let $\mathcal{C} \in G_2 \times G_2\text{–}\mathsf{mod}$, and let C denote the corresponding sheaf of categories on $\mathfrak{g}_{dR}/G_{dR} \times \mathfrak{g}_{dR}/G_{dR}$. The following commutative diagram provides a dictionary between these two perspectives:

$$\begin{aligned} \mathcal{C}^{N_2 \times N_2, (\psi, -\psi)} &\simeq \Gamma((f+\mathfrak{b})_{dR}/N_{dR} \times (-f+\mathfrak{b})_{dR}/N_{dR}, \mathsf{C}) \\ \mathcal{C}^{\Delta N_2} &\simeq \Gamma((\mathfrak{b} \times \mathfrak{b} + \Delta^- \mathfrak{g})_{dR}/N_{dR}, \mathsf{C}) \\ \mathcal{C}^{\Delta G_2} &\simeq \Gamma(\Delta^- \mathfrak{g}_{dR}/G_{dR}, \mathsf{C}). \end{aligned}$$

The averaging functor $\mathcal{C}^{N_2 \times N_2, (\psi, -\psi)} \to \mathcal{C}^{\Delta G_2}$ corresponds to !-pullback and then $*$-pushforward (in the D-module sense, which tautologically adapts to sheaves of categories on de Rham stacks) along the correspondence:

$$\begin{array}{ccc} & (f+\mathfrak{b})/N & \\ {\scriptstyle \Delta^-}\swarrow & & \searrow \\ (f+\mathfrak{b})/N \times (-f+\mathfrak{b})/N & & \mathfrak{g}/G. \end{array}$$

The left map Δ^- is a closed embedding because the Kostant slice $(f+\mathfrak{b})/N$ is an affine scheme, so !-pullback along it admits a left adjoint. The right map is smooth, so !-pullback along it equals $*$-pullback up to shift; in particular, the relevant $*$-pushforward admits a left adjoint.

This shows that our $*$-averaging functor admits a left adjoint in this case. That the comparison map $\mathrm{Av}_!^{\psi, -\psi} \to \mathrm{Av}_*^{\psi, -\psi}[2 \dim N_2]$ effects this isomorphism follows from the above analysis.

4.5. **Proof of Theorem 4.3.1 for $n > 2$.** The argument proceeds as in the proof of Theorem 3.4.1; we use the notation from that proof in what follows.

First, observe that it is equivalent to show that the left adjoint $\mathrm{Av}_!^{\psi} = \mathrm{Av}_!^{N_n, \psi}$ to $\mathrm{Av}_*^{\Delta(G_n)} : \mathcal{C}^{N_n \times 1, \psi} \to \mathcal{C}^{\Delta(G_n)}$ is defined, with the natural map $\mathrm{Av}_!^{\psi} \to \mathrm{Av}_*^{\psi}[2 \dim N_n]$ being an isomorphism; indeed, $\mathrm{Av}_*^{\Delta G_n}$ factors as:

$$\mathcal{C}^{N_n \times 1, \psi} \xrightarrow{\mathrm{Av}_*^{\Delta N_n}} (\mathcal{C}^{N_n \times 1, \psi})^{\Delta N_n} = \mathcal{C}^{N_n \times N_n, (\psi, -\psi)} \xrightarrow{\mathrm{Av}_*^{\Delta G_n}} \mathcal{C}^{\Delta G_n}$$

and the first functor admits the fully faithful left adjoint Oblv.

By induction on m, we will show that the appropriate left adjoint $\mathrm{Av}_!^{N_n^m, \psi} : \mathcal{C}^{\Delta G_n} \to \mathcal{C}^{N_n^m \times 1, \psi}$ is defined, and that the natural map $\mathrm{Av}_!^{N_n^m, \psi} \to \mathrm{Av}_*^{N_n^m, \psi}[2 \dim N_n^m]$ is an equivalence.

As in the proof of Theorem 3.4.1, the base case $m = 1$ is a consequence of the $n = 2$ case proved in §3.5. Moreover, as in §3.10, essentially the same argument applies for $m \leqslant \frac{n}{2}$. As in §3.14, the natural generalization of Lemma 4.5.1 vector bundles with alternating bilinear forms allows us to deduce the special case where $n = 2m - 1$; we omit the details, which are quite similar to §3.14.

Now in what follows, we assume $m \geqslant \frac{n+2}{2}$. By descending induction on r, we will show that the appropriate left adjoint $\mathrm{Av}_!^{N_n^{m,r},\psi} : \mathcal{C}^{\Delta G_n} \to \mathcal{C}^{N_n^{m,r}\times 1,\psi}$ is defined, and that the natural map $\mathrm{Av}_!^{N_n^{m,r},\psi} \to \mathrm{Av}_*^{N_n^{m,r},\psi}[2\dim N_n^{m,r}]$ is an equivalence. The base case $r \gg 0$ amounts to the inductive hypothesis for $m-1$.

To perform the induction, we use the following observation.

Lemma 4.5.1. *Let V be a finite-dimensional vector space over k and let $H = H(V)$ be the associated Heisenberg group, as in §3.6.*

Let $\mathcal{C} \in H\text{–}\mathsf{mod}_{reg}$. Then the functor $\mathrm{Av}_*^V : \mathcal{C}^{V^\vee} \to \mathcal{C}^V$ is an equivalence.

Moreover, if we (appropriately) denote the inverse functor $\mathrm{Av}_!^{V^\vee}$, then the natural map $\mathrm{Av}_!^{V^\vee} \to \mathrm{Av}_*^{V^\vee}[2\dim V]$ is an equivalence.

Proof. Immediate from the proof of Theorem 3.7.1.

□

The relevant Heisenberg group is constructed as follows. Here we use notation parallel to the proof of Theorem 3.4.1, but the meanings are different in the present context.

Define $\mathfrak{h}_0$ as $\mathrm{Lie}(N_n^{m,r}\times 1) + \Delta(t^{m-1}\mathfrak{g}_{1-r}) \subseteq \mathrm{Lie}(G_n \times G_n)$. Define $\mathfrak{h}_0'$ similarly, but with $\mathfrak{g}_{1-r}'$ in place of $\mathfrak{g}_{1-r}$ (in the notation of §3.13).

As in the proof of Theorem 3.4.1, these are nilpotent Lie subalgebras of $\mathrm{Lie}(G_n \times G_n)$, and there are associated unipotent subgroups $H_0' \subseteq H_0 \subseteq G_n \times G_n$. And again by the same argument as in *loc. cit.*, $(N_n^{m,r+1}\times 1) \subseteq H_0$ is normal, and its character is stabilized by the adjoint action of H_0. We again let $K \subseteq (N_n^{m,r+1}\times 1)$ denote the kernel of the character and $H := H_0'/K$; again, H is a Heisenberg group.

By induction, we have a !-averaging functor:

$$\mathrm{Av}_!^{N_n^{m,r+1},\psi} = \mathrm{Av}_*^{N_n^{m,r+1}}[2\dim N_n^{m,r+1}] : \mathcal{C}^{\Delta G_n} \to \mathcal{C}^{N_n^{m,r+1}\times 1,\psi}$$

which evidently lifts to invariants for the additive subgroup $\Delta(\mathfrak{g}_{1-r}') \subseteq H$. By Lemma 4.5.1, we can !-average $(\mathcal{C}^{N_n^{m,r+1}\times 1,\psi})^{\Delta(\mathfrak{g}_{1-r}')} \to \mathcal{C}^{N_n^{m,r}\times 1,\psi}$, and this coincides with $*$-averaging up to suitable shift (and moreover, the resulting functor gives an equivalence $(\mathcal{C}^{N_n^{m,r+1}\times 1,\psi})^{\Delta(\mathfrak{g}_{1-r}')} \xrightarrow{\simeq} \mathcal{C}^{N_n^{m,r}\times 1,\psi}$). This gives the claim.

4.6. **Proof of Theorem 4.3.1 for $n = 1$.** Let B^- be a Borel opposed to B with radical N^-.

Step 1. We have the functor:

$$\Psi : \mathcal{C}^{\Delta G} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^{\Delta B^-} \xrightarrow{\mathrm{Av}_*} \mathcal{C}^{N^-\times N^-\cdot \Delta T}.$$

This functor admits the left adjoint:

$$\Xi : \mathcal{C}^{N^-\times N^-\cdot \Delta T} \xrightarrow{\mathrm{Oblv}} \mathcal{C}^{\Delta B^-} \xrightarrow{\mathrm{Av}_!} \mathcal{C}^{\Delta G}$$

with $\mathrm{Av}_! = \mathrm{Av}_*[2\dim G/B^-]$ by properness of G/B^-.

Recall from [MV2] that the counit map $\Xi\Psi \to \mathrm{id}$ splits. Indeed, as in *loc. cit.*, $\Xi\Psi$ is computed as convolution with the Springer sheaf in $D(G)^{\mathrm{Ad}\, G} = D(\Delta G\backslash(G\times G)/\Delta G)$, and by an argument in *loc. cit.* using the decomposition theorem, the Springer sheaf admits the skyscraper sheaf at $1 \in G \overset{\mathrm{Ad}}{/} G$ as a summand.

In particular, every $\mathcal{F} \in \mathcal{C}^{\Delta G}$ is a summand of an object of the form $\Xi(\mathcal{F}')$.

Step 2. Next, we recall a key result of [BBM]. Theorem 1.1 (1) of *loc. cit.* implies that we can !-average N^--equivariant objects to be (N,ψ)-equivariant, and this !-average coincides with the $*$-averaging after shift by $2\dim N$. (Note that the authors work in the setting of perverse sheaves, but their argument works in this generality: c.f. the proof of [Ras5] Theorem 2.7.1.)

Applying this for $G\times G$ instead, we see that for $\mathcal{F}\in \mathcal{C}^{N^-\times N^-}$ (or $\mathcal{F}\in \mathcal{C}^{N^-\times N^-\cdot \Delta T}$), we can form $\mathrm{Av}_!^{(\psi,-\psi)}\mathcal{F}\in \mathcal{C}^{N\times N,(\psi,-\psi)}$, and the natural map:

$$\mathrm{Av}_!^{(\psi,-\psi)}\mathcal{F}\to \mathrm{Av}_*^{(\psi,-\psi)}\mathcal{F}[4\dim N]$$

is an isomorphism.

Step 3. Now suppose that $\mathcal{F}\in \mathcal{C}^{N^-\times N^-\cdot \Delta T}$. We claim that $\mathrm{Av}_!^{\psi,-\psi}\mathcal{F}$ coincides with $\mathrm{Av}_!^{\psi,-\psi}\Xi(\mathcal{F})$; in particular, the latter term is defined.

By base-change, $\mathrm{Av}_!^{\psi,-\psi}\Xi(\mathcal{F})$ should be computed as follows. We have a functor:

$$\mathrm{Av}_!=\mathrm{Av}_*[2\dim G/B^-]: D(N\backslash G)\overset{B^-}{\otimes}\mathcal{C}\to D(N\backslash G)\overset{G}{\otimes}\mathcal{C}=\mathcal{C}^{\Delta N}.$$

Also, $\mathcal{F}$ defines an object $\widetilde{\mathcal{F}}$ (i.e., $\omega_{N\backslash G}\widetilde{\boxtimes}\mathcal{F}$) in $D(N\backslash G)\overset{B^-}{\otimes}\mathcal{C}$. Finally, the recipe says that to compute $\mathrm{Av}_!^{\psi,-\psi}\Xi(\mathcal{F})$, we should form $\mathrm{Av}_!(\widetilde{\mathcal{F}})\in \mathcal{C}^{\Delta N}$ and then further !-average to $\mathcal{C}^{N\times N,(\psi,-\psi)}$.

Observe that $\widetilde{\mathcal{F}}$ carries a canonical Bruhat filtration. More precisely, for w an element of the Weyl group W, let i_w denote the locally closed embedding $N\backslash NwB^-\hookrightarrow N\backslash G$. Let $\mathcal{F}^w\in D(N\backslash NwB^-)\overset{B^-}{\otimes}\mathcal{C}$ be the object induced by $\mathcal{F}$, so $\widetilde{\mathcal{F}}$ is filtered with subquotients $i_{w,*,dR}(\mathcal{F}^w)$.

Let $N^w=N\cap \mathrm{Ad}_w(B^-)$. Then $D(N\backslash NwB^-)\overset{B^-}{\otimes}\mathcal{C}\simeq \mathcal{C}^{\Delta N^w}$, since $NwB^-=N\overset{N^w}{\times}B^-$, where N^w maps to B^- via $\mathrm{Ad}_{w^{-1}}$. The object $\mathcal{F}^w$ is then[15] $w\cdot\mathcal{F}$, which we note is equivariant for $\mathrm{Ad}_{\Delta w}(N^-\times N^-\cdot\Delta T)\supseteq N^w\times N^w\supseteq \Delta N^w$.

Then observe that up to cohomological shift, $\mathrm{Av}_!\, i_{w,*,dR}(\mathcal{F}^w)\in \mathcal{C}^{\Delta N}$ is obtained by $*$-averaging $w\cdot\mathcal{F}$ from ΔN^w to ΔN, since $\mathrm{Av}_!$ is !-averaging from B^- to G, and therefore coincides with $*$-averaging up to shift.

Now for $w\neq 1$, recall that the character ψ is non-trivial on $N\cap \mathrm{Ad}_w(N^-)$. Therefore, !-averaging to $(N\times N,(\psi,-\psi))$-equivariance vanishes on $\mathcal{C}^{N^w\times N^w}$. In particular, this !-averaging is defined. (The same applies for $*$-averaging.)

This vanishing implies:

$$\mathrm{Av}_!^{N\times N,(\psi,-\psi)}\Xi(\mathcal{F})=\mathrm{Av}_!^{N\times N,(\psi,-\psi)}\mathcal{F}^1.$$

(Here $1\in W$ is the unit in the Weyl group.) We note that $\mathcal{F}^1=\mathrm{Oblv}\,\mathcal{F}\in \mathcal{C}=D(NB^-)\overset{B^-}{\otimes}\mathcal{C}$. Since this last !-averaging is defined by [BBM] Theorem 1.1 (1), we obtain the result.

Step 4. We have now shown $\mathrm{Av}_!^{\psi,-\psi}\mathcal{F}$ is defined for $\mathcal{F}\in \mathcal{C}^{\Delta G}$. All that is left is to check that the natural map:

$$\mathrm{Av}_!^{\psi,\psi}\mathcal{F}\to \mathrm{Av}_*^{\psi,\psi}\mathcal{F}[2\dim N]$$

is an isomorphism.

We may assume $\mathcal{F}=\Xi\mathcal{G}$ for $\mathcal{G}\in \mathcal{C}^{N^-\times N^-\cdot\Delta T}$. In this case, the assertion is a straightforward verification in the above argument.

[15]Here $g\cdot\mathcal{F}$ is by definition $\delta_g * \mathcal{F}$, and we are using the diagonal action of G on $\mathcal{C}$.

4.7. **Application: construction of resolutions.** For the remainder of the section, we assume $n \geqslant 2$.

For $\mathcal{C} \in G_n$–mod, let $j^! : \mathcal{C} \rightleftarrows \mathcal{C}_{reg} : j_{*,dR}$ be as in §3.3.

For $\mathcal{C} = D(G_n)$, let $\delta_1 \in D(G_n)$ be the skyscraper D-module at the identity, and let $\delta_1^{reg} := j_{*,dR}j^!(\delta_1)$. Note that for any $\mathcal{C} \in G_n$–mod, the convolution functor $\delta_1^{reg} \star -$ is isomorphic to $j_{*,dR}j^!$ as endofunctors of $\mathcal{C}$.

Lemma 4.7.1. *δ_1^{reg} lies in the full subcategory of $D(G_n)$ generated by the essential image of the functor:*

$$D(G_n)^{N_n,-\psi,+} \times D(G_n)^{N_n,\psi,+} \to D(G_n)^{N_n,-\psi} \otimes D(G_n)^{N_n,\psi} \to D(G_n)$$

under finite colimits and direct summands. Here the first factor $D(G_n)^{N_n,-\psi}$ has invariants taken on the right, $D(G_n)^{N_n,\psi}$ has invariants on the left, and both terms are considered with their natural t-structures.

Proof. Suppose $\mathcal{C} \in G_n$–mod. By Theorem 4.2.1, the convolution functor:

$$D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi} \to \mathcal{C}$$

admits a left adjoint. Moreover, this left adjoint is a morphism in G_n–mod (where a priori, it is lax). Passing to (N_n, ψ)-invariants, we see that the functor:

$$\mathcal{H}_{N_n,\psi} \otimes \mathcal{C}^{N_n,\psi} \to \mathcal{C}^{N_n,\psi}$$

admits a left adjoint that is a morphism of $\mathcal{H}_{N_n,\psi}$-module categories (for $\mathcal{H}_{N_n,\psi}$ as in Corollary 3.4.2).

By the above remarks and [Gai6] Corollary C.2.3, the morphism:

$$D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi} \to D(G_n)^{N_n,-\psi} \underset{\mathcal{H}_{N_n,\psi}}{\otimes} \mathcal{C}^{N_n,\psi}$$

admits a monadic (discontinuous!) right adjoint. By Corollary 3.4.2, the right hand side maps isomorphically onto $\mathcal{C}_{reg}$.

Let $\mathrm{conv} : D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi} \to \mathcal{C}$ denote the convolution functor, let conv^R denote its (discontinuous!) right adjoint, and let $T = \mathrm{conv} \circ \mathrm{conv}^R : \mathcal{C} \to \mathcal{C}$ denote the corresponding monad. Clearly conv factors through $\mathcal{C}_{reg}$, and $\mathrm{conv}^R \circ j_{*,dR}$ is the right adjoint to the corresponding functor $D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi} \to \mathcal{C}_{reg}$.

Therefore, the monadic conclusion above shows that for any $\mathcal{F} \in \mathcal{C}_{reg} \overset{j_{*,dR}}{\subseteq} \mathcal{C}$, the geometric realization $|T^\bullet(\mathcal{F})| \in \mathcal{C}$ maps isomorphically onto $\mathcal{F}$.

We now specialize to the case $\mathcal{C} = D(G_n)$ and $\mathcal{F} = \delta_1^{reg}$. Note that δ_1^{reg} is holonomic in $D(G_n)$ and therefore compact. Therefore, as:

$$\delta_1^{reg} = |T^\bullet(\delta_1^{reg})| = \operatorname*{colim}_r |T^\bullet(\delta_1^{reg})|_{\leqslant r}$$

(for $|-|_{\leqslant r}$ the usual partial geometric realization, i.e., the colimit over $\mathbf{\Delta}^{op}_{\leqslant r}$), we obtain that δ_1^{reg} is a direct summand of $|T^\bullet(\delta_1^{reg})|_{\leqslant r}$ for some r.

We conclude in noting that T is left t-exact up to shift as conv is both left and right t-exact up to shift. Any object of $D(G_n)^{N_n,-\psi} \otimes D(G_n)^{N_n,\psi} = D(G_n \times G_n)^{N_n \times N_n,(-\psi,\psi)}$ bounded cohomologically bounded from below lies in the full subcategory generated by the image of $D(G_n)^{N_n,-\psi,+} \times D(G_n)^{N_n,\psi,+}$, so we obtain the claim.

□

We obtain the following result, which is a sort of effective version of Theorem 3.4.1.

Corollary 4.7.2. *Suppose that $n \geqslant 2$ and $\mathcal{C} \in G_n$–mod. Then for any $\mathcal{F} \in \mathcal{C}_{reg}$, $\mathcal{F}$ lies in the full subcategory of $\mathcal{C}$ generated under finite colimits and direct summands by the essential image of the convolution functor:*

$$D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi} \to \mathcal{C}.$$

Moreover, if $\mathcal{C}$ has a t-structure compatible with the action of G_n on it, and if $\mathcal{F} \in \mathcal{C}_{reg} \cap \mathcal{C}^+$, then $\mathcal{F}$ lies in the full subcategory of $\mathcal{C}$ generated under finite colimits and direct summands by the essential image of the convolution functor:

$$D(G_n)^{N_n,-\psi,+} \times \mathcal{C}^{N_n,\psi,+} \to D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi} \to \mathcal{C}.$$

Proof. Suppose $\mathcal{G}_1 \in D(G_n)^{N_n,-\psi,+}$ and $\mathcal{G}_2 \in D(G_n)^{N_n,\psi,+}$, with conventions for the actions as in Lemma 4.7.1. Then $\mathcal{G}_2 \star \mathcal{F} \in \mathcal{C}^{N_n,\psi}$, so $\mathcal{G}_1 \star \mathcal{G}_2 \star \mathcal{F} \in \mathcal{C}$ lies in the essential image of the convolution functor.

Moreover, in the presence of a t-structure on $\mathcal{C}$ as in the second part of the assertion, $\mathcal{G}_2 \star \mathcal{F} \in \mathcal{C}^{N_n,\psi,+}$ and $\mathcal{G}_1 \star \mathcal{G}_2 \star \mathcal{F} \in \mathcal{C}^+$ lies in the essential image of the functor considered in the second part.

Now we obtain the result by Lemma 4.7.1.

□

Corollary 4.7.3. *For any $\mathcal{C} \in G_n$–mod, the functor $\mathrm{Av}_!^{\psi,-\psi} : \mathcal{C} \to D(G_n)^{N_n,\psi} \otimes \mathcal{C}$ restricts to a conservative functor on $\mathcal{C}_{reg}$.*

Proof. Let $\mathcal{F} \in \mathcal{C}_{reg}$, and assume $\mathcal{F}$ is non-zero. We need to show that $\mathrm{Av}_!^{\psi,-\psi}(\mathcal{F}) \neq 0$.

By Corollary 4.7.2, there exists $\mathcal{G} \in D(G_n)^{N_n,\psi}$ with $\mathcal{G} \star \mathcal{F} \neq 0$ in $\mathcal{C}^{N_n,\psi}$. As $D(G_n)^{N_n,\psi}$ is compactly generated, we may assume that $\mathcal{G}$ is compact.

Note that $D(G_n)^{N_n,\psi}$ is canonically dual as a DG category to $D(G_n)^{N_n,-\psi}$. Let $\mathbb{D}\mathcal{G} : D(G_n)^{N_n,-\psi} \to \mathsf{Vect}$ denote the functor dual to the compact object $\mathcal{G}$ (explicitly, this functor is given as Hom out of the Verdier dual to $\mathcal{G}$).

Then the convolution $\mathcal{G} \star \mathcal{F}$ may be calculated by forming $\mathrm{Av}_*^{\psi,-\psi}(\mathcal{F}) \in D(G_n)^{N_n,-\psi} \otimes \mathcal{C}^{N_n,\psi}$ and then applying $\mathbb{D}\mathcal{G} \otimes \mathrm{id}_{\mathcal{C}^{N_n,\psi}}$. In particular, we deduce that $\mathrm{Av}_*^{\psi,-\psi}(\mathcal{F})$ is non-zero. As $\mathrm{Av}_*^{\psi,-\psi}(\mathcal{F})$ coincides with $\mathrm{Av}_!^{\psi,-\psi}(\mathcal{F})$ up to shift, we obtain the claim.

□

5. Most PGL_2-representations are generic

5.1. We now prove the following result.

Theorem 5.1.1. *Let $G = PGL_2$ and let $\mathcal{C}$ be acted on by $G(K)$ with level κ. Then $\mathcal{C}$ is generated under the action of $G(K)$ by $\mathsf{Whit}(\mathcal{C})$ and $\mathcal{C}^{\mathring{I}}$ where $\mathring{I} \subseteq G(K)$ is the radical of the Iwahori subgroup. That is, any subcategory of $\mathcal{C}$ that is closed under colimits, contains $\mathsf{Whit}(\mathcal{C})$ and $\mathcal{C}^{\mathring{I}}$, and is closed under the $G(K)$ action is $\mathcal{C}$ itself.*

Remark 5.1.2. This result is reminiscent of the existence of Whittaker models for those irreducible smooth representations of GL_2 over a locally compact non-Archimedean field with non-trivial restriction to SL_2.

However, in Theorem 5.1.1, $\mathring{I}$ cannot[16] be replaced by $G(K)$: this can be seen by applying Bezrukavnikov's theory [Bez] to local systems with non-trivial unipotent monodromy (c.f. with

[16]However, $\mathring{I}$ can be strengthened somewhat: one can take invariants with respect to the Iwahori subgroup of $SL_2(K)$, i.e., the canonical degree 2 cover of Iwahori.

the ideas of [AG1] in the spherical setting). Note that such local systems are outside the scope of arithmetic Langlands because they are not semisimple.

5.2. **Review of adolescent Whittaker theory.** We prove Theorem 5.1.1 using the theory of [Ras5] §2. For convenience, we review this here.

Let G be an adjoint[17] group and let $\mathcal{C} \in G(K)\text{–}\mathsf{mod}_\kappa$ be acted on by $G(K)$ with some level κ. We use the notation of §3. Let $K_n \subseteq G(O) \subseteq G(K)$ denote the nth congruence subgroup and observe that G_n acts on $\mathcal{C}^{K_n}$.

For $n > 0$, define $\mathsf{Whit}^{\leqslant n}(\mathcal{C}) := (\mathcal{C}^{K_n})^{N_n,\psi}$. There is a natural functor $\mathsf{Whit}^{\leqslant n+1}(\mathcal{C}) \to \mathsf{Whit}^{\leqslant n}(\mathcal{C})$:

$$\mathcal{F} \mapsto \mathrm{Av}_*^{K_n}(-\check{\rho}(t) \star \mathcal{F})$$

and which is denoted $\iota^!_{n,n+1}$ in *loc. cit.*

Theorem 5.2.1 ([Ras5] Theorem 2.7.1). *The functor $\iota^!_{n,n+1}$ admits a left adjoint $\iota_{n,n+1,!}$. This left adjoint is given by convolution with some D-module on $G(K)$.*

Moreover, there is a natural equivalence:

$$\operatorname*{colim}_{n,\iota_{n,n+1,!}} \mathsf{Whit}^{\leqslant n}(\mathcal{C}) \xrightarrow{\simeq} \mathsf{Whit}(\mathcal{C}) \in \mathsf{DGCat}_{cont}.$$

The structural functors $\mathsf{Whit}^{\leqslant n}(\mathcal{C}) \to \mathsf{Whit}(\mathcal{C})$ are left adjoint to the natural functors $\mathrm{Av}_*^{K_n} \circ (\delta_{n\check{\rho}(t)} \star -) : \mathsf{Whit}(\mathcal{C}) \to \mathsf{Whit}^{\leqslant n}(\mathcal{C})$. In particular, every object $\mathcal{F} \in \mathsf{Whit}(\mathcal{C})$ is canonically a colimit (in $\mathcal{C}$) of objects $\mathcal{F}_n$ with $\delta_{\check{\rho}(t^n)} \star \mathcal{F} \in \mathsf{Whit}^{\leqslant n}(\mathcal{C})$.

5.3. **Proof of Theorem 5.1.1.** We can now prove the main theorem of this section. Below, $G = PGL_2$.

Let $\mathcal{C}' \subseteq \mathcal{C}$ be a $G(K)$-subcategory containing $\mathsf{Whit}(\mathcal{C})$ and $\mathcal{C}^{\mathring{I}}$, and we wish to show that $\mathcal{C}' = \mathcal{C}$.

Recall that $\mathcal{C} = \mathrm{colim}_n\, \mathcal{C}^{K_n} \in \mathsf{DGCat}_{cont}$. Therefore, it suffices to show that $\mathcal{C}'$ contains $\mathcal{C}^{K_n}$ for all $n \geqslant 1$. We do this by induction on n.

In the base case $n = 1$, recall that for any $\mathcal{D}$ acted on by[18] G, $\mathcal{D}$ is the minimal cocomplete subcategory of itself closed under the G-action and containing $\mathcal{D}^N$; indeed, this follows from the main theorem of [BZGO].[19] Applying this to $\mathcal{D} = \mathcal{C}^{K_1}$, we find that $\mathcal{C}^{K_1}$ can be generated from $\mathcal{C}^{\mathring{I}}$ using the action of $G \subseteq G(K)$.

Now suppose the claim is true for n, and let us show it for $n+1$. Note that $n+1 \geqslant 2$, so we may apply the methods of §3 to $\mathcal{C}^{K_{n+1}}$ with its canonical G_{n+1}-action. In the notation of *loc. cit.*, we have adjoint functors:

$$j^! : \mathcal{C}^{K_{n+1}} \rightleftarrows (\mathcal{C}^{K_{n+1}})_{reg} : j_{*,dR}.$$

Note that $(\mathcal{C}^{K_{n+1}})_{reg} \subseteq \mathcal{C}'$ by Corollary 3.7.2, as $\mathsf{Whit}^{\leqslant n+1}(\mathcal{C}) = (\mathcal{C}^{K_{n+1}})^{N_{n+1},\psi} \subseteq \mathcal{C}'$ by hypothesis on $\mathcal{C}'$ (and Theorem 5.2.1).

Therefore, it suffices to show that $\mathrm{Ker}(j^!) \subseteq \mathcal{C}'$. Then we observe that $\mathfrak{g}_{reg} = \mathfrak{g}\backslash 0$ for $\mathfrak{g} = \mathfrak{sl}_2$, so (in the notation of §3.3), $\mathrm{Ker}(j^!) = \mathcal{C}^{K_{n+1}}|_0 = \mathcal{C}^{K_n}$ as we have the short exact sequence:

$$1 \to \mathfrak{g} \otimes \mathbb{G}_a \to K_{n+1} \to K_n \to 1.$$

[17]This is only for the convenience of using the action of $\check{\rho}(t) \in G(K)$ on $\mathcal{C}$. In fact, [Ras5] uses different indexing conventions than we use here, and which are better adapted to a general reductive group.

[18]Finite-dimensional, and here arbitrary reductive is fine.

[19]Or it follows from usual Beilinson-Bernstein localization theory: by reduction to the case $\mathcal{D} = D(G)$, one finds that $\mathcal{D}^{G,w}$ is a colocalization of $(\mathcal{D}^N)^{T,w}$, and then use conservativeness of weak invariants ([Gai6] Theorem 2.2.2).

But $\mathcal{C}^{K_n} \subseteq \mathcal{C}'$ by induction.

Remark 5.3.1. The above is the *descent* method discussed in the introduction. As this argument plays a key role in the paper, we reiterate the idea: with notation as above, for $\mathcal{C} \in G(K)\text{–}\mathsf{mod}_\kappa$, $\mathrm{Ker}(\mathrm{Av}_*^{K_n} : \mathcal{C}^{K_{n+1}} \to \mathcal{C}^{K_n})$ is the category $(\mathcal{C}^{K_{n+1}})_{reg}$, understood in the sense of §3.3 for the corresponding G_{n+1}-action. By Theorem 3.4.1 and Theorem 5.2.1, this kernel may therefore be functorially described in terms of the Whittaker model for $\mathcal{C}$.

One can then try to verify some property of objects of $\mathcal{C}$ as follows:

(1) Reduce to showing the property for objects in $\mathcal{C}^{K_n}$ for some n.
(2) Use the Whittaker model and the above observations to inductively reduce to the $n = 1$ case.
(3) Use [BZGO] to reduce the $n = 1$ case to a property of objects in $\mathcal{C}^{\mathring{I}}$.

6. Kac-Moody modules with central character

6.1. In this section, we study categories of critical level Kac-Moody representations with central character restrictions. We refer back to §1.9 for a review of standard notation at critical level.

First, for reductive G and any $n \geqslant 0$, we will introduce a certain category:

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \in \mathsf{DGCat}_{cont}$$

with a critical level $G(K)$-action.

In the above, the subscript ord_n indicates that we look at $\widehat{\mathfrak{g}}_{crit}$-modules on which the center $\mathfrak{Z}$ acts through a certain standard quotient $\mathfrak{Z} \twoheadrightarrow \mathfrak{z}_n$, and in a suitable derived sense. Equivalently, under Feigin-Frenkel, these can be thought of as representations *scheme-theoretically* supported on $\mathrm{Op}_{\check{G}}^{\leqslant n} \subseteq \mathrm{Op}_{\check{G}}$, where $\mathrm{Op}_{\check{G}}^{\leqslant n}$ are opers with singularity of order $\leqslant n$, c.f. [BD1] §3.8 or [FG2] §1.

For $n = 0$, $\mathfrak{z}_0 = \mathfrak{z}$. Here the central character condition is the regularity assumption from §1.10, so we use the notation *reg* in place of ord_0.

In the spirit of [Ras6], the subscript *naive* indicates that this is not the best derived category to consider. For instance, $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$ is not compactly generated. And for $n = 0$, the analogue of Conjecture 1.11.1 fails for it.

Following [FG4] §23, we introduce a somewhat better *renormalized* category $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$. This category will have a forgetful functor:

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$$

that is t-exact for suitable t-structures and an equivalence on eventually coconnective subcategories.

However, this renormalization procedure is somewhat subtle, and there are many basic questions about $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$ that I do not know how to answer. For instance, I cannot generally show that there is a $G(K)$-action on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$ compatible with the forgetful functor above. We refer to §6.10 for further discussion.

The material of this section is technical. Proposition 6.6.1 and Lemma 6.9.3 are the key points. After understanding the statements of these results, the reader should be equipped to move on to future sections.

Finally, we highlight that the material of this section relies on [Ras6] §11 and extends the material from *loc. cit.*

6.2. **Notation at critical level.** As in [Ras6] §11, we use the following notation. We refer to [FG2] §1 for background on opers.

First, $\mathrm{Op}_{\check{G}}$ denotes the indscheme of $\check{G}$-opers on the punctured disc. We let $\mathrm{Op}_{\check{G}}^{\leqslant n} \subseteq \mathrm{Op}_{\check{G}}$ denote the subscheme of opers with singularities of order $\leqslant n$.

We remind that $\mathrm{Op}_{\check{G}}^{\leqslant n}$ is affine for every n; we let $\mathfrak{z}_n$ denote the corresponding algebra of functions, so $\mathrm{Op}_{\check{G}}^{\leqslant n} = \mathrm{Spec}(\mathfrak{z}_n)$. We remind that $\mathfrak{z}_n$ is a polynomial algebra in infinitely many variables.

We let $\mathfrak{Z}$ denote the commutative $\overset{!}{\otimes}$-algebra $\lim_n \mathfrak{z}_n \in \mathsf{ProVect}^{\heartsuit}$, the limit being taken in $\mathsf{ProVect}^{\heartsuit}$; we refer to [Ras6] for the terminology on topological algebras used here. We remark that $\mathrm{Op}_{\check{G}} = \mathrm{Spf}(\mathfrak{Z})$.

By Feigin-Frenkel (see [FF] and [BD1] §3), $\mathfrak{Z}$ naturally identifies with $U(\widehat{\mathfrak{g}}_{crit})$, the twisted topological enveloping algebra of $\widehat{\mathfrak{g}}_{crit}$.

We let $\mathbb{V}_{crit,n} := \mathrm{ind}_{t^n\mathfrak{g}[[t]]}^{\widehat{\mathfrak{g}}_{crit}}(k) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}$.

6.3. **Naive categories.** We begin with some preliminary notations.

First, if $\mathcal{A} \in \mathsf{CoAlg}(\mathsf{DGCat}_{cont})$ and $\mathcal{M}$ (resp. $\mathcal{N}$) is a right (resp. left) comodule for $\mathcal{A}$, we let:

$$\mathcal{M} \overset{\mathcal{A}}{\otimes} \mathcal{N} \in \mathsf{DGCat}_{cont}$$

denote the cotensor product of these comodules. By definition, this means we regard $\mathcal{A}$ as an algebra in the opposite category $\mathsf{DGCat}_{cont}^{op}$ and form the usual tensor product there. This cotensor product may be calculated as a totalization in DGCat_{cont}:

$$\mathcal{M} \overset{\mathcal{A}}{\otimes} \mathcal{N} = \mathrm{Tot}\left(\mathcal{M} \otimes \mathcal{N} \rightrightarrows \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{N} \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \dots\right).$$

Next, for S a reasonable indscheme in the sense of [Ras6] §6, recall that we have the compactly generated DG category $\mathsf{IndCoh}^*(S) \in \mathsf{DGCat}_{cont}$. This construction is covariantly functorial in S. In particular, if S is a reasonable indscheme that is *strict*,[20] $\mathsf{IndCoh}^*(S)$ is canonically a cocommutative coalgebra in DGCat_{cont}.

6.4. Note that $\mathrm{Op}_{\check{G}}$ is a strict, reasonable indscheme. By [Ras6] Theorem 11.18.1, $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \in G(K)\text{–}\mathsf{mod}_{crit}$ is canonically an $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$-comodule (in $G(K)\text{–}\mathsf{mod}_{crit}$).

For $n \geqslant 0$, define:

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} := \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\leqslant n}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \in G(K)\text{–}\mathsf{mod}_{crit}.$$

Let i_n denote the embedding $\mathrm{Op}_{\check{G}}^{\leqslant n} \to \mathrm{Op}_{\check{G}}$. We abuse notation in letting $i_{n,*} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$ denote the functor $i_{n,*}^{\mathsf{IndCoh}} \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \mathrm{id}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}}$. By [Ras6] Lemma 6.17.1-2, this functor admits a continuous right adjoint $i_n^! \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \mathrm{id}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}}$, which we also denote $i_n^!$. Note that $i_{n,*}$ and $i_n^!$ are (by construction) morphisms of $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$-module categories.

Similarly, for $m \geqslant n$, we have a natural adjunction:

$$i_{n,m,*} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive} : i_{n,m}^!$$

with $i_{n,*} = i_{m,*} \circ i_{n,m,*}$. Note that $i_{n,m,*}$ actually admits a *left* adjoint $i_{n,m}^*$ as well as a right adjoint; this follows because the closed embedding $i_{n,m} : \mathrm{Op}_{\check{G}}^{\leqslant n} \hookrightarrow \mathrm{Op}_{\check{G}}^{\leqslant m}$ is a finitely presented regular embedding.

[20] See *loc. cit.* for the definition. The relevance here is that this condition implies e.g. that the natural functor $\mathsf{IndCoh}^*(S) \otimes \mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(S \times S)$ is an equivalence.

Remark 6.4.1. For a reasonable indscheme S, we let $\mathsf{IndCoh}^!(S)$ denote the dual DG category to $\mathsf{IndCoh}^*(S)$; this construction is contravariantly functorial in S. For strict S, $\mathsf{IndCoh}^!(S)$ is therefore a symmetric monoidal category.

In these terms, we can reformulate the above definition (to use monoidal categories instead of "comonoidal" categories):

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} = \mathsf{Hom}_{\mathsf{IndCoh}^!(\mathrm{Op}_{\check{G}})\text{–}\mathsf{mod}}(\mathsf{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\leqslant n}), \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}).$$

6.5. We record what symmetries the above construction provides.

As indicated above, there is an evident critical level $G(K)$-action on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$.

Moreover, $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$ is an $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\leqslant n})$-comodule category, or equivalently, an $\mathsf{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\leqslant n})$-module category. Because $\mathrm{Op}_{\check{G}}^{\leqslant n}$ is the spectrum of a polynomial algebra (on infinitely many generators), the natural symmetric monoidal functor $\mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\leqslant n}) \to \mathsf{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\leqslant n})$ is an equivalence. Therefore, we may as well regard $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$ as equipped with a $\mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\leqslant n})$-action commuting with the critical level $G(K)$-action.

In our notation, we regard $G(K)$ as acting on the left on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$ by convolution $-\star-$, and we regard $\mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\leqslant n})$ as acting on the right by an action functor:

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \otimes \mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{\leqslant n}) \xrightarrow{-\underset{\mathrm{Op}_{\check{G}}^{\leqslant n}}{\otimes}-} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}.$$

6.6. The following result summarizes the basic properties of the above construction.

Proposition 6.6.1. *(1) The functor $i_{n,*} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$ is comonadic, and in particular, conservative.*

(2) $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$ admits a unique t-structure for which $i_{n,*}$ is t-exact.

(3) The natural map:

$$\operatorname*{colim}_{m \geqslant n} i_{n,m}^! i_{n,m,*} \to i_n^! i_{n,*}$$

is an isomorphism.

(4) The natural functor:

$$\operatorname*{colim}_{n, i_{n,m,*}} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \in \mathsf{DGCat}_{cont}$$

is an equivalence.

Proof. Let $\mathbb{A}^\infty := \mathrm{colim}_r \mathbb{A}^r$, i.e., the ind-finite type indscheme version of infinite-dimensional affine space. Using standard choices of coordinates on $\mathrm{Op}_{\check{G}}$, one find an isomorphism $\mathrm{Op}_{\check{G}} = \mathrm{Op}_{\check{G}}^{\leqslant n} \times \mathbb{A}^\infty$ so that the diagram:

$$\begin{array}{ccc} \mathrm{Op}_{\check{G}}^{\leqslant n} & & \\ \downarrow{\scriptstyle i_n} & \searrow^{\mathrm{id}\times 0} & \\ \mathrm{Op}_{\check{G}} & \xrightarrow{\simeq} & \mathrm{Op}_{\check{G}}^{\leqslant n} \times \mathbb{A}^\infty \end{array}$$

commutes.

We then have:[21]

$$\mathsf{Hom}_{\mathsf{IndCoh}^!(\mathbb{A}^\infty)\text{–}\mathsf{mod}}(\mathsf{Vect}, \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) = \mathsf{Vect} \overset{\mathsf{IndCoh}^*(\mathbb{A}^\infty)}{\otimes} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \xrightarrow{\simeq} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n, naive} \in G(K)\text{–}\mathsf{mod}_{crit}.$$

Take $\mathcal{A} := \mathsf{IndCoh}^!(\mathbb{A}^\infty)$ as a monoidal category. Note that the monoidal product:

$$-\overset{!}{\otimes}- : \mathcal{A}\otimes\mathcal{A} \xrightarrow{\simeq} \mathsf{IndCoh}^!(\mathbb{A}^\infty \times \mathbb{A}^\infty) \xrightarrow{\Delta^!} \mathsf{IndCoh}^!(\mathbb{A}^\infty) = \mathcal{A}$$

admits a left adjoint $\Delta_*^{\mathsf{IndCoh}}$ that is a morphism of $\mathcal{A}$-bimodule categories (by the projection formula). It is easy to see in this setting that for any $\mathcal{A}$-module category $\mathcal{M}$, the action functor:

$$\mathrm{act} : \mathcal{A}\otimes\mathcal{M} \to \mathcal{M}$$

admits a continuous left adjoint act^L that is a morphism of $\mathcal{A}$-module categories, where the left hand side is regarded as an $\mathcal{A}$-module via the action on the first factor. It follows that for any pair of $\mathcal{A}$-module categories $\mathcal{M}, \mathcal{N}$, the cosimplicial category:

$$\mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathcal{M},\mathcal{N}) \rightrightarrows \mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathcal{A}\otimes\mathcal{M},\mathcal{N}) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \dots$$

satisfies the comonadic Beck-Chevalley conditions.[22] Applying this for $\mathcal{M} = \mathsf{Vect}$ and $\mathcal{N} = \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$, we obtain (1).

Next, we show (4). We calculate:

$$\operatorname*{colim}_{n, i_{n,m,*}} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n, naive} = \lim_{n, i^!_{n,m}} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n, naive} =$$

$$\lim_n \left(\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\leqslant n}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}\right) =$$

$$\lim_n \mathrm{Tot}\left(\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\leqslant n}) \otimes \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})^{\otimes\bullet} \otimes \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}\right) =$$

$$\mathrm{Tot}\lim_n\left(\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\leqslant n}) \otimes \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})^{\otimes\bullet} \otimes \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}\right) \overset{\star}{=}$$

$$\mathrm{Tot}\left(\left(\lim_n \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\leqslant n})\right) \otimes \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})^{\otimes\bullet} \otimes \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}\right) =$$

$$\mathrm{Tot}\left(\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})) \otimes \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})^{\otimes\bullet} \otimes \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}\right) =$$

$$\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} = \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$$

as desired; here the only non-trivial manipulations are the first, which expresses that a colimit in DGCat_{cont} under left adjoints is canonically isomorphic to the limit under right adjoints, and the one labeled $\star$, where the limit past tensor products is justified because we are tensoring with compactly generated, hence dualizable, DG categories.

We deduce (3) immediately from (4) and [Gai4] Lemma 1.3.6.

It remains to show (2). Given (1), a standard argument reduces us to checking that $i_{n,*}i_n^!$ is left t-exact.

By the above Beck-Chevalley analysis, $i_{n,*}i_n^!$ may be calculated by applying the composition:

[21] Of course, $\mathsf{IndCoh}^!(\mathbb{A}^\infty)$ and $\mathsf{IndCoh}^*(\mathbb{A}^\infty)$ coincide with usual IndCoh as $\mathbb{A}^\infty$ is locally of finite type. We include the notation to clarify whether this category is being viewed as an algebra or coalgebra in DGCat_{cont}.

[22] See [Lur2] §4.7.6 or [Gai6] §C for background on the Beck-Chevalley theory; our terminology here is taken from the latter source. We especially note [Gai6] Lemma C.2.2, which is essentially dual to the present assertion.

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \xrightarrow{\text{coact}} \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}) \otimes \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \xrightarrow{\pi_*^{\mathsf{IndCoh}} \otimes \mathrm{id}} \mathsf{IndCoh}^*(\mathbb{A}^\infty) \otimes \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$$

and then applying the right adjoint to this composition; here $\pi : \mathrm{Op}_{\check{G}} \to \mathbb{A}^\infty$ denotes the projection. It suffices to show the composition is t-exact (for the tensor product t-structure on the right hand side); we will show each of the functors appearing here is t-exact. The functor coact is t-exact by [Ras6] Lemma 11.13.1. Then π_*^{IndCoh} is t-exact because π is affine, and similarly for $\pi_*^{\mathsf{IndCoh}} \otimes \mathrm{id}$ by [Ras5] Lemma B.6.2.

□

6.7. We continue our study of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$.

Lemma 6.7.1. *Suppose $\mathcal{F} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit$. Then the adjunction map $H^0(i_{n,*}i_n^!(\mathcal{F})) \to \mathcal{F} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit$ is a monomorphism with image the maximal submodule of $\mathcal{F}$ on which $\mathfrak{Z}$ acts through $\mathfrak{z}_n$.*

Proof. The forgetful functor $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \to \mathsf{Vect}$ admits a unique lift $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \xrightarrow{\mathrm{Oblv}^{enh}} \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}) = \mathfrak{Z}\text{–}\mathsf{mod}_{ren} \xrightarrow{\mathrm{Oblv}} \mathsf{Vect}$ with Oblv^{enh} a morphism of $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$-comodule categories. By [Ras6] Lemma 11.13.1, Oblv^{enh} is t-exact, and on the hearts of the t-structure corresponds to restriction of modules along the homomorphism $\mathfrak{Z} \hookrightarrow U(\widehat{\mathfrak{g}}_{crit})$.

As Oblv^{enh} is a map of $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$-comodule categories, it intertwines $i_{n,*}i_n^!$ with the similar functor on $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$. It is clear that H^0 of that functor extracts the maximal submodule on which $\mathfrak{Z}$ acts through $\mathfrak{z}_n$, giving the claim.

□

Corollary 6.7.2. *The map $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit_{\mathrm{ord}_n,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit$ is fully faithful. Its essential image is the full subcategory of the target consisting of modules on which $\mathfrak{Z}$ acts through $\mathfrak{z}_n$.*

Proof. Immediate from Lemma 6.7.1 and Proposition 6.6.1 (1).

□

6.8. We use the notation:

$$\mathcal{P} := \lim_m \mathbb{V}_{crit,m} \in \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit) \subseteq \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}).$$

Here the limit is over the natural structure maps $\mathbb{V}_{crit,m+1} \to \mathbb{V}_{crit,m}$, and we emphasize that the limit occurs in the pro-category (or rather, in either pro-category). We remark that the pro-object $\mathcal{P}$ corepresents the forgetful functor $\mathrm{Oblv} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod} \to \mathsf{Vect}$: this is clear of its restriction to $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+$, and then the claim follows generally as the objects $\mathbb{V}_{crit,m}$ are compact in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$. Clearly $\mathrm{Oblv}(\mathcal{P}) \in \mathsf{Pro}(\mathsf{Vect}^\heartsuit)$ is $U(\widehat{\mathfrak{g}}_{crit})$; its $\overrightarrow{\otimes}$-algebra structure may be seen using [Ras6] Proposition 3.7.1.

For $m \geqslant 0$, let $\mathbb{V}_{\mathrm{ord}_n,m} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit_{\mathrm{ord}_n,naive}$ denote the minimal quotient of $\mathbb{V}_{crit,m}$ lying in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit_{\mathrm{ord}_n,naive} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit$, i.e., $\mathbb{V}_{\mathrm{ord}_n,m} = \mathbb{V}_{crit,m}/I_n$.

Define:

$$\mathcal{P}_{\mathrm{ord}_n} := \lim_m \mathbb{V}_{\mathrm{ord}_n,m} \in \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit_{\mathrm{ord}_n,naive}) \subseteq \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive})$$

to be the corresponding pro-object; we again emphasize that the displayed limit occurs in the pro-category.

There is an evident canonical morphism:

$$\pi : \mathcal{P} \to i_{n,*}\mathcal{P}_{\mathrm{ord}_n} \in \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit) \subseteq \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}).$$

Lemma 6.8.1. *As an object of* $\mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive})$, $\mathcal{P}_{\mathrm{ord}_n}$ *corepresents the composition:*

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive} \xrightarrow{i_{n,*}} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+} \xrightarrow{\mathrm{Oblv}} \mathsf{Vect}.$$

More precisely, for $\mathcal{F} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive}$, *the composite map:*

$$\underline{\mathrm{Hom}}_{\mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive})}(\mathcal{P}_{\mathrm{ord}_n},\mathcal{F}) \to \underline{\mathrm{Hom}}_{\mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})}(i_{n,*}\mathcal{P}_{\mathrm{ord}_n}, i_{n,*}\mathcal{F}) \xrightarrow{-\circ\pi}$$
$$\underline{\mathrm{Hom}}_{\mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})}(\mathcal{P}, i_{n,*}\mathcal{F}) \simeq \mathrm{Oblv}(i_{n,*}\mathcal{F})$$

is an isomorphism.

Proof.

Step 1. First, suppose $\mathcal{G} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\geqslant 0}_{\mathrm{ord}_n,naive}$ has the property that $i_{n,*}\mathcal{G}$ is compact in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$. Then we claim that for any $r \geqslant 0$, $\mathcal{G}$ is compact as an object of the category $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\geqslant -r}_{\mathrm{ord}_n,naive}$.

Indeed, this is standard from Proposition 6.6.1 (1)-(2): see the proof of [Ras6] Lemma 6.11.2.

Step 2. Suppose $\mathcal{G}$ as above, and let $\mathcal{F} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive}$. Then we claim that the natural map:

$$\operatorname*{colim}_{m\geqslant n} \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive}}(i_{n,m,*}\mathcal{G}, i_{n,m,*}\mathcal{F}) \to \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}}(i_{n,*}\mathcal{G}, i_{n,*}\mathcal{F}) \tag{6.8.1}$$

is an isomorphism.

Indeed, we have:

$$\operatorname*{colim}_{m\geqslant n} \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive}}(i_{n,m,*}\mathcal{G}, i_{n,m,*}\mathcal{F}) =$$
$$\operatorname*{colim}_{m\geqslant n} \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}}(\mathcal{G}, i^{!}_{n,m}i_{n,m,*}\mathcal{F}) \overset{\text{Step 1}}{=}$$
$$\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}}(\mathcal{G}, \operatorname*{colim}_{m\geqslant n} i^{!}_{n,m}i_{n,m,*}\mathcal{F}) \overset{\text{Prop. 6.6.1 (3)}}{=}$$
$$\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}}(\mathcal{G}, i^{!}_{n}i_{n,*}\mathcal{F}) = \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}}(i_{n,*}\mathcal{G}, i_{n,*}\mathcal{F}).$$

We remark that if $\mathcal{F}$ is in cohomological degrees $\geqslant -r$, then each $i^{!}_{n,m}i_{n,m,*}(\mathcal{F})$ is as well (because the functors $i_{n,m,*}$ are t-exact); this justifies the reference to Step 1. We also note that the composite identification here is easily seen to be given by the map considered above.

Step 3. Next, recall the functors $i^{*}_{n,m}$ from §6.4. We claim that $i^{*}_{n,m}(\mathbb{V}_{crit,m}) = \mathbb{V}_{\mathrm{ord}_n,m}$. Clearly the right hand side is the top (= degree 0) cohomology of the left hand side, so this amounts to arguing that the lower cohomology groups vanish.

As in the argument for Lemma 6.7.1, we have a commutative diagram:[23]

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive} & \xrightarrow{i^{*}_{n,m}} & \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \\ \downarrow & & \downarrow \\ \mathsf{IndCoh}^{*}(\mathrm{Op}^{\leqslant m}_{\check{G}}) & \xrightarrow{i^{*}_{n,m}} & \mathsf{IndCoh}^{*}(\mathrm{Op}^{\leqslant n}_{\check{G}}). \end{array}$$

[23]To be explicit, we remind that by the definition from [Ras6] §6, $\mathsf{IndCoh}^{*}(\mathrm{Op}^{\leqslant n}_{\check{G}})$ is $\mathsf{Ind}(Coh(\mathrm{Op}^{\leqslant n}_{\check{G}}))$. As $\mathrm{Op}^{\leqslant n}_{\check{G}}$ is the spectrum of a (infinitely generated) polynomial algebra, $\mathsf{Coh}(\mathrm{Op}^{\leqslant n}_{\check{G}}) = \mathsf{Perf}(\mathrm{Op}^{\leqslant n}_{\check{G}})$. Therefore, IndCoh^{*} in the bottom row may be replaced by the more familiar QCoh. The functor $i^{*}_{n,m}$ in that bottom row is then the standard pullback functor.

The vertical arrows are the natural restriction maps, and arise from Oblv^{enh} (from the proof of Lemma 6.7.1) and the evident identification $\mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) = \mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$, and similarly for m. These vertical arrows are t-exact and conservative on bounded below subcategories as this is true for Oblv^{enh}.

The functor $i^*_{n,m} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$ is easily[24] seen to be left t-exact up to shift. Therefore, it suffices to see that the underlying object of $\mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}})$ defined by $i^*_{n,m}(\mathbb{V}_{crit,m})$ lies in cohomological degree 0.

This follows from the commutativity of the above diagram and the fact that $\mathbb{V}_{crit,m} \in \mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant m}_{\check{G}})^{\heartsuit}$ defines a *flat* sheaf by [FG2] Lemma 7.2.2 (which is based on [EF]).

Step 4. We now deduce the claim.

In what follows, we consider $\mathbb{V}_{crit,m}$ as an object of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{\mathrm{ord}_m,naive}$; we let $i_{m,*}\mathbb{V}_{crit,m}$ denote the corresponding object of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}$.

For $\mathcal{F} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive}$, we calculate:

$$\underline{\mathrm{Hom}}_{\mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive})}(\mathcal{P}_{\mathrm{ord}_n},\mathcal{F}) = \operatorname*{colim}_{m\geqslant n} \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}}(\mathbb{V}_{\mathrm{ord}_n,m},\mathcal{F}) \overset{\text{Step 3}}{=}$$
$$\operatorname*{colim}_{m\geqslant n} \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}}(i^*_{n,m}\mathbb{V}_{crit,m},\mathcal{F}) = \operatorname*{colim}_{m\geqslant n} \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive}}(\mathbb{V}_{crit,m}, i_{n,m,*}\mathcal{F}) =$$
$$\operatorname*{colim}_{m\geqslant n}\operatorname*{colim}_{r\geqslant m} \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_r,naive}}(i_{m,r,*}\mathbb{V}_{crit,m}, i_{n,r,*}\mathcal{F}) \overset{\text{Step 2}}{=}$$
$$\operatorname*{colim}_{m\geqslant n} \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}}(i_{m,*}\mathbb{V}_{crit,m}, i_{n,*}\mathcal{F}) = \underline{\mathrm{Hom}}_{\mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})}(\mathcal{P}, i_{n,*}\mathcal{F})$$

as desired.

□

In what follows, we let $\mathrm{Oblv} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \to \mathsf{Vect}$ denote the forgetful functor considered above, i.e., $\mathrm{Oblv}\, i_{n,*}$.

Corollary 6.8.2. *The (non-cocomplete) DG category $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive}$ is the bounded below derived category of its heart.*

Proof. Note that $U(\widehat{\mathfrak{g}}_{crit})_{\mathrm{ord}_n} := \mathrm{Oblv}(\mathcal{P}_{\mathrm{ord}_n}) \in \mathsf{Pro}(\mathsf{Vect}^{\heartsuit})$ by construction. Therefore, the result follows from [Ras6] Proposition 3.7.1.

□

It follows that $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive}$ identifies with the similar category considered in the works of Frenkel-Gaitsgory, e.g. in [FG2] §23.

[24]For one, it is (non-canonically) isomorphic to $i^!_{n,m}$ up to shift. Alternatively, $i_{n,m,*}i^*_{n,m}$ is calculated as the composition:

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive} \xrightarrow{\mathrm{coact}} \mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant m}_{\check{G}}) \otimes \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive} \xrightarrow{i_{n,m,*}i^*_{n,m}\otimes \mathrm{id}}$$
$$\mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant m}_{\check{G}}) \otimes \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive} \xrightarrow{\Gamma^{\mathsf{IndCoh}}(\mathrm{Op}^{\leqslant m}_{\check{G}},-)\otimes \mathrm{id}} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive}$$

giving the claim by considering the standard finite Koszul filtration on the endofunctor $i_{n,m,*}i^*_{n,m}$ of $\mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant m}_{\check{G}})$.

6.9. **Renormalization.** We now introduce a renormalized version of the above categories following [FG4] §23.

Define $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n} \subseteq \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n,naive}$ as the full subcategory of objects $\mathcal{F}$ such that $i_{n,*}(\mathcal{F})$ is compact in $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}$. By Proposition 6.6.1 and the similar fact for $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}$, $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n} \subseteq \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^+_{\mathrm{ord}_n,naive}$.

Example 6.9.1. For $m \geqslant n$, Koszul resolutions for the finitely presented regular embedding $\mathrm{Op}^{\leqslant n}_{\check{G}} \hookrightarrow \mathrm{Op}^{\leqslant m}_{\check{G}}$ imply that the functors $i^!_{n,m}$ and $i^*_{n,m}$ map $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_m}$ to $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n}$.

Example 6.9.2. The objects $\mathbb{V}_{\mathrm{ord}_n,m}$ lie in $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n}$. Indeed, for $0 \leqslant m \leqslant n$, $i_{n,*}\mathbb{V}_{\mathrm{ord}_n,m} = \mathbb{V}_{crit,m}$, clearly giving the claim in this case. In general, for $m \geqslant n$, we have $i^*_{n,m}\mathbb{V}_{crit,m} = \mathbb{V}_{\mathrm{ord}_n,m}$ as in Step 3 from the proof of Lemma 6.8.1, clearly giving the claim.

Define $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n} = \mathsf{Ind}(\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n})$. Define a t-structure on $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n}$ by taking $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\leqslant 0}_{\mathrm{ord}_n}$ to be generated under colimits by objects in $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n} \cap \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\leqslant 0}_{\mathrm{ord}_n,naive}$.

We have a canonical functor $\rho : \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n} \to \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n,naive}$: this is the unique continuous functor with $\rho|_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^c_{\mathrm{ord}_n}}$ the canonical embedding.

Lemma 6.9.3 (C.f. [FG4] §23.2.2). *The functor ρ is t-exact and induces an equivalence on eventually coconnective subcategories.*

Proof.

Step 1. We collect some observations we will need later.

Note that for any m, $\mathbb{V}_{\mathrm{ord}_n,m} \in \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n} \subseteq \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n}$ lies in the heart of the t-structure; indeed, it is connective by definition, and it is clear that any object in $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n}$ that is coconnective in $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n,naive}$ is also coconnective in $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n}$.

In addition, the canonical map $\mathbb{V}_{\mathrm{ord}_n,m+1} \to \mathbb{V}_{\mathrm{ord}_n,m}$ is an epimorphism in $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\heartsuit}_{\mathrm{ord}_n}$. Indeed, it suffices to show that the (homotopy) kernel of this map is in cohomological degree 0, and the above logic applies just as well to see this.

Step 2. Define $\mathrm{Oblv} : \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{\mathrm{ord}_n} \to \mathsf{Vect}$ as $\mathrm{Oblv} \circ \rho$. We claim that $\mathrm{Oblv}\,|_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{\mathrm{ord}_n}}$ is conservative and t-exact.

Suppose $\mathcal{F} \in \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\geqslant 0}_{\mathrm{ord}_n}$ with $\mathrm{Oblv}(\mathcal{F}) = 0$; it suffices to show that $H^0(\mathcal{F}) = 0$. To this end, it suffices to show that any morphism $\eta : \mathcal{G} \to \mathcal{F}$ is nullhomotopic for a connective object $\mathcal{G} \in \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{\mathrm{ord}_n}$.

Note that the top cohomology group $H^0(\mathcal{G})$ is finitely generated as a module over $U(\widehat{\mathfrak{g}}_{crit})$, say by $v_1, \ldots, v_N \in H^r(\mathcal{G})$. By Lemma 6.8.1, for each $i = 1, \ldots, N$ we can find $m_i \gg 0$ and a map $\alpha_i : \mathbb{V}_{\mathrm{ord}_n,m_i} \to \mathcal{G}$ such that $H^0(\alpha_i)$ maps the vacuum vector in $\mathbb{V}_{\mathrm{ord}_n,m_i}$ to v_i.

Let $\alpha : \oplus_{i=1}^N \mathbb{V}_{\mathrm{ord}_n,m_i} \to \mathcal{G}$ be the induced map; α is surjective on H^0 by design, so $\mathrm{Coker}(\alpha)$ is in cohomological degrees $\leqslant -1$. It follows that $\mathcal{G} \to \mathcal{F}$ is nullhomotopic if and only if its composition with α is. Therefore, it suffices to show that any map $\mathbb{V}_{\mathrm{ord}_n,m} \to \mathcal{F}$ is nullhomotopic.

The map:

$$H^0\big(\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}}(\mathbb{V}_{\mathrm{ord}_n,m}, \mathcal{F})\big) = \mathrm{Hom}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{\mathrm{ord}_n}}(\mathbb{V}_{\mathrm{ord}_n,m}, H^0(\mathcal{F})) \to$$

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{\mathrm{ord}_n}}(\mathbb{V}_{\mathrm{ord}_n,m+1}, H^0(\mathcal{F})) = H^0\big(\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}}(\mathbb{V}_{\mathrm{ord}_n,m+1}, \mathcal{F})\big) \in \mathsf{Vect}^{\heartsuit}$$

is injective by Step 1. But we have:

$$\operatorname*{colim}_m \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}}(\mathbb{V}_{\mathrm{ord}_n,m}, \mathcal{F}) = \mathrm{Oblv}(\mathcal{F}) = 0$$

by Lemma 6.8.1 (and compactness of $\mathbb{V}_{\mathrm{ord}_n,m}$), giving the claim.

Step 3. We now show t-exactness of ρ. Right t-exactness follows immediately from the construction, so we show left t-exactness.

Let $m \geqslant n$ be fixed. It what follows, we regard $\mathbb{V}_{crit,m}$ as an object of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^c_{\mathrm{ord}_m} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_m,naive}$.

As $r \geqslant m$ varies, we have natural maps:

$$\ldots \to i^*_{n,r+1} i_{m,r+1,*}(\mathbb{V}_{crit,m}) \to i^*_{n,r} i_{m,r,*} \mathbb{V}_{crit,m} \to \ldots \to i^*_{n,m} \mathbb{V}_{crit,m} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^c_{\mathrm{ord}_n}.$$

We claim that for $\mathcal{F} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$, the natural map:

$$\operatorname*{colim}_r \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}}(i^*_{n,r} i_{m,r,*} \mathbb{V}_{crit,m}, \mathcal{F}) \to \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}}(\mathbb{V}_{crit,m}, i_{n,*}\rho(\mathcal{F})) \tag{6.9.1}$$

is an isomorphism. Indeed, both sides commute with colimits in $\mathcal{F}$ by compactness, so we are reduced to the case where $\mathcal{F} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^c_{\mathrm{ord}_n}$. For such $\mathcal{F}$, the claim follows from (6.8.1).

Now suppose that $\mathcal{F} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\geqslant 0}_{\mathrm{ord}_n}$. As each object $i^*_{n,r} i_{m,r,*} \mathbb{V}_{crit,m}$ is connective in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$, (6.9.1) implies that $\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}}(\mathbb{V}_{crit,m}, i_{n,*}\rho(\mathcal{F})) \in \mathsf{Vect}^{\geqslant 0}$. As the objects $\mathbb{V}_{crit,m}$ generate $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$ under colimits, this implies that $i_{n,*}\rho(\mathcal{F})$ lies in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\geqslant 0}$, i.e., $i_{n,*}\rho$ is left t-exact.

Finally, as $i_{n,*}$ is t-exact and conservative by Proposition 6.6.1, ρ itself must be left t-exact.

Step 4. Finally, we show that ρ induces an equivalence on eventually coconnective subcategories.

By t-exactness of ρ, we have a commutative diagram:

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{\mathrm{ord}_n} & \xrightarrow{\quad\rho\quad} & \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{\mathrm{ord}_n,naive} \\ & \searrow \quad \swarrow & \\ & \mathsf{Vect}^+ & \end{array}$$

with the diagonal arrows being the forgetful functors. Each of these functors is conservative.

Moreover, the forgetful functor $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n} \to \mathsf{Vect}$ is corepresented by the pro-object:

$$\lim_m \mathbb{V}_{\mathrm{ord}_n,m} \in \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^c_{\mathrm{ord}_n}) \subseteq \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}).$$

Indeed, this follows immediately from Lemma 6.8.1 and compactness of $\mathbb{V}_{\mathrm{ord}_n,m} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$.

Applying Lemma 6.8.1 again, we see that $\rho : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{\mathrm{ord}_n} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{\mathrm{ord}_n,naive}$ intertwines the pro-left adjoints to the forgetful functors in the above diagram. Therefore, it induces an equivalence on the corresponding $\overrightarrow{\otimes}$-algebras, so we obtain the claim from [Ras6] Proposition 3.7.1.

□

Remark 6.9.4. Unlike $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$, we are not aware of an explicit description of compact generators of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$. For instance, does $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$ admit compact generators that admit weakly $G(O)$-equivariant structures? Does it admit compact generators lying in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{\mathrm{ord}_n}$? (For $G = PGL_2$ and $n = 0$, the answer to both questions is yes by Theorem 7.14.1.)

This general issue is closely related to the technical problems highlighted in §6.10.

6.10. **Equivariant renormalization.** We now highlight a technical problem: there is not an evident critical level $G(K)$-action on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$. (Similarly, we cannot construct a weak $G(K)$-action in the sense of [Ras6].)

Conjecture 6.10.1. *For any $\mathcal{F} \in D^*_{crit}(G(K))$ compact, define a functor:*

$$\chi_{\mathcal{F}} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$$

whose restriction to $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{c}_{\mathrm{ord}_n}$ *is calculated as the composition:*

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{c}_{\mathrm{ord}_n} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive} \xrightarrow{\mathcal{F}\star -} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n,naive} \simeq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{\mathrm{ord}_n} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}.$$

Then we conjecture that $\chi_{\mathcal{F}}$ *is left t-exact up to shift.*

Remark 6.10.1. Assuming Conjecture 6.10.1 if K is prounipotent, say, then we obtain $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{K}_{\mathrm{ord}_n} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$ as the essential image of χ_{δ_K}. Without assuming the conjecture, we are not otherwise aware of a good definition of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{K}_{\mathrm{ord}_n}$.

Remark 6.10.2. In the language of [Ras6] §4.4, the above conjecture asserts that the functor $\mathcal{F}\star - : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}$ *renormalizes*.

Remark 6.10.3. Suppose Conjecture 6.10.1 holds for a reductive group G and an integer $n \geqslant 0$. Then there exists a unique critical level $G(K)$-action on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$ such that:

- The functor ρ upgrades to a morphism of categories with critical level $G(K)$-actions.
- The (critical level) $G(K)$-action on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n}$ is strongly compatible with the t-structure in the sense of [Ras6] §10.12.

Indeed, this is essentially immediate from [Ras6] Lemma 8.16.4.

Remark 6.10.4. The technical issue associated with the above conjecture appears implicitly in [FG4].

In §4.1.4 of *loc. cit.*, Frenkel and Gaitsgory suggest a definition of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{K}_{\mathrm{ord}_n}$ (adapted to their particular setting). But their definition is not clearly a good one: for example, it is not clear that their category carries the expected Hecke symmetries. This issue is discussed somewhat in the remark in that same section. Related to that discussion, Main Theorem 2 from *loc. cit.* in effect verifies the above conjecture in a special case.

Combined with our proof of Theorem 6.10.5, it may be fair to expect verifying Conjecture 6.10.1 in a given instance requires substantial input from local geometric Langlands.

As an immediate consequence of our main theorem, Theorem 7.14.1, we may deduce:

Theorem 6.10.5. *Suppose* $G = PGL_2$ *and* $n = 0$. *Then Conjecture 6.10.1 holds.*

Conversely, if we a priori knew Theorem 6.10.5, then the proof that the functor in Theorem 7.14.1 is an equivalence could be substantially simplified: the proof of Lemma 7.17.1 would be applicable and would directly give the essential surjectivity of Γ^{Hecke} (c.f. the outline from §1.17).

7. The localization theorem

7.1. This section begins our study of the Frenkel-Gaitsgory conjecture.

First, we recall the constructions underlying the Frenkel-Gaitsgory localization conjecture, following [FG2]. We include more attention to derived issues than *loc. cit.*, so our discussion distinguishes between naive and renormalized categories of regular Kac-Moody modules.

We then formulate our main result, Theorem 7.14.1.

Next, we recall the main results of Frenkel-Gaitsgory. We include some details on how to deduce the corresponding results in the DG framework from the exact results that they showed.

Finally, in §7.17, we formulate three lemmas from which we deduce Theorem 7.14.1. The proofs of these lemmas occupy the remainder of the section.

7.2. **Regular Kac-Moody representations.** In the setting of §6, for $n = 0$, we prefer the notation reg to ord_0. So we let:

$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive} := \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_0,naive}$$
$$\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg} := \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_0}.$$

We highlight that the subscript reg is being used in a completely different way than it was in §3. In the Kac-Moody context, this terminology rather follows [FG1]. (We believe that this point should not cause confusion in navigating the paper.)

Finally, we let $\mathbb{V}_{crit} := \mathbb{V}_{0,crit}$ denote the critical level vacuum representation.

7.3. **Notation regarding geometric Satake.** Let $\mathcal{H}_{sph} = D_{crit}(\mathrm{Gr}_G)^{G(O)}$, considered as a monoidal category via convolution. Recall that for any $\mathcal{C} \in G(K)\text{–}\mathsf{mod}_{crit}$, there is a canonical action of $\mathcal{H}_{sph}$ on $\mathcal{C}^{G(O)}$ coming from the identifications $\mathcal{H}_{sph} = \mathsf{End}_{G(K)\text{–}\mathsf{mod}_{crit}}(D_{crit}(\mathrm{Gr}_G))$ and $\mathcal{C}^{G(O)} = \mathsf{Hom}_{G(K)\text{–}\mathsf{mod}_{crit}}(D_{crit}(\mathrm{Gr}_G), \mathcal{C})$.

In particular, $\mathcal{H}_{sph}$ acts canonically on $D_{crit}(\mathrm{Gr}_G) = D_{crit}(G(K))^{G(O)}$.

Next, recall that there is a canonical monoidal functor $\mathsf{Rep}(\check{G}) \to \mathcal{H}_{sph}$. This functor is characterized by the fact that it is t-exact and the monoidal equivalence on abelian categories defined by [MV1]. As in [GL], this functor is actually *more* naturally defined when the critical twisting is included, unlike in [MV1].

We refer to the above functor as the geometric Satake functor and denote it by $V \mapsto \mathcal{S}_V$.

In what follows, whenever we consider $D_{crit}(\mathrm{Gr}_G)$ as a $\mathsf{Rep}(\check{G})$-module category, it is via this construction.

7.4. **The canonical torsor.** Let $\mathcal{P}_{\mathrm{Op}^{reg}_{\check{G}}}$ denote the canonical $\check{G}$-bundle on $\mathrm{Op}^{reg}_{\check{G}}$; by definition, it corresponds to the forgetful map $\mathrm{Op}^{reg}_{\check{G}} \to \mathrm{LocSys}_{\check{G}}(\mathcal{D}) = \mathbb{B}\check{G}$.

We obtain a symmetric monoidal functor $\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$. We denote this functor $V \mapsto V_{\mathcal{P}_{\mathrm{Op}^{reg}_{\check{G}}}}$. Note that for $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ finite-dimensional, $V_{\mathcal{P}_{\mathrm{Op}^{reg}_{\check{G}}}}$ is a vector bundle on $\mathrm{Op}^{reg}_{\check{G}}$.

Throughout this section, whenever we consider $\mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$ as a $\mathsf{Rep}(\check{G})$-module category, it is via this construction.

7.5. **Hecke D-modules.** Define $D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)$ as:

$$D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G) := D_{crit}(\mathrm{Gr}_G) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}).$$

By construction, $D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)$ is canonically a $D_{crit}(G(K)) \otimes \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$-module category.

Remark 7.5.1. The above may be considered as a variant of the category:

$$D^{\mathrm{Hecke}}_{crit}(\mathrm{Gr}_G) := D_{crit}(\mathrm{Gr}_G) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{Vect}$$

that is suitably parametrized by regular opers. The category $D^{\mathrm{Hecke}}_{crit}(\mathrm{Gr}_G)$ is the category of Hecke eigenobjects in $D_{crit}(\mathrm{Gr}_G)$; its Iwahori equivariant subcategory was studied in detail in [ABB$^+$].

7.6. There is a natural functor:

$$\mathrm{ind}^{\mathrm{Hecke}_\mathfrak{Z}} : D_{crit}(\mathrm{Gr}_G) \to D^{\mathrm{Hecke}_\mathfrak{Z}}_{crit}(\mathrm{Gr}_G)$$

defined as the composition:

$$D_{crit}(\mathrm{Gr}_G) = D_{crit}(\mathrm{Gr}_G) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{Rep}(\check{G}) \to D_{crit}(\mathrm{Gr}_G) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}) = D^{\mathrm{Hecke}_\mathfrak{Z}}_{crit}(\mathrm{Gr}_G).$$

Because because $\mathrm{Op}^{reg}_{\check{G}} \to \mathbb{B}\check{G}$ is affine, $\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$ admits a continuous, conservative, right adjoint that is a morphism of $\mathsf{Rep}(\check{G})$-module categories. By functoriality, the same is true of $\mathrm{ind}^{\mathrm{Hecke}_\mathfrak{Z}}$; we denote this right adjoint by $\mathrm{Oblv}^{\mathrm{Hecke}_\mathfrak{Z}}$.

In particular, we deduce that $D^{\mathrm{Hecke}_\mathfrak{Z}}_{crit}(\mathrm{Gr}_G)$ is compactly generated with compact generators of the form $\mathrm{ind}^{\mathrm{Hecke}_\mathfrak{Z}}(\mathcal{F})$ for $\mathcal{F} \in D_{crit}(\mathrm{Gr}_G)$ compact.

7.7. The DG category $D^{\mathrm{Hecke}_\mathfrak{Z}}_{crit}(\mathrm{Gr}_G)$ carries a canonical t-structure that plays an important role.

We construct the t-structure by setting connective objects to be generated under colimits by objects of the form $\mathrm{ind}^{\mathrm{Hecke}_\mathfrak{Z}}(\mathcal{F})$ for $\mathcal{F} \in D_{crit}(\mathrm{Gr}_G)^{\leqslant 0}$.

By construction, the composition $\mathrm{Oblv}^{\mathrm{Hecke}_\mathfrak{Z}} \mathrm{ind}^{\mathrm{Hecke}_\mathfrak{Z}} : D_{crit}(\mathrm{Gr}_G) \to D_{crit}(\mathrm{Gr}_G)$ is given by convolution with a spherical D-module in the heart of the t-structure, namely, the object corresponding under Satake to functions on $\mathcal{P}_{\mathrm{Op}^{reg}_{\check{G}}}$ (considered as an object of $\mathsf{Rep}(\check{G})$ in the obvious way). Therefore, by [Gai1] (or [FG2] §8.4), this monad is t-exact on $D_{crit}(\mathrm{Gr}_G)$.

One deduces by standard methods that $\mathrm{Oblv}^{\mathrm{Hecke}_\mathfrak{Z}}$ and $\mathrm{ind}^{\mathrm{Hecke}_\mathfrak{Z}}$ are t-exact. In particular, because $\mathrm{Oblv}^{\mathrm{Hecke}_\mathfrak{Z}}$ is t-exact, conservative, and $G(K)$-equivariant, we find that the t-structure on $D^{\mathrm{Hecke}_\mathfrak{Z}}_{crit}(\mathrm{Gr}_G)$ is strongly compatible with the (critical level) $G(K)$-action in the sense of [Ras6] §10.12.

7.8. In §7.9-7.12, following Frenkel-Gaitsgory, we will construct canonical global sections functors:

$$\begin{array}{ccc} D^{\mathrm{Hecke}_\mathfrak{Z}}_{crit}(\mathrm{Gr}_G) & & \\ \Big\downarrow{\scriptstyle \Gamma^{\mathrm{Hecke}}} & \searrow^{\Gamma^{\mathrm{Hecke},naive}} & \\ \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg} & \xrightarrow{\rho} & \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive} \end{array}$$

that are our central objects of study.

7.9. **The Hecke property of the vacuum representation.** The construction of global sections functors as above is based on the following crucial construction of Beilinson-Drinfeld.

Theorem 7.9.1 (Beilinson-Drinfeld). *For $\mathbb{V}_{crit} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{reg} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive} \in G(K)\text{–}\mathsf{mod}_{crit}$ the vacuum representation and $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ finite-dimensional, the convolution $\mathcal{S}_V \star \mathbb{V}_{crit} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{G(O)}_{reg,naive}$ lies in the heart of the t-structure.*

Moreover, there is a canonical isomorphism:

$$\beta_V : \mathcal{S}_V \star \mathbb{V}_{crit} \xrightarrow{\simeq} \mathbb{V}_{crit} \underset{\mathrm{Op}^{reg}_{\check{G}}}{\otimes} V_{\mathcal{P}_{\mathrm{Op}^{reg}_{\check{G}}}} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{G(O),\heartsuit}_{reg}.$$

For $V, W \in \mathsf{Rep}(\check{G})^{\heartsuit}$ finite-dimensional, the following diagram in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{G(O),\heartsuit}_{reg} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{G(O)}_{reg,naive}$ commutes:

$$\begin{array}{ccccc} \mathcal{S}_W \star \mathcal{S}_V \star \mathbb{V}_{crit} & \xrightarrow{\mathcal{S}_W \star \beta_V} & \mathcal{S}_W \star \mathbb{V}_{crit} \underset{\mathrm{Op}_{\check{G}}^{reg}}{\otimes} V_{\mathcal{P}_{\mathrm{Op}_{\check{G}}^{reg}}} & \xrightarrow{\beta_W} & \mathbb{V}_{crit} \underset{\mathrm{Op}_{\check{G}}^{reg}}{\otimes} W_{\mathcal{P}_{\mathrm{Op}_{\check{G}}^{reg}}} \underset{\mathrm{Op}_{\check{G}}^{reg}}{\otimes} V_{\mathcal{P}_{\mathrm{Op}_{\check{G}}^{reg}}} \\ \downarrow \simeq & & & & \| \\ \mathcal{S}_{W\otimes V} \star \mathbb{V}_{crit} & & \xrightarrow{\beta_{W\otimes V}} & & \mathbb{V}_{crit} \underset{\mathrm{Op}_{\check{G}}^{reg}}{\otimes} (W\otimes V)_{\mathcal{P}_{\mathrm{Op}_{\check{G}}^{reg}}} \end{array}$$

Here the left isomorphism comes from geometric Satake.

We refer to [BD1] §5.5-6 and [Ras1] for proofs and further discussion.

7.10. Let us reformulate the Hecke property more categorically.

For any $\mathcal{C} \in G(K)$–mod_{crit}, $\mathsf{Rep}(\check{G})$ acts on $\mathcal{C}^{G(O)}$ via the monoidal functor $\mathsf{Rep}(\check{G}) \to \mathcal{H}_{sph} \curvearrowright \mathcal{C}^{G(O)}$, where the first functor is the geometric Satake functor.

For $\mathcal{C} = \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg,naive}$, we also have an action of $\mathsf{Rep}(\check{G})$ on $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg,naive}$ via the (symmetric) monoidal functor $\mathsf{Rep}(\check{G}) \to \mathsf{QCoh}(\mathrm{Op}_{\check{G}}^{reg})$ defined by $\mathcal{P}_{\mathrm{Op}_{\check{G}}^{reg}}$. By construction, this action commutes with the $G(K)$-action.

Therefore, $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg,naive}^{G(O)}$ is canonically a $\mathsf{Rep}(\check{G})$-bimodule category.

Corollary 7.10.1. *There is a unique morphism:*

$$\lambda : \mathsf{Rep}(\check{G}) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}^{G(O)} \in \mathsf{Rep}(\check{G})\text{–}\mathsf{bimod}$$

of $\mathsf{Rep}(\check{G})$*-bimodule categories sending the trivial representation* $k \in \mathsf{Rep}(\check{G})$ *to* $\mathbb{V}_{crit}$ *and such that for any finite-dimensional representation* $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$*, the isomorphism:*

$$\mathcal{S}_V \star \mathbb{V}_{crit} = \lambda(V \otimes k) = \lambda(k \otimes V) = \mathbb{V}_{crit} \underset{\mathrm{Op}_{\check{G}}^{reg}}{\otimes} V_{\mathcal{P}_{\mathrm{Op}_{\check{G}}^{reg}}}$$

is the isomorphism β_V *of Theorem 7.9.1.*

Proof. Suppose $H_1 \subseteq H_2$ are affine algebraic groups with H_2/H_1 affine, and let $\mathcal{C} \in \mathsf{Rep}(H_2)$–$\mathsf{mod}$. Then the functor:

$$\mathsf{Hom}_{\mathsf{Rep}(H_2)\text{–}\mathsf{mod}}(\mathsf{Rep}(H_1), \mathcal{C}) \to \mathcal{C}$$

of evaluation on the trivial representation is monadic, with the corresponding monad on $\mathcal{C}$ being given by $\mathrm{Fun}(H_2/H_1) \in \mathsf{ComAlg}(\mathsf{Rep}(H_2))$.[25]

We apply the above to $H_1 = \check{G}$ diagonally embedded into $H_2 = \check{G} \times \check{G}$. We then have $\mathrm{Fun}((\check{G} \times \check{G})/\check{G}) = \mathrm{Fun}(\check{G}) \in \mathsf{Rep}(\check{G} \times \check{G})^{\heartsuit}$, where we consider $\check{G}$ as equipped with its left and right $\check{G}$-actions. We are trying to show that $\mathbb{V}_{crit} \in \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg,naive}^{G(O)}$ admits a unique $\mathrm{Fun}(\check{G})$-module structure satisfying the stated compatibility. In particular, this structure corresponds to certain maps in the *abelian* category $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg,naive}^{G(O),\heartsuit}$, so there are no homotopical issues.

From here, the claim is standard. For example, for V a finite-dimensional representation of $\check{G}$, we have a map $\mu_V : V \otimes V^\vee \to \mathrm{Fun}(\check{G})$ of $\check{G}$-bimodules. The composition of μ_V with the action map for the $\mathrm{Fun}(\check{G})$-module structure on $\mathbb{V}_{crit}$ is given by the map:

[25]This construction extends for H_2/H_1 quasi-affine as well as long as $\mathrm{Fun}(H_2/H_1)$ is replaced by the (derived) global sections $\Gamma(H_2/H_1, \mathcal{O}_{H_2/H_1})$.

$$\mathcal{S}_V \star \mathbb{V}_{crit} \underset{\mathrm{Op}^{reg}_{\check{G}}}{\otimes} V^{\vee}_{\mathcal{P}_{\mathrm{Op}^{reg}_{\check{G}}}} \simeq \mathcal{S}_V \star \mathcal{S}_{V^\vee} \star \mathbb{V}_{crit} = \mathcal{S}_{V\otimes V^\vee} \star \mathbb{V}_{crit} \to \mathbb{V}_{crit}$$

where the first isomorphism is induced by $\beta_{V^\vee}$ and the second isomorphism and the last map is induced by the pairing $V\otimes V^\vee \to k \in \mathsf{Rep}(\check{G})$ (for k the trivial representation). It is straightforward from Theorem 7.9.1 that this defines an action of $\mathrm{Fun}(\check{G})$ as desired.

□

7.11. **Construction of the naive functor.** For any $\mathcal{C} \in G(K)\text{–}\mathsf{mod}_{crit}$, we have a canonical identification:

$$\mathsf{Hom}_{G(K)\text{–}\mathsf{mod}_{crit}}(D_{crit}(\mathrm{Gr}_G), \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}^{G(O)}$$

given by evaluation on $\delta_1 \in D_{crit}(\mathrm{Gr}_G)^{G(O)}$. (Explicitly, the functor $D_{crit}(\mathrm{Gr}_G) \to \mathcal{C}$ corresponding to an object $\mathcal{F} \in \mathcal{C}^{G(O)}$ is given by convolution with $\mathcal{F}$.)

For $\mathcal{C} = \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$ and $\mathbb{V}_{crit} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$, we denote the corresponding functor by $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -) : D_{crit}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$. Note that the composition with the forgetful functor $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$ is the usual (IndCoh-)global sections functor by Appendix A.

Now observe that $D_{crit}(\mathrm{Gr}_G)$ and $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$ are each $D^*_{crit}(G(K)) \otimes \mathsf{Rep}(\check{G})$-module categories. We claim that Corollary 7.10.1 naturally upgrades $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -)$ to a morphism of $D^*_{crit}(G(K)) \otimes \mathsf{Rep}(\check{G})$-module categories.

Indeed, suppose more generally that $\mathcal{C}$ is a $D^*_{crit}(G(K)) \otimes \mathsf{Rep}(\check{G})$-module category. We then have:

$$\mathsf{Hom}_{D^*_{crit}(G(K))\otimes\mathsf{Rep}(\check{G})\text{–}\mathsf{mod}}(D_{crit}(\mathrm{Gr}_G), \mathcal{C}) = \mathsf{Hom}_{\mathsf{Rep}(\check{G})\text{–}\mathsf{bimod}}(\mathsf{Rep}(\check{G}), \mathsf{Hom}_{G(K)\text{–}\mathsf{mod}_{crit}}(D_{crit}(\mathrm{Gr}_G), \mathcal{C})) =$$
$$\mathsf{Hom}_{\mathsf{Rep}(\check{G})\text{–}\mathsf{bimod}}(\mathsf{Rep}(\check{G}), \mathcal{C}^{G(O)}).$$

Therefore, Corollary 7.10.1 has the claimed effect.

Because the action of $\mathsf{Rep}(\check{G})$ on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$ comes from an action of $\mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$, we obtain an induced functor:

$$D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G) = D_{crit}(\mathrm{Gr}_G) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive} \in D^*_{crit}(G(K))\otimes\mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})\text{–}\mathsf{mod}.$$

In what follows, we denote[26] this functor by:

$$\Gamma^{\mathrm{Hecke},naive} = \Gamma^{\mathrm{Hecke},naive}(\mathrm{Gr}_G, -).$$

7.12. **Construction of the renormalized functor.** Next, we construct a functor Γ^{Hecke} valued in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$.

First, we need the following observation.

[26]A comment on the notation:

We use $\mathrm{Hecke}_{\mathfrak{z}}$ rather than Hecke in $D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)$ to distinguish this category from $D^{\mathrm{Hecke}}_{crit}(\mathrm{Gr}_G)$. But the global sections functor is defined only on $D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)$, not on $D^{\mathrm{Hecke}}_{crit}(\mathrm{Gr}_G)$, so we simplify the notation here by omitting the subscript $\mathfrak{z}$.

Lemma 7.12.1. *Suppose H is a Tate group indschenme, $K \subseteq H$ is a polarization (i.e., a compact open subgroup with H/K ind-proper). Let $\mathcal{F} \in D(H/K)$ be compact. Then for any $\mathcal{C} \in H$–mod, the functor:*

$$\mathcal{F} \star - : \mathcal{C}^K \to \mathcal{C}$$

admits a continuous right adjoint.

Proof. Let $\mathbb{D}\mathcal{F} \in D(H/K)$ denote the Verdier dual to $\mathcal{F}$, and let $\mathrm{inv}\, \mathbb{D}\mathcal{F} \in D(K\backslash H)$ denote the pullback along the inversion map. As in [FG2] Proposition 22.10.1, the functor:

$$\mathrm{inv}\, \mathbb{D}\mathcal{F} \star - : \mathcal{C} \to \mathcal{C}^K$$

canonically identifies with the desired right adjoint.

Alternatively, we may write convolution as a composition:

$$D(H)^K \otimes \mathcal{C}^K \to D(H) \overset{K}{\otimes} \mathcal{C} \to \mathcal{C}$$

and each of these functors admits a continuous right adjoint (the former because K is a group scheme, and the latter because H/K is ind-proper). This formally implies the claim.

□

By Lemma 7.12.1, the global sections functor $\rho \circ \Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -) : D_{crit}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg,naive}$ preserves compact objects; indeed, it is given as convolution with $\mathbb{V}_{crit} \in \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{G(O)}$, which is compact.

Therefore, the functor $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -)$ maps $D_{crit}(\mathrm{Gr}_G)^c$ to $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{reg}$.

From §7.6, we deduce that $\Gamma^{\mathrm{Hecke},naive}$ maps compact objects in $D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)$ into $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^c_{reg}$.

We now define:

$$\Gamma^{\mathrm{Hecke}} = \Gamma^{\mathrm{Hecke}}(\mathrm{Gr}_G, -) : D^{\mathrm{Hecke}_\mathfrak{z}}_{crit} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$$

as the ind-extension of:

$$\Gamma^{\mathrm{Hecke},naive}|_{D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^c} : D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G))^c \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^c_{reg}.$$

7.13. By abuse of notation, we let $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -)$ denote the induced functor $\Gamma^{\mathrm{Hecke}} \circ \mathrm{ind}^{\mathrm{Hecke}_\mathfrak{z}}$, so we have a commutative diagram:

$$\begin{array}{ccc} D_{crit}(\mathrm{Gr}_G) & \xrightarrow{\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G,-)} & \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg} \\ & \searrow^{\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G,-)} & \downarrow \rho \\ & & \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}. \end{array}$$

(This abuse is mild because of Corollary 7.15.2 below.)

7.14. **Main result.** We can now state the main theorem of this paper in its precise form.

Theorem 7.14.1. *For G of semisimple rank* 1*, the functor Γ^{Hecke} is a t-exact equivalence.*

In the remainder of this section, we review some general results of Frenkel-Gaitsgory on Γ^{Hecke} and then formulate some intermediate results in this case from which we will deduce Theorem 7.14.1. The proofs of those intermediate results occupy the remainder of the paper.

7.15. **Review of some results of Frenkel-Gaitsgory.** The following exactness result was essentially shown in [FG1].

Theorem 7.15.1 ([FG1], Theorem 1.2). *The functor:*

$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -) = \Gamma^{\mathrm{Hecke},naive} \circ \mathrm{ind}^{\mathrm{Hecke}_!} : D_{crit}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$$

is t-exact.

There is something to do to properly deduce this from the Frenkel-Gaitsgory result, so we include a few comments.

Because $D_{crit}(\mathrm{Gr}_G)$ is compactly generated and compact objects are closed under truncations, it suffices to show that compact objects in $D_{crit}(\mathrm{Gr}_G)$ lying in the heart of the t-structure map into $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{reg,naive}$.

By Proposition 6.6.1, we are reduced to verifying this result after composing with the functor $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$.

By Lemma 9.2.2, for $\mathcal{F} \in D_{crit}(\mathrm{Gr}_G)$ compact, $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}) = \mathcal{F} \star \mathbb{V}_{crit} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$ is eventually coconnective. Therefore, it suffices to show that when considered as an object of Vect, $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F})$ lies in $\mathsf{Vect}^{\heartsuit}$.

Now the result follows from [FG1] Theorem 1.2 and the comparison results of Appendix A.[27,28]

Corollary 7.15.2. *The functor* $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -) : D_{crit}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$ *is t-exact.*

Proof. For $\mathcal{F} \in D_{crit}(\mathrm{Gr}_G)^{\heartsuit}$ compact and hence, compact in $D_{crit}(\mathrm{Gr}_G)$, $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F})$ is compact in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$ by construction, so lies in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{reg}$. Therefore, by Theorem 7.15.1, we deduce that $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{reg}$.

Because $D_{crit}(\mathrm{Gr}_G)^{\heartsuit}$ is compactly generated and our t-structures are compatible with filtered colimits, we obtain the claim.

□

Corollary 7.15.3. *The functor* $\Gamma^{\mathrm{Hecke}} : D^{\mathrm{Hecke}_!}_{crit}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$ *is right t-exact.*

Proof. By construction, $D^{\mathrm{Hecke}_!}_{crit}(\mathrm{Gr}_G)^{\leqslant 0}$ is generated under colimits by objects of the form $\mathrm{ind}^{\mathrm{Hecke}_!}(\mathcal{F})$ for $\mathcal{F} \in D_{crit}(\mathrm{Gr}_G)^{\leqslant 0}$. Then $\rho\Gamma^{\mathrm{Hecke}}(\mathrm{ind}^{\mathrm{Hecke}_!}(\mathcal{F})) = \Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F})$ lies in degrees $\leqslant 0$ by Theorem 7.15.1, so $\Gamma^{\mathrm{Hecke}}(\mathrm{ind}^{\mathrm{Hecke}_!}(\mathcal{F}))$ lies in degrees $\leqslant 0$ and we obtain the claim.

□

[27] In fact, that $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, -)$ as a functor $D_{crit}(\mathrm{Gr}_G)^{\heartsuit} \to \mathsf{Vect}$ coincides with the standard global sections functor is one of the easier results in Appendix A; it is shown directly in §A.9.

[28] Formally, [FG1] Theorem 1.2 only asserts that the non-derived global sections functor is exact on $D_{crit}(\mathrm{Gr}_G)^{\heartsuit}$, not exactly that higher cohomology groups vanish. As the argument is missing in the literature, we indicate the details here.

For any formally smooth $\aleph_0$-indscheme S of ind-finite type, we claim that if $H^0\Gamma^{\mathsf{IndCoh}}(S, -) : D(S)^{\heartsuit} \to \mathsf{Vect}^{\heartsuit}$ is exact, then $\Gamma^{\mathsf{IndCoh}}(S, -) : D(S) \to \mathsf{Vect}$ is t-exact, and similarly for twisted D-modules.

Indeed, we are reduced to showing that the restriction to $D(S)^+$ is t-exact. This category is the bounded below derived category of its heart by [Ras5] Lemma 5.4.3 and the corresponding assertion for finite type schemes. Therefore, it suffices to show that $\Gamma^{\mathsf{IndCoh}}(S, -)$ is the derived functor of $H^0\Gamma^{\mathsf{IndCoh}}(S, -)$, or equivalently, that $\Gamma^{\mathsf{IndCoh}}(S, -)$ a priori maps injective objects in $D(S)^{\heartsuit}$ into $\mathsf{Vect}^{\heartsuit}$.

Formal smoothness of S implies that $\mathrm{ind} : \mathsf{IndCoh}(S) \to D(S)$ is t-exact, so its t-exact right adjoint $\mathrm{Oblv} : D(S) \to \mathsf{IndCoh}(S)$ preserves injective objects. Therefore, we are reduced to showing that $\Gamma^{\mathsf{IndCoh}}(S, -)$ maps injective objects in $\mathsf{IndCoh}(S)^{\heartsuit}$ into $\mathsf{Vect}^{\heartsuit}$.

As S is a classical indscheme by [GR2], an argument along the lines of the proof of [Ras5] Lemma 5.4.3 reduces us to the corresponding assertion for finite type classical schemes. As $\mathsf{IndCoh}(S)^{\heartsuit} = \mathsf{QCoh}(S)^{\heartsuit}$ with Γ^{IndCoh} corresponding to Γ, the assertion here is standard.

7.16. **Fully faithfulness.** Next, we review the fully faithfulness of Γ^{Hecke}, which was essentially shown in [FG2] Theorem 8.7.1. For the sake of completeness, we include the reduction to a calculation performed in [FG2].

Theorem 7.16.1 (Modified Frenkel-Gaitsgory). *For any reductive G, the functor Γ^{Hecke} is fully faithful.*

This result can be deduced from [FG2] Theorem 8.7.1. As the argument in *loc. cit.* is quite involved, we present a simpler one in Appendix B based on the ideas of the current paper (especially the use of Whittaker categories).

7.17. **Intermediate results.** We now formulate three results whose proofs we defer to subsequent sections.

For each of the following results, we assume G has semisimple rank 1; we do not do not know how to prove any of these lemmas for GL_3.

Lemma 7.17.1. *Let $\widehat{\mathfrak{g}}_{crit}\text{-}\widetilde{\mathsf{mod}}_{reg,naive} \subseteq \widehat{\mathfrak{g}}_{crit}\text{-}\mathsf{mod}_{reg,naive}$ be the full subcategory generated by $\widehat{\mathfrak{g}}_{crit}\text{-}\mathsf{mod}^{+}_{reg,naive}$ under colimits.*[29]

Then the essential image of $\Gamma^{\mathrm{Hecke},naive}$ lies in $\widehat{\mathfrak{g}}_{crit}\text{-}\widetilde{\mathsf{mod}}_{reg,naive}$ and generates it under colimits.

Lemma 7.17.2. *The functor $\Gamma^{\mathrm{Hecke},naive}$ is t-exact.*

Lemma 7.17.3. *For every $K \subseteq G(O)$ a compact open subgroup, the composition:*

$$D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)^K \to D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G) \xrightarrow{\Gamma^{\mathrm{Hecke}}} \widehat{\mathfrak{g}}_{crit}\text{-}\mathsf{mod}_{reg}$$

is left t-exact up to shift.

Assuming these results, let us show Theorem 7.14.1.

Proof of Theorem 7.14.1.

Step 1. First, we show that Γ^{Hecke} is t-exact.

By Theorem 7.15.1 and the definition of the t-structure on $D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)$, Γ^{Hecke} is right t-exact.

To see left t-exactness, it suffices to see that for any compact open subgroup $K \subseteq G(O)$, $\Gamma^{\mathrm{Hecke}}|_{D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)^K}$ is left t-exact. Indeed, for any $\mathcal{F} \in D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)$, $\mathcal{F} = \mathrm{colim}_K \mathrm{Oblv}\,\mathrm{Av}^K_*(\mathcal{F})$, and $\mathrm{Oblv}\,\mathrm{Av}^K_*$ is left t-exact by the discussion of §7.7.

By Lemma 7.17.3, $\Gamma^{\mathrm{Hecke}}|_{D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)^K}$ is left t-exact up to shift. Because $\rho : \widehat{\mathfrak{g}}_{crit}\text{-}\mathsf{mod}^{+}_{reg} \to \widehat{\mathfrak{g}}_{crit}\text{-}\mathsf{mod}^{+}_{reg,naive}$ is a t-exact equivalence, it suffices to see that $\rho \circ \Gamma^{\mathrm{Hecke}}|_{D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)^K}$ is left t-exact. But this is immediate from Lemma 7.17.2.

Step 2. By Theorem 7.16.1, it suffices to show that Γ^{Hecke} is essentially surjective.

First, the composition:

$$D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G) \xrightarrow{\Gamma^{\mathrm{Hecke},naive}} \widehat{\mathfrak{g}}_{crit}\text{-}\widetilde{\mathsf{mod}}_{reg,naive} \xrightarrow{\tau^{\geqslant 0}} \widehat{\mathfrak{g}}_{crit}\text{-}\mathsf{mod}^{\geqslant 0}_{reg,naive} \tag{7.17.1}$$

generates the target under colimits. Indeed, the first functor generates under colimits by Lemma 7.17.1, and the second functor is essentially surjective because $\widehat{\mathfrak{g}}_{crit}\text{-}\widetilde{\mathsf{mod}}_{reg,naive}$ contains $\widehat{\mathfrak{g}}_{crit}\text{-}\mathsf{mod}^{\geqslant 0}_{reg,naive}$ by definition.

[29] This is a technical distinction. It may perfectly well be the case that $\widehat{\mathfrak{g}}_{crit}\text{-}\widetilde{\mathsf{mod}}_{reg,naive}$ coincides with $\widehat{\mathfrak{g}}_{crit}\text{-}\mathsf{mod}_{reg,naive}$. But we do not see an argument and do not need to consider this question for the application to Theorem 7.14.1.

By the previous step, if we identify $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\geqslant 0}_{reg,naive}$ with $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\geqslant 0}_{reg}$ via ρ, then (7.17.1) factors through $D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{\geqslant 0}$, where it coincides with $\Gamma^{\mathrm{Hecke}}|_{D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{\geqslant 0}}$.

It therefore follows that the essential image of Γ^{Hecke} contains $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\geqslant 0}_{reg}$. Because $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg}$ is compactly generated with compact objects lying in $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{+}_{reg}$, we deduce that the essential image of Γ^{Hecke} is all of $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg}$.

□

8. Equivariant categories

8.1. In this section, we collect some results about $\Gamma^{\mathrm{Hecke},naive}$ and Γ^{Hecke} in the presence of $\mathring{I}$ and Whittaker invariants. These results will be used to establish the results formulated in §7.17.

We emphasize that we have nothing new to say about $\mathring{I}$-invariants; our proofs here consist only of references to [FG6].

Remark 8.1.1. All of the results of this section are valid for a general reductive group G.

8.2. **Iwahori equivariance.** The main result in this setting is the following.

Theorem 8.2.1 (Frenkel-Gaitsgory, [FG6] Theorem 1.7). *The functor* $\Gamma^{\mathrm{Hecke},naive}$ *induces a* t*-exact equivalence:*

$$D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{\mathring{I},+} \xrightarrow{\simeq} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\mathring{I},+}_{reg,naive}$$

on eventually coconnective $\mathring{I}$*-equivariant categories.*

Proof. Because our setting is slightly different from that of [FG6], especially as regards derived categories and derived functors, we indicate the deduction from the results of *loc. cit.*

First, we show t-exactness. By [FG6] Lemma 3.6 and Proposition 3.18, every object $\mathcal{F} \in D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ can be written as a filtered colimit $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ for $\mathcal{F}_i \in D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ admitting a finite filtration with subquotients of the form $\mathrm{ind}^{\mathrm{Hecke}_\mathfrak{z}}(\mathcal{F}_{i,j}) \otimes_{\mathrm{Op}^{reg}_{\check{G}}} \mathcal{H}_{i,j}$ for $\mathcal{F}_{i,j} \in D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ and $\mathcal{H}_{i,j} \in \mathsf{QCoh}(\mathrm{Op}^{\heartsuit}_{\check{G}})$.

We then have:

$$\Gamma^{\mathrm{Hecke},naive}(\mathrm{ind}^{\mathrm{Hecke}_\mathfrak{z}}(\mathcal{F}_{i,j}) \underset{\mathrm{Op}^{reg}_{\check{G}}}{\otimes} \mathcal{H}_{i,j}) = \Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}_{i,j}) \underset{\mathrm{Op}^{reg}_{\check{G}}}{\otimes} \mathcal{H}_{i,j}.$$

By *loc. cit.* Proposition 3.17, $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}_{i,j})$ is flat as an $\mathfrak{z}$-module, so the displayed tensor product is concentrated in cohomological degree 0. This shows that $\Gamma^{\mathrm{Hecke},naive}(\mathcal{F})$ is in degree 0 as well, providing the t-exactness.

Next, observe that fully faithfulness follows from Theorem 7.16.1.

Finally, we show essential surjectivity. By [FG6] Theorem 1.7, Lemma 3.6, Proposition 3.17, and Proposition 3.18, any $\mathcal{G} \in \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\mathring{I},\heartsuit}_{reg,naive}$ can be written as a filtered colimit $\mathcal{G} = \mathrm{colim}_i \mathcal{G}_i$ with $\mathcal{G}_i \in \widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\mathring{I},\heartsuit}_{reg,naive}$ and such that $\mathcal{G}_i$ admits a finite filtration with associated graded terms of the form:

$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \widetilde{\mathcal{G}}) \underset{\mathrm{Op}^{reg}_{\check{G}}}{\otimes} \mathcal{H}$$

for $\widetilde{\mathcal{G}} \in D_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ and $\mathcal{H} \in \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})^{\heartsuit}$ (and where we are using the notation of §6.5), and where the displayed (derived) tensor product is concentrated in cohomological degree 0. Clearly

each associated graded term lies in the essential image of $\Gamma^{\mathrm{Hecke},naive}$, so $\mathcal{G}$ does as well. This implies the essential image of $\Gamma^{\mathrm{Hecke},naive}$ contains $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\mathring{I},\heartsuit}_{reg,naive}$, so all of $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{\mathring{I},+}_{reg,naive}$.

□

We include one other result in a similar spirit.

Proposition 8.2.2. *The functor:*

$$\Gamma^{\mathrm{Hecke}}|_{D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)^{\mathring{I}}} : D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)^{\mathring{I}} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$$

is t-exact.

Proof. By Corollary 7.15.3, the functor is right t-exact. Therefore, we need to show left t-exactness. By Theorem 8.2.1, it suffices to show that objects in $D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ map to eventually coconnective objects.

Suppose $\mathcal{F} \in D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$. As in the proof of Theorem 8.2.1, the results of [FG6] imply that $\mathcal{F}$ can be written as a filtered colimit $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ for $\mathcal{F}_i \in D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ admitting a finite filtration with subquotients of the form $\mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{z}}}(\mathcal{F}_{i,j}) \otimes_{\mathrm{Op}^{reg}_{\check{G}}} \mathcal{H}_{i,j}$ for $\mathcal{F}_{i,j}$ and $\mathcal{H}_{i,j}$ as in the proof of Theorem 8.2.1.

Therefore, we are reduced to showing that:

$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}) \underset{\mathrm{Op}^{reg}_{\check{G}}}{\otimes} \mathcal{H} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg} \tag{8.2.1}$$

is eventually coconnective for any $\mathcal{F} \in D_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ and $\mathcal{H} \in \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})^{\heartsuit}$.

If $\mathcal{F}$ is compact, then $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$ is compact by construction of the functor. In particular, this object is eventually coconnective. By Theorem 8.2.1, we deduce $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{reg}$ in this case. As the t-structures are compatible with filtered colimits, and every object of $D_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ is a filtered colimit of objects in $D_{crit}(\mathrm{Gr}_G)^{\mathring{I},\heartsuit}$ that are compact in $D_{crit}(\mathrm{Gr}_G)$, we obtain the claim for general $\mathcal{F}$ and $\mathcal{H}$ being the structure sheaf.

Now if $\mathcal{H}$ is coherent,[30] then because $\mathrm{Op}^{reg}_{\check{G}} = \mathrm{Spec}(\mathfrak{z})$ with $\mathfrak{z}$ an (infinite) polynomial algebra, $\mathcal{H}$ is perfect. Therefore, the object (8.2.1) is eventually coconnective for general $\mathcal{F}$ and coherent $\mathcal{H}$. Applying Theorem 8.2.1 again, we deduce that (8.2.1) lies in the heart of the t-structure under these same assumptions. Finally, the general case follows as any $\mathcal{H} \in \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})^{\heartsuit}$ is a filtered colimit of coherent objects.

□

8.3. **Whittaker equivariance.** We now study the behavior of $\Gamma^{\mathrm{Hecke},naive}$ under the Whittaker functor, following [Ras5] and [Ras6].

Our main result is the following.

Theorem 8.3.1. *(1) The functor $\Gamma^{\mathrm{Hecke},naive}$ induces an equivalence:*

$$\mathsf{Whit}(D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)) \xrightarrow{\simeq} \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}).$$

(2) For $n > 0$, the functor:

$$\Gamma^{\mathrm{Hecke},naive} : \mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)) \to \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive})$$

[30] I.e., $\mathcal{H}$ corresponds to a finitely presented $\mathfrak{z}$-module.

is a t-exact equivalence for the natural[31] *t-structures on both sides.*

We will verify the above result in what follows after recalling some results on Whittaker categories in this setting.

8.4. We recall the following result, which appears as [Ras6] Theorem 11.19.1 and is an enhancement of the affine Skryabin theorem [Ras5] Theorem 5.1.1.

Theorem 8.4.1. *There is a canonical equivalence of* $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$*-comodule categories:*

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) \simeq \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}).$$

Under this equivalence, the full subcategory (c.f. §5.2) $\mathsf{Whit}^{\leqslant m}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) \subseteq \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})$ *identifies with the full subcategory* $\mathsf{IndCoh}^*_{\mathrm{Op}^{\leqslant m}_{\check{G}}}(\mathrm{Op}_{\check{G}}) \subseteq \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$ *generated under colimits by pushforwards from* $\mathsf{QCoh}(\mathrm{Op}_{\check{G}}) \simeq \mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant m}_{\check{G}}) \to \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$.

Corollary 8.4.2. *For any* n*, there is a canonical equivalence of* $\mathsf{QCoh}(\mathrm{Op}^{\leqslant n}_{\check{G}})$*-module categories:*

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}) \simeq \mathsf{QCoh}(\mathrm{Op}^{\leqslant m}_{\check{G}}).$$

Moreover, for any positive m *with* $m \geqslant n$*, the embedding:*

$$\mathsf{Whit}^{\leqslant m}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}) \to \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive})$$

is an equivalence.

Proof. By [Ras5] Theorem 2.1.1 (or its refinement Theorem 2.7.1, which we recalled above as Theorem 5.2.1), the functor $G(K)\text{–}\mathsf{mod}_{crit} \xrightarrow{\mathcal{C}\mapsto\mathsf{Whit}(\mathcal{C})} \mathsf{DGCat}_{cont}$ is a morphism of DGCat_{cont}-module categories that commutes with limits and colimits.

Therefore, from the definitions, we have:

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}) = \mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) \overset{Thm.8.4.1}{\simeq}$$

$$\mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}) \simeq \mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) \simeq \mathsf{QCoh}(\mathrm{Op}^{\leqslant n}_{\check{G}}).$$

The stabilization of adolescent Whittaker models is proved similarly. For m positive, we have:

$$\mathsf{Whit}^{\leqslant m}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{\mathrm{ord}_n,naive}) = \mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \mathsf{Whit}^{\leqslant m}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) \overset{Thm.8.4.1}{\simeq}$$

$$\mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \mathsf{IndCoh}^*_{\mathrm{Op}^{\leqslant m}_{\check{G}}}(\mathrm{Op}_{\check{G}}) \subseteq$$

$$\mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) \overset{\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})}{\otimes} \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}}) \simeq \mathsf{IndCoh}^*(\mathrm{Op}^{\leqslant n}_{\check{G}}) \simeq \mathsf{QCoh}(\mathrm{Op}^{\leqslant n}_{\check{G}}).$$

The functor at the end of the second line is indeed fully faithful because $\mathsf{IndCoh}^*_{\mathrm{Op}^{\leqslant m}_{\check{G}}}(\mathrm{Op}_{\check{G}}) \to \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$ is fully faithful (by definition) and admits a right adjoint that is a morphism of $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$-module categories. Clearly this functor is essentially surjective for $m \geqslant n$.

□

[31] We remind that $\mathsf{Whit}^{\leqslant n}$ is defined as equivariance against a character for a compact open subgroup. For our two categories, the t-structures are compatible with the $G(K)$-action, so there are natural t-structures on such equivariant categories.

8.5. Before proceeding, we recall that for $\mathcal{C} \in G(K)\text{–}\mathsf{mod}_{crit}$, the functor:

$$\mathsf{Whit}(\mathcal{C}) \xrightarrow{\mathrm{Oblv}} \mathcal{C} \xrightarrow{\mathrm{Av}_*} \mathcal{C}^{G(O)}$$

admits a left adjoint, which we denote $\mathrm{Av}^{\psi}_!$ in what follows. That this left adjoint is defined is the special case $n = 0$, $m = \infty$ of [Ras5] Theorem 2.7.1.

8.6. We now recall the following result.

Theorem 8.6.1 (Frenkel-Gaitsgory-Vilonen, [FGV]). *The composition:*

$$\mathsf{Rep}(\check{G}) \xrightarrow{V \mapsto \mathcal{S}_V} \mathcal{H}_{sph} = D_{crit}(\mathrm{Gr}_G)^{G(O)} \xrightarrow{\mathrm{Av}^{\psi}_!} \mathsf{Whit}(D_{crit}(\mathrm{Gr}_G))$$

is an equivalence.

Remark 8.6.2. Formally, the setting of [FGV] is somewhat different. We refer to [Gai7] for the necessary comparison results.

8.7. We now can prove the main result on Whittaker categories.

Proof of Theorem 8.3.1. We begin with (1). We first construct some equivalence, and then we show that $\Gamma^{\mathrm{Hecke},naive}$ induces the corresponding functor.

By Corollary 8.4.2 (for $n = 0$), we have:

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}) \simeq \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}).$$

Moreover, as Whittaker invariants coincide with coinvariants by [Ras5] Theorem 2.1.1, we can calculate:

$$\mathsf{Whit}(D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)) = \mathsf{Whit}(D_{crit}(\mathrm{Gr}_G)) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}).$$

By Theorem 8.6.1, $\mathsf{Whit}(D_{crit}(\mathrm{Gr}_G))$ identifies canonically with $\mathsf{Rep}(\check{G})$ as a $\mathsf{Rep}(\check{G})$-module category. Therefore, we obtain:

$$\mathsf{Whit}(D^{\mathrm{Hecke}_{\mathfrak{z}}}_{crit}(\mathrm{Gr}_G)) = \mathsf{Rep}(\check{G}) \underset{\mathsf{Rep}(\check{G})}{\otimes} \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}) = \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}).$$

We now show that $\Gamma^{\mathrm{Hecke},naive}$ induces the evident equivalence on Whittaker categories. By construction, $\Gamma^{\mathrm{Hecke},naive}$ is a morphism of $\mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$-module categories. Therefore, it suffices to show that it sends the structure sheaf $\mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}}$ to itself.

This follows from the following diagram, which is commutative by functoriality:

$$\begin{array}{ccccc}
D_{crit}(\mathrm{Gr}_G)^{G(O)} & \xrightarrow{\Gamma^{\mathrm{Hecke},naive}} & \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{G(O)}_{reg,naive} & & \\
\downarrow{\scriptstyle \mathrm{Av}^{\psi}_!} & & \downarrow{\scriptstyle \mathrm{Av}^{\psi}_!} & \searrow & \\
\mathsf{Whit}(D_{crit}(\mathrm{Gr}_G)) & \xrightarrow{\Gamma^{\mathrm{Hecke},naive}} & \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}) & = & \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}).
\end{array}$$

By construction of the equivalence of Theorem 8.4.1, the diagonal arrow in the diagram above is the Drinfeld-Sokolov functor Ψ. Therefore, if we consider the δ D-module $\delta_1 \in D_{crit}(\mathrm{Gr}_G)^{G(O)}$ supported at the origin $1 \in \mathrm{Gr}_G$, apply Hecke induction and the above diagram, we find:

$$\Gamma^{\mathrm{Hecke},naive}(\mathrm{Av}^{\psi}_! \operatorname{ind}^{\mathrm{Hecke}_{\mathfrak{z}}} \delta_1) = \Psi(\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \delta_1)) = \Psi(\mathbb{V}_{crit}).$$

Clearly $\mathrm{Av}^{\psi}_{!}\,\mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}\,\delta_1 \in \mathsf{Whit}(D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G))$ corresponds to $\mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}} \in \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$. Moreover, $\Psi(\mathbb{V}_{crit})$ corresponds to the structure sheaf $\mathcal{O}_{\mathrm{Op}^{reg}_{\check{G}}}$ by design.

We now verify (2). For $n > 0$, we have natural functors:

$$\begin{aligned} \mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)) &\to \mathsf{Whit}(D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)) \\ \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}) &\to \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}) \end{aligned} \tag{8.7.1}$$

as in Theorem 5.2.1, and that we claim are equivalences. In the second case, this assertion is part of Corollary 8.4.2. In the first case, this follows from the fact that:

$$\mathsf{Whit}^{\leqslant 1}(D_{crit}(\mathrm{Gr}_G)) \to \mathsf{Whit}(D_{crit}(\mathrm{Gr}_G))$$

is an equivalence; see [Ras4] Theorem 7.3.1 for a stronger assertion.

It now follows by functoriality and (1) that $\Gamma^{\mathrm{Hecke},naive}$ is an equivalence on $\mathsf{Whit}^{\leqslant n}$ for all $n > 0$.

Finally, we need to show that $\Gamma^{\mathrm{Hecke},naive}$ is t-exact on $\mathsf{Whit}^{\leqslant n}$ for all n.

In [Ras5], the functors:

$$\iota_{n,n+1,!}[-2(\check{\rho},\rho)] : \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) \to \mathsf{Whit}^{\leqslant n+1}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})$$

were shown to be t-exact. Moreover, by the proof of the affine Skryabin theorem Theorem 8.4.1, the resulting t-structure on $\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})$ identifies with the canonical one on $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$. We deduce parallel results for $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$ in place of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$ in the setting of Corollary 8.4.2.

Similarly, the functors:

$$\iota_{n,n+1,!}[-2(\check{\rho},\rho)] : \mathsf{Whit}^{\leqslant n}(D_{crit}(\mathrm{Gr}_G)) \to \mathsf{Whit}^{\leqslant n+1}(D_{crit}(\mathrm{Gr}_G))$$

are t-exact. The resulting t-structure on $\mathsf{Whit}(D_{crit}(\mathrm{Gr}_G))$ identifies with the canonical one on $\mathsf{Rep}(\check{G})$ under Theorem 8.6.1; indeed, the geometric Satake functor $\mathsf{Rep}(\check{G}) \to \mathcal{H}_{sph}$ is t-exact by construction, and $\mathrm{Av}^{\psi}_{!}$ is t-exact by [Ras5] Remark B.7.1. As $\mathrm{Oblv}^{\mathrm{Hecke}_{\mathfrak{Z}}}$ is t-exact, we obtain similar results for $D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)$.

Finally, we deduce t-exactness. Indeed, we have equivalences:

$$\mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)) \simeq \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}) \simeq \mathsf{Whit}^{\leqslant n}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive})$$

with the t-structures on the left and right hand sides corresponding to the canonical t-structure on $\mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$, and the composition being given by $\Gamma^{\mathrm{Hecke},naive}$.

□

8.8. **Exactness of renormalized global sections.** We will also need the following parallel to Proposition 8.2.2.

Proposition 8.8.1. *(1) For any $n \geqslant 1$, the functor:*

$$\Gamma^{\mathrm{Hecke}}|_{\mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G))} : \mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)) \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$$

is t-exact.

(2) More generally, suppose $\mathcal{G} \in D_{crit}(G(K))$ has the following properties:

- *For $m \gg 0$, $\mathcal{G}$ is right K_m-equivariant (where $K_m \subseteq G(O)$ is the mth congruence subgroup).*
- *There exists a K_m-stable closed subscheme $S \subseteq G(K)$ such that $\mathcal{G}$ is supported on S.*
- *As an object of $D(S/K_m)$, $\mathcal{G}$ is eventually coconnective.*

Then for every $\mathcal{F} \in \mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G))^+$, $\Gamma^{\mathrm{Hecke}}(\mathrm{Gr}_G, \mathcal{G} \star \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{reg}$.

Proof. We begin with (1).

As above, we have a t-exact equivalence:

$$\mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)) \simeq \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}}). \tag{8.8.1}$$

As $\mathrm{Op}^{reg}_{\check{G}}$ is the spectrum of a polynomial algebra (however infinitely generated), we deduce that every object of $\mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G))^\heartsuit$ is a filtered colimit of objects that are compact in $\mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G))$, hence in $D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)$.

By construction, Γ^{Hecke} maps compact objects to compact objects, and in particular maps compact objects to $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{reg}$. By Theorem 8.3.1, we deduce that it maps compact objects of $\mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G))$ that lie in the heart of the t-structure into $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^\heartsuit_{reg}$. As every object of $\mathsf{Whit}^{\leqslant n}(D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G))^\heartsuit$ is a filtered colimit of such (by the above), we obtain the result.

We now proceed to (2). We begin by noting that our assumptions imply that for any $\mathcal{C} \in G(K)\text{–}\mathsf{mod}_{crit}$ equipped with a t-structure that is strongly compatible with the $G(K)$-action (in the sense of [Ras6] §10.12), the functor $\mathcal{G} \star - : \mathcal{C} \to \mathcal{C}$ is left t-exact up to shift (see the proof of Lemma 9.2.2 below). This is the key property we will use about $\mathcal{G}$. By [Ras6] Lemma 10.14.1, this property is true for $\mathcal{C} = \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$.

Next, if $\mathcal{F}$ is the object corresponding under (8.8.1) to the structure sheaf on $\mathrm{Op}^{reg}_{\check{G}}$, then $\mathcal{F} = \mathrm{ind}^{\mathrm{Hecke}_\mathfrak{z}}(\delta_n)$ for $\delta_n \in \mathsf{Whit}^{\leqslant n}(D_{crit}(\mathrm{Gr}_G))^\heartsuit \simeq \mathsf{Rep}(\check{G})^\heartsuit$ corresponding to the trivial representation (by construction of (8.8.1)). Therefore, $\Gamma^{\mathrm{Hecke}}(\mathrm{Gr}_G, \mathcal{G} \star \mathcal{F}) = \Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G, \mathcal{G} \star \delta_n)$. As $\mathcal{G} \star \delta_n \in D_{crit}(\mathrm{Gr}_G)$ is eventually coconnective by the above, the resulting object of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}$ is eventually coconnective as well by Corollary 7.15.2.

We deduce from (8.8.1) that for $\mathcal{F} \in \mathsf{Whit}^{\leqslant n}(D_{crit}(\mathrm{Gr}_G))$ compact, $\Gamma^{\mathrm{Hecke}}(\mathrm{Gr}_G, \mathcal{G}\star\mathcal{F})$ is eventually coconnective. We claim that in fact there is a universal integer r such that for compact $\mathcal{F}$ lying in $\mathsf{Whit}^{\leqslant n}(D_{crit}(\mathrm{Gr}_G))^{\geqslant 0}$, we have:

$$\Gamma^{\mathrm{Hecke}}(\mathrm{Gr}_G, \mathcal{G} \star \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\geqslant -r}_{reg}.$$

Indeed, choose r such that $\mathcal{G} \star -$ maps $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\geqslant 0}$ into $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\geqslant -r}$. As we know the above object is eventually coconnective, it suffices to verify the boundedness after applying ρ. Then the resulting object is $\mathcal{G} \star \Gamma^{\mathrm{Hecke},naive}(\mathrm{Gr}_G, \mathcal{F})$, which lies in degrees $\geqslant -r$ by construction of r and Theorem 8.3.1.

Finally, the same claim for general (possibly non-compact) $\mathcal{F} \in \mathsf{Whit}^{\leqslant n}(D_{crit}(\mathrm{Gr}_G))^{\geqslant 0}$ follows by the same argument as in (1): such $\mathcal{F}$ is a filtered colimit of objects of $\mathsf{Whit}^{\leqslant n}(D_{crit}(\mathrm{Gr}_G))^{\geqslant 0}$ that are compact in $\mathsf{Whit}^{\leqslant n}(D_{crit}(\mathrm{Gr}_G))$.

□

9. Generation under colimits

9.1. In this section, we prove Lemma 7.17.1.

9.2. **Preliminary observations.** We begin with the following basic result.

Lemma 9.2.1. *The subcategory* $\widehat{\mathfrak{g}}_{crit}\widetilde{\text{–}\mathsf{mod}}_{reg,naive} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$ *is a* $D^*_{crit}(G(K))$*-submodule category.*

Proof. By definition of $\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive}$, we need to show that for $\mathcal{F} \in D^*_{crit}(G(K))$, the functor $\mathcal{F} \star -$ maps $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{reg,naive}$ into $\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive}$. As $D^*_{crit}(G(K))$ is compactly generated, we are reduced to the case where $\mathcal{F}$ is compact. In that case, we claim that $\mathcal{F} \star -$ maps $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{reg,naive}$ into itself.

Indeed, this follows from Lemma 9.2.2 and the observation that the action of $G(K)$ on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$ is strongly compatible with the t-structure; the latter claim reduces via Lemma 6.9.3 to the same claim for $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}$, which is shown as [Ras6] Lemma 10.14.1 (3).

□

Above, we used the following result.

Lemma 9.2.2. *Let H be a Tate group indscheme with prounipotent tail acting strongly on $\mathcal{C} \in \mathsf{DGCat}_{cont}$. Suppose $\mathcal{C}$ is equipped with a t-structure strongly compatible with the H-action in the sense of* [Ras6] *§10.12. Then for any $\mathcal{F} \in D^*(H)$ compact, the functor $\mathcal{F} \star - : \mathcal{C} \to \mathcal{C}$ is left t-exact up to shift.*

Proof. Because $\mathcal{F}$ is compact and H has a prounipotent tail, $\mathcal{F} \in D^*(H)^K \simeq D(H/K)$ for some prounipotent compact open subgroup $K \subseteq H$. Again because $\mathcal{F}$ is compact, as an object of $D(H/K)$, it is supported on a closed subscheme $S \subseteq H/K$. By [DG], $\mathcal{F}$ has a bounded resolution by compact objects of the form $\mathrm{ind}(i^{\mathsf{IndCoh}}_*(\mathcal{G}))$ for $i : S \to H/K$ the embedding, $\mathcal{G} \in \mathsf{IndCoh}(S)$ compact, and ind the functor of (right) D-module induction. Therefore, we may consider $\mathcal{F}$ of this form.

The functor $\mathcal{F} \star -$ then factors as:

$$\mathcal{C} \xrightarrow{\mathrm{Av}^K_*} \mathcal{C}^K \xrightarrow{\mathrm{Oblv}} \mathcal{C}^{K,w} \xrightarrow{i^{\mathsf{IndCoh}}_*(\mathcal{G}) \overset{K_0,w}{\star} -} \mathcal{C}$$

where $- \overset{K_0,w}{\star} -$ indicates the appropriate relative convolution functor $\mathsf{IndCoh}(H/K)^{K_0/K,w} \otimes \mathcal{C}^{K_0,w} = \mathsf{IndCoh}(H/K_0) \otimes \mathcal{C}^{K_0,w} \to \mathcal{C}$.

As the H-action on $\mathcal{C}$ is compatible with the t-structure, $\mathcal{C}^K \subseteq \mathcal{C}$ is closed under truncations; it follows that Av^K_* is left t-exact. By [Ras6] §10.13, $\mathcal{C}^{K,w}$ has a canonical t-structure for which $\mathrm{Oblv} : \mathcal{C}^K \to \mathcal{C}^{K,w}$ is t-exact. Finally, the functor of convolution with $\mathcal{G}$ is left t-exact by [Ras6] Proposition 10.16.1.[32]

□

Corollary 9.2.3. $\Gamma^{\mathrm{Hecke},naive}$ *factors through* $\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive}$.

Proof. By §7.6, it suffices to show $\Gamma^{\mathsf{IndCoh}} = \Gamma^{\mathrm{Hecke},naive} \circ \mathrm{ind}^{\mathrm{Hecke}}$ factors through $\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive}$. This functor is given by convolution with $\mathbb{V}_{crit} \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\heartsuit}_{reg,naive} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive}$, so the claim follows from Lemma 9.2.1.

□

Corollary 9.2.4. *Let $K \subseteq G(O)$ be a prounipotent*[33] *group subscheme. Then $\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}^K_{reg,naive}$ is the subcategory of $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^K_{reg,naive}$ generated under colimits by $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{K,+}_{reg,naive}$.*

Proof. We have a commutative diagram:

[32] There is a polarizability assumption at this point in *loc. cit* that we have omitted here. This assumption is only needed in *loc. cit.* to deduce a stronger result. The beginning of that argument from *loc. cit.* is all that is needed here, and for that the polarizability is not needed. (Regardless, we only apply this result to $G(K)$, which is polarizable.)

[33] This assumption can be omitted, but the argument requires some additional details.

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{reg,naive} & \hookrightarrow & \widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive} \\ \downarrow{\scriptstyle \mathrm{Av}^K_*} & & \downarrow{\scriptstyle \mathrm{Av}^K_*} \\ \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{K,+}_{reg,naive} & \hookrightarrow & \widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}^{K}_{reg,naive}. \end{array}$$

The top and right functors generate under colimits, so the same is true of their composition. This implies that the bottom arrow generates under colimits, as desired.

□

9.3. **Proof for PGL_2.** To simplify the discussion, we first assume $G = PGL_2$. We indicate the necessary modifications for general G of semisimple rank 1 in §9.4.

By construction, $\Gamma^{\mathrm{Hecke},naive}$ is a $G(K)$-equivariant functor (at critical level). In particular, the subcategory of $\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive}$ generated under colimits by its essential image is closed under the $G(K)$-action.

Therefore, by Theorem 5.1.1, to prove Lemma 7.17.1 it suffices to show that the essential image of $\Gamma^{\mathrm{Hecke},naive}$ contains $\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}^{\mathring{I}}_{reg,naive}$ and $\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive})$. The former follows from Theorem 8.2.1, while the latter follows from Theorem 8.3.1.[34]

9.4. **Generalization to groups of semisimple rank 1.** We briefly indicate the argument for general G of semisimple rank 1.

First, for $\varphi : G_1 \to G_2$ an isogeny of reductive groups, the natural functor:

$$D^{\mathrm{Hecke}_\Im}_{crit}(\mathrm{Gr}_{G_1}) \to D^{\mathrm{Hecke}_\Im}_{crit}(\mathrm{Gr}_{G_2})$$

is an equivalence. Indeed, this follows as:

$$D_{crit}(\mathrm{Gr}_{G_1}) \underset{\mathsf{Rep}(\check{G}_1)}{\otimes} \mathsf{Rep}(\check{G}_2) \to D_{crit}(\mathrm{Gr}_{G_2})$$

and:

$$\mathrm{Op}^{reg}_{\check{G}_2} \to \mathrm{Op}^{reg}_{\check{G}_1}$$

are equivalences (the latter being a consequence of Remark 1.9.2).

In particular, one deduces that $G(K)$ acts (with critical level) on $D^{\mathrm{Hecke}_\Im}_{crit}(\mathrm{Gr}_G)$ through $G^{ad}(K)$ (e.g., it is easy to see directly that the action is trivial for G a torus). The same is evidently true for the action on $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$. Moreover, $\Gamma^{\mathrm{Hecke},naive}$ is $G^{ad}(K)$-equivariant.

Next, one observes that the Whittaker category with respect to the $G^{ad}(K)$ action coincides with the Whittaker category for the $G(K)$ action, and similarly for the radical of Iwahori. For later reference, we also highlight that for $n > 0$, the invariants for the nth subgroup of $G(K)$ coincide with the similar invariants for the $G^{ad}(K)$-action.

Finally, we observe that for G of semisimple rank 1, $G^{ad} = PGL_2$, so we can apply the above observations and Theorem 5.1.1.

[34] In the latter case, it is shown that $\Gamma^{\mathrm{Hecke},naive}$ even induces an equivalence on Whittaker categories with $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$, i.e., the distinction with $\widehat{\mathfrak{g}}_{crit}\text{–}\widetilde{\mathsf{mod}}_{reg,naive}$ is not necessary for the Whittaker part of the argument.

10. Exactness

10.1. In this section, we prove Lemma 7.17.2. The main idea is Proposition 10.4.1.

10.2. **t-structures on quotient categories.** We will need the following construction.

Suppose $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is equipped with a t-structure that is compatible with filtered colimits. Let $i_* : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be a fully faithful functor admitting a continuous right adjoint $i^!$. We suppose the full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is closed under truncation functors for the t-structure; in particular, $\mathcal{C}_0$ admits a unique t-structure for which i_* is t-exact.

Define $\mathring{\mathcal{C}}$ as $\mathrm{Ker}(i^! : \mathcal{C} \to \mathcal{C}_0)$. We denote the embedding of $\mathring{\mathcal{C}}$ into $\mathcal{C}$ by j_*. This embedding admits a left adjoint $\mathcal{F} \mapsto \mathrm{Coker}(i_* i^! \mathcal{F} \to \mathcal{F})$, which we denote by $j^* : \mathcal{C} \to \mathring{\mathcal{C}}$.

Lemma 10.2.1. *Suppose that the functor $j_* j^* : \mathcal{C} \to \mathcal{C}$ is left t-exact. Then there is a unique t-structure on $\mathring{\mathcal{C}}$ such that $j^* : \mathcal{C} \to \mathcal{C}$ is t-exact.*

Remark 10.2.2. The hypothesis of the lemma is equivalent to the assertion that for $\mathcal{F} \in \mathcal{C}^\heartsuit$, the map $H^0(i_* i^! \mathcal{F}) \to \mathcal{F}$ is a monomorphism in the abelian category $\mathcal{C}^\heartsuit$. In turn, this assertion is well-known to be equivalent to $\mathcal{C}_0^\heartsuit \subseteq \mathcal{C}^\heartsuit$ being closed under subobjects.

Proof of Lemma 10.2.1. Define $\mathring{\mathcal{C}}^{>0} \subseteq \mathring{\mathcal{C}}$ as the full subcategory of $\mathcal{F} \in \mathring{\mathcal{C}}$ with $j_*(\mathcal{F}) \in \mathcal{C}^{>0}$. Define $\mathring{\mathcal{C}}^{\leqslant 0} \subseteq \mathring{\mathcal{C}}$ as the left orthogonal to $\mathcal{C}^{>0}$.

The functor $j^* : \mathcal{C} \to \mathring{\mathcal{C}}$ maps $\mathcal{C}^{\leqslant 0}$ to $\mathring{\mathcal{C}}^{\leqslant 0}$ immediately from the definition, and maps $\mathcal{C}^{>0}$ to $\mathring{\mathcal{C}}^{>0}$ by our assumption that $j_* j^*$ is left t-exact.

In particular, for $\mathcal{F} \in \mathring{\mathcal{C}}$, $j^* \tau^{>0} j_*(\mathcal{F}) \in \mathring{\mathcal{C}}^{>0}$ and $j^* \tau^{\leqslant 0} j_*(\mathcal{F}) \in \mathring{\mathcal{C}}^{\leqslant 0}$. As $j^* j_*(\mathcal{F}) \xrightarrow{\simeq} \mathcal{F}$, we see that we have in fact defined a t-structure on $\mathring{\mathcal{C}}$. By the previous paragraph, the functor j^* is t-exact as desired.

□

10.3. **Subobjects in equivariant categories.** To apply the previous material, we use the following result.

Proposition 10.3.1. *Suppose H is a connected, affine algebraic group acting strongly on $\mathcal{C} \in \mathsf{DGCat}_{cont}$. Suppose that $\mathcal{C}$ is equipped with a t-structure compatible with the H-action.*

Then the functor $\mathcal{C}^{H,\heartsuit} \to \mathcal{C}^\heartsuit$ is fully faithful and the resulting subcategory is closed under subobjects.

Proof. In what follows, we let $\iota : \mathrm{Spec}(k) \to H$ denote the unit for the group structure and we let $\mathring{H} := H \backslash 1$ the complementary open with embedding $\jmath : \mathring{H} \to H$.

We let $\delta_1 = \iota_{*,dR}(k) \in D(H)$ denote the δ D-module on G supported at $1 \in H$, and we let $k_H \in D(H)$ (resp. $k_{\mathring{H}} \in D(\mathring{H})$) denote the constant D-module (i.e., the $*$-dR pullback of $k \in D(\mathrm{Spec}(k)) = \mathsf{Vect}$).

Step 1. We begin with reductions.

First, that $\mathcal{C}^{H,\heartsuit} \to \mathcal{C}^\heartsuit$ is fully faithful for H connected is well-known.[35]

By Remark 10.2.2, it suffices to show that for $\mathcal{F} \in \mathcal{C}^{H,\heartsuit}$, the map:

[35] We recall the argument for the reader's convenience. For $\mathcal{F} \in \mathcal{C}^{H,\heartsuit}$, we need to show that $\mathcal{F} \to \mathrm{Av}^H_* \mathrm{Oblv}(\mathcal{F})$ gives an isomorphism after applying H^0. Moreover, it suffices to do so after applying Oblv.

The resulting map is obtained by (H-equivariant) convolution with the canonical map $k_H \to k_H \star k_H \in D(H)^H$. Under the identification $D(H)^H = \mathsf{Vect}$ with $k \in \mathsf{Vect}$ corresponding to $k_H \in D(H)^H$, the resulting map corresponds to $k \to \Gamma_{dR}(H, k_H)$. Because H is connected (hence, geometrically connected), this map is an isomorphism in degree 0, giving the claim.

$$\mathrm{Oblv}\,\mathrm{Av}^H_*(\mathcal{F}) \to \mathcal{F}$$

induces a monomorphism on H^0, or equivalently, the (homotopy) cokernel of this map is coconnective. As the above map is obtained by convolution with the map $k_H \to \delta_1 \in D(H)$, it suffices to show that convolution with its cokernel, which is $\jmath_!(k_{\mathring{H}})[1]$, is left t-exact.

Step 2. Let $\mathcal{F} \in D(H)$ be given. Suppose the functor $\mathcal{F} \star - : D(H) \to D(H)$ is left t-exact. We claim that the functor $\mathcal{F} \star - : \mathcal{C} \to \mathcal{C}$ is left t-exact.

Indeed, by definition of the t-structure on $\mathcal{C}$ being compatible with the H-action, the functor coact : $\mathcal{C} \to D(H) \otimes \mathcal{C}$ is t-exact up to shift. The functor coact is H-equivariant for the H-action on $D(H) \otimes \mathcal{C}$ on the first factor alone. Moreover, coact is conservative: its composition with !-restriction along the origin $\mathrm{Spec}(k) \hookrightarrow H$ is the identity functor for $\mathcal{C}$.

Therefore, the claim follows from [Ras5] Lemma B.6.2.

Step 3. By Step 2, we are reduced to showing that convolution with $\jmath_!(k_{\mathring{H}})[1]$ defines a left t-exact functor $D(H) \to D(H)$. By the reasoning of Step 1, it is equivalent to say that the essential image of the functor $D(H)^{H,\heartsuit} = \mathsf{Vect}^\heartsuit \hookrightarrow D(H)^\heartsuit$ is closed under subobjects, which is evident: a sub D-module of a constant one is itself constant.

□

10.4. **An exactness criterion.** We begin with a scheme for checking that a functor between categories with (finite jets into) PGL_2-actions is t-exact.

Proposition 10.4.1. *Let $G = PGL_2$ and let G_n be as in §3.2.*

Let $\mathcal{C}, \mathcal{D} \in G_n$–$\mathsf{mod}$ be equipped with t-structures compatible with the G_n-actions.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is a G_n-equivariant functor.

Then F is left t-exact if and only if the functors:

$$\mathcal{C}^{N_n,\psi} \to \mathcal{D}^{N_n,\psi} \text{ and } \begin{cases} \mathcal{C}^N \to \mathcal{D}^N & n = 1 \\ \mathcal{C}^{\mathfrak{g}\otimes\mathbb{G}_a} \to \mathcal{D}^{\mathfrak{g}\otimes\mathbb{G}_a} & n \geqslant 2 \end{cases}$$

are left t-exact, where $\mathfrak{g} \otimes \mathbb{G}_a$ is embedded into G_n via (3.3.1).

Below, we give the proofs separately for $n = 1$ and $n \geqslant 2$. We remark that in both cases, the "only if" direction is obvious.

Proof of Proposition 10.4.1 for $n = 1$. As we will see, in this case we only need the action of the Borel $B = T \ltimes N = \mathbb{G}_m \ltimes \mathbb{G}_a$ of $G = PGL_2$.

Define $\mathring{\mathcal{C}}$ as $\mathrm{Ker}(\mathcal{C} \xrightarrow{\mathrm{Av}^{\mathbb{G}_a}_*} \mathcal{C}^{\mathbb{G}_a})$. The embedding $\mathring{\mathcal{C}} \hookrightarrow \mathcal{C}$ admits a left adjoint calculated as $\mathcal{F} \mapsto \mathrm{Coker}(\mathrm{Oblv}\,\mathrm{Av}^{\mathbb{G}_a}_*(\mathcal{F}) \to \mathcal{F})$. By Lemma 10.2.1 and Proposition 10.3.1, $\mathring{\mathcal{C}}$ admits a unique t-structure such that this functor $\mathcal{C} \to \mathring{\mathcal{C}}$ is t-exact.

The action functor act : $D(\mathbb{G}_m) \otimes \mathcal{C} \to \mathcal{C}$ maps $D(\mathbb{G}_m) \otimes \mathcal{C}^{\mathbb{G}_a,\psi}$ into $\mathring{\mathcal{C}}$, and the resulting functor is an equivalence (by Fourier transforming, c.f. [Ber]). We claim that this equivalence is t-exact, where $D(\mathbb{G}_m) \otimes \mathcal{C}^{\mathbb{G}_a,\psi}$ is given the tensor product t-structure.

To verify this, we will need the following commutative diagram:

$$\begin{array}{ccc} D(\mathbb{G}_m) \otimes \mathcal{C}^{\mathbb{G}_a,\psi} & \xrightarrow{\text{act}} & \mathring{\mathcal{C}} \\ \downarrow{\scriptstyle \mathrm{id}_{D(\mathbb{G}_m)} \otimes \mathrm{Oblv}} & & \downarrow \\ D(\mathbb{G}_m) \otimes \mathring{\mathcal{C}} & \longrightarrow & D(\mathbb{G}_m) \otimes \mathring{\mathcal{C}} \end{array} \tag{10.4.1}$$

with morphisms as follows. The top arrow is induced by the action functor from above. The left arrow is $\mathrm{id}_{\mathbb{G}_m}$ tensored with the embedding $\mathcal{C}^{\mathbb{G}_a,\psi} \hookrightarrow \mathring{\mathcal{C}}$ ($\subseteq \mathcal{C}$). For the right arrow, note that $\mathring{\mathcal{C}} \subseteq \mathcal{C}$ is closed under the $\mathbb{G}_a$-action, and the corresponding coaction functor coact : $\mathring{\mathcal{C}} \to D(\mathbb{G}_a) \otimes \mathring{\mathcal{C}}$ composed with the Fourier transform $D(\mathbb{G}_a) \simeq D(\mathbb{A}^1)$ (tensored with $\mathrm{id}_{\mathcal{C}}$) maps into $D(\mathbb{A}^1\backslash 0) \otimes \mathring{\mathcal{C}}$; we have identified $\mathbb{A}^1\backslash 0$ with $\mathbb{G}_m$ here. Finally, the bottom arrow is the unique map of $D(\mathbb{G}_m)$-comodule categories whose composition with $\Gamma_{dR}(\mathbb{G}_m, -) \otimes \mathrm{id}_{\mathring{\mathcal{C}}}$ is act (the action functor for the $\mathbb{G}_m$-action on $\mathring{\mathcal{C}}$); here $D(\mathbb{G}_m)$ is a coalgebra in DGCat_{cont} via diagonal pushforwards, and both sides are considered as cofree comodules coinduced from $\mathring{\mathcal{C}}$. (That the diagram commutes is immediate.)

Now in (10.4.1), the bottom arrow is t-exact by [Ras5] Lemma B.6.2. By [Ras5] Lemma B.6.2, the left arrow is t-exact because $\mathcal{C}^{\mathbb{G}_a,\psi} \hookrightarrow \mathring{\mathcal{C}}$ is (as this functor coincides with the composition $\mathcal{C}^{\mathbb{G}_a,\psi} \hookrightarrow \mathcal{C} \to \mathring{\mathcal{C}}$ of t-exact functors). The right functor is t-exact because the t-structure on $\mathcal{C}$ is compatible with the $\mathbb{G}_a$-action. As the vertical arrows are fully faithful and the bottom arrow is an equivalence, we obtain that the top arrow is a t-exact equivalence as well.

We can now conclude the argument. By assumption and [Ras5] Lemma B.6.2, the functor:

$$\mathring{\mathcal{C}} \simeq D(\mathbb{G}_m) \otimes \mathcal{C}^{\mathbb{G}_a,\psi} \to D(\mathbb{G}_m) \otimes \mathcal{D}^{\mathbb{G}_a,\psi} \simeq \mathring{\mathcal{D}}$$

is left t-exact.

Suppose $\mathcal{F} \in \mathcal{C}^{\geqslant 0}$. Then $\mathrm{Oblv}\,\mathrm{Av}_*^{\mathbb{G}_a}(\mathcal{F}) \in \mathcal{C}^{\mathbb{G}_a,\geqslant 0}$, so $F(\mathrm{Oblv}\,\mathrm{Av}_*^{\mathbb{G}_a}(\mathcal{F})) \in \mathcal{D}^{\mathbb{G}_a,\geqslant 0}$. Moreover, defining:

$$\mathring{\mathcal{F}} := \mathrm{Coker}(\mathrm{Oblv}\,\mathrm{Av}_*^{\mathbb{G}_a}(\mathcal{F}) \to \mathcal{F}) \in \mathring{\mathcal{C}}$$

we have $\mathring{\mathcal{F}} \in \mathring{\mathcal{C}}^{\geqslant 0}$ by definition of the t-structure on $\mathring{\mathcal{C}}$. Therefore, $F(\mathring{\mathcal{F}}) \in \mathring{\mathcal{D}}^{\geqslant 0}$. Because the embedding $\mathring{\mathcal{D}} \hookrightarrow \mathcal{D}$ is left t-exact (being right adjoint to a t-exact functor), we obtain:

$$\mathrm{Oblv}\,\mathrm{Av}_*^{\mathbb{G}_a} F(\mathcal{F}),\ \mathrm{Coker}(\mathrm{Oblv}\,\mathrm{Av}_*^{\mathbb{G}_a} F(\mathcal{F}) \to F(\mathcal{F})) \in \mathcal{D}^{\geqslant 0}$$

implying $F(\mathcal{F}) \in \mathcal{D}^{\geqslant 0}$.

□

Proof of Proposition 10.4.1 for $n \geqslant 2$. Let $\mathcal{C}_{reg} \subseteq \mathcal{C}$ be defined as in §3.3. The embedding $\mathcal{C}_{reg} \hookrightarrow \mathcal{C}$ admits a left adjoint $j^!$ as in *loc. cit.* Moreover, because $G = PGL_2$, the argument from §5.3 shows that $\mathrm{Ker}(j^!) = \mathcal{C}^{\mathfrak{g}\otimes\mathfrak{G}_a}$. Applying Lemma 10.2.1 and Proposition 10.3.1, we find that $\mathcal{C}_{reg}$ admits a unique t-structure for which $j^!$ is t-exact.

By Theorem 4.2.1, the convolution functor:

$$D(G_n)^{N_n,\psi} \otimes \mathcal{C}^{N_n,\psi} \to \mathcal{C}$$

admits a left adjoint $\mathrm{Av}_!^{\psi,-\psi} = \mathrm{Av}_*^{\psi,-\psi}[2\dim N_n]$. By [Ras5] Lemma B.6.1, $\mathrm{Av}_!^{\psi,-\psi}[-\dim N_n] = \mathrm{Av}_*^{\psi,-\psi}[\dim N_n]$ is t-exact.

Because the above convolution functor factors through $\mathcal{C}_{reg}$, $\mathrm{Av}_!^{\psi,-\psi} : \mathcal{C} \to D(G_n)^{N_n,\psi} \otimes \mathcal{C}^{N_n,\psi}$ coincides with $\mathrm{Av}_!^{\psi,-\psi} j_{*,dR} j^!$. By the above, we find that $\mathrm{Av}_!^{\psi,-\psi} \circ j_{*,dR}$ is t-exact. Moreover, by Corollary 4.7.3, $\mathrm{Av}_!^{\psi,-\psi} \circ j_{*,dR}$ is conservative.

Therefore, as:

$$D(G_n)^{N_n,\psi} \otimes \mathcal{C}^{N_n,\psi} \xrightarrow{\mathrm{id}\otimes F} D(G_n)^{N_n,\psi} \otimes \mathcal{D}^{N_n,\psi}$$

is left t-exact by assumption and [Ras5] Lemma B.6.2, the resulting functor $\mathcal{C}_{reg} \to \mathcal{D}_{reg}$ is left t-exact.

As $\mathcal{C}^{\mathfrak{g}\otimes\mathbb{G}_a} \to \mathcal{D}_{\mathfrak{g}\otimes\mathbb{G}_a}$ is left t-exact by assumption, the argument concludes as in the $n = 1$ case. □

10.5. **Exactness of $\Gamma^{\mathrm{Hecke},naive}$.** We can now show t-exactness.

Proof of Lemma 7.17.2. For simplicity, we take $G = PGL_2$; the argument for general G of semisimple rank 1 follows by the considerations of §9.4.

By Corollary 7.15.3, it remains to show left t-exactness. It suffices to show that for every $n \geqslant 1$, the functor:

$$\Gamma^{\mathrm{Hecke},naive} : D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{K_n} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{K_n}_{reg,naive}$$

is left t-exact; here $K_n \subseteq G(O)$ is the nth congruence subgroup. We show this by induction on n.

First, we treat the $n = 1$ case. By Proposition 10.4.1, it suffices to show (left) t-exactness for the corresponding functors:

$$D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{\mathring{I}} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{\mathring{I}}_{reg,naive}$$
$$\mathsf{Whit}^{\leqslant 1}(D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)) \to \mathsf{Whit}^{\leqslant 1}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive})$$

These results follow from Theorems 8.2.1 and 8.3.1.

We now proceed by induction; we suppose the result is true for $n \geqslant 1$ and deduce it for $n + 1$. By Proposition 10.4.1, it suffices to show that the functors:

$$D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)^{K_n} \to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{K_n}_{reg,naive}$$
$$\mathsf{Whit}^{\leqslant n+1}(D^{\mathrm{Hecke}_\mathfrak{z}}_{crit}(\mathrm{Gr}_G)) \to \mathsf{Whit}^{\leqslant n+1}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive})$$

are (left) t-exact. The former is the inductive hypothesis and the latter is Theorem 8.3.1.

□

11. The renormalized category

11.1. In this section, we prove Lemma 7.17.3. The argument is quite similar to the proof of Lemma 7.17.2.

11.2. **A boundedness criterion.** The following result is a cousin of Proposition 10.4.1.

Proposition 11.2.1. *Let $G = PGL_2$ and let G_n be as in §3.2.*

Let $\mathcal{C} \in G_n\text{–}\mathsf{mod}$ be equipped with a t-structure compatible with the G_n-action. Suppose that $\mathcal{D}$ is equipped with a t-structure compatible with filtered colimits. Suppose $F : \mathcal{C} \to \mathcal{D} \in \mathsf{DGCat}_{cont}$ is given.

Then F is left t-exact up to shift if and only if:

$$\begin{cases} F|_{\mathcal{C}^N} & n = 1 \\ F|_{\mathcal{C}^{\mathfrak{g}\otimes\mathbb{G}_a}} & n \geqslant 2 \end{cases}$$

is left t-exact up to shift, and $F(\mathcal{G} \star \mathcal{F}) \in \mathcal{D}^+$ for every:

$$\mathcal{G} \in D(G_n)^+,\ \mathcal{F} \in \mathcal{C}^{N_n,\psi,+}.$$

Remark 11.2.2. We emphasize that there is no assumption here that G_n acts on $\mathcal{D}$, in contrast to Proposition 10.4.1.

Remark 11.2.3. The "only if" direction of Proposition 11.2.1 is clear, as $\mathcal{G} \star - : \mathcal{C} \to \mathcal{C}$ is left t-exact up to shift for $\mathcal{G} \in D(G_n)^+$.

Proof of Proposition 11.2.1 for $n = 1$. As the t-structures on $\mathcal{C}$ and $\mathcal{D}$ are compatible with filtered colimits, F is left t-exact up to shift if and only if $F(\mathcal{C}^+) \subseteq \mathcal{D}^+$. We verify the result in this form.

Suppose $\mathcal{F} \in \mathcal{C}^+$. Then $\mathrm{Oblv}\,\mathrm{Av}^N_*\,\mathcal{F} \in \mathcal{C}^{N,+}$, so by assumption $F(\mathrm{Oblv}\,\mathrm{Av}^N_*\,\mathcal{F}) \in \mathcal{D}^+$. Therefore, setting $\mathring{\mathcal{F}} := \mathrm{Coker}(\mathrm{Oblv}\,\mathrm{Av}^N_*\,\mathcal{F} \to \mathcal{F})$, it suffices to show that $F(\mathring{\mathcal{F}}) \in \mathcal{D}^+$.

As in the proof of Proposition 10.4.1 (for $n = 1$), $\mathring{\mathcal{F}}$ is in the essential image of the fully faithful, t-exact convolution functor $D(T) \otimes \mathcal{C}^{N,\psi} \to \mathcal{C}$. Therefore, it suffices to show that the composition:

$$D(T) \otimes \mathcal{C}^{N,\psi} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is left t-exact up to shift. For convenience, in what follows, we identify $\mathring{\mathcal{F}}$ with the correpsonding object of $D(T) \otimes \mathcal{C}^{N,\psi}$.

For this, we observe that any object $\mathring{\mathcal{F}} \in D(T) \otimes \mathcal{C}^{N,\psi}$ lies in the full subcategory of $D(T) \otimes \mathcal{C}^{N,\psi}$ generated under finite colimits and direct summands by objects of the form $D_T \boxtimes (\Gamma(T, -) \otimes \mathrm{id})(\mathring{\mathcal{F}})$, where $D_T \in D(T)^\heartsuit$ is the sheaf of differential operators; c.f. Lemma 11.2.4 below. Then by [Ras5] Lemma B.6.2, for $\mathring{\mathcal{F}} \in D(T) \otimes \mathcal{C}^{N,\psi}$, we have:

$$(\Gamma(T, -) \otimes \mathrm{id})(\mathring{\mathcal{F}}) \in \mathcal{C}^{N,\psi,+}.$$

Therefore, by assumption, $F(D_T \star (\Gamma(T, -) \otimes \mathrm{id})(\mathring{\mathcal{F}})) \in \mathcal{D}^+$, so we find that the same is true of $F(\mathring{\mathcal{F}})$.

□

Above, we used the following result.

Lemma 11.2.4. *Let S be a smooth affine scheme (over* $\mathrm{Spec}(k)$*).*

As is standard, let $\mathrm{Oblv} : D(S) \to \mathsf{IndCoh}(S) \overset{\Psi}{\simeq} \mathsf{QCoh}(S)$ *denote the "right" D-module forgetful functor from* [GR3] *and let* $\mathrm{ind} : \mathsf{QCoh}(S) \to D(S)$ *denote its left adjoint. Let $D_S := \mathrm{ind}(\mathcal{O}_S) \in D(S)^\heartsuit$. Let $\Gamma(S, -) : D(S) \to \mathsf{Vect}$ denote the composition of* Oblv *with the usual global sections functor on* $\mathsf{QCoh}(S)$*.*

Then for any $\mathcal{C} \in \mathsf{DGCat}_{cont}$ and any $\mathcal{F} \in D(S) \otimes \mathcal{C}$, $\mathcal{F}$ lies in the full subcategory of $D(S) \otimes \mathcal{C}$ generated under finite colimits and direct summands by objects of the form:

$$D_S \boxtimes (\Gamma(S, -) \otimes \mathrm{id}_{\mathcal{C}})(\mathcal{F}).$$

Proof. As S is affine, $D(S \times S)$ is compactly generated by objects of the form $D_S \boxtimes D_S$. As $\Delta_{dR,*}(\omega_S)$ is compact and connective, it lies in the full subcategory generated under finite colimits and direct summands by objects of the form $D_S \boxtimes D_S$.

Identifying $D(S \times S)$ in the usual way with $\mathsf{End}_{\mathsf{DGCat}_{cont}}(D(S))$ (c.f. [GR3]), the object $\Delta_{dR,*}(\omega_S)$ corresponds to the identity functor, while $D_S \boxtimes D_S$ corresponds to $D_S \otimes \Gamma(S, -)$.

Therefore, $\mathrm{id}_{D(S) \otimes \mathcal{C}}$ lies in the full subcategory of $\mathsf{End}_{\mathsf{DGCat}_{cont}}(D(S) \otimes \mathcal{C})$ generated under finite colimits and direct summands by endofunctors of the form $(D_S \otimes \Gamma(S, -)) \otimes \mathrm{id}_{\mathcal{C}}$. Applying such a resolution to the object $\mathcal{F}$, we obtain the claim.

□

We now turn to the higher n case.

Proof of Proposition 11.2.1 for $n \geqslant 2$. Suppose $\mathcal{F} \in \mathcal{C}^+$. Then $\mathrm{Oblv}\,\mathrm{Av}_*^{\mathfrak{g}\otimes\mathbb{G}_a}\mathcal{F} \in \mathcal{C}^{\mathfrak{g}\otimes\mathbb{G}_a,+}$, so by assumption $F(\mathrm{Oblv}\,\mathrm{Av}_*^{\mathfrak{g}\otimes\mathbb{G}_a}\mathcal{F}) \in \mathcal{D}^+$. Therefore, setting $\mathring{\mathcal{F}} := \mathrm{Coker}(\mathrm{Oblv}\,\mathrm{Av}_*^{\mathfrak{g}\otimes\mathbb{G}_a}\mathcal{F} \to \mathcal{F})$, it suffices to show that $F(\mathring{\mathcal{F}}) \in \mathcal{D}^+$.

As $G = PGL_2$, $\mathring{\mathcal{F}} \in \mathcal{C}_{reg}$, hence in $\mathcal{C}^+_{reg}$. Now the claim follows as in the $n = 1$ case from our assumption and Corollary 4.7.2.

□

11.3. **Boundedness of Γ^{Hecke}.** We now boundedness of the non-naive version of the Hecke global sections functor.

Proof of Lemma 7.17.3. We again assume $G = PGL_2$ for simplicity, referring to §9.4 for indications on general G of semisimple rank 1.

It suffices to show the result for K being the nth congruence subgroup of $G(O)$ for some $n \geqslant 1$. We proceed by induction on n.

For $\mathcal{F} \in D^{\mathrm{Hecke}_!}_{crit}(\mathrm{Gr}_G)^{\mathring{I},+}$, $\Gamma^{\mathrm{Hecke}}(\mathrm{Gr}_G, \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{reg}$: this follows from Proposition 8.2.2.

Next, suppose that $\mathcal{G} \in D(G)^+$ and $\mathcal{F} \in \mathsf{Whit}^{\leqslant 1}(D^{\mathrm{Hecke}_!}_{crit}(\mathrm{Gr}_G))^+$. Then $\Gamma^{\mathrm{Hecke}}(\mathrm{Gr}_G, \mathcal{G} \star \mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{reg}$ by Proposition 8.8.1.

Therefore, Proposition 11.2.1 implies the $n = 1$ case of the claim.

We now suppose the result is true for some n and deduce it for $n+1$. The inductive hypothesis states that $\Gamma^{\mathrm{Hecke}}(\mathrm{Gr}_G, \mathcal{G}\star\mathcal{F})$ is eventually coconnective for $\mathcal{F} \in D^{\mathrm{Hecke}_!}_{crit}(\mathrm{Gr}_G)^{K_n,+}$, while Proposition 8.8.1 implies the result if $\mathcal{F} \in \mathsf{Whit}^{\leqslant n+1}(D^{\mathrm{Hecke}_!}_{crit}(\mathrm{Gr}_G))^+$. Therefore, Proposition 11.2.1 gives the result for general $\mathcal{F} \in D^{\mathrm{Hecke}_!}_{crit}(\mathrm{Gr}_G)^{K_{n+1},+}$.

□

Appendix A. The global sections functor

A.1. Let κ be a level for $\mathfrak{g}$. In this appendix, we define a global sections functor:

$$\Gamma(G(K), -) : D^*_\kappa(G(K)) \to \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod}.$$

Moreover, we show the following basic property:

Proposition A.1.1. *The functor $\Gamma(G(K), -)$ is t-exact for the natural t-structure on $D^*(G(K))$.*

To define both Γ and the "natural" t-structure mentioned above, there is an implicit choice of compact open subgroup of $G(K)$ (or rather, its Tate extension) that goes into the definitions. For definiteness, we choose $G(O)$ in what follows.

Abelian categorically, this construction is well-known from [AG2]. Our setup is a little different from *loc. cit.*, so we indicate basic definitions and properties. We compare our construction to theirs in Proposition A.10.1.

A.2. **Definition of the functor.** By [Ras6] §11.9, we have a canonical isomorphism:

$$\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^\vee \simeq \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod}.$$

Here the left hand side is the dual in DGCat_{cont}. This isomorphism depends (mildly) on our choice $G(O)$ of compact open subgroup of $G(K)$. This isomorphism is a refinement of the usual semi-infinite cohomology construction; more precisely, by *loc. cit.*, the pairing:

$$\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod} \to \mathsf{Vect}$$

is calculated by tensoring Kac-Moody representations and then taking semi-infinite cohomology for the diagonal action.

In addition, by [Ras6] §8, we have a level κ $G(K)$-action on $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod}$.

Therefore, we obtain a functor:

$$D^*_\kappa(G(K)) \to \mathsf{End}_{\mathsf{DGCat}_{cont}}(\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}) \simeq \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod}.$$

By definition, the resulting functor is $\Gamma(G(K), -)$.

A.3. **Definition of the t-structure.** The choice of $G(O)$ also defines a t-structure on $D^*_\kappa(G(K))$: we write $D^*_\kappa(G(K))$ as $\mathrm{colim}_n D_\kappa(G(K)/K_n)$ under $*$-pullback functors; the structure functors are t-exact up to shift by smoothness of the structure maps, so there is a unique t-structure such that the pullback functor $\pi_n^{*,dR}[-\dim G(O)/K_n] : D_\kappa(G(K)/K_n) \to D^*_\kappa(G(K))$ is t-exact for all n.

A.4. **t-exactness.** Below, we prove Proposition A.1.1.

A.5. Because compact objects in $D^*_\kappa(G(K))$ are bounded in the t-structure and closed under truncations, it suffices to show that for $\mathcal{F} \in D^*_\kappa(G(K))^\heartsuit$ compact in $D^*_\kappa(G(K))$, $\Gamma(G(K), \mathcal{F}) \in (\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod})^\heartsuit$.

We fix such an $\mathcal{F}$ in what follows.

A.6. Because $\mathcal{F}$ is compact, there exists a positive integer r such that $\mathcal{F}$ is K_r-equivariant on the right. Moreover, by compactness again, $\mathcal{F}$ is supported on some closed subscheme $S \subseteq G(K)$, which we may assume is preserved under the right K_r-action.

Note that S is necessarily affine as $G(K)$ is ind-affine. We have $S = \lim S/K_{r+r'}$, so by Noetherian approximation, $S/K_{r+r'}$ is affine for some $r' \geqslant 0$. Up to replacing r by $r + r'$, we may assume S/K_r itself is affine.

A.7. For any two integers $m_1, m_2 > 0$, we have:

$$\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}\otimes\widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod}}(\mathbb{V}_{\kappa,m_1} \boxtimes \mathbb{V}_{-\kappa+2\cdot crit,m_2}, \Gamma(G(K), \mathcal{F})) = \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}}(\mathbb{V}_{\kappa,m_1}, \mathcal{F} \star \mathbb{D}\mathbb{V}_{-\kappa+2\cdot crit,m_2}).$$

by definition of Γ. Here $\mathbb{D} : (\widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod}^c)^{op} \simeq \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^c$ is the isomorphism defined by the (semi-infinite) duality $\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \simeq \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^\vee$ used above.

To see that $\Gamma(G(K), \mathcal{F})$ is in degrees $\geqslant 0$, it suffices to see that the above complex is in degrees $\geqslant 0$ for all m_1, m_2. Moreover, it suffices to check this for all sufficiently large m_1, m_2; we will do so for $m_1, m_2 \geqslant r$.

Then to see that $\Gamma(G(K), \mathcal{F})$ is in degree 0, it suffices to show that when we pass to the limits $m_1, m_2 \to \infty$ (using the standard structure maps between our modules as we vary these parameters), we obtain a complex in degree 0. In fact, we will see that already $\mathcal{F} \star \mathcal{F} \star \mathbb{D}\mathbb{V}_{-\kappa+2\cdot crit,m_2}$ is in the heart of the t-structure (for $m_2 \geqslant r$), which clearly suffices.

A.8. By [Ras6] Lemma 9.17.1, $\mathbb{D}\mathbb{V}_{-\kappa+2\cdot crit,m_2} \simeq \mathbb{V}_{\kappa,m_2}[\dim G(O)/K_{m_2}] = \mathbb{V}_{\kappa,m_2}[m_2 \cdot \dim G]$.

We then have $\mathcal{F} \star \mathbb{V}_{\kappa,r} = \Gamma^{\mathsf{IndCoh}}(G(K)/K_r, \mathcal{F})$; here we have descended $\mathcal{F}$ by K_r-equivariance to a D-module on $G(K)/K_r$ and then we have calculated its IndCoh-global sections.

Putting these together, we find:

$$\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}}(\mathbb{V}_{\kappa,m_1}, \mathcal{F} \star \mathbb{D}\mathbb{V}_{\kappa,m_2}) = \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}}(\mathbb{V}_{\kappa,m_1}, \mathcal{F} \star \mathbb{V}_{\kappa,r}[m_2 \cdot \dim G - (m_2 - r)\dim G]) = \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}}(\mathbb{V}_{\kappa,m_1}, \mathcal{F} \star \mathbb{V}_{\kappa,r}[r \cdot \dim G]).$$

By Lemma 9.2.2 (and [Ras6] Proposition 10.16.1), $\mathcal{F} \star \mathbb{V}_{\kappa,r} \in \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^+$. Moreover, by §A.9 below, $\mathcal{F} \star \mathbb{V}_{\kappa,r}$ maps under the forgetful functor to Vect to $\Gamma^{\mathsf{IndCoh}}(G(K)/K_r, \mathcal{F}) \in \mathsf{Vect}$ (i.e., descend $\mathcal{F}$ to $G(K)/K_r$ and take IndCoh-global sections).

As $\mathcal{F} \in D^*_\kappa(G(K))^\heartsuit$, when we consider $\mathcal{F}$ as an object of $D_\kappa(G(K)/K_r)$, it lies in cohomological degree $\dim(G(O)/K_r) = r \cdot \dim G$. Therefore, the same is true when we forget to $\mathsf{IndCoh}(G(K)/K_r)$, as that forgetful functor is t-exact (c.f. [GR3]). Finally, as $\mathcal{F}$ is supported on an affine subscheme of $G(K)/K_r$ by construction, $\Gamma^{\mathsf{IndCoh}}(G(K)/K_r, \mathcal{F})$ is in cohomological degree $r \cdot \dim G$.

Combining this with the above, we find that $\mathcal{F} \star \mathbb{V}_{\kappa,r}[r \cdot \dim G]) \in \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^\heartsuit$. This gives the desired claims, proving Proposition A.1.1 modulo the above assertion.

A.9. Above, we needed the following observation.

Suppose $\mathcal{F} \in D^*_\kappa(G(K))^{K_r}$. We claim that $\mathrm{Oblv}(\mathcal{F} \star \mathbb{V}_{\kappa,r}) = \Gamma^{\mathsf{IndCoh}}(G(K)/K_r, \mathcal{F}) \in \mathsf{Vect}$, where we implicitly descend $\mathcal{F}$ to $G(K)/K_r$ through equivariance.

To simplify the notation, we omit the level κ and work with a general Tate group indscheme H and a compact open subgroup K. (Then the level may easily be reincorporated in a standard way by taking H to be the Tate extension of $G(K)$, c.f. [Ras6] §11.3.)

For any $\mathcal{C} \in H\text{–}\mathsf{mod}_{weak}$, suppose $\mathcal{G} \in \mathcal{C}^{K,w}$ and $\mathcal{F} \in D(H/K)$. As in [Ras6] §8, $D(H/K)$ is canonically isomorphic to $\mathsf{IndCoh}^*(H/K)_{H^\wedge_K}$, with $H^\wedge_K$ the formal completion of H along K. Moreover, the functor $\mathrm{Oblv} : \mathcal{C}^{H^\wedge_K,w} \to \mathcal{C}^{K,w}$ admits a left adjoint, which we denote by $\mathrm{Av}^w_!$.

Then we claim that we have isomorphisms:

$$\mathcal{F} \overset{H^\wedge_K,w}{\star} \mathrm{Av}^w_!(\mathcal{G}) = \mathrm{Oblv}(\mathcal{F}) \overset{K,w}{\star} \mathcal{G} \in \mathcal{C}$$

functorial in $\mathcal{F}$ and $\mathcal{G}$ (i.e., an isomorphism of functors $D(H/K) \otimes \mathcal{C}^{K,w} \to \mathcal{C}$). Here for the convolution on the left, we regard $\mathcal{F}$ as an object of $\mathsf{IndCoh}^*(H)_{H^\wedge_K}$ as above. The notation $\overset{H^\wedge_K,w}{\star}$ means we convolve (in the setting of weak actions) over $H^\wedge_K$, and similarly on the right hand side. Then $\mathrm{Av}^w_!(\mathcal{G}) = \omega_{H^\wedge_K/K} \overset{K,w}{\star} \mathcal{G}$, and $\mathcal{F} \overset{H^\wedge_K,w}{\star} \omega_{H^\wedge_K/K} = \mathrm{Oblv}(\mathcal{F})$, so we obtain the claim.

Now taking $\mathcal{C} = \mathsf{Vect}$ and $\mathcal{G} = k$ the trivial representation in $\mathsf{Vect}^{K,w} = \mathsf{Rep}(K)$, we obtain:

$$\mathcal{F} \star \mathrm{ind}^{\mathfrak{h}}_{\mathfrak{k}}(k) = \mathrm{Oblv}(\mathcal{F}) \star k \in \mathsf{Vect}.$$

The right hand side calculates $\Gamma^{\mathsf{IndCoh}}(H/K, \mathrm{Oblv}(\mathcal{F}))$ as desired.

A.10. **Comparison with Arkhipov-Gaitsgory.** To conclude, we observe that our construction above recovers the one given by Arkhipov-Gaitsgory.

More precisely, $D^*(G(K))^\heartsuit$ manifestly coincides with the abelian category denoted $D\text{–}\mathsf{mod}(G((t)))$ in §6.10 of [AG2], and similarly with a level κ included (which they discuss only in passing).

Below, we outline the proof of the following comparison result.

Proposition A.10.1. *The functor:*

$$\Gamma(G(K), -) : D^*_\kappa(G(K))^\heartsuit \to (\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod})^\heartsuit = \widehat{\mathfrak{g} \times \mathfrak{g}}_{(\kappa,-\kappa+2\cdot crit)}\text{–}\mathsf{mod}^\heartsuit$$

constructed above coincides with the one constructed in [AG2].

Proof.

Step 1. Define $\mathrm{CDO}_{G,\kappa} \in \mathsf{Vect}$ as $\Gamma(G(K), \delta_{G(O)})$, where $\delta_{G(O)} \in D^*_\kappa(G(K))$ is the $*$-pullback of $\delta_1 \in D_\kappa(\mathrm{Gr}_G)$.

As $\delta_{G(O)} \in D^*(G(K))^\heartsuit$, $\mathrm{CDO}_{G,\kappa} \in \mathsf{Vect}^\heartsuit$.

The object $\delta_{G(O)}$ manifestly upgrades to a factorization algebra in the factorization category with fiber $D^*_\kappa(G(K))$ (defined using the standard unital factorization structure on $G(K)$, c.f. [Ras4] §2). Therefore, by [BD2], $\mathrm{CDO}_{G,\kappa}$ has a natural vertex algebra structure.

Note that $\mathrm{CDO}_{G,\kappa}$ has commuting $\widehat{\mathfrak{g}}_\kappa$ and $\widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}$-actions.

There is a tautological map $\mathrm{Fun}(G(O)) \to \mathrm{CDO}_{G,\kappa} \in \mathsf{Vect}^\heartsuit$, which is compatible with factorization and is a morphism of $\mathfrak{g}[[t]]$-bimodules. Regarding $\mathrm{CDO}_{G,\kappa}$ as a $\widehat{\mathfrak{g}}_\kappa$-module, we obtain an induced map:

$$\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{g}[[t]]}(\mathrm{Fun}(G(O))) \to \mathrm{CDO}_{G,\kappa} \in \mathsf{Vect}^\heartsuit.$$

In [AG2], a natural vertex algebra structure is defined on the left hand side. We claim that this map is an isomorphism of vertex algebras.

Indeed, the construction of the vertex algebra structure from [AG2] exactly uses factorization geometry, showing that the map above is a map of vertex algebras.

This map is an isomorphism because both sides have standard filtrations and the map is an isomorphism at the associated graded level.

Step 2. Next, [AG2] constructs a $\widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}$-action on $\mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{g}[[t]]}(\mathrm{Fun}(G(O)))$. We claim that the above isomorphism is an isomorphism of $\widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}$-modules as well.

We regard both sides as objects of:

$$(\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod})^\heartsuit \subseteq \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod} \simeq \mathsf{End}_{\mathsf{DGCat}_{cont}}(\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}).$$

By construction, $\mathrm{CDO}_{G,\kappa}$ corresponds to the endofunctor $\mathrm{Oblv}\,\mathrm{Av}^{G(O)}_* : \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \to \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}$.

By [Ras6] Theorem 9.16.1, the functor:

$$\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^+ \to \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}$$

corresponding to an object:

$$M \in (\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod})^\heartsuit$$

is the functor:

$$N \mapsto C^{\frac{\infty}{2}}(\mathfrak{g}((t)), \mathfrak{g}[[t]]; M \otimes N).$$

Here the right hand side is the functor of $G(O)$-integrable semi-infinite cohomology, which is defined because $M \otimes N$ is a Kac-Moody module with level $2 \cdot crit$.

By [AG2] Theorem 5.5, we have:

$$C^{\frac{\infty}{2}}(\mathfrak{g}((t)), \mathfrak{g}[[t]]; \mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{g}[[t]]}(\mathrm{Fun}(G(O))) \otimes N) = \mathrm{Oblv}\,\mathrm{Av}^{G(O)}_*(N)$$

as desired. (More precisely, one needs to upgrade [AG2] a bit; this is done in [FG2] Lemma 22.6.2, where we note that the definition of convolution in *loc. cit.* involves tensoring and forming semi-infinite cohomology.)

This gives the desired isomorphism of modules with two commuting Kac-Moody symmetries; this isomorphism is readily seen to coincide with the one constructed earlier.

Step 3. The functor:

$$\Gamma(G(K), -) : D^*_\kappa(G(K)) \to \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod}$$

canonically upgrades to a functor between factorization categories. This induces a canonical morphism of vertex algebras:

$$\mathbb{V}_{\mathfrak{g},\kappa} \otimes \mathbb{V}_{\mathfrak{g},-\kappa+2\cdot crit} \to \mathrm{CDO}_{G,\kappa}\,.$$

This map coincides with the one constructed in [AG2]; indeed, both are given by acting on the unit vector $1 \in \mathrm{Fun}(G(O)) \subseteq \mathrm{CDO}_{G,\kappa}$ using the Kac-Moody action, and we have shown that our Kac-Moody action coincides with the one in [AG2].

Step 4. Now suppose $\mathcal{F} \in D^*_\kappa(G(K))^\heartsuit$. By construction, $\Gamma(G(K),\mathcal{F}) \in \mathsf{Vect}^\heartsuit$ carries an action of $\widehat{\mathfrak{g}}_\kappa$ and of $\mathrm{Fun}(G(K))$ (considered as a topological algebra).

These two actions coincide with the ones considered in [AG2]. Indeed, this is tautological for $\mathrm{Fun}(G(K))$.

For $\widehat{\mathfrak{g}}_\kappa$, we are reduced to showing that for $K_n \subseteq G(O)$ the nth congruence subgroup and $\mathcal{F} \in D_\kappa(G(K)/K_n)^\heartsuit$, the two actions of $\widehat{\mathfrak{g}}_\kappa$ on $H^0(\Gamma(G(K)/K_n,\mathcal{F}))$ coincide.

This is a general assertion about Tate Lie algebras: for H a Tate group indscheme and S a classical indscheme with an action of H, the above logic defines $\Gamma^{\mathsf{IndCoh}}(S,-) : D(S) \to \mathfrak{h}\text{–}\mathsf{mod}$, and we claim that $\mathcal{F} \in D(S)^\heartsuit$, this action of $\mathfrak{h}$ on $H^0\Gamma(S,\mathcal{F})$ coincides with the standard one. This can be checked element by element in $\mathfrak{h}$, so reduces to the case where $\mathfrak{h}$ is 1-dimensional. There it follows by the construction of the comparison results in [GR1].

Step 5. Because $\delta_{G(O)}$ is the *unit* object in the unital factorization category $D^*_\kappa(G(K))$ (see [Ras4] §2), [BD2] Proposition 8.14.1 shows that Γ upgrades to a functor:

$$\Gamma(G(K),-) : D^*_\kappa(G(K)) \to \mathrm{CDO}_{G,\kappa}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}.$$

Here we use the notation from [BD2], and are not distinguishing in the notation between our factorization algebra and its fiber at a point.

Comparing with the construction in [AG2] and applying Step 4, we find that on abelian categories that the functor:

$$D^*_\kappa(G(K))^\heartsuit \to \mathrm{CDO}_{G,\kappa}\text{–}\mathsf{mod}^{\mathrm{fact},\heartsuit}_{un} \simeq \mathrm{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{g}[[t]]}(\mathrm{Fun}(G(O)))\text{–}\mathsf{mod}^{\mathrm{fact},\heartsuit}_{un}$$

coincides with the one constructed in *loc. cit.*

Now the assertion follows from Step 3.

□

Corollary A.10.2. *For every $\mathcal{F} \in D^*_\kappa(G(K))^\heartsuit$ compact, the functor:*

$$\mathcal{F} \star - : \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^+ \to \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^+$$

coincides with the similarly-named functor constructed in [FG2] §*22.*

Proof. By construction of $\Gamma(G(K),-)$, the following diagram commutes:

$$\begin{array}{ccccc} D^*_\kappa(G(K)) \otimes \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} & \xrightarrow{\Gamma(G(K),-)\otimes \mathrm{id}} & \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\cdot crit}\text{–}\mathsf{mod} \otimes \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} & & \\ & \searrow^{-\star-} & & \downarrow{\scriptstyle \mathrm{id}\otimes\langle -,-\rangle} & \\ & & & \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}. & \end{array}$$

By [Ras6] Theorem 9.16.1, this means that for $M \in \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^+$, we have:

$$C^{\frac{\infty}{2}}(\mathfrak{g}((t)),\mathfrak{g}[[t]];\Gamma(G(K),\mathcal{F}) \otimes M) \in \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}.$$

Here we tensor and use the diagonal action mixing the level $-\kappa+2\cdot crit$ action on $\Gamma(G(K),\mathcal{F})$ and the given level κ action on M, and then we form the semi-infinite cochain complex, which retains a level κ action from the corresponding action on $\Gamma(G(K),\mathcal{F})$.

By Proposition A.10.1, the latter amounts to the definition of convolution given in [FG2] §22.5 (see also *loc. cit.* §22.7).

□

One can similarly show that this isomorphism is compatible with the associativity isomorphisms constructed in *loc. cit.* §22.9.

Appendix B. Fully faithfulness

B.1. In this appendix, we present a different proof of Theorem 7.16.1 (fully faithfulness of Γ^{Hecke}) than the one given in [FG2].

B.2. We have the following general criterion.

Proposition B.2.1. *Suppose $\mathcal{C}_i \in G(K)$–mod_{crit} are given for $i=1,2$. Suppose that each $\mathcal{C}_i$ is equipped with a t-structure such that:*

- *The t-structure that is strongly compatible with the $G(K)$-action.*
- *The functor $\mathrm{Av}_!^{\psi}:\mathcal{C}_i^{G(O)}\to \mathsf{Whit}(\mathcal{C}_i)$ is t-exact for $\mathrm{Av}_!^{\psi}$ as in §8.5. Here $\mathsf{Whit}(\mathcal{C}_i)$ is equipped with the t-structure coming from* [Ras5] *Theorem 2.7.1 and §B.7.*
- *The functor $\mathrm{Av}_!^{\psi}:\mathcal{C}_i^{G(O),\heartsuit}\to \mathsf{Whit}(\mathcal{C}_i)^{\heartsuit}$ is conservative.*

Suppose that $F:\mathcal{C}_1\to\mathcal{C}_2\in G(K)$–$\mathsf{mod}_{crit}$ is given. We suppose that the induced functor $\mathcal{C}_1^{G(O)}\to\mathcal{C}_2^{G(O)}$ is t-exact.

Then if the induced functor $\mathsf{Whit}(\mathcal{C}_1)\to\mathsf{Whit}(\mathcal{C}_2)$ is a t-exact equivalence, the functor $\mathcal{C}_1^{G(O),+}\to\mathcal{C}_2^{G(O),+}$ is as well.

Proof. For $i=1,2$, the functor $\mathrm{Av}_!^{\psi}:\mathcal{C}_i^{G(O),+}\to\mathsf{Whit}(\mathcal{C}_i)^+$ is t-exact and conservative by assumption. Moreover, this functor admits the right adjoint $\mathrm{Av}_*^{G(O)}$. By [Ras6] Lemma 3.7.2, the functor $\mathcal{C}_i^{G(O),+}\to\mathsf{Whit}(\mathcal{C}_i)^+$ is comonadic.

Being $G(K)$-equivariant, the functor F intertwines the comonads $\mathrm{Av}_!^{\psi}\,\mathrm{Av}_*^{G(O)}$ on $\mathsf{Whit}(\mathcal{C}_1)$ and $\mathsf{Whit}(\mathcal{C}_2)$. Therefore, as we have assumed F induces an equivalence $\mathsf{Whit}(\mathcal{C}_1)^+\xrightarrow{\simeq}\mathsf{Whit}(\mathcal{C}_2)^+$, we obtain the result.

□

B.3. We now deduce the following result.

Corollary B.3.1. *The functor:*

$$\Gamma^{\mathrm{Hecke},naive}: D_{crit}^{\mathrm{Hecke}_3}(\mathrm{Gr}_G)^{G(O),+}\to \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}^{G(O),+}$$

is a t-exact equivalence.

Proof. We apply Proposition B.2.1 with $\mathcal{C}_1=D_{crit}^{\mathrm{Hecke}_3}(\mathrm{Gr}_G)$, $\mathcal{C}_2=\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}_{reg,naive}$, and $F=\Gamma^{\mathrm{Hecke},naive}$. It remains to check the hypotheses.

Both t-structures are strongly compatible with t-structures by [Ras6].

The functor $\mathrm{Av}_!^{\psi}: D_{crit}(\mathrm{Gr}_G)^{G(O)}\to\mathsf{Whit}(D_{crit}(\mathrm{Gr}_G))$ is t-exact and an equivalence (in particular, conservative) on the hearts of the t-structures by Theorem 8.6.1. We deduce the same for $D_{crit}^{\mathrm{Hecke}_3}(\mathrm{Gr}_G)$ by §7.6-7.7.

The functor $\mathrm{Av}^{\psi}_{!} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{G(O)} \to \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})$ is t-exact by [Ras5] Theorem 7.2.1. We immediately deduce the same for $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}$. The functor:

$$\mathrm{Av}^{\psi}_{!} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{G(O),\heartsuit}_{reg,naive} \to \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive})^{\heartsuit} \overset{Cor.8.4.2}{=} \mathsf{QCoh}(\mathrm{Op}^{reg}_{\check{G}})^{\heartsuit}$$

is an equivalence by [FG1] Theorem 5.3.

Finally, $\Gamma^{\mathrm{Hecke},naive}$ restricted to $D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)^{G(O)}$ is t-exact by Theorem 8.2.1, and similarly for $\mathsf{Whit}(D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G))$ by Theorem 8.3.1.

□

B.4. We now prove Theorem 7.16.1. The reductions follow [FG2]; only the last step differs.

Proof of Theorem 7.16.1.

Step 1. Recall from §7.6 that $D^{\mathrm{Hecke}}_{crit}(\mathrm{Gr}_G)$ is compactly generated by objects of the form $\mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F})$ for $\mathcal{F} \in D_{crit}(\mathrm{Gr}_G)$ compact. Moreover, Γ^{Hecke} preserves compact objects by construction. Therefore, it suffices to show that the map:

$$\underline{\mathrm{Hom}}_{D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)}(\mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F}), \mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{G})) \to \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg}}(\Gamma^{\mathsf{IndCoh}}(\mathcal{F}), \Gamma^{\mathsf{IndCoh}}(\mathcal{G}))$$

is an equivalence for $\mathcal{F}, \mathcal{G} \in D_{crit}(\mathrm{Gr}_G)$ compact.

As $\Gamma^{\mathsf{IndCoh}}(\mathcal{F}) \in \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{c}_{reg} \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{+}_{reg}$, it suffices to show that if we apply ρ, then the induced map:

$$\underline{\mathrm{Hom}}_{D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)}(\mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{F}), \mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathcal{G})) \to \underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}_{reg,naive}}(\Gamma^{\mathsf{IndCoh}}(\mathcal{F}), \Gamma^{\mathsf{IndCoh}}(\mathcal{G}))$$

is an equivalence.

We will show this below with the weaker assumption that $\mathcal{G} \in D_{crit}(\mathrm{Gr}_G)^{+}$.

Step 2. By Lemma 7.12.1 (and its proof), we can rewrite the above terms as:

$$\underline{\mathrm{Hom}}_{D^{\mathrm{Hecke}_{\mathfrak{Z}}}_{crit}(\mathrm{Gr}_G)^{G(O)}}(\mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\delta_1), \mathrm{ind}^{\mathrm{Hecke}_{\mathfrak{Z}}}(\mathrm{inv}\,\mathbb{D}(\mathcal{F}) \star \mathcal{G})) \to$$
$$\underline{\mathrm{Hom}}_{\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^{G(O)}_{reg,naive}}(\Gamma^{\mathsf{IndCoh}}(\delta_1), \Gamma^{\mathsf{IndCoh}}(\mathrm{inv}\,\mathbb{D}(\mathcal{F}) \star \mathcal{G})).$$

Noting that all the terms that appear here are eventually coconnective in the relevant t-structures, the claim follows from Corollary B.3.1.

□

References

[AB] Sergey Arkhipov and Roman Bezrukavnikov. Perverse sheaves on affine flags and Langlands dual group. *Israel J. Math.*, 170:135–183, 2009. With an appendix by Bezrukavrikov and Ivan Mirković.

[ABB$^+$] S. Arkhipov, A. Braverman, R. Bezrukavnikov, D. Gaitsgory, and I. Mirković. Modules over the small quantum group and semi-infinite flag manifold. *Transform. Groups*, 10(3-4):279–362, 2005.

[ABG] Sergey Arkhipov, Roman Bezrukavnikov, and Victor Ginzburg. Quantum groups, the loop Grassmannian, and the Springer resolution. *J. Amer. Math. Soc.*, 17(3):595–678, 2004.

[AF] Tomoyuki Arakawa and Peter Fiebig. On the restricted Verma modules at the critical level. *Trans. Amer. Math. Soc.*, 364(9):4683–4712, 2012.

[AG1] Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Mathematica*, 21(1):1–199, 2015.

[AG2] S. Arkhipov and D. Gaitsgory. Differential operators on the loop group via chiral algebras. *Int. Math. Res. Not.*, (4):165–210, 2002.

[Ari] Dima Arinkin. Irreducible connections admit generic oper structures. *arXiv preprint arXiv:1602.08989*, 2016.

[Bar] Jonathan Barlev. Opers, 2010. Lecture notes, available at `http://people.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/March2-9(Opers).pdf`.

[BB] Sasha Beilinson and Joseph Bernstein. Localisation de $\mathfrak{g}$-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981.

[BBM] Roman Bezrukavnikov, Alexander Braverman, and Ivan Mirkovic. Some results about geometric Whittaker model. *Adv. Math.*, 186(1):143–152, 2004.

[BD1] Sasha Beilinson and Vladimir Drinfeld. Quantization of Hitchin's integrable system and Hecke eigensheaves. Available at `http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf`.

[BD2] Sasha Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[Bei] Alexander Beilinson. Langlands parameters for Heisenberg modules. In *Studies in Lie theory*, pages 51–60. Springer, 2006.

[Ber] Dario Beraldo. Loop group actions on categories and Whittaker invariants. *arXiv preprint arXiv:1310.5127*, 2013.

[Bez] Roman Bezrukavnikov. On two geometric realizations of an affine Hecke algebra. *Publ. Math. Inst. Hautes Études Sci.*, 123:1–67, 2016.

[BZGO] David Ben-Zvi, Sam Gunningham, and Hendrik Orem. Highest Weights for Categorical Representations. *arXiv preprint arXiv:1608.08273*, 2016.

[DG] Vladimir Drinfeld and Dennis Gaitsgory. On some finiteness questions for algebraic stacks. *Geom. Funct. Anal.*, 23(1):149–294, 2013.

[EF] D Eisenbud and E Frenkel. Appendix to *Jet schemes of locally complete intersection canonical singularities*, by M. Mustata. *Inv. Math*, 145:397–424, 2001.

[FF] Boris Feigin and Edward Frenkel. Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras. *International Journal of Modern Physics A*, 7:197–215, 1992.

[FG1] Edward Frenkel and Dennis Gaitsgory. D-modules on the affine Grassmannian and representations of affine Kac-Moody algebras. *Duke Math. J.*, 125(2):279–327, 2004.

[FG2] Edward Frenkel and Dennis Gaitsgory. Local geometric Langlands correspondence and affine Kac-Moody algebras. In *Algebraic geometry and number theory*, pages 69–260. Springer, 2006.

[FG3] Edward Frenkel and Dennis Gaitsgory. Geometric realizations of Wakimoto modules at the critical level. *Duke Mathematical Journal*, 143(1):117–203, 2008.

[FG4] Edward Frenkel and Dennis Gaitsgory. D-modules on the affine flag variety and representations of affine Kac-Moody algebras. *Represent. Theory*, 13:470–608, 2009.

[FG5] Edward Frenkel and Dennis Gaitsgory. Local geometric Langlands correspondence: the spherical case. In *Algebraic analysis and around*, volume 54 of *Adv. Stud. Pure Math.*, pages 167–186. Math. Soc. Japan, Tokyo, 2009.

[FG6] Edward Frenkel and Dennis Gaitsgory. Localization of $\widehat{\mathfrak{g}}$-modules on the affine Grassmannian. *Annals of Mathematics*, pages 1339–1381, 2009.

[FGV] Edward Frenkel, Dennis Gaitsgory, and Kari Vilonen. Whittaker patterns in the geometry of moduli spaces of bundles on curves. *The Annals of Mathematics*, 153(3):699–748, 2001.

[Fre] Edward Frenkel. Wakimoto modules, opers and the center at the critical level. *Advances in Mathematics*, 195(2):297–404, 2005.

[FT] Edward Frenkel and Constantin Teleman. Self-extensions of Verma modules and differential forms on opers. *Compos. Math.*, 142(2):477–500, 2006.

[FZ] Edward Frenkel and Xinwen Zhu. Any flat bundle on a punctured disc has an oper structure. *Math. Res. Lett.*, 17(1):27–37, 2010.

[Gai1] D. Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Invent. Math.*, 144(2):253–280, 2001.

[Gai2] Dennis Gaitsgory. Quantum Langlands correspondence. *arXiv preprint arXiv:1601.05279*, 2007.

[Gai3] Dennis Gaitsgory. Whittaker categories. Available at `http://math.harvard.edu/~gaitsgde/GL/LocalWhit.pdf`. Formerly titled *Les jours et les travaux*, 2008.

[Gai4] Dennis Gaitsgory. Generalities on DG categories. Available at `http://math.harvard.edu/~gaitsgde/GL/textDG.pdf`, 2012.

[Gai5] Dennis Gaitsgory. Kac-Moody representations (Day IV, Talk 3). Available at `https://sites.google.com/site/geometriclanglands2014/notes`. Unpublished conference notes, 2014.

[Gai6] Dennis Gaitsgory. Sheaves of categories and the notion of 1-affineness. In T. Pantev, C. Simpson, B. Toën, M. Vaquié, and G. Vezzosi, editors, *Stacks and Categories in Geometry, Topology, and Algebra:*, Contemporary Mathematics. American Mathematical Society, 2015.

[Gai7] Dennis Gaitsgory. The local and global versions of the Whittaker category. *arXiv preprint arXiv:1811.02468*, 2018.

[Gin] Victor Ginzburg. Nil-Hecke algebras and Whittaker $\mathscr{D}$-modules. In *Lie groups, geometry, and representation theory*, volume 326 of *Progr. Math.*, pages 137–184. Birkhäuser/Springer, Cham, 2018.

[GL] D. Gaitsgory and S. Lysenko. Parameters and duality for the metaplectic geometric Langlands theory. *Selecta Math. (N.S.)*, 24(1):227–301, 2018.

[GR1] Dennis Gaitsgory and Nick Rozenblyum. Crystals and D-modules. *Pure and Applied Mathematics Quarterly*, 10(1):57–154, 2014.

[GR2] Dennis Gaitsgory and Nick Rozenblyum. DG indschemes. *Contemp. Math*, 610:139–251, 2014.

[GR3] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry*. American Mathematical Soc., 2017.

[KT] Masaki Kashiwara and Toshiyuki Tanisaki. Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. *Duke Mathematical Journal*, 77(1):21–62, 1995.

[Lur1] Jacob Lurie. *Higher Topos Theory*, volume 189. Princeton University Press, 2009.

[Lur2] Jacob Lurie. Higher algebra. Available at `http://math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`, 2012.

[MV1] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[MV2] I Mirković and Kari Vilonen. Characteristic varieties of character sheaves. *Inventiones mathematicae*, 93(2):405–418, 1988.

[Ras1] Sam Raskin. A geometric proof of the Feigin-Frenkel theorem. *Representation Theory of the American Mathematical Society*, 16(13):489–512, 2012.

[Ras2] Sam Raskin. D-modules on infinite dimensional varieties. Available at `https://web.ma.utexas.edu/users/sraskin/dmod.pdf`, 2015.

[Ras3] Sam Raskin. A generalization of the b-function lemma. Available at `https://web.ma.utexas.edu/users/sraskin/b-function.pdf`, 2016.

[Ras4] Sam Raskin. Chiral principal series categories II: the factorizable Whittaker category. Available at `https://web.ma.utexas.edu/users/sraskin/cpsii.pdf`, 2016.

[Ras5] Sam Raskin. $\mathcal{W}$-algebras and Whittaker categories. Available at `https://web.ma.utexas.edu/users/sraskin/whit.pdf`, 2016.

[Ras6] Sam Raskin. Homological methods in semi-infinite contexts. Available at `https://web.ma.utexas.edu/users/sraskin/topalg.pdf`, 2019.

The University of Texas at Austin, Department of Mathematics, RLM 8.100, 2515 Speedway Stop C1200, Austin, TX 78712

Email address: `sraskin@math.utexas.edu`